Symmetries in General Relativity

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2. Participants

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3. Brief description

3.1. Spacetime splitting techniques in General Relativity

Spacetime splitting techniques play a central role and have fundamental interest in general relativity in view of extracting from the unified notion of spacetime the separate classical notions of space and time, at the foundation of all of our experience and intuition. Studying all the existing different approaches scattered in the literature has allowed the creation a unique framework encompassing all of them [1] and a more clear geometrical interpretation of the underlying "measurement process" for tensors and tensorial equations. "Gravitoelectromagnetism" is a convenient name for this framework because it helps explain the close relation between gravity and electromagnetism represented by the Coriolis and centrifugal forces on one side and the Lorentz force on the other side.

3.1.1. "1+3" splitting of the spacetime

During the last century, the various relativistic schools: Zelmanov, Landau, Lifshitz and the Russian school, Lichnerowicz in France, the British school, the Italian school (Cattaneo and Ferrarese), scattered Europeans (Ehlers and Trautman, for example) and the Americans (Wheeler, Misner, etc.), developed a number of different independent approaches to spacetime splitting almost without reference to each other.

R. Ruffini [2], a former student of Cattaneo and a collaborator of Wheeler, looking for a better understanding of black holes and their electromagnetic properties, stimulated Jantzen, Carini and Bini to approach the problem and to make an effort to clarify the interrelationships between these various approaches as well as to shed some light on the then confusing works of Abramowicz and others on relativistic centrifugal and Coriolis forces. By putting them all in a common framework and clarifying the related geometrical aspects, some order was brought to the field [1, 3, 4].

3.1.2. Measurement process in general relativity

The investigations on the underlying geometrical structure of any spacetime splitting approach show that it is not relevant to ask which of these various splitting formalisms is the "best" or "correct" one, but to instead ask what exactly each one of them "measures" and which is specially suited to a particular application.

For instance, in certain situations a given approach can be more suitable than another to provide intuition about or simplify the presentation of the invariant spacetime geometry, even if all of them may always be used. These ideas were then used to try to understand better the geometry of circular orbits in stationary spacetimes and their physical properties where the connection between general relativity and its Newtonian progenitor are most natural.

The list of problems addressed and results obtained together can be found in Appendix A.

3.2. Motion of particles and extended bodies in general relativity

The features of test particle motion along a given orbit strongly depend on the nature of the background spacetime as well as on the model adopted for the description of the intrinsic properties of the particle itself (e.g., its charge or spin). As a basic assumption, the dimensions of the test particle are supposed to be very small compared with the characteristic length of the background field in such a way that the background metric is not modified by the presence of the particle (i.e., the back reaction is neglected), and that the gravitational radiation emitted by the particle in its motion is negligible. The particle can in turn be thought as a small extended body described by its own energy-momentum tensor, whose motion in a given background may be studied by treating the body via a multipole expansion. Thus, a single-pole particle is a test particle without any internal structure; a pole-dipole particle instead is a test particle whose internal structure is expressed by its spin, and so on. The equations of motion are then obtained by applying Einsteins field equations together with conservation of the energy-momentum tensor describing the body. For a single-pole particle this leads to a free particle moving along the geodesics associated with the given background geometry. The motion of a pole-dipole particle is instead described by the Mathisson-Papapetrou-Dixon equations which couple background curvature and the spin tensor of the field. The motion of particles with an additional quadrupolar structure has been developed mostly by Dixon; because of its complexity, there are very few applications in the literature. Finally, the discussion of the case in which the test particle also has charge in addition to spin or mass quadrupole moment is due to Dixon and Souriau and this situation has been very poorly studied as well.

A complete list of the original results obtained and a deeper introduction to the models can be found in Appendix B.

3.2.1. Test particles

Since the 1990s we have been investigating the geometrical as well as physical properties of circular orbits in black hole spacetimes, selecting a number of special orbits for various reasons. These were already reviewed in a previous ICRANet report on activities. A recent work has instead been to consider a given gravitational background a (weak) radiation field superposed on it and a test particle interacting with both fields. Interesting effects arise like the Poynting-Robertson effect which have been considered in the framework of the full general relativistic theory for the first time.

Poynting-Robertson effect can be briefly described as follows.

For a small body orbiting a star, the radiation pressure of the light emitted by the star in addition to the direct effect of the outward radial force exerts a drag force on the body's motion which causes it to fall into the star unless the body is so small that the radiation pressure pushes it away from the star. Called the Poynting-Robertson effect, it was first investigated by J.H. Poynting in 1903 using Newtonian gravity and then later calculated in linearized general relativity by H.P. Robertson in 1937. These calculations were revisited by Wyatt and Whipple in 1950 for applications to meteor orbits, making more explicit Robertson's calculations for slowly evolving elliptical orbits and slightly extending them.

The drag force is easily naively understood as an aberration effect: if the body is in a circular orbit, for example, the radiation pressure is radially outward from the star, but in the rest frame of the body, the radiation appears to be coming from a direction slightly towards its own direction of motion, and hence a backwards component of force is exerted on the body which acts as a drag force. If the drag force dominates the outward radial force, the body falls into the star. For the case in which a body is momentarily at rest, a critical luminosity similar to the Eddington limit for a star exists at which the inward gravitational force balances the outward radiation force, a critical value separating radial infall from radial escape. Similarly for a body initially in a circular orbit, there are two kinds of solutions: those in which the body spirals inward or spirals outward, depending on the strength of the radiation pressure.

We have considered [136, 139, 138] this problem in the context of a test body in orbit in a spherically symmetric Schwarzschild spacetime without the restriction of slow motion, and then in the larger context of an axially symmetric Kerr spacetime while developing the equations for a more general stationary axially symmetric spacetime. The finite size of the radiating body is ignored.

We have also developed applications to cylindrically symmetric Weyl class spacetimes (exhibiting a typical naked singularity structure) as well as to the Vaidya radiating spacetime where the photon field is not a test field but the source of the spacetime itself.

3.2.2. Spinning test particles

During the last five years we have investigated the motion of spinning test particles along special orbits in various spacetimes of astrophysical interest: black hole spacetimes as well as more "exotic" background fields representing naked singularities or the superposition of two or more axially symmetric bodies kept apart in a stable configuration by gravitationally inert singular structures.

In particular, we have focused on the so called "clock effect," defined by the difference in the arrival times between two massive particles (as well as photons) orbiting around a gravitating source in opposite directions after one complete loop with respect to a given observer [5, 6, 7].

We have also analyzed the motion of massless spinning test particles, according to an extended version of the Mathisson-Papapetrou model in a general vacuum algebraically special spacetime using the Newman-Penrose formalism in the special case in which the multipole reduction world line is aligned with a principal null direction of the spacetime. Recent applications concern instead the study of the Poynting-Robertson effect for spinning particles.

3.2.3. Particles with both dipolar and quadrupolar structure (Dixon's model)

We have studied the motion of particles with both dipolar and quadrupolar structure in several different gravitational backgrounds (including Schwarzschild, Kerr, weak and strong gravitational waves, etc.) following Dixon's model and within certain restrictions (constant frame components for the spin and the quadrupole tensor, center of mass moving along a circular orbit, etc.).

We have found a number of interesting situations in which deviations from geodesic motion due to the internal structure of the particle can give rise to measurable effects.

3.2.4. Exact solutions representing extended bodies with quadrupolar structure

We have investigated geometrical as well as physical properties of exact solutions of Einstein's field equations representing extended bodies with structure up to the quadrupole mass moment, generalizing so the familiar black hole spacetimes of Schwarzschild and Kerr.

Recent results involve the use of the equivalence principle to compare geodesic motion in these spacetimes with nongeodesic motion of structured particles in Schwarzschild and Kerr spacetimes, allowing an interesting analysis which strongly support Dixon's model.

3.3. Perturbations

A discussion of curvature perturbations of black holes needs many different approaches and mathematical tools. For example, the Newman-Penrose formalism in the tetradic and spinor version, the Cahen-Debever-Defrise self-dual theory, the properties of the spin-weighted angular harmonics, with particular attention to the related differential geometry and the group theory, some tools of complex analysis, etc. Furthermore, even using any of the above mentioned approaches, this remains a difficult problem to handle. It is not by chance, for instance, that the gravitational and electromagnetic perturbations of the Kerr-Newman rotating and charged black hole still represent an open problem in general relativity.

During the last years, however, modern computers and software have reached an exceptional computational level and one may re-visit some of these still open problems, where technical difficulties stopped the research in the past. Details can be found in Appendix C.

3.3.1. Curvature and metric perturbations in algebraically special spacetimes

Most of the work done when studying perturbations in General Relativity concerns curvature perturbations from one side or metric perturbations from the other side. In the first case, one can easily deal with gauge invariant quantities but the problem of finding frame-independent objects arises. Furthermore, the reconstruction of the metric once the curvature perturbations are known is a very difficult task. In the second case, instead, in order to start working with an explicit metric since the beginning, the choice of a gauge condition is necessary. Gauge independent quantities should therefore be determined properly.

There exist very few examples of works considering both cases of curvature and metric perturbations on the same level so that we have been motivated to start working in this direction.

3.3.2. Curvature perturbations in type D spacetimes

In the Kerr spacetime Teukolsky [8] has given a single "master equation" to deal with curvature perturbations by a field of any spin ("spin-weight," properly speaking). Then the problem of extending the results of Teukolsky to other spacetimes is raised.

Actually, a very important result that we have obtained framed the Teukolsky equation in the form of a linearized de-Rham Laplacian equation for the perturbing field [9, 10]. In addition, in all the cases (type D spacetime:

Taub-NUT, type D-Kasner, etc) in which an equation similar to the Teukolsky equation can be written down, one can study the various couplings between the spin of the perturbing field and the background parameters, i.e., spin-rotation, spin-acceleration couplings, etc., which can also be relevant in different contexts and from other points of view. We have obtained important results considering explicit applications to the Taub-NUT, Kerr-Taub-NUT, C-metric, spinning C-metric, Kasner and de Sitter spacetimes. For example in the Taub-NUT spacetime we have shown that the perturbing field acquires an effective spin which is simply related to the gravitomagnetic monopole parameter ℓ of the background [11]; in the C-metric case (uniformly accelerated black hole spacetime) we have been able to introduce a gravitational analog of the Stark effect, etc.

3.3.3. Curvature perturbations due to spinning bodies on a Kerr background

A new scheme for computing dynamical evolutions and gravitational radiations for intermediate-mass-ratio inspirals (IMRIs) based on an effective one-body (EOB) dynamics plus Teukolsky perturbation theory has been recently derived by Wen-Biao Han and collaborators [141]. This research line is very promising in view of many possible applications ranging from the Post-Newtonian physics of binary systems to numerical relativity.

3.3.4. Metric perturbations due to spinning bodies on a Schwarzschild background

The full reconstruction of the perturbed metric by a pinning particle moving on a Schwarzschild background is possibile following the original Zerilli and Ruffini approach, at least perturbatively at various Post-Newtonian orders. This research project is expected to add corrections due to spin to the relativistic two body problem within the effective one-body formalism introduced by Damour.

3.3.5. Gravitational Self-force and Effective-One-Body model: synergies

In recent years, it has been understood that a useful strategy for studying the strong-field aspects of the dynamics of compact binaries is to combine, in a synergetic manner, information gathered from several different approximation methods, namely: the post-Newtonian (PN) formalism, the post-Minkowskian one, the gravitational self-force (SF) formalism, full numerical relativity simulations, and, the effective one-body (EOB) formalism. In particular, the EOB formalism appears to define a useful framework which can combine, in an efficient and accurate manner, information coming from all the other approximation schemes, while also adding genuinely new information coming from EOB theory.

A main motivation to pursue this synergetic effort is certainly the current development of gravitational wave detectors gives, which makes it urgent to improve our theoretical understanding of the general relativistic dynamics of compact binary systems, i.e., systems comprising black holes and/or neutron stars. Recent work has shown that tidal interactions have a significant influence on the late dynamics of coalescing neutron star binaries.

3.4. Cosmology

3.4.1. Mixmaster universe and the spectral index

We have recently revisited the Mixmaster dynamics in a new light, revealing a series of transitions in the complex scale invariant scalar invariant of the Weyl curvature tensor best represented by the speciality index S, which gives a 4-dimensional measure of the evolution of the spacetime independent of all the 3-dimensional gauge-dependent variables except the time used to parametrize it.

Its graph versus time with typical spikes in its real and imaginary parts corresponding to curvature wall collisions serves as a sort of electrocardiogram of the Mixmaster universe, with each such spike pair arising from a single circuit or "pulse" around the origin in the complex plane. These pulses in the speciality index seem to invariantly characterize some of the so called spike solutions in inhomogeneous cosmology and should play an important role in the current investigations of inhomogeneous Mixmaster dynamics.

3.4.2. Wave equations in de Sitter spacetime

Wave propagation on a de Sitter background spacetime can be considered for both the electromagnetic and the gravitational case under the preliminar choice of a gauge conditions. Usually, even in the recent literature, the discussion is limited to special gauge conditions only, like the harmonic one. Recently, some interest has been raised instead for the development of a systematic study in terms of the de Donder gauge since this is close to the Lorentz gauge of the electromagnetic case. Due to the particular symmetries of the de Sitter spacetime we expect to be able to reconstruct the perturbed metric by the wave propagation, at least in the PN scheme.

3.4.3. Fluids obeying non-ideal equations of state

Recently, we have proposed a new class of cosmological models consisting of a FRW universe with a fluid source obeying a non-ideal equation of state, with the suitable property to support a phase transition between low and high density regimes, both characterized by an ideal gas behavior, i.e., pressure and density change in linear proportion to each other. This kind of equation of state was first introduced by Shan and Chen [149] in the context of lattice kinetic theory. We have first investigated the possibility to explain the growth of the dark energy component of the present universe, as a natural consequence of the fluid evolution equations. We have then developed an inflationary model based on a dark energy field described by a Shan-Chen-like equation of state. The results are summarized in Appendix D.

4. Publications (2018 – 2023)

Refereed journals

1. Bini D. , Geralico A.,
   Relative-observer definition of the Simon tensor
   Class. Quantum. Grav. vol. 35, 105003 (2018)
   e-print arXiv:1808.05830 [gr-qc]
   Abstract
   The definition of the Simon tensor, originally given only in Kerr spacetime and associated with the static family of observers, is generalized to any spacetime and to any possible observer family. Such generalization is obtained by a standard 3+1 splitting of the Bianchi identities, which are rewritten here as a 'balance equation' between various spatial fields, associated with the kinematical properties of the observer congruence and representing the spacetime curvature.

2. Bini D., Damour T., A. Geralico
   Spin-orbit precession along eccentric orbits: improving the knowledge of self-force corrections and of their effective-one-body counterparts
   Phys. Rev. D, **97** no.10, 104046 (2018)
   e-print arXiv: 1801.03704 [gr-qc]
   Abstract
   The (first-order) gravitational self-force correction to the spin-orbit precession of a spinning compact body along a slightly eccentric orbit around a Schwarzschild black hole is computed through the ninth post-Newtonian order and to second order in the eccentricity, improving recent results by Kavanagh et al. [Phys. Rev. D 96, 064012 (2017), 10.1103/PhysRevD.96.064012]. We show that our higher-accurate theoretical estimates of the spin precession exhibits an improved agreement with corresponding numerical self-force data. We convert our new theoretical results into its corresponding effective-one-body counterpart, thereby determining several new post-Newtonian terms in the gyrogravitomagnetic ratio gS*.

3. Bini D., Chicone C., Mashhoon B.,
   Twisted Gravitational Waves
   Phys. Rev. D, **97**, no. 6, 064022 (2018).
   e-Print arXiv:1801.06003 [gr-qc]
   Abstract
   In general relativity (GR), linearized gravitational waves propagating in empty Minkowski spacetime along a fixed spatial direction have the property that the wave front is the Euclidean plane. Beyond the linear regime, exact plane waves in GR have been studied theoretically for a long time and many exact vacuum solutions of the gravitational field equations are known that represent plane gravitational waves. These have parallel rays and uniform wave fronts. It turns out, however, that GR also admits exact solutions representing gravitational waves propagating along a fixed direction that are nonplanar. The wave front is then nonuniform and the bundle of rays is twisted. We find a class of solutions representing nonplanar unidirectional gravitational waves and study some of the properties of these twisted waves.

4. Bini D., Damour T., Geralico A., Kavanagh C.
   Detweiler's redshift invariant for spinning particles along circular orbits on a Schwarzschild background
   Phys. Rev. D, **97** no.10, 104022 (2018).
   e-print arXiv:1801.09616 [gr-qc]
   Abstract
   We study the metric perturbations induced by a classical spinning particle moving along a circular orbit on a Schwarzschild background, limiting the analysis to effects which are first order in spin. The particle is assumed to move on the equatorial plane and has its spin aligned with the z axis. The metric perturbations are obtained by using two different approaches, i.e., by working in two different gauges: the Regge-Wheeler gauge (using the Regge-Wheeler-Zerilli formalism) and a radiation gauge (using the Teukolsky formalism). We then compute the linear-in-spin contribution to the first-order self-force contribution to Detweiler's redshift invariant up to the 8.5 post-Newtonian order. We check that our result is the same in both gauges, as appropriate for a gauge-invariant quantity, and agrees with the currently known 3.5 post-Newtonian results.

5. Bini D., Geralico A.
   On the energy content of electromagnetic and gravitational plane waves through super-energy tensors
   Class. Quantum Grav., **35** no.16, 165006 (2018).
   e-print arXiv:1809.05305
   Abstract
   The energy content of (exact) electromagnetic and gravitational plane waves is studied in terms of super-energy tensors (the Bel, Bel-Robinson and the -less familiar- Chevreton tensors) and natural observers. Starting from the case of single waves, the more interesting situation of colliding waves is then discussed, where the nonlinearities of the Einsteins theory play an important role. The causality properties of the super-momentum four vectors associated with each of these tensors are also investigated when passing from the single-wave regions to the interaction region.

6. Bini D., Chicone C., Mashhoon B., Rosquist K.
   Spinning particles in Twisted Gravitational Wave Spacetimes
   Phys. Rev. D, **98**, 024043 (2018).
   e-print arXiv:1805.07080
   Abstract
   Twisted gravitational waves (TGWs) are nonplanar waves with twisted rays that move along a fixed direction in space. We study further the physical characteristics of a recent class of Ricci-flat solutions of general relativity representing TGWs with wave fronts that have negative Gaussian curvature. In particular, we investigate the influence of TGWs on the polarization of test electromagnetic waves and on the motion of classical spinning test particles in such radiation fields. To distinguish the polarization effects of twisted waves from plane waves, we examine the theoretical possibility of existence of spin-twist coupling and show that this interaction is generally consistent with our results.

7. Bini D., Esposito G.,
   On the local isometric embedding of trapped surfaces into three-dimensional Riemannian manifolds
   Class. quantum Grav., **35** no.19, 195003 (2018).
   e-print arXiv:1805.08723
   Abstract

We study trapped surfaces from the point of view of local isometric embedding into 3D Riemannian manifolds. When a two-surface is embedded into 3D Euclidean space, the problem of finding all surfaces applicable upon it gives rise to a non-linear partial differential equation of the MongeAmp type, first discovered by Darboux, and later reformulated by Weingarten. Even today, this problem remains very difficult, despite some remarkable results. We find an original way of generalizing the Darboux technique, which leads to a coupled set of six non-linear partial differential equations. For the 3-manifolds occurring in Friedmann(Lemaitre)RobertsonWalker cosmologies, we show that the local isometric embedding of trapped surfaces into them can be proved by solving just one non-linear equation. Such an equation is here solved for the three kinds of Friedmann model associated with positive, zero, negative curvature of spatial sections, respectively.

8. Bini D., Damour T.,
Gravitational spin-orbit coupling in binary systems at the second post-Minkowskian approximation
Phys. Rev. D, **98**, 044036 (2018).
e-print arXiv:1805.10809
Abstract
We compute the rotations, during a scattering encounter, of the spins of two gravitationally interacting particles at second order in the gravitational constant (second post-Minkowskian order). Following a strategy introduced by us [D. Bini and T. Damour, Phys. Rev. D 96, 104038 (2017), 10.1103/PhysRevD.96.104038], we transcribe our result into a correspondingly improved knowledge of the spin-orbit sector of the effective one-body (EOB) Hamiltonian description of the dynamics of spinning binary systems. We indicate ways of resumming our results for defining improved versions of spinning EOB codes which might help in providing a better analytical description of the dynamics of coalescing spinning binary black holes.

9. Bini D., Geralico A.,
High-energy hyperbolic scattering by neutron stars and black holes
Phys. Rev. D, **98**, 024049 (2018).
e-print arXiv:1806.02085
Abstract

We investigate the hyperbolic scattering of test particles, spinning test particles, and particles with spin-induced quadrupolar structure by a Kerr black hole in the ultrarelativistic regime. We also study how the features of the scattering process modify if the source of the background gravitational field is endowed with a nonzero mass quadrupole moment as described by the (approximate) Hartle-Thorne solution. We compute the scattering angle either in closed analytical form, when possible, or as a power series of the (dimensionless) inverse impact parameter. It is a function of the parameters characterizing the source (intrinsic angular momentum and mass quadrupole moment) as well as the scattered body (spin and polarizability constant). Measuring the scattering angle thus provides useful information to determine the nature of the two components of the binary system undergoing high-energy scattering processes.

10. Bini D., Geralico A.,
Gravitational self-force corrections to tidal invariants for spinning particles on circular orbits in a Schwarzschild spacetime
Phys. Rev. D, **98**, 084021 (2018).
e-print arXiv:1806.03495
Abstract
We compute gravitational self-force (conservative) corrections to tidal invariants for spinning particles moving along circular orbits in a Schwarzschild spacetime. In particular, we consider the square and the cube of the gravitoelectric quadrupolar tidal tensor and the square of the gravitomagnetic quadrupolar tidal tensor. Our results are accurate to first order in spin and through the 9.5 post-Newtonian order. We also compute the associated electric-type and magnetic-type eigenvalues.

11. Bini D., Geralico A.,
Gravitational self-force corrections to tidal invariants for particles on eccentric orbits in a Schwarzschild spacetime
Phys. Rev. D, **98**, 064026 (2018).
e-print arXiv:1806.06635
Abstract
We study tidal effects induced by a particle moving along a slightly eccentric equatorial orbit in a Schwarzschild spacetime within the gravitational self-force framework. We compute the first-order (conserva-

tive) corrections in the mass ratio to the eigenvalues of the electric-type and magnetic-type tidal tensors up to the second order in eccentricity and through the 9.5 post-Newtonian order. Previous results on circular orbits are thus generalized and recovered in a proper limit.

12. Bini D., Geralico A.,
Gravitational self-force corrections to tidal invariants for particles on circular orbits in a Kerr spacetime
Phys. Rev. D, **98**, 064040 (2018).
e-print arXiv:1806.08765
Abstract
We generalize to the Kerr spacetime existing self-force results on tidal invariants for particles moving along circular orbits around a Schwarzschild black hole. We obtain linear-in-mass-ratio (conservative) corrections to the quadratic and cubic electric-type invariants and the quadratic magnetic-type invariant in series of the rotation parameter up to the fourth order and through the ninth and eighth post-Newtonian orders, respectively. We then analytically compute the associated eigenvalues of both electric and magnetic tidal tensors.

13. Rosquist K., Bini D., Mashhoon B.,
Twisted Gravitational Waves of Petrov type D
Phys. Rev. D **98**, 064039 (2018).
e-print arXiv:1807.09214
Abstract
Twisted gravitational waves (TGWs) are nonplanar unidirectional Ricci-flat solutions of general relativity. Thus far only TGWs of Petrov type II are implicitly known that depend on a solution of a partial differential equation and have wave fronts with negative Gaussian curvature. A special Petrov type D class of such solutions that depends on an arbitrary function is explicitly studied in this paper and its Killing vectors are worked out. Moreover, we concentrate on two solutions of this class, namely, the Harrison solution and a simpler solution we call the w -metric and determine their Penrose plane-wave limits. The corresponding transition from a nonplanar TGW to a plane gravitational wave is elucidated.

14. Bini D., Geralico A., Jantzen R.T.
Black hole geodesic parallel transport and the Marck recipe for isolating cumu-

lative precession effects
Phys. Rev. D, submitted (2018).
e-print arXiv:1807.10085
Abstract
The Wigner rotations arising from the combination of boosts along two different directions are rederived from a relative boost point of view and applied to gyroscope spin precession along timelike geodesics in a Kerr spacetime, clarifying the geometrical properties of Marck's recipe for describing parallel transport along such world lines expressed in terms of the constants of the motion. His final angular velocity isolates the cumulative spin precession angular velocity independent of the spacetime tilting required to keep the spin 4-vector orthogonal to the gyro 4-velocity. As an explicit example the cumulative precession effects are computed for a test gyroscope moving along both bound and unbound equatorial plane geodesic orbits.

15. Bini D., Damour T., Geralico A., Kavanagh C., van de Meent M.,
Gravitational self-force corrections to gyroscope precession along circular orbits in the Kerr spacetime
Phys. Rev. D, **98**, 104062 (2018).
e-print arXiv: 1809.02516
Abstract
We generalize to Kerr spacetime previous gravitational self-force results on gyroscope precession along circular orbits in the Schwarzschild spacetime. In particular we present high order post-Newtonian expansions for the gauge invariant precession function along circular geodesics valid for an arbitrary Kerr spin parameter and show agreement between these results and those derived from the full post-Newtonian conservative dynamics. Finally we present strong field numerical data for a range of the Kerr spin parameter, showing agreement with the gravitational self-force-post-Newtonian results, and the expected light-ring divergent behavior. These results provide useful testing benchmarks for self-force calculations in Kerr spacetime, and provide an avenue for translating self-force data into the spin-spin coupling in effective-one-body models.

16. Bini D., Geralico A., Jantzen R.T.
Black hole geodesic parallel transport and the Marck reduction procedure

Phys. Rev. D, **99** , 064041 (2019).
e-print arXiv:1807.10085
DOI:10.1103/PhysRevD.99.064041 *Abstract*
The Wigner rotations arising from the combination of boosts along two different directions are rederived from a relative boost point of view and applied to study gyroscope spin precession along timelike geodesics in a Kerr spacetime. First this helps to clarify the geometrical properties of Marcks recipe for reducing the equations of parallel transport along such world lines expressed in terms of the constants of the motion to a single differential equation for the essential planar rotation. Second this shows how to bypass Marcks reduction procedure by direct boosting of orthonormal frames associated with natural observer families. Wigner rotations mediate the relationship between these two descriptions for reaching the same parallel transported frame along a geodesic. The comparison is particularly straightforward in the case of equatorial plane motion of a test gyroscope, where Marcks scalar angular velocity captures the essential cumulative spin precession relative to the spherical frame locked to spatial infinity. These cumulative precession effects are computed explicitly for both bound and unbound equatorial plane geodesic orbits. The latter case is of special interest in view of recent applications to the dynamics of a two-body system with spin. Our results are consistent with the point-particle limit of such two-body results and also pave the way for similar computations in the context of gravitational self-force.

17. Bini D., Geralico A., Plastino W.,
Cylindrical gravitational waves: C-energy, super-energy and associated dynamical effects
Class. Quantum Grav., **36**, no. 9, 095012 (2019).
e-print arXiv:1812.07938 [gr-qc]
DOI: 10.1088/1361-6382/ab10ec
Abstract
The energy content of cylindrical gravitational wave spacetimes is analyzed by considering two local descriptions of energy associated with the gravitational field, namely those based on the C-energy and the Bel-Robinson super-energy tensor. A PoyntingRobertson-like effect on the motion of massive test particles, beyond the geodesic approximation, is discussed, allowing them to interact with the background field through

an external force which accounts for the exchange of energy and momentum between particles and waves. In addition, the relative strains exerted on a bunch of particles displaced orthogonally to the direction of propagation of the wave are examined, providing invariant information on spacetime curvature effects caused by the passage of the wave. The explicit examples of monochromatic waves with either a single or two polarization states as well as pulses of gravitational radiation are discussed.

18. Nagar A., Messina F., Rettegno P., Bini D., Damour T., Geralico A., Akcay S., Bernuzzi S.,
Nonlinear-in-spin effects in effective-one-body waveform models of spin-aligned, inspiralling, neutron star binaries
Phys. Rev. D **99**, no. 4, 044007 (2019)
DOI:10.1103/PhysRevD.99.044007
[arXiv:1812.07923 [gr-qc]].
Abstract
Spinning neutron stars acquire a quadrupole moment due to their own rotation. This quadratic-in-spin, self-spin effect depends on the equation of state (EOS) and affects the orbital motion and rate of inspiral of neutron star binaries. Building upon circularized post-Newtonian results, we incorporate the EOS-dependent self-spin (or monopole-quadrupole) terms in the spin-aligned effective-one-body (EOB) waveform model TEOBResumS at next-to-next-to-leading (NNLO) order, together with other (bilinear, cubic and quartic) nonlinear-in-spin effects (at leading order, LO). We point out that the structure of the Hamiltonian of TEOBResumS is such that it already incorporates, in the binary black hole case, the recently computed [Levi and Steinhoff, arXiv:1607.04252] quartic-in-spin LO term. Using the gauge-invariant characterization of the phasing provided by the function $Q_\omega = \omega^2/\dot{\omega}$ of $\omega = 2\pi f$, where f is the gravitational wave frequency, we study the EOS dependence of the self-spin effects and show that: (i) the next-to-leading order (NLO) and NNLO monopole-quadrupole corrections yield increasingly phase-accelerating effects compared to the corresponding LO contribution; (ii) the standard TaylorF2 post-Newtonian (PN) treatment of NLO (3PN) EOS-dependent self-spin effects makes their action stronger than the corresponding EOB description; (iii) the addition to the standard 3PN TaylorF2 post-Newtonian phasing description of self-spin tail effects at

LO allows one to reconcile the self-spin part of the TaylorF2 PN phasing with the corresponding TEOBResumS one up to dimensionless frequencies $M \approx 0.040.06$. Such a tail-augmented TaylorF2 approximant then yields an analytically simplified, EOB-faithful, representation of the EOS-dependent self-spin phasing that can be useful to improve current PN-based (or phenomenological) waveform models for inspiralling neutron star binaries. Finally, by generating the inspiral dynamics using the post-adiabatic approximation, incorporated in a new implementation of TEOBResumS, one finds that the computational time needed to obtain a typical waveform (including all multipoles up to $l = 8$) from 10 Hz is of the order of 0.4 sec.

19. Bini D., Geralico A., Jantzen R.T., Plastino W.,
Gödel spacetime: elliptic-like geodesics and gyroscope precession
Phys. Rev. D, **100**, 084051, (2019)
DOI:10.1103/PhysRevD.100.084051
e-print arXiv:1905.04917 [gr-qc]
Abstract
Using standard cylindrical-like coordinates naturally adapted to the cylindrical symmetry of the Gdel spacetime, we study ellipticlike geodesic motion on hyperplanes orthogonal to the symmetry axis through an eccentricity-semi-latus rectum parametrization which is familiar from the Newtonian description of a two-body system. We compute several quantities which summarize the main features of the motion, namely the coordinate time and proper time periods of the radial motion, the frequency of the azimuthal motion, the full variation of the azimuthal angle over a period, and so on. Exact as well as approximate (i.e., Taylor-expanded in the limit of small eccentricity) analytic expressions of all these quantities are obtained. Finally, we consider their application to the gyroscope precession frequency along these orbits, generalizing the existing results for the circular case.

20. Bini D., Geralico A., Gionti G., Plastino W., Velandia N.
Scattering of uncharged particles in the field of two extremely charged black holes
Gen. Rel. Gravitation, vol. 51, 153, (2019)
e-print arXiv:1906.01991 [gr-qc]
DOI:doi.org/10.1007/s10714-019-2642-y *Abstract*

We investigate the motion of uncharged particles scattered by a binary system consisting of extremely charged black holes in equilibrium as described by the MajumdarPapapetrou solution. We focus on unbound orbits confined to the plane containing both black holes. We consider the two complementary situations of particles approaching the system along a direction parallel to the axis where the black holes are displaced and orthogonal to it. We numerically compute the scattering angle as a function of the particles conserved energy parameter, which provides a gauge-invariant information of the scattering process. We also study the precession of a test gyroscope along such orbits and evaluate the accumulated precession angle after a full scattering, which is another gauge-invariant quantity.

21. Bini D. and Geralico A.
New gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole: redshift invariant
Phys. Rev. D, 100, 104002, (2019)
DOI:10.1103/PhysRevD.100.104002
e-print arXiv:1907.11080 [gr-qc]
Abstract
The Detweiler-Barack-Sago redshift function for particles moving along slightly eccentric equatorial orbits around a Kerr black hole is currently known up to the second order in eccentricity, second order in spin parameter, and the 8.5 post-Newtonian order. We improve the analytical computation of such a gauge-invariant quantity by including terms up to the fourth order in eccentricity at the same post-Newtonian approximation level. We also check that our results agrees with the corresponding post-Newtonian expectation of the same quantity, calculated by using the currently known Hamiltonian for spinning binaries.

22. Bini D. and Geralico A.
New gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole: gyroscope precession
Phys. Rev. D, 100, 104003, (2019)
DOI:10.1103/PhysRevD.100.104003
e-print arXiv:1907.11082 [gr-qc]
Abstract
We analytically compute the gravitational self-force correction to the gy-

roscope precession along slightly eccentric equatorial orbits in the Kerr spacetime, generalizing previous results for the Schwarzschild spacetime. Our results are accurate through the 9.5 post-Newtonian order and to second order in both eccentricity and rotation parameter. We also provide a post-Newtonian check of our results based on the currently known Hamiltonian for spinning binaries.

23. Bini D. and Geralico A.
Analytical determination of the periastron advance in spinning binaries from self-force computations
Phys. Rev. D, to appear, (2019)
e-print arXiv:1907.11083 [gr-qc]
Abstract
We present the first analytical computation of the (conservative) gravitational self-force correction to the periastron advance around a spinning black hole. Our result is accurate to the second order in the rotational parameter and through the 9.5 post-Newtonian level. It has been obtained as the circular limit of the correction to the gyroscope precession invariant along slightly eccentric equatorial orbits in the Kerr spacetime. The latter result is also new and we anticipate here the first few terms only of the corresponding post-Newtonian expansion.

24. Bini D., Damour T. and Geralico A.
Novel approach to binary dynamics: application to the fifth post-Newtonian level
Phys. Rev. Lett., **123**, 231104, (2019)
DOI:10.1103/PhysRevLett.123.231104
e-print arXiv:1909.02375 [gr-qc]
Abstract
We introduce a new methodology for deriving the conservative dynamics of gravitationally interacting binary systems. Our approach combines, in a novel way, several theoretical formalisms: post-Newtonian, post-Minkowskian, multipolar-post-Minkowskian, gravitational self-force, and effective one body. We apply our method to the derivation of the fifth post-Newtonian dynamics. By restricting our results to the third post-Minkowskian level, we give the first independent confirmation of the recent result of Bern et al. [Phys. Rev. Lett. 122, 201603 (2019)PRLTAO0031-900710.1103/PhysRevLett.122.201603]. We also offer checks for future

fourth post-Minkowskian calculations. Our technique can, in principle, be extended to higher orders of perturbation theory.

25. Bini D. , Damour T. and Geralico A.
Scattering of tidally interacting bodies in post-Minkowskian gravity
Phys. Rev. D **101**, no. 4, 044039 (2020)
DOI:10.1103/PhysRevD.101.044039
e-Print: arXiv:2001.00352 [gr-qc] .
Abstract
The post-Minkowskian approach to gravitationally interacting binary systems (*i.e.*, perturbation theory in G, without assuming small velocities) is extended to the computation of the dynamical effects induced by the tidal deformations of two extended bodies, such as neutron stars. Our derivation applies general properties of perturbed actions to the effective field theory description of tidally interacting bodies. We compute several tidal invariants (notably the integrated quadrupolar and octupolar actions) at the first post-Minkowskian order. The corresponding contributions to the scattering angle are derived.

26. Bini D. , Geralico A. Jantzen R. T., Plastino W.,
Gödel spacetime, planar geodesics and the Möbius map
Gen Relativ Gravit vol. 52, 73 (2020)
doi: doi.org/10.1007/s10714-020-02731-w e-print: arXiv:2002.11432
Abstract
Timelike geodesics on a hyperplane orthogonal to the symmetry axis of the Gdel spacetime appear to be elliptic-like if standard coordinates naturally adapted to the cylindrical symmetry are used. The orbit can then be suitably described through an eccentricity-semi-latus rectum parametrization, familiar from the Newtonian dynamics of a two-body system. However, changing coordinates such planar geodesics all become explicitly circular, as exhibited by Kundt's form of the Gdel metric. We derive here a one-to-one correspondence between the constants of the motion along these geodesics as well as between the parameter spaces of elliptic-like versus circular geodesics. We also show how to connect the two equivalent descriptions of particle motion by introducing a pair of complex coordinates in the 2-planes orthogonal to the symmetry axis, which brings the metric into a form which is invariant under Mbius transformations preserving the symmetries of the orbit,

i.e., taking circles to circles.

27. Rettegno P., Martinetti F., Nagar A., Bini D. , Riemenschneider G., and Damour T.
Comparing effective One Body Hamiltonians for spin-aligned coalescing binaries
Physical Review D ,Vol. 101, No. 10 (2020)
DOI: 10.1103/PhysRevD.101.104027
e-Print: arXiv:1911.10818 [gr-qc]
Abstract
TEOBResumS and SEOBNRv4 are the two existing semi-analytical gravitational waveform models for spin-aligned coalescing black hole binaries based on the effective-one-body approach.They are informed by numerical relativity simulations and provide the relative dynamics and waveforms from early inspiral to plunge, merger and ringdown The central building block of each model is the EOB resummed Hamiltonian.The two models implement different Hamiltonians that are both deformations of the Hamiltonian of a test spinning black hole moving around a Kerr black hole.Here we analytically compare, element by element, the two Hamiltonians. In particular: we illustrate that one can introduce a centrifugal radius SEOBNRv4, so to rewrite the Hamiltonian in a more compact form that is analogous to the one of TEOBResumS.The latter centrifugal radius cannot, however, be identified with the one used in TEOBResumS because the two models differ in their ways of incorporating spin effects in their respective deformations of the background Kerr Hamiltonian. We performed extensive comparisons between the energetics corresponding to the two Hamiltonians using gauge-invariant quantities. Finally, as an exploratory investigation, we apply the post-adiabatic approximation to the newly rewritten SEOBNRv4 Hamiltonian, illustrating that it is possible to generate long-inspiral waveforms with negligible computational cost.

28. Bini D. and Esposito G.
New solutions of the Ermakov-Pinney equation in curved spacetime
General Relativity and Gravitation, Vol. 52, No. 60, 2020
doi: 10.1007/s10714-020-02713-y
e-Print: arXiv:1912.01869 [gr-qc]
Abstract

An Ermakov-Pinney-like equation associated with the scalar wave equation in curved space-time is here studied. The example of Schwarzschild space-time considered in the present work shows that this equation can be viewed more as a model equation, with interesting applications in black hole physics. Other applications studied involve cosmological space-times (de Sitter) and pulse of plane gravitational waves. In all these cases the evolution of the Ermakov-Pinney field seems to be consistent with a rapid blow-up, unlike the Schwarzschild case where spatially damped oscillations are allowed. Eventually, the phase function is also evaluated in many of the above space-time models.

29. Bini D. , Geralico A. and Steinhoff J.,
Detweiler's redshift invariant for extended bodies orbiting a Schwarzschild black hole
Phys. Rev. D, vol. 102, 024091, (2020)
doi: 10.1103/PhysRevD.102.024091
e-print: arXiv:2003.12887 [gr-qc]
Abstract
We compute the first-order self-force contribution to Detweiler's redshift invariant for extended bodies endowed with both dipolar and quadrupolar structure (with spin-induced quadrupole moment) moving along circular orbits on a Schwarzschild background. Our analysis includes effects which are second order in spin, generalizing previous results for purely spinning particles. The perturbing body is assumed to move on the equatorial plane, the associated spin vector being orthogonal to it. The metric perturbations are obtained by using a standard gravitational self-force approach in a radiation gauge. Our results are accurate through the 6.5 post-Newtonian order, and are shown to reproduce the corresponding post-Newtonian expression for the same quantity computed by using the available Hamiltonian from an effective field theory approach for the dynamics of spinning binaries.

30. Bini D. , Damour T. and Geralico A.
Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders
Phys. Rev. D, vol. 102, 024062 (2020)
e-print: arXiv:2003.11891 [gr-qc]
DOI: 10.1103/PhysRevD.102.024062
Appeared as Editor Suggestion paper

Abstract
Using the new methodology introduced in a recent Letter [Phys. Rev. Lett. **123**, 231104 (2019)], we present the details of the computation of the conservative dynamics of gravitationally interacting binary systems at the fifth post-Newtonian (5PN) level, together with its extension at the fifth-and-a-half post-Newtonian (5.5PN) level. We present also the sixth post-Newtonian (6PN) contribution to the third-post-Minkowskian (3PM) dynamics. Our strategy combines several theoretical formalisms: post-Newtonian, post-Minkowskian, multipolar-post-Minkowskian, gravitational self-force, effective one-body, and Delaunay averaging. We determine the full functional structure of the 5PN Hamiltonian (which involves 95 non-zero numerical coefficients), except for two undetermined coefficients proportional to the cube of the symmetric mass ratio, and to the fifth and sixth power of the gravitational constant, G. We present not only the 5PN-accurate, 3PM contribution to the scattering angle, but also its 6PN-accurate generalization. Both results agree with the corresponding truncations of the recent 3PM result of Bern et al. [Phys. Rev. Lett. **122**, 201603 (2019)]. We also compute the 5PN-accurate, fourth-post-Minkowskian (4PM) contribution to the scattering angle, including its nonlocal contribution, thereby offering checks for future 4PM calculations. We point out a remarkable hidden simplicity of the gauge-invariant functional relation between the radial action and the effective-one-body energy and angular momentum.

31. Bini D. , Damour T. and Geralico A.
Sixth Post-Newtonian local-in-time dynamics of binary systems
Phys. Rev. D, vol 102, 024061 (2020)
e-print: arXiv:2004.05407 [gr-qc]
DOI: 10.1103/PhysRevD.102.024061
Appeared as Editor Suggestion paper
Abstract
Using a recently introduced method [Phys. Rev. Lett. **123**, 231104 (2019)], which splits the conservative dynamics of gravitationally interacting binary systems into a non-local-in-time part and a local-in-time one, we compute the local part of the dynamics at the sixth post-Newtonian (6PN) accuracy. Our strategy combines several theoretical formalisms: post-Newtonian, post-Minkowskian, multipolar-post-Minkowskian, effective-field-theory, gravitational self-force, effective one-body, and Delaunay

averaging. The full functional structure of the local 6PN Hamiltonian (which involves 151 numerical coefficients) is derived, but contains four undetermined numerical coefficients. Our 6PN-accurate results are complete at orders G3 and G4, and the derived O(G3) scattering angle agrees, within our 6PN accuracy, with the computation of [Phys. Rev. Lett. **122**, no. 20, 201603 (2019)]. All our results are expressed in several different gauge-invariant ways. We highlight, and make a crucial use of, several aspects of the hidden simplicity of the mass-ratio dependence of the two-body dynamics.

32. Bini D. , Damour T. and Geralico A.
    Sixth post-Newtonian nonlocal-in-time dynamics of binary systems
    Phys. Rev. D, **102**, no.8, 084047 (2020)
    e-print: arXiv:2007.11239 [gr-qc, hep-th]
    DOI: 10.1103/PhysRevD.102.084047
    Abstract
    We complete our previous derivation, at the sixth post-Newtonian (6PN) accuracy, of the local-in-time dynamics of a gravitationally interacting two-body system by giving two gauge-invariant characterizations of its complementary nonlocal-in-time dynamics. On the one hand, we compute the nonlocal part of the scattering angle for hyberboliclike motions; and, on the other hand, we compute the nonlocal part of the averaged (Delaunay) Hamiltonian for ellipticlike motions. The former is computed as a large-angular-momentum expansion (given here to next-to-next-to-leading order), while the latter is given as a small-eccentricity expansion (given here to the tenth order). We note the appearance of $\zeta(3)$ in the nonlocal part of the scattering angle. The averaged Hamiltonian for ellipticlike motions then yields two more gauge-invariant observables: the energy and the periastron precession as functions of orbital frequencies. We point out the existence of a hidden simplicity in the mass-ratio dependence of the gravitational-wave energy loss of a two-body system.

33. Salucci P, et al., *Einstein, Planck and Vera Rubin: Relevant Encounters Between the Cosmological and the Quantum Worlds.*
    Frontiers in Astronomy and Space Sciences, 8:603190, 2021
    doi: 10.3389/fphy.2020.603190
    White Paper of the INFN collaboration QGSKY

e-print: arXiv:2011.09278

Abstract
In Cosmology and in Fundamental Physics there is a crucial question like: where the elusive substance that we call Dark Matter is hidden in the Universe and what is it made of?, that, even after 40 years from the Vera Rubin seminal discovery does not have a proper answer. Actually, the more we have investigated, the more this issue has become strongly entangled with aspects that go beyond the established Quantum Physics, the Standard Model of Elementary particles and the General Relativity and related to processes like the Inflation, the accelerated expansion of the Universe and High Energy Phenomena around compact objects. Even Quantum Gravity and very exotic DM particle candidates may play a role in framing the Dark Matter mystery that seems to be accomplice of new unknown Physics. Observations and experiments have clearly indicated that the above phenomenon cannot be considered as already theoretically framed, as hoped for decades. The Special Topic to which this review belongs wants to penetrate this newly realized mystery from different angles, including that of a contamination of different fields of Physics apparently unrelated. We show with the works of this ST that this contamination is able to guide us into the required new Physics. This review wants to provide a good number of these "paths or contamination" beyond/among the three worlds above; in most of the cases, the results presented here open a direct link with the multi-scale dark matter phenomenon, enlightening some of its important aspects. Also in the remaining cases, possible interesting contacts emerges.

34. Bini D. , Damour T., Geralico A., Laporta S. and Mastrolia P.
Gravitational scattering at the seventh order in G: nonlocal contribution at the sixth post-Newtonian accuracy,
Phys. Rev. D, Vol. 103, No. 4, 044038 (2021)
e-print: [arXiv:2012.12918 [gr-qc]].
DOI: 10.1103/PhysRevD.103.044038
Abstract

A recently introduced approach to the classical gravitational dynamics

of binary systems involves intricate integrals (linked to a combination of nonlocal-in-time interactions with iterated $1/r$-potential scattering) which have so far resisted attempts at their analytical evaluation. By using computing techniques developed for the evaluation of multiloop Feynman integrals (notably harmonic polylogarithms and Mellin transform) we show how to analytically compute all the integrals entering the nonlocal-in-time contribution to the classical scattering angle at the sixth post-Newtonian accuracy, and at the seventh order in Newtons constant, G (corresponding to six-loop graphs in the diagrammatic representation of the classical scattering angle).

35. Bini D. , Esposito G.,
Investigating new forms of gravity-matter couplings in the gravitational field equations
Phys. Rev. D, vol. 103, 064030 (2021)
e-print: [arXiv:2101.09771 [gr-qc]]
DOI: 10.1103/PhysRevD.103.064030
Abstract

This paper proposes a toy model where, in the Einstein equations, the right-hand side is modified by the addition of a term proportional to the symmetrized partial contraction of the Ricci tensor with the energy-momentum tensor, while the left-hand side remains equal to the Einstein tensor. Bearing in mind the existence of a natural length scale given by the Planck length, dimensional analysis shows that such a term yields a correction linear in $\hbar$ to the classical term that is instead just proportional to the energy-momentum tensor. One then obtains an effective energy-momentum tensor that consists of three contributions: pure energy part, mechanical stress, and thermal part. The pure energy part has the appropriate property for dealing with the dark sector of modern relativistic cosmology. Such a theory coincides with general relativity in vacuum, and the resulting field equations are here solved for a Dunn and Tupper metric, for departures from an interior Schwarzschild solution as well as for a Friedmann-Lemaitre-Robertson-Walker universe.

36. Bini D. , Damour T., Geralico A.
Radiative contributions to gravitational scattering,

Phys. Rev. D **104**, no.8, 084031 (2021)
doi:10.1103/PhysRevD.104.084031
e-print: [arXiv:2107.08896 [gr-qc]].
Abstract

The linear-order effects of radiation-reaction on the classical scattering of two point masses, in general relativity, are derived by a variation-of-constants method. Explicit expressions for the radiation-reaction contributions to the changes of 4-momentum during scattering are given to linear order in the radiative losses of energy, linear-momentum, and angular momentum. The polynomial dependence on the masses of the 4-momentum changes is shown to lead to nontrivial identities relating the various radiative losses. At order G^3 our results lead to a streamlined classical derivation of results recently derived within a quantum approach. At order G^4 we compute the needed radiative losses to next-to-next-to-leading-order in the post-Newtonian expansion, thereby reaching the absolute fourth and a half post-Newtonian level of accuracy in the 4-momentum changes. We also provide explicit expressions, at the absolute sixth post-Newtonian accuracy, for the radiation-graviton contribution to conservative $O(G^4)$ scattering. At orders G^5 and G^6 we derive explicit theoretical expressions for the last two hitherto undetermined parameters describing the fifth-post-Newtonian dynamics. Our results at the fifth-post-Newtonian level confirm results of [Nucl. Phys. B965, 115352 (2021)] but exhibit some disagreements with results of [Phys. Rev. D 101, 064033 (2020)].

37. Bini D., Geralico A.
Frequency domain analysis of the gravitational wave energy loss in hyperbolic encounters
Phys. Rev. D, vol. 104, 104019 (2021)
doi:10.1103/PhysRevD.104.104019
e-print: [arXiv:2108.02472 [gr-qc]].
Abstract

The energy radiated (without the 1.5PN tail contribution which requires a different treatment) by a binary system of compact objects moving in a hyperboliclike orbit is computed in the frequency domain through the

second post-Newtonian level as an expansion in the large-eccentricity parameter up to next-to-next-to-leading order, completing the time domain corresponding information (fully known in closed form at the second post-Newtonian of accuracy). The spectrum contains quadratic products of the modified Bessel functions of the first kind (Bessel K functions) with frequency-dependent order (and argument) already at Newtonian level, so preventing the direct evaluation of Fourier integrals. However, as the order of the Bessel functions tends to zero for large eccentricities, a large-eccentricity expansion of the spectrum allows for analytical computation beyond the lowest order.

38. Bini D., Geralico A.
Higher-order tail contributions to the energy and angular momentum fluxes in a two-body scattering process
Phys. Rev. D, 104, 104020 (2021)
doi:10.1103/PhysRevD.104.104020
e-print: [arXiv:2108.05445 [gr-qc]].
Abstract

The need for more and more accurate gravitational-wave templates requires taking into account all possible contributions to the emission of gravitational radiation from a binary system. Therefore, working within a multipolar-post-Minkowskian framework to describe the gravitational-wave field in terms of the source multipole moments, the dominant instantaneous effects should be supplemented by hereditary contributions arising from nonlinear interactions between the multipoles. The latter effects include tails and memories and are described in terms of integrals depending on the past history of the source. We compute higher-order tail (i.e., tail-of-tail, tail-squared, and memory) contributions to both energy and angular momentum fluxes and their averaged values along hyperboliclike orbits at the leading post-Newtonian approximation, using harmonic coordinates and working in the Fourier domain. Because of the increasing level of accuracy recently achieved in the determination of the scattering angle in a two-body system by several complementary approaches, the knowledge of these terms will provide useful information to compare results from different formalisms.

39. Bini D. , Mashhoon B., Obukhov Y. N.

Gravitomagnetic Helicity
Phys. Rev. D **105**, no.6, 064028 (2022)
doi:10.1103/PhysRevD.105.064028
[arXiv:2112.07550 [gr-qc]].
Abstract

Mass currents in astrophysics generate gravitomagnetic fields of enormous complexity. Gravitomagnetic helicity, in direct analogy with magnetic helicity, is a measure of entwining of the gravitomagnetic field lines. We discuss gravitomagnetic helicity within the gravitoelectromagnetic (GEM) framework of linearized general relativity. Furthermore, we employ the spacetime curvature approach to GEM in order to determine the gravitomagnetic helicity for static observers in Kerr spacetime.

40. Bini D. and Geralico A.,
Momentum recoil in the relativistic two-body problem: Higher-order tails
Phys. Rev. D **105**, no.8, 084028 (2022)
doi:10.1103/PhysRevD.105.084028
[arXiv:2202.03037 [gr-qc]].
Abstract

In the description of the relativistic two-body interaction, together with the effects of energy and angular momentum losses due to the emission of gravitational radiation, one has to take into account also the loss of linear momentum, which is responsible for the recoil of the center-of-mass of the system. We compute higher-order tail (i.e., tail-of-tail and tail-squared) contributions to the linear momentum flux for a non-spinning binary system either along hyperboliclike or ellipticlike orbits. The corresponding orbital averages are evaluated at their leading post-Newtonian approximation, using harmonic coordinates and working in the Fourier domain. The final expressions are given in a large-eccentricity (or large-angular momentum) expansion along hyperboliclike orbits and in a small-eccentricity expansion along ellipticlike orbits. We thus complete a previous analysis focusing on both energy and angular momentum losses [Phys. Rev. D 104, 104020 (2021)], providing brick-type results which will be useful, e.g., in the high-accurate

determination of the radiated impulses of the two bodies undergoing a scattering process.

41. Bini D. and Geralico A.,
Multipolar invariants and the eccentricity enhancement function parametrization of gravitational radiation
Phys. Rev. D **105**, no.12, 124001 (2022)
doi: 10.1103/PhysRevD.105.124001
[arXiv:2204.08077 [gr-qc]].
Abstract

Gravitational radiation can be decomposed as an infinite sum of radiative multipole moments, which parametrize the waveform at infinity. The multipolar-post-Minkowskian formalism provides a connection between these multipoles and the source multipole moments, known as explicit integrals over the matter source. The gravitational wave energy, angular momentum, and linear momentum fluxes are then expressed as multipolar expansions containing certain combinations of the source moments. We compute several gauge-invariant quantities as "building blocks" entering the multipolar expansion of both radiated energy and angular momentum at the 2.5 post-Newtonian (PN) level of accuracy in the case of hyperboliclike motion, by completing previous studies through the calculation of tail effects up to the fractional 1PN order. We express such multipolar invariants in terms of certain eccentricity enhancement factor functions, which are the counterpart of the well-known enhancement functions already introduced in the literature for ellipticlike motion. Finally, we use the complete 2.5PN-accurate averaged energy and angular momentum fluxes to study the associated adiabatic evolution of orbital elements under gravitational radiation reaction.

42. Bini D. and Mashhoon B.,
Static and Dynamic Melvin Universes
Phys. Rev. D **105**, no.12, 124012 (2022)
doi: 10.1103/PhysRevD.105.124012
[arXiv:2202.02033 [gr-qc]].
Abstract

We briefly review the known properties of Melvins magnetic universe and study the propagation of test charged matter waves in this static spacetime. Moreover, the possible correspondence between the wave perturbations on the background Melvin universe and the motion of charged test particles is discussed. Next, we explore a simple scenario for turning Melvins static universe into one that undergoes gravitational collapse. In the resulting dynamic gravitational field, the formation of cosmic double-jet configurations is emphasized.

43. Bini D. , Kauffman S., Succi S., Tello P. G. ,
First Post-Minkowskian approach to turbulent gravity
Phys. Rev. D **106**, no.10, 104007 (2022)
doi: 10.1103/PhysRevD.106.104007
[arXiv:2208.03572 [gr-qc]].
Abstract

We compute the metric fluctuations induced by a turbulent energy-matter tensor within the first order post-Minkowskian approximation. It is found that the turbulent energy cascade can in principle interfere with the process of black hole formation, leading to a potentially strong coupling between these two highly nonlinear phenomena. It is further found that a power-law turbulent energy spectrum $E(k) \sim k^{-n}$ generates metric fluctuations scaling as x^{n-2}, where x is the four-dimensional spacelike distance from an arbitrary origin in Minkowski spacetime, highlighting the onset of metric singularities whenever $n < 2$. Finally, the effect of metric fluctuations on the geodesic motion of test particles is also discussed as a potential technique to extract information on the spectral characteristics of fluctuating spacetime.

44. Bini D., Damour T.,
Radiation-reaction and angular momentum loss at the second Post-Minkowskian order
Phys.Rev.D 106 **12**, 124049 (2022)
[arXiv:2211.06340 [gr-qc]].
Abstract

We compute the variation of the Fokker-Wheeler-Feynman total linear and angular momentum of a gravitationally interacting binary system under the second post-Minkowskian retarded dynamics. The resulting $O(G^2)$ equations-of-motion-based, total change in the systems angular momentum is found to agree with existing computations that assumed balance with angular momentum fluxes in the radiation zone.

45. P. G. Tello, Bini D., S. Kauffman, S. Succi,
Predicting today's cosmological constant via the Zel'dovich-Holographic connection
EPL, **141**, 19002 (2023)
doi: 10.1209/0295-5075/acae01
[arXiv:2208.08129 [gr-qc]].
Abstract

This Letter proposes a solution of the Vacuum Energy and the Cosmological Constant (CC) paradox based on the Zel'dovich's ansatz, which states that the observable contribution to the vacuum energy density is given by the gravitational energy of virtual particle-antiparticle pairs, continually generated and annihilated in the vacuum state. The novelty of this work is the use of an ultraviolet cut-off length based on the Holographic Principle, which is shown to yield current values of the CC in good agreement with experimental observations.

46. Bini D., Damour T., Geralico A.
Radiated momentum and radiation reaction in gravitational two-body scattering including time-asymmetric effects
Phys. Rev. D **107**, no.2, 024012 (2023)
doi:10.1103/PhysRevD.107.024012
[arXiv:2210.07165 [gr-qc]].
Abstract

We compute to high post-Newtonian accuracy the 4-momentum (linear momentum and energy), radiated as gravitational waves in a two-body system undergoing gravitational scattering. We include, for the first time, all the relevant time-asymmetric effects that arise when consistently going three post-Newtonian orders beyond the leading post-Newtonian order. We find that the inclusion of time-asymmetric radiative effects (both in tails and in the radiation-reacted hyperbolic motion) is crucial to ensure the mass polynomiality of the post-Minkowskian expansion (G-expansion) of the radiated 4-momentum. Imposing the mass polynomiality of the corresponding individual impulses determines the conservativelike radiative contributions at the fourth post-Minkowskian order and strongly constrains them at the fifth post-Minkowskian order.

47. Bini D., Geralico A., R. T. Jantzen
Petrov type I spacetime curvature: principal null vector spanning dimension
IJGMMP, Vol. 20, No. 05, 2350087 (2023)
doi: 10.1142/S0219887823500871
e-Print: [arXiv:2111.01283 [gr-qc]]
Abstract

The class of Petrov type I curvature tensors is further divided into those for which the span of the set of distinct principal null directions has dimension four (maximally spanning type I) or dimension three (nonmaximally spanning type I). Explicit examples are provided for both vacuum and nonvacuum spacetimes.

48. Bini D., Geralico A., R. T. Jantzen
Wedging spacetime principal null directions
IJGMMP, Vol. 20, No. 9 (2023) 2350149 (24 pages)
doi: 10.1142/S0219887823501499
e-Print: [arXiv:2302.03367 [gr-qc]]
Abstract

Taking wedge products of the p distinct principal null directions associated with the eigen-bivectors of the Weyl tensor associated with the Petrov classification, when linearly independent, one is able to express them in terms of the eigenvalues governing this decomposition. We study here algebraic and differential properties of such p-forms by completing previous geometrical results concerning type I spacetimes and extending that analysis to algebraically special spacetimes with at least 2 distinct principal null directions. A number of vacuum and nonvacuum spacetimes are examined to illustrate the general treatment.

49. P. G. Tello, S. Succi, Bini D., S. Kauffman,
From quantum foam to graviton condensation: the Zeldovich route

EPL, 143 (2023) 39002
doi: 10.1209/0295-5075/acec95
e-Print: [arXiv:2306.17168 [physics.gen-ph]]
Abstract

Based on a previous ansatz by Zeldovich for the gravitational energy of virtual particle-antiparticle pairs, supplemented with the Holographic Principle, we estimate the vacuum energy in a fairly reasonable agreement with the experimental values of the Cosmological Constant. We further highlight a connection between Wheelers quantum foam and graviton condensation, as contemplated in the quantum N-portrait paradigm, and show that such connection also leads to a satisfactory prediction of the value of the cosmological constant. The above results suggest that the unnaturally small value of the cosmological constant may find a quite natural explanation once the nonlocal perspective of the large N-portrait gravitational condensation is endorsed.

50. Bini D., Geralico A., Rettegno P.
Spin-orbit contribution to radiative losses for spinning binaries with aligned spins
Phys. Rev. D **108**, no.6, 064049 (2023)
doi:10.1103/PhysRevD.108.064049
e-Print: [arXiv:2307.12670 [gr-qc]]
Abstract

We compute the leading order contribution to radiative losses in the case of spinning binaries with aligned spins due to their spin-orbit interaction. The orbital average along hyperboliclike orbits is taken through an appropriate spin-orbit modification to the quasi-Keplerian parametrization for nonspinning bodies, which maintains the same functional form, but with spin-dependent orbital elements. We perform consistency checks with existing post-Newtonian-based and post-Minkowskian (PM)-based results. In the former case, we compare our expressions for both radiated energy and angular momentum with those obtained in [G. Cho et al., From boundary data to bound states. Part III. Radiative effects, J. High Energy Phys. 04 (2022) 154] by applying the boundary-to-bound correspondence to known results for ellipticlike orbits, finding agreement. The linear momentum loss is instead newly computed here. In the latter case, we also find agreement with the low-velocity limit of recent calculations of the total radiated energy, angular momentum and linear momentum in the framework of an extension of the worldline quantum field theory approach to the classical scattering of spinning bodies at the leading PM order [G. U. Jakobsen et al., Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies, Phys. Rev. Lett. 128, 011101 (2022), M. M. Riva et al., Gravitational bremsstrahlung from spinning binaries in the post-Minkowskian expansion, Phys. Rev. D 106, 044013 (2022)]. We get exact expressions of the radiative losses in terms of the orbital elements, even if they are at the leading post-Newtonian order, so that their expansion for large values of the eccentricity parameter (or equivalently of the impact parameter) provides higher-order terms in the corresponding PM expansion, which can be useful for future crosschecks of other approaches.

51. Bini D., Damour T., Geralico A.
Comparing One-loop Gravitational Bremsstrahlung Amplitudes to the Multipolar-Post-Minkowskian Waveform
Phys.Rev.D **108** (2023) 12, 124052 doi:10.1103/PhysRevD.108.124052 [arXiv:2309.14925 [gr-qc]].
Abstract

We compare recent one-loop-level, scattering-amplitude-based, computations of the classical part of the gravitational bremsstrahlung waveform to the frequency-domain version of the corresponding multipolar-post-Minkowskian waveform result. When referring the one-loop result to the classical averaged momenta $\bar{p}_a = \frac{1}{2}(p_a + p'_a)$, the two waveforms are found to agree at the Newtonian and first post-Newtonian levels, as well as at the first-and-a-half post-Newtonian level, i.e., for the leading-order quadrupolar tail. However, we find that there are significant differences at the second-and-a-half post-Newtonian level, $O(G^2/c^5)$, i.e., when reaching he following: (i) the first post-Newtonian correction to the linear quadrupole tail; (ii) Newtonian-level linear tails of higher multipolarity (odd octupole and even hexadecapole); (iii) radiation-reaction effects on the worldlines; and (iv) various contributions of cubically nonlinear origin (notably linked to the quadrupole$\times$quadrupole$\times$ quadrupole coupling in the wave zone). These differences are reflected

at the sub-sub-sub-leading level in the soft expansion, $\sim \omega \ln \omega$, i.e., $O(\frac{1}{t^2})$ in the time domain. Finally, we computed the first four terms of the low-frequency expansion of the multipolar-post-Minkowskian waveform and checked that they agree with the corresponding existing classical soft graviton results.

52. Astesiano D., Bini D., Geralico A., Ruggiero M.L.
Particle motion in a rotating dust spacetime: the Bonnor solution Class. Quantum Grav. submitted (2023)
[arXiv:2310.04157 [gr-qc]].
Abstract

We investigate the geometrical properties, spectral classification, geodesics, and causal structure of the Bonnors spacetime [Journal of Physics A Math. Gen., 10, 1673 (1977)], i.e., a stationary axisymmetric solution with a rotating dust as a source. This spacetime has a directional singularity at the origin of the coordinates (related to the diverging vorticity field of the fluid there), which is surrounded by a toroidal region where closed timelike curves (CTCs) are allowed, leading to chronology violations. We use the effective potential approach to provide a classification of the different kind of orbits on the symmetry plane as well as to study the motion parallel to the symmetry axis. In the former case we find that as a general feature test particles released from a fixed space point and directed towards the singularity are repelled and scattered back as soon as they approach the CTC boundary, without reaching the central singularity.

53. Bini D., Geralico A., R. T. Jantzen, R. Ruffini
On Fermis resolution of the "4/3 problem" in the classical theory of the electron Foundation of Physics, submitted (2023)
Abstract

We discuss the solution proposed by Fermi to the so called 4/3 problem in the classical theory of the electron, a problem which puzzled the physics community for many decades before and after his contribution. Unfortunately his early resolution of the problem in 19221923 published in three versions in Italian and German journals (after three preliminary articles on the topic) went largely unnoticed. Even more

recent texts devoted to classical electron theory still do not present his argument or acknowledge the actual content of those articles. The calculations initiated by Fermi at the time are completed here and finally brought to their logical conclusion.

Books and book chapters

1. (**Book**) Ferrarese G., Bini D.,
   Introduction to relativistic continuum mechanics,
   Lecture Notes in Physics **727**, Ed. Springer, 2007.

2. (**Book**) De Felice F., Bini D.,
   Classical Measurements in Curved Space-Times
   Series: Cambridge Monographs on Mathematical Physics, Cambridge, UK, 2010

   Brief description

   The theory of relativity describes the laws of physics in a given space-time. However, a physical theory must provide observational predictions expressed in terms of measurements, which are the outcome of practical experiments and observations. Ideal for readers with a mathematical background and a basic knowledge of relativity, this book will help readers understand the physics behind the mathematical formalism of the theory of relativity. It explores the informative power of the theory of relativity, and highlights its uses in space physics, astrophysics and cosmology. Readers are given the tools to pick out from the mathematical formalism those quantities that have physical meaning and which can therefore be the result of a measurement. The book considers the complications that arise through the interpretation of a measurement, which is dependent on the observer who performs it. Specific examples of this are given to highlight the awkwardness of the problem.
   Provides a large sample of observers and reference frames in space-times that can be applied to space physics, astrophysics and cosmology. Tackles the problems encountered in interpreting measurements, giving specific examples. Features advice to help readers understand the logic of a given theory and its limitations.

Contents

APPENDICES

A. Spacetime splitting techniques in general relativity

The concept of a "gravitational force" modeled after the electromagnetic Lorentz force was born in the Newtonian context of centrifugal and Coriolis "fictitious" forces introduced by a rigidly rotating coordinate system in a flat Euclidean space. Bringing this idea first into linearized general relativity and then into its fully nonlinear form, it has found a number of closely related but distinct generalizations. Regardless of the details, this analogy between gravitation and electromagnetism has proven useful in interpreting the results of spacetime geometry in terms we can relate to, and has been illustrated in many research articles and textbooks over the past half century.

ICRANet has itself devoted a workshop and its proceedings to aspects of this topic in 2003 [2]. In the lengthy introduction to these proceedings, R. Ruffini has discussed a number of related topics, like "the gravitational analogue of the Coulomb-like interactions, of Hertz-like wave solutions, of the Oersted-Ampére-like magnetic interaction, etc.," supporting the thesis that treating gravitation in analogy with electromagnetism may help to better understand the main features of certain gravitational phenomena, at least when the gravitational field may be considered appropriately described by its linearized approximation [12, 13, 14, 15, 16, 17, 18]. A particularly long bibliography surveying most of the relevant literature through 2001 had been published earlier in the Proceedings of one of the annual Spanish Relativity Meetings [19].

In the 1990s, working in fully nonlinear general relativity, all of the various notions of "noninertial forces" (centrifugal and Coriolis forces) were put into a single framework by means of a unifying formalism dubbed "gravitoelectromagnetism" [1, 3, 4] which is a convenient framework to deal with these and curvature forces and related questions of their effect on test bodies moving in the gravitational field. More precisely, such a language is based on the splitting of spacetime into "space plus time," accomplished locally by means of an observer congruence, namely a congruence of timelike worldlines with (future-pointing) unit tangent vector field u which may be interpreted as the 4-velocity field of a family of test observers filling some region of spacetime. The orthogonal decomposition of each tangent space into a local time direction along u and the orthogonal local rest space (LRS) is used to decompose all spacetime tensors and tensor equations into a "space plus time" representation; the latter representation is somehow equivalent to a geometrical "measurement" process. This leads to a family of "spatial" spacetime tensor fields which represent each spacetime field and a family of spatial equations which represent each spacetime equation. Dealing with spacetime splitting techniques as well as 3-dimensional-like quantities clearly permits a better interface of our intuition and experience with the 4-dimensional geometry in certain gravitational problems. It can be particularly useful in spacetimes which have a geometrically defined timelike congruence, either explicitly given or defined implicitly as the congruence of orthogonal trajectories to a slicing or foliation of spacetime by a family of privileged spacelike hypersurfaces.

For example, splitting techniques are useful in the following spacetimes:

1. Stationary spacetimes, having a preferred congruence of Killing trajectories associated with the stationary symmetry, which is timelike on a certain region of spacetime (usually an open region, the boundary of which corresponding to the case in which the Killing vector becomes null so that in the exterior region Killing trajectories are spacelike).

2. Stationary axially symmetric spacetimes having in addition a preferred slicing whose orthogonal trajectories coincide with the worldlines of locally nonrotating test observers.

3. Cosmological spacetimes with a spatial homogeneity subgroup, which have a preferred spacelike slicing by the orbits of this subgroup.

From the various schools of relativity that blossomed during the second half of the last century a number of different approaches to spacetime splitting were developed without reference to each other. During the 1950s efforts were initiated to better understand general relativity and the mathematical tools needed to flush out its consequences. Lifshitz and the Russian school, Lichnerowicz in France, the British school, scattered Europeans (Ehlers and Trautman, for example) and the Americans best represented by Wheeler initiated this wave of relativity which blossomed in the 1960s. The textbook of Landau and Lifshitz and articles of Zelmanov [20, 21, 22, 23] presented the "threading point of view" of the Russian school and of Moller [21] which influenced Cattaneo in Rome and his successor Ferrarese [24, 25, 26, 27, 28], while a variation of this approach not relying on a complementary family of hypersurfaces (the "congruence point of view) began from work initially codified by Ehlers [17] and then taken up by Ellis [29, 30] in analyzing cosmological issues.

However, issues of quantum gravity lead to the higher profile of the "slicing point of view" in the 1960s initiated earlier by Lichnerowicz and developed by Arnowitt, Deser and Misner and later promoted by the influential textbook "Gravitation" by Misner, Thorne and Wheeler [31, 32, 33, 34] represents a splitting technique which is complementary to the threading point of view and its congruence variation, and proved quite useful in illuminating properties of black hole spacetimes.

R. Ruffini, a former student of Cattaneo and a collaborator of Wheeler, in his quest to better understand electromagnetic properties of black holes, awakened the curiosity of Jantzen and Carini at the end of the 1980s, later joined by Bini, who together made an effort to clarify the interrelationships between these various approaches as well as shed some light on the then confusing work of Abramowicz and others on relativistic centrifugal and Coriolis forces. By putting them all in a common framework, and describing what each measured in geometrical terms, and how each related to the others, some order was brought to the field [1, 3, 4].

The ICRANet people working on this subject have applied the main ideas underlying spacetime splitting techniques to concrete problems arising when studying test particle motion in black hole spacetimes. Among the various results obtained it is worth mentioning the relativistic and geometrically correct definition of inertial forces in general relativity [35, 36, 37, 38, 39], the definition of special world line congruences, relevant for the description of the motion of test particles along circular orbits in the Kerr spacetime (geodesic meeting point observers, extremely accelerated observers, etc.), the specification of all the geometrical properties concerning observer-adapted frames to the above mentioned special world line congruences [40, 41], the characterization of certain relevant tensors in black hole spacetimes (Simon tensor, Killing-Yano tensor) in terms of gravitoelectromagnetism [42, 43], etc. This research line is still ongoing and productive.

Over a period of several decades Jantzen, Bini and a number of students at the University of Rome "La Sapienza" under the umbrella of the Rome ICRA group have been working on this problem under the supervision of Ruffini.

The collaborators involved have been already listed and the most relevant papers produced are indicated in the references below [44]–[88]. In the present year 2010 a book by F. de Felice and D. Bini, including a detailed discussion of this and related topics, has been published by Cambridge University Press [137].

Let us now describe some fundamental notions of gravitoelectromagnetism.

A.1. Observer-orthogonal splitting

Let ${}^{(4)}$g (signature −+++ and components ${}^{(4)}g_{\alpha\beta}$, $\alpha, \beta, \ldots = 0, 1, 2, 3$) be the spacetime metric, ${}^{(4)}\nabla$ its associated covariant derivative operator, and ${}^{(4)}\eta$ the unit volume 4-form which orients spacetime (${}^{(4)}\eta_{0123} = {}^{(4)}g^{1/2}$ in an oriented frame, where ${}^{(4)}g \equiv |\det({}^{(4)}g_{\alpha\beta})|$). Assume the spacetime is also time oriented and let u be a future-pointing unit timelike vector field ($u^\alpha u_\alpha = -1$) representing the 4-velocity field of a family of test observers filling the spacetime (or some open submanifold of it).

If S is an arbitrary tensor field, let $S^\flat$ and $S^\sharp$ denote its totally covariant and totally contravariant forms with respect to the metric index-shifting operations. It is also convenient to introduce the right contraction notation $[S \llcorner X]^\alpha = S^\alpha{}_\beta X^\beta$ for the contraction of a vector field and the covariant index of a $\binom{1}{1}$-tensor field (left contraction notation being analogous).

A.1.1. The measurement process

The observer-orthogonal decomposition of the tangent space, and in turn of the algebra of spacetime tensor fields, is accomplished by the temporal projection operator $T(u)$ along u and the spatial projection operator $P(u)$ onto LRS_u, which may be identified with mixed second rank tensors acting by contraction

$$\begin{aligned} \delta^\alpha{}_\beta &= T(u)^\alpha{}_\beta + P(u)^\alpha{}_\beta \,, \\ T(u)^\alpha{}_\beta &= -u^\alpha u_\beta \,, \\ P(u)^\alpha{}_\beta &= \delta^\alpha{}_\beta + u^\alpha u_\beta \,. \end{aligned} \tag{A.1.1}$$

These satisfy the usual orthogonal projection relations $P(u)^2 = P(u)$, $T(u)^2 = T(u)$, and $T(u) \llcorner P(u) = P(u) \llcorner T(u) = 0$. Let

$$[P(u)S]^{\alpha\cdots}{}_{\beta\ldots} = P(u)^{\alpha}{}_{\gamma} \cdots P(u)^{\delta}{}_{\beta} \cdots S^{\gamma\cdots}{}_{\delta\ldots} \tag{A.1.2}$$

denote the spatial projection of a tensor S on all indices.

The *measurement of* S by the observer congruence is the family of spatial tensor fields which result from the spatial projection of all possible contractions of S by any number of factors of u. For example, if S is a $\binom{1}{1}$-tensor, then its measurement

$$S^{\alpha}{}_{\beta} \leftrightarrow (\underbrace{u^{\delta} u_{\gamma} S^{\gamma}{}_{\delta}}_{scalar}, \underbrace{P(u)^{\alpha}{}_{\gamma} u^{\delta} S^{\gamma}{}_{\delta}}_{vector}, \underbrace{P(u)^{\delta}{}_{\alpha} u_{\gamma} S^{\gamma}{}_{\delta}}_{vector}, \underbrace{P(u)^{\alpha}{}_{\gamma} P(u)^{\delta}{}_{\beta} S^{\gamma}{}_{\delta}}_{tensor}) \tag{A.1.3}$$

results in a scalar field, a spatial vector field, a spatial 1-form and a spatial $\binom{1}{1}$-tensor field. It is exactly this family of fields which occur in the orthogonal "decomposition of S" with respect to the observer congruence

$$\begin{aligned} S^{\alpha}{}_{\beta} &= [T(u)^{\alpha}{}_{\gamma} + P(u)^{\alpha}{}_{\gamma}][T(u)^{\delta}{}_{\beta} + P(u)^{\delta}{}_{\beta}]S^{\gamma}{}_{\delta} \\ &= [u^{\delta} u_{\gamma} S^{\gamma}{}_{\delta}] u^{\alpha} u_{\beta} + \cdots + [P(u)S]^{\alpha}{}_{\beta} \,. \end{aligned} \tag{A.1.4}$$

A.2. Examples

1. Measurement of the spacetime metric and volume 4-form
   - spatial metric $\quad [P(u)^{(4)}\mathrm{g}]_{\alpha\beta} = P(u)_{\alpha\beta}$
   - spatial unit volume 3-form $\quad \eta(u)_{\alpha\beta\gamma} = u^{\delta\,(4)}\eta_{\delta\alpha\beta\gamma}$;
     In a compact notation: $\eta(u) = [P(u)\, u \lrcorner\, ^{(4)}\eta]$
2. Measurement of the Lie, exterior and covariant derivative
   - spatial Lie derivative $£(u)_X = P(u)£_X$
   - the spatial exterior derivative $d(u) = P(u)d$
   - the spatial covariant derivative $\nabla(u) = P(u)^{(4)}\nabla$
   - the spatial Fermi-Walker derivative (or Fermi-Walker temporal derivative) $\nabla_{(\mathrm{fw})}(u) = P(u)^{(4)}\nabla_u$ (when acting on spatial fields)

- the Lie temporal derivative $\nabla_{(\mathrm{lie})}(u) = P(u)\pounds_u = \pounds(u)_u$

Note that spatial differential operators do not obey the usual product rules for nonspatial fields since undifferentiated factors of u are killed by the spatial projection.

3. Notation for 3-dimensional operations

It is convenient to introduce 3-dimensional vector notation for the spatial inner product and spatial cross product of two spatial vector fields X and Y. The inner product is just

$$X \cdot_u Y = P(u)_{\alpha\beta} X^\alpha Y^\beta \tag{A.2.1}$$

while the cross product is

$$[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma} X^\beta Y^\gamma \,. \tag{A.2.2}$$

With the "vector derivative operator" $\nabla(u)^\alpha$ one can introduce spatial gradient, curl and divergence operators for functions f and spatial vector fields X by

$$\begin{aligned} \mathrm{grad}_u\, f &= \nabla(u) f = [d(u) f]^\sharp \,, \\ \mathrm{curl}_u\, X &= \nabla(u) \times_u X = [{}^{*(u)} d(u) X^\flat]^\sharp \,, \\ \mathrm{div}_u\, X &= \nabla(u) \cdot_u X = {}^{*(u)}[d(u)^{*(u)} X^\flat] \,, \end{aligned} \tag{A.2.3}$$

where ${}^{*(u)}$ is the spatial duality operation for antisymmetric tensor fields associated with the spatial volume form $\eta(u)$ in the usual way. These definitions enable one to mimic all the usual formulas of 3-dimensional vector analysis. For example, the spatial exterior derivative formula for the curl has the index form

$$[\mathrm{curl}_u\, X]^\alpha = \eta(u)^{\alpha\beta\gamma\,(4)}\nabla_\beta X_\gamma \tag{A.2.4}$$

which also defines a useful operator for nonspatial vector fields X.

4. Measurement of the covariant derivative of the observer four velocity

Measurement of the covariant derivative $[{}^{(4)}\nabla u]^\alpha{}_\beta = u^\alpha{}_{;\beta}$ leads to two spatial fields, the acceleration vector field $a(u)$ and the kinematical mixed tensor field $k(u)$

$$\begin{aligned} u^\alpha{}_{;\beta} &= -a(u)^\alpha u_\beta - k(u)^\alpha{}_\beta \,, \\ a(u) &= \nabla_{(\mathrm{fw})}(u) u \,, \\ k(u) &= -\nabla(u) u \,. \end{aligned} \tag{A.2.5}$$

The kinematical tensor field may be decomposed into its antisymmetric and symmetric parts:

$$k(u) = \omega(u) - \theta(u)\,, \tag{A.2.6}$$

with

$$\begin{aligned} [\omega(u)^\flat]_{\alpha\beta} &= P(u)^\sigma_\alpha P(u)^\delta_\beta u_{[\delta;\sigma]} \\ &= \tfrac{1}{2}[d(u)u^\flat]_{\alpha\beta} \,, \\ [\theta(u)^\flat]_{\alpha\beta} &= P(u)^\sigma_\alpha P(u)^\delta_\beta u_{(\delta;\sigma)} \\ &= \tfrac{1}{2}[\nabla_{(\mathrm{lie})}(u) P(u)^\flat]_{\alpha\beta} = \tfrac{1}{2}\pounds(u)u^{(4)}g_{\alpha\beta} \,, \end{aligned} \tag{A.2.7}$$

defining the mixed rotation or vorticity tensor field $\omega(u)$ (whose sign depends on convention) and the mixed expansion tensor field $\theta(u)$, the latter of which may itself be decomposed into its tracefree and pure trace parts

$$\theta(u) = \sigma(u) + \frac{1}{3}\Theta(u)P(u)\,, \tag{A.2.8}$$

where the mixed shear tensor field $\sigma(u)$ is tracefree ($\sigma(u)^\alpha{}_\alpha = 0$) and the expansion scalar is

$$\Theta(u) = u^\alpha{}_{;\alpha} = {}^{*(u)}[\nabla_{(\mathrm{lie})}(u)\eta(u)]\,. \tag{A.2.9}$$

Define also the rotation or vorticity vector field $\omega(u) = \frac{1}{2}\,\mathrm{curl}_u\, u$ as the spatial dual of the spatial rotation tensor field

$$\omega(u)^\alpha = \tfrac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} = \tfrac{1}{2}{}^{(4)}\eta^{\alpha\beta\gamma\delta}u_\beta u_{\gamma;\delta}\,. \tag{A.2.10}$$

5. Lie, Fermi-Walker and co-Fermi-Walker derivatives

The kinematical tensor describes the difference between the Lie and Fermi-Walker temporal derivative operators when acting on spatial tensor fields. For example, for a spatial vector field X

$$\begin{aligned}\nabla_{(\mathrm{fw})}(u)X^\alpha &= \nabla_{(\mathrm{lie})}(u)X^\alpha - k(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha - \omega(u)^\alpha{}_\beta X^\beta + \theta(u)^\alpha{}_\beta X^\beta \,,\end{aligned} \tag{A.2.11}$$

where

$$\omega(u)^\alpha{}_\beta X^\beta = -\eta(u)^\alpha{}_{\beta\gamma}\omega(u)^\beta X^\gamma = -[\omega(u) \times_u X]^\alpha \,. \tag{A.2.12}$$

The kinematical quantities associated with u may be used to introduce two spacetime temporal derivatives, the Fermi-Walker derivative and the co-rotating Fermi-Walker derivative along u

$$\begin{aligned}{}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha &= {}^{(4)}\nabla_u X^\alpha + [a(u) \wedge u]^{\alpha\beta} X_\beta \,, \\ {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \,.\end{aligned} \tag{A.2.13}$$

These may be extended to arbitrary tensor fields in the usual way (so that they commute with contraction and tensor products) and they both commute with index shifting with respect to the metric and with duality operations on antisymmetric tensor fields since both ${}^{(4)}\mathrm{g}$ and ${}^{(4)}\eta$ have zero derivative with respect to both operators (as does u itself). For an arbitrary vector field X the following relations hold

$$\begin{aligned}\pounds_u X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + [\omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta] X^\beta \\ &= {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha + [-\theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta] X^\beta \,.\end{aligned} \tag{A.2.14}$$

A spatial co-rotating Fermi-Walker derivative $\nabla_{(\mathrm{cfw})}(u)$ ("co-rotating Fermi-Walker temporal derivative") may be defined in a way analogous to the ordinary one, such that the three temporal derivatives have the following relation when acting on a spatial vector field X

$$\begin{aligned}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= \nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha + \theta(u)^\alpha{}_\beta X^\beta \,,\end{aligned} \tag{A.2.15}$$

while $\nabla_{(\mathrm{cfw})}(u)[fu] = fa(u)$ determines its action on nonspatial fields.

It has been introduced an index notation to handle these three operators simultaneously

$$\{\nabla_{(\mathrm{tem})}(u)\}_{\mathrm{tem=fw,cfw,lie}} = \{\nabla_{(\mathrm{fw})}(u), \nabla_{(\mathrm{cfw})}(u), \nabla_{(\mathrm{lie})}(u)\}\ . \quad \text{(A.2.16)}$$

A.3. Comparing measurements by two observers in relative motion

Suppose U is another unit timelike vector field representing a different family of test observers. One can then consider relating the "observations" of each to the other. Their relative velocities are defined by

$$\begin{aligned} U &= \gamma(U,u)[u + \nu(U,u)]\ , \\ u &= \gamma(u,U)[U + \nu(u,U)]\ , \end{aligned} \quad \text{(A.3.1)}$$

where the relative velocity $\nu(U,u)$ of U with respect to u is spatial with respect to u and vice versa, both of which have the same magnitude $||\nu(U,u)|| = [\nu(U,u)_\alpha \nu(U,u)^\alpha]^{1/2}$, while the common gamma factor is related to that magnitude by

$$\gamma(U,u) = \gamma(u,U) = [1 - ||\nu(U,u)||^2]^{-1/2} = -U_\alpha u^\alpha\ . \quad \text{(A.3.2)}$$

Let $\hat{\nu}(U,u)$ be the unit vector giving the direction of the relative velocity $\nu(U,u)$. In addition to the natural parametrization of the worldlines of U by the proper time τ_U, one may introduce two new parametrizations: by a (Cattaneo) relative standard time $\tau_{(U,u)}$

$$d\tau_{(U,u)}/d\tau_U = \gamma(U,u)\ , \quad \text{(A.3.3)}$$

which corresponds to the sequence of proper times of the family of observers from the u congruence which cross paths with a given worldline of the U congruence, and by a relative standard lenght $\ell_{(U,u)}$

$$d\ell_{(U,u)}/d\tau_U = \gamma(U,u)||\nu(U,u)|| = ||\nu(U,u)||d\tau_{(U,u)}/d\tau_U\ , \quad \text{(A.3.4)}$$

which corresponds to the spatial arc lenght along U as observed by u.

Eqs. (A.3.1) describe a unique active "relative observer boost" $B(U,u)$ in the "relative observer plane" spanned by u and U such that

$$B(U,u)u = U\,, \quad B(U,u)\nu(U,u) = -\nu(u,U) \tag{A.3.5}$$

and which acts as the identity on the common subspace of the local rest spaces $LRS_u \cap LRS_U$ orthogonal to the direction of motion.

A.3.1. Maps between the LRSs of different observers

The projection $P(U)$ restricts to an invertible map when combined with $P(u)$ as follows

$$P(U,u) = P(U) \circ P(u) : LRS_u \to LRS_U \tag{A.3.6}$$

with inverse $P(U,u)^{-1} : LRS_U \to LRS_u$ and vice versa, and these maps also act as the identity on the common subspace of the local rest spaces.

Similarly the boost $B(U,u)$ restricts to an invertible map

$$B_{(\mathrm{lrs})}(U,u) \equiv P(U) \circ B(U,u) \circ P(u) \tag{A.3.7}$$

between the local rest spaces which also acts as the identity on their common subspace. The boosts and projections between the local rest spaces differ only by a gamma factor along the direction of motion.

An expression for the inverse projection

If $Y \in LRS_u$, then the orthogonality condition $0 = u_\alpha Y^\alpha$ implies that Y has the form

$$Y = [\nu(u,U) \cdot_U P(U,u)Y]U + P(U,u)Y\,. \tag{A.3.8}$$

If $X = P(U,u)Y \in LRS_U$ is the field seen by U, then $Y = P(U,u)^{-1}X$ and

$$P(U,u)^{-1}X = [\nu(u,U) \cdot_U X]U + X = [P(U) + U \otimes \nu(u,U)^\flat] \llcorner X\,, \tag{A.3.9}$$

which gives a useful expression for the inverse projection.

This map appears in the transformation law for the electric and magnetic fields:

$$\begin{aligned} E(u) &= \gamma P(U,u)^{-1}[E(U) + \nu(u,U) \times_U B(U)]\,, \\ B(u) &= \gamma P(U,u)^{-1}[B(U) - \nu(u,U) \times_U E(U)]\,. \end{aligned} \tag{A.3.10}$$

A.4. Comparing measurements by three or more observers in relative motion

A typical situation is that of a fluid/particle whis is observed by two diferrent families of observers. In this case one deal with three timelike congruences (or two congruences and a single line): the rest frame of the fluid U and the two observer families u e u'.

All the previous formalism can be easily generalized. One has

$$\begin{aligned} U &= \gamma(U,u)[u + \nu(U,u)] \, , \\ U &= \gamma(U,u')[u' + \nu(U,u')] \, , \\ u' &= \gamma(u',u)[u + \nu(u',u)] \, , \\ u &= \gamma(u,u')[u' + \nu(u,u')] \, . \end{aligned} \tag{A.4.1}$$

and mixed projectors involving the various four-velocities can be introduced. They are summarized in the following table:

PROJECTORS	
$P(u,U,u)$	$P(u) + \gamma(U,u)^2\nu(U,u) \otimes \nu(U,u)$
$P(u,U,u)^{-1}$	$P(u) - \nu(U,u) \otimes \nu(U,u)$
$P(u,U,u')$	$P(u,u') + \gamma(U,u)\gamma(U,u')\nu(U,u) \otimes \nu(U,u')$
$P(u,U,u')^{-1}$	$P(u',u) + \gamma(u,u')[(\nu(u,u') - \nu(U,u')) \otimes \nu(U,u) + \nu(U,u') \otimes \nu(u',u)]$
$P(U,u)^{-1}P(U,u')$	$P(u,u') + \gamma(u,u')\nu(U,u) \otimes \nu(u,u')$
$P(u',u)P(U,u)^{-1}P(U,u')$	$P(u') + \delta(U,u,u')\nu(U,u') \otimes \nu(u,u')$
$P(u',u)P(u',U,u)^{-1}$	$P(u') + \delta(U,u,u')\nu(U,u') \otimes [\nu(u,u') - \nu(U,u')]$

where

$$\delta(U,u,u') = \frac{\gamma(U,u')\gamma(u',u)}{\gamma(U,u)}, \qquad \delta(U,u,u')^{-1} = \delta(u,U,u') \, , \tag{A.4.2}$$

and

$$P(u,U,u') = P(u,U)P(U,u')$$

A.5. Derivatives

Suppose one uses the suggestive notation

$$ {}^{(4)}D(U)/d\tau_U = {}^{(4)}\nabla_U \qquad \text{(A.5.1)} $$

for the "total covariant derivative" along U. Its spatial projection with respect to u and rescaling corresponding to the reparametrization of Eq. (A.3.4) is then given by the "Fermi-Walker total spatial covariant derivative," defined by

$$ \begin{aligned} D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)} &= \gamma^{-1} D_{(\mathrm{fw},U,u)}/d\tau_U = \gamma^{-1} P(u)^{(4)}D(U)/d\tau_U \\ &= \nabla_{(\mathrm{fw})}(u) + \nabla(u)_{\nu(U,u)}\,. \end{aligned} \qquad \text{(A.5.2)} $$

Extend this to two other similar derivative operators (the co-rotating Fermi-Walker and the Lie total spatial covariant derivatives) by

$$ D_{(\mathrm{tem},U,u)}/d\tau_{(U,u)} = \nabla_{(\mathrm{tem})}(u) + \nabla(u)_{\nu(U,u)}\,, \qquad \mathrm{tem{=}fw,cfw,lie}\,, \qquad \text{(A.5.3)} $$

which are then related to each other in the same way as the corresponding temporal derivative operators

$$ \begin{aligned} D_{(\mathrm{cfw},U,u)}X^\alpha/d\tau_{(U,u)} &= D_{(\mathrm{fw},U,u)}X^\alpha/d\tau_{(U,u)} + \omega(u)^\alpha{}_\beta X^\beta \\ &= D_{(\mathrm{lie},U,u)}X^\alpha/d\tau_{(U,u)} + \theta(u)^\alpha{}_\beta X^\beta \end{aligned} \qquad \text{(A.5.4)} $$

when acting on a spatial vector field X. All of these derivative operators reduce to the ordinary parameter derivative $D/d\tau_{(U,u)} \equiv d/d\tau_{(U,u)}$ when acting on a function and extend in an obvious way to all tensor fields.

Introduce the ordinary and co-rotating Fermi-Walker and the Lie "relative accelerations" of U with respect to u by

$$ a_{(\mathrm{tem})}(U,u) = D_{(\mathrm{tem})}(U,u)\nu(U,u)/d\tau_{(U,u)}\,, \qquad \mathrm{tem{=}fw,cfw,lie}\,. \qquad \text{(A.5.5)} $$

These are related to each other in the same way as the corresponding derivative operators in Eq. (A.2.15).

The total spatial covariant derivative operators restrict in a natural way to a single timelike worldline with 4-velocity U, where the $D/d\tau$ notation is most appropriate; ${}^{(4)}D(U)/d\tau_U$ is often called the absolute or intrinsic derivative along the worldline of U (associated with an induced connection along such a worldline).

A.6. Applications

A.6.1. Test-particle motion

Let's consider the motion of a unit mass test-particle with four velocity U, accelerated by an external force $f(U)$: $a(U) = f(U)$. A generic observer u can measure the particle four velocity U, obtaining its relative energy $E(U,u) = \gamma(U,u)$ and momentum $p(U,u) = \gamma(U,u)\nu(U,u)$,

$$U = E(U,u)[u + p(U,u)] = \gamma(U,u)[u + \nu(U,u)]. \tag{A.6.1}$$

Splitting the acceleration equation gives the evolution (along U) of the relative energy and momentum of the particle

$$\begin{aligned} \frac{dE(U,u)}{d\tau_{(U,u)}} &= [F^{(G)}_{(\text{tem},U,u)} + F(U,u)] \cdot \nu(U,u) \\ &\quad + \epsilon_{(\text{tem})}\gamma(U,u)\nu(U,u) \cdot (\theta(u) \llcorner \nu(U,u)) \\ \frac{D_{(\text{tem})}p(U,u)}{d\tau_{(U,u)}} &= F^{(G)}_{(\text{tem},U,u)} + F(U,u)\,, \end{aligned} \tag{A.6.2}$$

where tem=fw,cfw,lie,lie$^\flat$ refers to the various possible (i.e. geometrically meaningful) transport of vectors along U, $\epsilon_{(\text{tem})} = (0,0,-1,1)$ respectively and

$$\begin{aligned} d\tau_{(U,u)} &= \gamma(U,u)d\tau_U \\ F^{(G)}_{(\text{tem},U,u)} &= \gamma(U,u)[g(u) + H_{(\text{tem},u)} \llcorner \nu(U,u)] \\ F(U,u) &= \gamma(U,u)^{-1}P(u,U)f(U) \end{aligned}$$

with

$$\begin{aligned} H_{(\text{fw},u)} &= \omega(u) - \theta(u) \qquad & H_{(\text{cfw},u)} &= 2\omega(u) - \theta(u) \\ H_{(\text{lie},u)} &= 2\omega(u) - 2\theta(u) \qquad & H_{(\text{lie}^\flat,u)} &= 2\omega(u)\,. \end{aligned} \tag{A.6.3}$$

The gravitoelectric vector field $g(u) = -a(u) = -\nabla_u u$ and the gravitomagnetic vector field $H(u) = 2[^{*(u)}\omega(u)^\flat]^\sharp$ of the observer u (sign-reversed acceleration and twice the vorticity vector field) are defined by the exterior derivative of u

$$du^\flat = [u \wedge g(u) + {}^{*(u)}H(u)]^\flat \,. \tag{A.6.4}$$

and will be essential in showing the analogy between the gravitational force $F^{(G)}_{(\text{tem},U,u)}$ and the Lorentz force. The expansion scalar $\Theta(u) = \operatorname{Tr}\theta(u)$ appears in an additional term in the covariant derivative of u as the trace of the (mixed) expansion tensor $\theta(u)$, of which the shear tensor $\sigma(u) = \theta(u) - \frac{1}{3}\Theta(u)P(u)$ is its tracefree part

$$\nabla u = -a(u) \otimes u^\flat + \theta(u) - \omega(u) \,. \tag{A.6.5}$$

The term $D_{(\text{tem})}p(U,u)/d\tau_{(U,u)}$ contains itself the "spatial geometry" contribution which must be added to the gravitational and the external force to reconstruct the spacetime point of view. Actually, this term comes out naturally and is significant all along the line of the particle: the single terms $\nabla_{(\text{fw},u)}$ and $\nabla(u)_{\nu(U,u)}$, in which it can be further decomposed, are not individually meaningful unless one defines some extension for the spatial momentum $p(U,u)$ off the line of the particle, which of course is unnecessary at all.

From this spatial geometry contribution a general relativistic version of inertial forces can be further extracted.

A.6.2. Maxwell's equations

Maxwell's equations can be expressed covariantly in many ways. For instance, in differential form language one has

$$dF = 0 \,, \qquad d^*F = -4\pi{}^*J^\flat \,, \tag{A.6.6}$$

where F is the Faraday electromagnetic 2-form and J is the current vector field, obeying the conservation law

$$\delta J^\flat = {}^*d^*J^\flat = 0 \,. \tag{A.6.7}$$

The splitting of the electromagnetic 2-form F by any observer family (with unit 4-velocity vector field u) gives the associated electric and magnetic vector fields $E(u)$ and $B(u)$ as measured by those observers through the Lorentz force law on a test charge, and the relative charge and current density $\rho(u)$ and $J(u)$. The "relative observer decomposition" of F and its dual 2-form ${}^*F$ is

$$\begin{aligned} F &= [u \wedge E(u) + {}^{*(u)}B(u)]^\flat , \\ {}^*F &= [-u \wedge B(u) + {}^{*(u)}E(u)]^\flat , \end{aligned}$$

while J has the representation

$$J = \rho(u)u + J(u) . \tag{A.6.8}$$

If U is the 4-velocity of any test particle with charge q and nonzero rest mass m, it has the orthogonal decomposition

$$U = \gamma(U,u)[u + \nu(U,u)] . \tag{A.6.9}$$

Its absolute derivative with respect to a proper time parametrization of its world line is its 4-acceleration $a(U) = DU/d\tau_U$. The Lorentz force law then takes the form

$$ma(U) = q\gamma(U,u)[E(u) + \nu(U,u) \times_u B(u)] . \tag{A.6.10}$$

The relative observer formulation of Maxwell's equations is well known. Projection of the differential form equations (A.6.6) along and orthogonal to u gives the spatial scalar (divergence) and spatial vector (curl) equations:

$$\begin{aligned} &\mathrm{div}_u B(u) + \vec{H}(u) \cdot_u E(u) = 0 , \\ &\mathrm{curl}_u E(u) - \vec{g}(u) \times_u E(u) + [\pounds(u)_u + \Theta(u)]B(u) = 0 , \\ &\mathrm{div}_u E(u) - \vec{H}(u) \cdot_u B(u) = 4\pi\rho(u) , \\ &\mathrm{curl}_u B(u) - \vec{g}(u) \times_u B(u) - [\pounds(u)_u + \Theta(u)]E(u) = 4\pi J(u) , \end{aligned} \tag{A.6.11}$$

This representation of Maxwell's equations differs from the Ellis representation only in the use of the spatially projected Lie derivative rather than the spatially projected covariant derivative along u (spatial Fermi-Walker derivative). These two derivative operators are related by the following identity for a spatial vector field X (orthogonal to u)

$$[\pounds(u)_u + \Theta(u)]X = [\nabla(u)_u + \{-\sigma(u) + \omega(u)\}\llcorner]X\,. \qquad \text{(A.6.12)}$$

It is clear, at this point, that for any spacetime tensor equation the "1+3" associated version allows one to read it in a Newtonian form and to interpret it quasi-classically.

For instance one can consider motion of test fields in a given gravitational background (i.e. neglecting backreaction) as described by spacetime equations and look at their "1+3" counterpart. Over the last ten years, in a similar way in which we have discussed the splitting of Maxwell's equations in integral formulation, we have studied scalar field, spinorial field (Dirac fields), fluid motions, etc.

B. Motion of particles and extended bodies in General Relativity

The motion of an extended body in a given background may be studied by treating the body via a multipole expansion. The starting point of this method is the covariant conservation law

$$\nabla_\mu T^\mu{}_\nu = 0\,, \tag{B.0.1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor describing the body. The body sweeps out a narrow tube in spacetime as it moves. Let L be a line inside this tube representing the motion of the body. Denote the coordinates of the points of this line by X^α, and define the displacement $\delta x^\alpha = X^\alpha - x^\alpha$, where x^α are the coordinates of the points of the body. Let us consider now the quantities

$$\int T^{\mu\nu}\mathrm{d}V\,, \qquad \int \delta x^\lambda T^{\mu\nu}\mathrm{d}V\,, \qquad \int \delta x^\lambda \delta x^\rho T^{\mu\nu}\mathrm{d}V\,, \quad \ldots \tag{B.0.2}$$

where the integrations are carried out on the 3-dimensional hypersurfaces of fixed time $t = X^0 = const$, the tensor $T^{\mu\nu}$ being different from zero only inside the world tube: these are the successive terms of the multipole expansion. A single-pole particle is defined as a particle that has nonvanishing at least some of the integrals in the first (monopole) term, assuming that all the integrals containing δx^μ vanish. A pole-dipole particle, instead, is defined as a particle for which all the integrals with more than one factor of δx^μ (dipole term) vanish. Higher order approximations may be defined in a similar way. Thus, a single-pole particle is a test particle without any internal structure. A pole-dipole particle, instead, is a test particle whose internal structure is expressed by its spin, an antisymmetric second-rank tensor defined by

$$S^{\mu\nu} \equiv \int \left[\delta x^{\mu} T^{0\nu} - \delta x^{\nu} T^{0\mu}\right] \mathrm{d}V \, . \tag{B.0.3}$$

The equations of motion are, then, obtained by applying the Einstein's field equations together with conservation of the energy-momentum tensor (B.0.1) describing the body. For a single-pole particle this leads to a free particle moving along the geodesics associated with the given background field. For the motion of a pole-dipole particle, instead, the corresponding set of equations was derived by Papapetrou [90] by using the above procedure. Obviously, the model is worked out under the assumption that the dimensions of the test particle are very small compared with the characteristic length of the basic field (i.e., with backreaction neglected), and that the gravitational radiation emitted by the particle in its motion is negligible. As a final remark, note that this model can be extended to charged bodies by considering in addition the conservation law of the current density.

B.1. The Mathisson-Papapetrou model

The equations of motion for a spinning (or pole-dipole) test particle in a given gravitational background were deduced by Mathisson and Papapetrou [90, 89] and read

$$\frac{DP^{\mu}}{\mathrm{d}\tau_U} = -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} U^{\nu} S^{\alpha\beta} \equiv F^{(\mathrm{spin})\mu} \, , \tag{B.1.1}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = P^{\mu} U^{\nu} - P^{\nu} U^{\mu} \, , \tag{B.1.2}$$

where P^{μ} is the total four-momentum of the particle, and $S^{\mu\nu}$ is a (antisymmetric) spin tensor; U is the timelike unit tangent vector of the "center of mass line" used to make the multipole reduction. Equations (B.1.1) and (B.1.2) define the evolution of P and S only along the world line of U, so a correct interpretation of U is that of being tangent to the *true* world-line of the spinning particle. The 4-momentum P and the spin tensor S are then defined as vector fields along the trajectory of U. By contracting both sides of Eq. (B.1.2) with U_ν, one obtains the following expression for the total 4-momentum

$$P^\mu = -(U \cdot P) U^\mu - U_\nu \frac{DS^{\mu\nu}}{d\tau_U} \equiv m U^\mu + P_s^\mu \ , \tag{B.1.3}$$

where $m = -U \cdot P$ reduces to the ordinary mass in the case in which the particle is not spinning, and P_s is a 4-vector orthogonal to U.

The test character of the particle under consideration refers to its mass as well as to its spin, since both quantities should not be large enough to contribute to the background metric. In what follows, with the magnitude of the spin of the particle, with the mass and with a natural lengthscale associated with the gravitational background we will construct a dimensionless parameter as a smallness indicator, which we retain to the first order only so that the test character of the particle be fully satisfied. Moreover, in order to have a closed set of equations Eqs. (B.1.1) and (B.1.2) must be completed with supplementary conditions (SC), whose standard choices in the literature are the following

1. Corinaldesi-Papapetrou [91] conditions (CP): $S^{\mu\nu}(e_0)_\nu = 0$, where e_0 is the coordinate timelike direction given by the background;
2. Pirani [92] conditions (P): $S^{\mu\nu}U_\nu = 0$;
3. Tulczyjew [93] conditions (T): $S^{\mu\nu}P_\nu = 0$;

all of these are algebraic conditions.

Detailed studies concerning spinning test particles in General Relativity are due to Dixon [94, 95, 96, 97, 98], Taub [99], Mashhoon [100, 101] and Ehlers and Rudolph [102]. The Mathisson-Papapetrou model does not give *a priori* restrictions on the causal character of U and P and there is no agreement in the literature on how this point should be considered. For instance, Tod, de Felice and Calvani [103] consider P timelike, assuming that it represents the total energy momentum content of the particle, while they do not impose any causality condition on the world line U, which plays the role of a mere mathematical "tool" to perform the multipole reduction. Differently, according to Mashhoon [101], P can be considered analogously to the canonical momentum of the particle: hence, there should be not any meaning for its causality character, while the world line U has to be timelike (or eventually null) because it represents the center of mass line of the particle. This uncertainty in the model itself then reflects in the need for a supplementary condition, whose choice among the three mentioned above is arbitrary, making the general relativistic description of a spinning test particle somehow unsatisfactory. When both U and P are timelike vectors as e_0, all of them can be taken as the 4-velocity field of a preferred observer family, and all the SC above state that for the corresponding observer the spin tensor is purely spatial. In a sense, only P and T supplementary conditions give "intrinsic" relations between the various unknown of the model and they should be somehow more physical conditions. In fact, the CP conditions are "coordinate dependent," being e_0 the coordinate timelike vector. It is worth to mention that grounded on physical reasons, Dixon has shown that the T conditions should be preferred with respect to the others.

B.2. The Dixon-Souriau model

The equations of motion for a charged spinning test particle in a given gravitational as well as electromagnetic background were deduced by Dixon-Souriau [104, 105, 106, 107]. They have the form

$$\frac{DP^\mu}{\mathrm{d}\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} + qF^\mu{}_\nu U^\nu - \frac{\lambda}{2}S^{\rho\sigma}\nabla^\mu F_{\rho\sigma} \equiv F^{(\mathrm{tot})\mu}\,, \quad \text{(B.2.1)}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = P^\mu U^\nu - P^\nu U^\mu + \lambda[S^{\mu\rho}F_\rho{}^\nu - S^{\nu\rho}F_\rho{}^\mu]\,, \quad \text{(B.2.2)}$$

where $F^{\mu\nu}$ is the electromagnetic field, P^μ is the total 4-momentum of the particle, and $S^{\mu\nu}$ is the spin tensor (antisymmetric); U is the timelike unit tangent vector of the "center of mass line" used to make the multipole reduction. As it has been shown by Souriau, the quantity λ is an arbitrary electromagnetic coupling scalar constant. We note that the special choice $\lambda = -q/m$ (see [46]) in flat spacetime corresponds to the Bargman-Michel-Telegdi [108] spin precession law.

B.3. Particles with quadrupolar structure

The equations of motion for an extended body in a given gravitational background were deduced by Dixon [94, 95, 96, 97, 98] in multipole approximation to any order. In the quadrupole approximation they read

$$\frac{DP^\mu}{d\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}{}^{;\mu} \equiv F^{(\text{spin})\mu} + F^{(\text{quad})\mu} \tag{B.3.1}$$

$$\frac{DS^{\mu\nu}}{d\tau_U} = 2P^{[\mu}U^{\nu]} - \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]}{}_{\alpha\beta\gamma}\,, \tag{B.3.2}$$

where $P^\mu = mU_p^\mu$ (with $U_p \cdot U_p = -1$) is the total four-momentum of the particle, and $S^{\mu\nu}$ is a (antisymmetric) spin tensor; U is the timelike unit tangent vector of the "center of mass line" $\mathcal{C}_U$ used to make the multipole reduction, parametrized by the proper time τ_U. The tensor $J^{\alpha\beta\gamma\delta}$ is the quadrupole moment of the stress-energy tensor of the body, and has the same algebraic symmetries as the Riemann tensor. Using standard spacetime splitting techniques it can be reduced to the following form

$$J^{\alpha\beta\gamma\delta} = \Pi^{\alpha\beta\gamma\delta} - \bar{u}^{[\alpha}\pi^{\beta]\gamma\delta} - \bar{u}^{[\gamma}\pi^{\delta]\alpha\beta} - 3\bar{u}^{[\alpha}Q^{\beta][\gamma}\bar{u}^{\delta]}\,, \tag{B.3.3}$$

where $Q^{\alpha\beta} = Q^{(\alpha\beta)}$ represents the quadrupole moment of the mass distribution as measured by an observer with 4-velocity $\bar{u}$. Similarly $\pi^{\alpha\beta\gamma} = \pi^{\alpha[\beta\gamma]}$ (with the additional property $\pi^{[\alpha\beta\gamma]} = 0$) and $\Pi^{\alpha\beta\gamma\delta} = \Pi^{[\alpha\beta][\gamma\delta]}$ are essentially the body's momentum and stress quadrupoles. Moreover the various fields $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma}$ and $\Pi^{\alpha\beta\gamma\delta}$ are all spatial (i.e. give zero after any contraction by $\bar{u}$). The number of independent components of $J^{\alpha\beta\gamma\delta}$ is 20: 6 in $Q^{\alpha\beta}$, 6 in $\Pi^{\alpha\beta\gamma\delta}$ and 8 in $\pi^{\alpha\beta\gamma}$. When the observer $\bar{u} = U_p$, i.e. in the frame associated with the momentum of the particle, the tensors $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma}$ and $\Pi^{\alpha\beta\gamma\delta}$ have an intrinsic meaning.

There are no evolution equations for the quadrupole as well as higher multipoles as a consequence of the Dixon's construction, so their evolution is completely free, depending only on the considered body. Therefore the system of equations is not self-consistent, and one must assume that all unspecified quantities are known as intrinsic properties of the matter under consideration.

In order the model to be mathematically correct the following additional condition should be imposed to the spin tensor:

$$S^{\mu\nu}U_{p\nu} = 0. \tag{B.3.4}$$

Such supplementary conditions (or Tulczyjew-Dixon conditions [93, 94]) are necessary to ensure the correct definition of the various multipolar terms.

Dixon's model for structured particles originated to complete and give a rigorous mathematical support to the previously introduced Mathisson-Papapetrou model [90, 89, 91, 92], i.e. a multipole approximation to any order which includes evolutional equations along the "center line" for all the various structural quantities. The models are then different and a comparison between the two is possible at the dipolar order but not once the involved order is the quadrupole.

Here we limit our considerations to Dixon's model under the further simplifying assumption[99, 102] that the only contribution to the complete quadrupole moment $J^{\alpha\beta\gamma\delta}$ stems from the mass quadrupole moment $Q^{\alpha\beta}$, so that $\pi^{\alpha\beta\gamma} = 0 = \Pi^{\alpha\beta\gamma\delta}$ and

$$J^{\alpha\beta\gamma\delta} = -3U_p^{[\alpha}Q^{\beta][\gamma}U_p^{\delta]}\,, \qquad Q^{\alpha\beta}U_{p\beta} = 0\,; \tag{B.3.5}$$

The assumption that the particle under consideration is a test particle means that its mass, its spin as well as its quadrupole moments must all be small enough not to contribute significantly to the background metric. Otherwise, backreaction must be taken into account.

B.4. Null multipole reduction world line: the massless case

The extension of the Mathisson-Papapetrou model to the case of a null multipole reduction world line l has been considered by Mashhoon [101]: the model equations have exactly the same form as (B.1.1) and (B.1.2), with U (timelike) replaced by l (null) for what concerns the multipole reduction world line and τ_U (proper time parametrization of the U line) replaced by λ (affine parameter along the l line):

$$\frac{DP^\alpha}{\mathrm{d}\lambda} = -\frac{1}{2}R^\alpha{}_{\beta\rho\sigma}l^\beta S^{\rho\sigma} \equiv F^{(\mathrm{spin})\alpha}\,, \tag{B.4.1}$$

$$\frac{DS^{\alpha\beta}}{\mathrm{d}\lambda} = [P \wedge l]^{\alpha\beta}\,. \tag{B.4.2}$$

Equations (B.4.1) and (B.4.2) should be then solved assuming some SC. Let us limit ourselves to the case of "intrinsic" SC, i.e. Pirani and Tulczyjew, with

Pirani's conditions now naturally generalized as $S^{\alpha\beta}l_\beta = 0$. Furthermore, we require $P \cdot l = 0$: in fact, we are interested to the massless limit of the Mathisson-Papapetrou equations, and as the mass of the particle is defined by $m = -P \cdot U$ the massless limit implies $-P \cdot l = 0$.

By denoting with $\{l = e_1, n = e_2, m = e_3, \bar{m} = e_4\}$ a complex null frame along the center line l, such that $l \cdot l = n \cdot n = m \cdot m = 0$, $l \cdot n = 1$, $l \cdot m = l \cdot m = 0$ and $m \cdot \bar{m} = -1$, it is possible to parametrize the path so that

$$\begin{aligned} \frac{Dl^\mu}{\mathrm{d}\lambda} &= \bar{b}m^\mu + b\bar{m}^\mu \, , \\ \frac{Dn^\mu}{\mathrm{d}\lambda} &= \bar{a}m^\mu + a\bar{m}^\mu \, , \\ \frac{Dm^\mu}{\mathrm{d}\lambda} &= al^\mu + bn^\mu + icm^\mu \, , \end{aligned} \tag{B.4.3}$$

where a, b, c are functions of λ and c is real. The metric signature is assumed now $+---$ in order to follow standard notation of Newman-Penrose formalism, and the bar over a quantity denotes complex conjugation. Equations (B.4.3) are the analogous of the FS relations for null lines so that, repeating exactly the above procedure, one gets the final set of equations. Since for a massless spinning test particle we have $m = -P \cdot l = 0$, the total 4-momentum P has the following decomposition:

$$P^\mu = -[Bl^\mu + Am^\mu + \bar{A}\bar{m}^\mu] \, . \tag{B.4.4}$$

Following Mashhoon [101], Tulczyjew's conditions $S^{\alpha\beta}P_\beta = 0$ are in general inconsistent in the presence of a gravitational background if in addition one has P lightlike: $P \cdot P = 0$. Thus, even if these inconsistencies concern only the case of null P, we are clearly forced to consider Pirani's SC as the only physically meaningful supplementary conditions. Using the P supplementary conditions (implying $b = 0$), Mashhoon has shown that l is necessarily geodesics: $Dl^\mu/\mathrm{d}\lambda = 0$ and

$$S^{\mu\nu} = f(\lambda)[l \wedge m]^{\mu\nu} + \bar{f}(\lambda)[l \wedge \bar{m}]^{\mu\nu} + ig(\lambda)[m \wedge \bar{m}]^{\mu\nu} \, , \tag{B.4.5}$$

with B real and

$$A = \frac{\mathrm{d}f}{\mathrm{d}\lambda} + icf - ig\bar{a} \, , \qquad P^\mu P_\mu = -2|A|^2 \, . \tag{B.4.6}$$

so that P is in general spacelike or eventually null. Furthermore, he has shown that the spin vector defined by

$$\mathcal{S}^\mu = \frac{1}{2}\eta^{\mu\nu\alpha\beta} l_\nu S_{\alpha\beta} \tag{B.4.7}$$

is constant along l and either parallel or antiparallel to l.

Finally, the generalized momentum of the particle should be determined by solving equations (B.4.1) and (B.4.2) supplemented by $S^{\alpha\beta}l_\beta = 0$. The other components of the spin tensor not summarized by the spin vector should be determined too. By assuming $a = 0$ (n parallel propagated along l) without any loss of the physical content of the solution, Mashhoon has obtained for f and B the following differential equations:

$$\begin{aligned} \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} + ic\right]^2 f &= fR_{1413} + \bar{f}R_{1414} + igR_{1434} \,, \\ -\frac{\mathrm{d}B}{\mathrm{d}\lambda} &= fR_{1213} + \bar{f}R_{1214} + igR_{1234} \,, \end{aligned} \tag{B.4.8}$$

which determine the total 4-momentum and the spin tensor along the path once they have been specified initially.

B.5. Applications

B.5.1. The special case of constant frame components of the spin tensor

Due to the mathematical complexity in treating the general case of non-constant frame components of the spin tensor, we have considered first the simplest case of massive spinning test particles moving uniformly along circular orbits with constant frame components of the spin tensor with respect to a naturally geometrically defined frame adapted to the stationary observers in the Schwarzschild spacetime [109] as well as in other spacetimes of astrophysical interest: Reissner-Nordström spacetime [110], Kerr spacetime [5], superposed static Weyl field [111], vacuum C metric [112]. A static spin vector is a very strong restriction on the solutions of the Mathisson-Papapetrou equations of motion. However, this assumption not only greatly simplifies the calculation, but seems to be not so restrictive, since, as previously demonstrated at least in the Schwarzschild case, the spin tensor components still remain constant under the CP an T choices of supplementary conditions, starting from the more general non-constant case.

We have confined our attention to spatially circular equatorial orbits in Schwarzschild, Reissner-Nordström and Kerr spacetimes, and searched for observable effects which could eventually discriminate among the standard supplementary conditions. We have found that if the world line chosen for the multipole reduction and whose unit tangent we denote as U is a circular orbit, then also the generalized momentum P of the spinning test particle is tangent to a circular orbit even though P and U are not parallel 4-vectors. These orbits are shown to exist because the spin induced tidal forces provide the required acceleration no matter what supplementary condition we select. Of course, in the limit of a small spin the particle's orbit is close of being a circular geodesic and the (small) deviation of the angular velocities from the geodesic values can be of an arbitrary sign, corresponding to the possible spin-up and spin-down alignment to the z-axis. When two massive particles (as well as photons) orbit around a gravitating source in opposite directions, they make one loop with respect to a given static observer with different arrival times. This difference is termed clock effect (see [50, 113, 114, 115, 116] and references therein). Hereafter we shall refer the co/counter-rotation as with respect to a fixed sense of variation of the azimuthal angular coordinate. In the case of a static observer and of timelike spatially circular geodesics the coordinate time delay is given by

$$\Delta t_{(+,-)} = 2\pi \left(\frac{1}{\zeta_+} + \frac{1}{\zeta_-} \right) , \tag{B.5.1}$$

where $\zeta_\pm$ denote angular velocities of two opposite rotating geodesics. In the case of spinless neutral particles in geodesic motion on the equatorial plane of both Schwarzschild and Reissner-Nordström spacetimes one has $\zeta_+ = -\zeta_-$, and so the clock effect vanishes; in the Kerr case, instead, the clock effect reads $\Delta t_{(+,-)} = 4\pi a$, where a is the angular momentum per unit mass of the Kerr black hole. These results are well known in the literature. We have then extended the notion of clock effect to non geodesic circular trajectories considering co/counter-rotating spinning-up/spinning-down particles. In this case we have found that the time delay is nonzero for oppositely orbiting both spin-up or spin-down particles even in both Schwarzschild and Reissner-Nordström cases, and can be measured. In addition, we have found that a nonzero gravitomagnetic clock effect appears in the Reissner-Nordström spacetime for spinless (oppositely) charged particles as well.

An analogous effect is found in the case of superposed Weyl fields corresponding to Chazy-Curzon particles and Schwarzschild black holes when the circular motion of spinning test particles is considered on particular symmetry hyperplanes, where the orbits are close to a geodesic for small values of the spin. In the case of the C metric, instead, we have found that the orbital frequency is in general spin-dependent, but there is no clock effect, in contrast to the limiting Schwarzschild case.

B.5.2. Spin precession in Schwarzschild and Kerr spacetimes

We have then studied the behaviour of spinning test particles moving along equatorial circular orbits in the Schwarzschild [6] as well as Kerr [7] spacetimes within the framework of the Mathisson-Papapetrou approach supplemented by standard conditions, in the general case in which the components of the spin tensor are not constant along the orbit. We have found that precession effects occur only if the Pirani's supplementary conditions are imposed, where one finds a Fermi-Walker transported spin vector along an accelerated center of mass world line. The remaining two supplementary conditions apparently force the test particle center of mass world line to deviate from a circular orbit because of the feedback of the precessing spin vector; in addition, under these choices of supplementary conditions the spin tensor components still remain constant. In reaching these conclusions, we only considered solutions for which both U and P are timelike vectors, in order to have a meaningful interpretation describing a spinning test particle with nonzero rest mass.

B.5.3. Massless spinning test particles in vacuum algebraically special spacetimes

As a final application, we have derived the equations of motion for massless spinning test particles in general vacuum algebraically special spacetimes, using the Newman-Penrose formalism, in the special case in which the multipole reduction world line is aligned with a principal null direction of the spacetime [117]. This situation gives very simple equations and their complete integration is straightforward. Explicit solutions corresponding to some familiar Petrov type D and type N spacetimes (including Schwarzschild, Taub-NUT, Kerr, C metric, Kasner, single exact gravitational wave) are derived and discussed. Furthermore, we have investigated the motion along (null) circular orbits, providing explicit solutions in black hole spacetimes.

B.5.4. Quadrupole effects in black hole spacetimes

We have studied the motion of quadrupolar particles on a Schwarzschild as well as Kerr backgrounds [118, 119] following Dixon's model. In the simplified situation of constant frame components (with respect to a natural orthonormal frame) of both the spin and the quadrupole tensor of the particle we have found the kinematical conditions to be imposed to the particle's structure in order the orbit of the particle itself be circular and confined on the equatorial plane. Co-rotating and counter-rotating particles result to have a non-symmetric speed in spite of the spherical symmetry of the background, due to their internal structure. This fact has been anticipated when studying spinning particles only, i.e. with vanishing quadrupole moments. We show modifications due to the quadrupole which could be eventually observed in experiments. Such experiment, however, cannot concern standard clock effects, because in this case we have shown that there are no contributions arising from the quadrupolar structure of the body. In contrast, the effect of the quadrupole terms could be important when considering the period of revolution of an extended body around the central source: measuring the period will provide an estimate of the quantities determining the quadrupolar structure of the body, if its spin is known.

It would be of great interest to extend this analysis to systems with varying quadrupolar structure and emitting gravitational waves without perturbing significantly the background spacetime.

B.5.5. Quadrupole effects in gravitational wave spacetimes

We have studied how a small extended body at rest interacts with an incoming single plane gravitational wave. The body is spinning and also endowed with a quadrupolar structure, so that due to the latter property it can be thus considered as a good model for a gravitational wave antenna.

We have first discussed the motion of such an extended body by assuming that it can be described according to Dixon's model and that the gravitational field of the wave is weak, i.e. the "reaction" (induced motion) of a "gravitational wave antenna" (the extended body) to the passage of the wave, and then the case of an exact plane gravitational wave. We have found that in general, even if initially absent, the body acquires a dipolar moment induced by the quadrupole tensor, a property never pointed out before in the literature.

Special situations may occur in which certain spin components change their magnitude leading to effects (e.g. spin-flip) which can be eventually observed. This interesting feature recalls the phenomenon of glitches observed in pulsars: a sudden increase in the rotation frequency, often accompanied by an increase in slow-down rate. The physical mechanism triggering glitches is not well understood yet, even if these are commonly thought to be caused by internal processes. If one models a pulsar by a Dixon's extended body, then the present analysis shows that a sort of glitch can be generated by the passage of a strong gravitational wave, due to the pulsar quadrupole structure. In fact, we have found that the profile of a polarization function can be suitably selected in order to fit observed glitches and in particular to describe the post-glitch behavior.

B.5.6. Quadrupolar particles and the equivalence principle

We have compared the two "reciprocal " situations of motion of an extended body endowed with structure up to the mass quadrupole moment in a Shwarzschild background spacetime (as described by Dixon's model) with that of a test particle in geodesic motion in the background of an exact solution of Einstein's field equations describing a source with quadrupolar structure (for a more detailed study of this kind of solutions generalizing Schwarzschild, Kerr and Kerr-Newman spacetimes see also the section "Generalizations of the Kerr-Newman solution," included in the present report). Under certain conditions the two situations give perfect corresponding results a fact which has been interpreted as an argument in favour of the validity of Dixon's model.

B.5.7. Poynting-Robertson-like effects

Test particle motion in realistic gravitational fields is of obvious astrophysical importance and at the same time it provides reliable evidence of the properties of those gravitational fields. However, in many actual astrophysical systems the particles are not moving freely but are influenced by ambient matter, electromagnetic fields and radiation. In typical situations, these "physical" effects are probably even more important than fine details of the spacetime geometry alone. The most remarkable conditions, from the point of view of general relativity as well as astrophysics, appear near very compact objects where both the pure gravitational and other "physical" effects typically become extraordinarily strong.

In a series of papers, recently, we have focused on the motion of test particles in a spherically symmetric gravitational field, under the action of a Thomson-type interaction with radiation emitted or accreted by a compact center. This kind of problem was first investigated by Poynting using Newtonian gravity and then in the framework of linearized general relativity by Robertson (see [140] and the references therein). It involves competition between gravity and radiation drag, which may lead to interesting types of motion which do not occur in strictly vacuum circumstances. In particular, there arises the question of whether equilibrium behavior like circular orbit motion or even "staying at rest" are possible in some cases. Theoretical aspects of the Poynting-Robertson effect as well as its astrophysical relevance in specific situations have been studied by many authors since the original pioneering work. We first considered this effect on test particles orbiting in the equatorial plane of a Schwarzschild or Kerr black hole, assuming that the source of radiation is located symmetrically not far from the horizon (in the case of outgoing flux). Successively, we have generalized these results by including in our discussion other relevant spacetimes, e.g. Vaidya, or considering spinning particles undergoing Poynting-Robertson-like effect.

C. Metric and curvature perturbations in black hole spacetimes

C.1. Perturbations of charged and rotating black hole

The gravitational and electromagnetic perturbations of the Kerr-Newman metric represent still an open problem in General Relativity whose solution could have an enormous importance for the astrophysics of charged and rotating collapsed objects. A complete discussion about this problems needs a plenty of different mathematical tools: the Newman-Penrose formalism in the tetradic and spinor version, the Cahen-Debever-Defrise self dual theory, the properties of the spin-weighted angular harmonics, with particular attention to the related differential geometry and the group theory, some tools of complex analysis, etc, but in any case it is difficult to handle with the perturbative equations. Fortunately, during the last years, the modern computers and software have reached an optimal computational level which allows now to approach this problem from a completely new point of view.

The Kerr-Newman solution in Boyer-Lindquist coordinates is represented by the metric:

$$\begin{aligned} ds^2 &= \left(1 - \frac{V}{\Sigma}\right) dt^2 + \frac{2a\sin^2\theta}{\Sigma} V dt d\phi - \frac{\Sigma}{\Delta} dr^2 \\ & -\Sigma d\theta^2 - \left[r^2 + a^2 + \frac{a^2 \sin^2\theta}{\Sigma} V\right] \sin^2\theta d\phi^2 \end{aligned} \tag{C.1.1}$$

where as usual:

$$\begin{aligned} V &\equiv 2Mr - Q^2 \\ \Delta &\equiv r^2 - 2Mr + a^2 + Q^2 \\ \Sigma &\equiv r^2 + a^2 \cos^2\theta \end{aligned} \tag{C.1.2}$$

and by the vector potential:

$$A^\flat = A_\mu \, dx^\mu = \frac{Qr}{\Sigma}(dt - a\sin^2\theta \, d\phi) \ . \tag{C.1.3}$$

To investigate the geometrical features of this metric it is convenient to introduce a symmetry-adapted tetrad. For any type D metric, and in particular for the Kerr-Newman solution, the best choice is a null tetrad with two "legs" aligned along the two repeated principal null directions of the Weyl tensor. The standard theory for analyzing different spin massless wave fields in a given background is represented by the spinorial tetradic formalism of Newman-Penrose (hereafter N-P)[120]. Here we follow the standard approach, pointing out that a more advanced reformulation of this formalism, called "GHP" [121] exists, allowing a more geometric comprehension of the theory. In the N-P formalism, this solution is represented by the following quantities [122] (in this section we use an A label over all quantities for a reason which will be clear later). The Kinnersley tetrad [123]:

$$\begin{aligned} (l^\mu)^A &= \frac{1}{\Delta}[r^2 + a^2, \Delta, 0, a] \\ (n^\mu)^A &= \frac{1}{2\Sigma}[r^2 + a^2, -\Delta, 0, a] \\ (m^\mu)^A &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\,[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}] \ , \end{aligned} \tag{C.1.4}$$

with the 4^{th} leg represented by the conjugate $(m^{*\mu})^A$, gives the metric tensor of Kerr-Newman spacetime the form:

$$\eta_{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{C.1.5}$$

The Weyl tensor is represented by:

$$\begin{gathered}\Psi_0^A = \Psi_1^A = \Psi_3^A = \Psi_4^A = 0 \\ \Psi_2^A = M\rho^3 + Q^2\rho^*\rho^3\end{gathered} \tag{C.1.6}$$

and the electromagnetic field is given by:

$$\phi_0^A = \phi_2^A = 0 \ , \ \phi_1^A = \frac{Q}{2(r - ia\cos\theta)^2} \ . \tag{C.1.7}$$

For the Ricci tensor and the curvature scalar we have:

$$\Lambda^A = 0 \ \ , \ \ \Phi^A_{nm} = 2\phi^A_m \phi^{*A}_n \quad (\mathrm{m}, \mathrm{n} = 0, 1, 2) \tag{C.1.8}$$

so in Kerr-Newman, the only quantity different from zero is:

$$\Phi^A_{11} = \frac{Q^2}{2\Sigma^2} \ . \tag{C.1.9}$$

The spin coefficients, which are linear combination of the Ricci rotation coefficients, are given by:

$$\begin{aligned}
&\kappa^A = \sigma^A = \lambda^A = \nu^A = \epsilon^A = 0 \,, \\
&\rho^A = \frac{-1}{(r - ia\cos\theta)}\,, \qquad \tau^A = \frac{-ia\rho^A\rho^{*A}\sin\theta}{\sqrt{2}}\,, \\
&\beta^A = \frac{-\rho^{*A}\cot\theta}{2\sqrt{2}}\,, \qquad \pi^A = \frac{ia(\rho^A)^2\sin\theta}{\sqrt{2}}\,, \\
&\mu^A = \frac{(\rho^A)^2\rho^{*A}\Delta}{2}\,, \qquad \gamma^A = \mu^A + \frac{\rho^A\rho^{*A}(r - M)}{2}\,, \\
&\alpha^A = \pi^A - \beta^{*A} \,.
\end{aligned} \tag{C.1.10}$$

The directional derivatives are expressed by:

$$D = l^\mu \partial_\mu \,, \quad \Delta = n^\mu \partial_\mu \,, \quad \delta = m^\mu \partial_\mu \,, \quad \delta^* = m^{*\mu}\partial_\mu \,. \tag{C.1.11}$$

Unfortunately in the literature the same letter for (C.1.2) and for the directional derivative along **n** it is used. However the meaning of Δ will always be clear from the context. The study of perturbations in the N-P formalism is achieved splitting all the relevant quantities in the form $l = l^A + l^B, \Psi_4 = \Psi_4^A + \Psi_4^B,\ \sigma = \sigma^A + \sigma^B, D = D^A + D^B$, etc., where the A terms are the background and the B's are small perturbations. The full set of perturbative equations is obtained inserting these quantities in the basic equations of the theory (Ricci and Bianchi identities, Maxwell, Dirac, Rarita-Schwinger equations etc.) and keeping only first order terms. After certain standard algebraic manipulations one usually obtains coupled linear PDE's involving curvature quantities. In the following, we will omit the A superscript for the background quantities. Comparing with the standard Regge-Wheeler-Zerilli [124, 125] approach which gives the equation for the metric, here one gets the equations for Weyl tensor components. This theory is known as *curvature perturbations*. In the case of Einstein-Maxwell perturbed metrics, one gets as in R-W-Z the well known phenomenon of the "gravitationally induced electromagnetic radiation and vice versa" [126], which couples gravitational and electromagnetic fields. In the first formulation, one gets a coupled system for $F^B_{\mu\nu}$ and $g^B_{\mu\nu}$ quantities. In the N-P approach one has the coupling between perturbed Weyl and Maxwell tensor components, although it's possible to recover the metric perturbations using the curvature one [127]. A discussion about the connections between these two approaches can be found in [128]. To make a long story short, taking in account the two Killing vectors of this spacetime, one can write the unknown functions in the form:

$$F(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} f(r, \theta) \ . \tag{C.1.12}$$

In the easier cases of Kerr, Reissner-Nordstrom and Schwarzchild, writing $f(r,\theta) = R(r)Y(\theta)$ one gets separability of the problem. For instance, the Reissner-Nordström case [129] is separable using the spin-weighted spherical harmonics:

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \left(\frac{m^2 + s^2 + 2ms\cos\theta}{\sin^2\theta}\right)\right]{}_sY^m{}_l(\theta) = -l(l+1){}_sY^m{}_l(\theta) \tag{C.1.13}$$

and their related laddering operators:

$$\left(\frac{d}{d\theta} - \frac{m}{\sin\theta} - s\frac{\cos\theta}{\sin\theta}\right){}_sY^m{}_l(\theta) = -\sqrt{(l-s)(l+s+1)}{}_{s+1}Y^m{}_l(\theta) \tag{C.1.14}$$

$$\left(\frac{d}{d\theta}+\frac{m}{\sin\theta}+s\frac{\cos\theta}{\sin\theta}\right){}_sY^m{}_l(\theta)=+\sqrt{(l+s)(l-s+1)}{}_{s-1}Y^m{}_l(\theta)\ . \quad \text{(C.1.15)}$$

The unknown functions can be cast in the form:

$$\begin{aligned}
\Psi_0^B &= e^{-i\omega t}e^{im\phi}{}_2Y^m{}_l(\theta)R_l^{(2)}(r)\\
\chi_1^B &= e^{-i\omega t}e^{im\phi}{}_1Y^m{}_l(\theta)R_l^{(1)}(r)\\
\chi_{-1}^B &= e^{-i\omega t}e^{im\phi}{}_{-1}Y^m{}_l(\theta)\frac{\Delta}{2r^2}R_l^{(-1)}(r)\\
\Psi_4^B &= e^{-i\omega t}e^{im\phi}{}_{-2}Y^m{}_l(\theta)\frac{\Delta^2}{4r^4}R_l^{(-2)}(r)
\end{aligned} \quad \text{(C.1.16)}$$

where $\Delta = r^2 - 2Mr + Q^2$, and after manipulations, one gets two sets of coupled ODE's. The first set is:

$$\begin{aligned}
&\left[-\omega^2\frac{r^4}{\Delta}+4i\omega r\left(-2+\frac{r(r-M)}{\Delta}+\frac{Q^2}{3Mr-4Q^2}\right)-\Delta\frac{d^2}{dr^2}\right.\\
&-\left\{6(r-M)-\frac{4Q^2\Delta}{r(3Mr-4Q^2)}\right\}\frac{d}{dr}-4-\frac{2Q^2}{r^2}\\
&\left.+\frac{4Q^2(r^2+2Mr-3Q^2)}{r^2(3Mr-4Q^2)}+\frac{3Mr-4Q^2}{3Mr-2Q^2}(l-1)(l+2)\right]R_l^{(2)} \quad \text{(C.1.17)}\\
&=\frac{2\sqrt{2}Q\sqrt{(l-1)(l+2)}r^3}{3Mr-2Q^2}\left(-i\omega\frac{r^2}{\Delta}+\frac{d}{dr}+\frac{4}{r}\right.\\
&\left.-\frac{4Q^2}{r(3Mr-4Q^2)}\right)R_l^{(1)}
\end{aligned}$$

$$\begin{aligned}
&\left[-\omega^2\frac{r^4}{\Delta}+2i\omega r\left(-2+\frac{r(r-M)}{\Delta}-\frac{Q^2}{3Mr-2Q^2}\right)-\Delta\frac{d^2}{dr^2}\right.\\
&-\left\{\frac{6\Delta}{r}+4(r-M)-\frac{2Q^2\Delta}{r(3Mr-2Q^2)}\right\}\frac{d}{dr}-\frac{18r^2-24Mr+2Q^2}{r^2}\\
&\left.+\frac{12Q^2\Delta}{r^2(3Mr-2Q^2)}+\frac{3Mr-2Q^2}{3Mr-4Q^2}(l-1)(l+2)\right]R_l^{(1)}\\
&=\frac{-\sqrt{2}Q^2\sqrt{(l-1)(l+2)}\Delta}{r^3(3Mr-4Q^2)}\left(i\omega\frac{r^2}{\Delta}+\frac{d}{dr}-\frac{2}{r}+\frac{4(r-M)}{\Delta}\right.\\
&\left.-\frac{2Q^2}{r(3Mr-2Q^2)}\right)R_l^{(2)}\quad .
\end{aligned} \tag{C.1.18}$$

The quantities $(R_l^{(-1)})^*$ e $(R_l^{(-2)})^*$ (from χ_{-1}^{B} e Ψ_4^B), satisfy the same equations of $R_l^{(1)}$ and $R_l^{(2)}$. At this point decoupling this system of ordinary differential equations is straightforward.

Similarly, the Kerr case is separable using but the so-called spin-weighted spheroidal harmonics [8, 130]:

$$(H_0+H_1)\Theta(\theta)=-E\Theta(\theta) \tag{C.1.19}$$

where:

$$H_0=\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-\left(\frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta}\right)\right] \tag{C.1.20}$$

$$H_1=a^2\omega^2\cos^2\theta-2a\omega s\cos\theta \tag{C.1.21}$$

and E is the eigenvalue. We have factorized the spherical and the spheroidal parts to give the problem the form of a typical Quantum Mechanics exercise. In fact depending if the H_1 term is small or not, the way to approach the problem is very different. Unfortunately, in this case the laddering operators are not know [131] and this does not allow the same strategy used in the case of the Reissner-Nordström spacetime. In the case of the Kerr spacetime instead, this is not a problem because laddering operators are unnecessary to solve completely the problem. In the case of the Kerr-Newman spacetime this creates a "formal" problem. In fact the presence of the charge Q generates "ugly" terms which don't allow the separation of variables in all known coordinates.

A hypothetical separation of variables in these coordinates would have been stopped by the explicit absence of laddering operators. During the last 25 years there have been various attempts to solve this problem. One idea, proposed in Chandrasekhar's monography [131], is to decouple the PDE's before the separation of variables, obtaining 4th order or higher linear PDE's. This task could be accomplished only using a super-computer, because of the 4th order derivatives. Another formulation was developed using de Cahen-Debever-Defrise formalism, but a part some elegant conservative equations, the problem has not been solved [132, 133]. In conclusion the problem remains still open. A new approach has been developed [9, 10] for vacuum spacetimes which gives directly the full set of perturbative equations. The direct extension of this work to the case of Einstein-Maxwell or more complicate spacetimes can put in a new light this difficult problem.

After this short historical overview we can discuss the results obtained by ICRANet researchers in this field. In [134], due to Cherubini and Ruffini, gravitational and electromagnetic perturbations to the Kerr-Newman spacetime using Maple tensor package are shown; a detailed analysis for slightly charged, rotating and oblate black hole is presented too. Subsequent to this article there have been various studies regarding the Teukolsky Master Equations (TMEs) in General Relativity. To this aim, a new form is found for the Teukolsky Master Equation in Kerr and interpreted in terms of de Rham-Lichenrowicz laplacians. The exact form of these generalized wave equations in any vacuum spacetime is given for the Riemann and Maxwell tensors, and the equations are linearized at any order, obtaining a hierarchy. It is shown that the TME for any Petrov type D spacetime is nothing more than a component of this laplacian linearized and that the TME cannot be derived by variational principles [9, 10]. More in detail, the Teukolsky Master Equation in the Kerr case, can be cast in a more compact form (Bini-Cherubini-Jantzen-Ruffini form) by introducing a "connection vector" whose components are:

$$
\begin{aligned}
\Gamma^t &= -\frac{1}{\Sigma}\left[\frac{M(r^2-a^2)}{\Delta}-(r+ia\cos\theta)\right]\\
\Gamma^r &= -\frac{1}{\Sigma}(r-M)\\
\Gamma^\theta &= 0\\
\Gamma^\phi &= -\frac{1}{\Sigma}\left[\frac{a(r-M)}{\Delta}+i\frac{\cos\theta}{\sin^2\theta}\right]\,.
\end{aligned}
\tag{C.1.22}
$$

It's easy to prove that:

$$\nabla^\mu \Gamma_\mu = -\frac{1}{\Sigma} \qquad , \qquad \Gamma^\mu \Gamma_\mu = \frac{1}{\Sigma}\cot^2\theta + 4\psi_2^A \tag{C.1.23}$$

and consequently the Teukolsky Master Equation assumes the form:

$$[(\nabla^\mu + s\Gamma^\mu)(\nabla_\mu + s\Gamma_\mu) - 4s^2\psi_2^A]\psi^{(s)} = 4\pi T \tag{C.1.24}$$

where ψ_2^A is the only non vanishing NP component of the Weyl tensor in the Kerr background in the Kinnersley tetrad (C.1.5) (with $Q = 0$). Equation (C.1.24) gives a common structure for these massless fields in the Kerr background varying the "s" index. In fact, the first part in the lhs represents (formally) a D'Alembertian, corrected by taking into account the spin-weight, and the second part is a curvature (Weyl) term linked to the "s" index too. This particular form of the Teukolsky Master Equation forces us to extend this analysis in the next sections because it suggests a connection between the perturbation theory and a sort of generalized wave equations which differ from the standard ones by curvature terms. In fact generalized wave operators are know in the mathematical literature as De Rham-Lichnerowicz Laplacians and the curvature terms which make them different from the ordinary ones are given by the Weitzenböck formulas. Mostly known examples in electromagnetism are

- the wave equation for the vector potential A_μ:

$$\nabla_\alpha \nabla^\alpha A_\mu - R_\mu{}^\lambda A_\lambda = -4\pi J_\mu \qquad , \qquad \nabla^\alpha A_\alpha = 0 \tag{C.1.25}$$

- the wave equation for the Maxwell tensor :

$$\nabla^\mu \nabla_\mu F_{\nu\lambda} + R_{\rho\mu\nu\lambda}F^{\rho\mu} - R^\rho{}_\lambda F_{\nu\rho} + R^\rho{}_\nu F_{\lambda\rho} = -8\pi \nabla_{[\mu} J_{\nu]} \tag{C.1.26}$$

while for the gravitational case one has

- the wave equation for the metric perturbations:

$$\nabla_\alpha \nabla^\alpha \bar{h}_{\mu\nu} + 2R_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} - 2R_{\alpha(\mu}\bar{h}_{\nu)}{}^\alpha = 0,$$

$$\nabla_\alpha \bar{h}_\mu{}^\alpha = 0, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h_\alpha{}^\alpha \tag{C.1.27}$$

- the wave equation for the Riemann Tensor

$$
\begin{aligned}
R^{\alpha\beta}{}_{\gamma\delta;\epsilon}{}^{\epsilon} &= 4R^{[\alpha}{}_{[\gamma;\delta]}{}^{\beta]} - 2R^{[\alpha}{}_{\epsilon}R^{\beta]\epsilon}{}_{\gamma\delta} - 2R^{\alpha\mu\beta\nu}R_{\mu\nu\gamma\delta} \\
&\quad -4R^{[\alpha}{}_{\mu\nu[\gamma}R^{\beta]\mu\nu}{}_{\delta]} \,. \qquad \text{(C.1.28)}
\end{aligned}
$$

These equations are "non minimal," in the sense that they cannot be recovered by a minimal substitution from their flat space counterparts. A similar situation holds in the standard Quantum Field Theory for the electromagnetic Dirac equation. In fact, applying for instance to the Dirac equation an "ad hoc" first order differential operator one gets the second order Dirac equation

$$
\begin{aligned}
&(i\partial\!\!\!/ - eA\!\!\!/ + m)(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = \\
&\left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right]\psi = 0, \qquad \text{(C.1.29)}
\end{aligned}
$$

where the notation is obvious. It is easy to recognize in equation (C.1.29) a generalized Laplacian and a curvature (Maxwell) term applied to the spinor. Moreover this equation is "non minimal", in the sense that the curvature (Maxwell) term cannot be recovered by electromagnetic minimal substitution in the standard Klein-Gordon equation for the spinor components. The analogous second order Dirac equation in presence of a gravitational field also has a non minimal curvature term and reduces to the form:

$$
(\nabla_\alpha \nabla^\alpha + m^2 + \frac{1}{4}R)\psi = 0 \quad . \qquad \text{(C.1.30)}
$$

The general TME formalism is applied to other exact solutions of the vacuum Einstein field equations of Petrov type D. A new analysis of the Kerr-Taub-NUT black hole is given, focussing on Mashhoon spin-coupling and superradiance [135, 59].

More in detail, in [135] Bini, Cherubini and Jantzen studied a single master equation describing spin $s = 0 - 2$ test field gauge and tetrad-invariant perturbations of the Taub-NUT spacetime. This solution of vacuum Einstein field equations describes a black hole with mass M and gravitomagnetic monopole moment ℓ. This equation can be separated into its radial and angular parts. The behaviour of the radial functions at infinity and near the horizon is studied. The angular equation, solved in terms of hypergeometric functions, can be related both to spherical harmonics of suitable weight, resulting from the coupling of the spin-weight of the field and the gravitomagnetic monopole moment of the spacetime, and to the total angular momentum operator associated with the spacetime's rotational symmetry. The results are compared with the Teukolsky master equation for the Kerr spacetime.

In [59] instead Bini, Cherubini, Jantzen and Mashhoon have studied a single master equation describing spin $s \leq 2$ test fields that are gauge- and tetrad-invariant perturbations of the Kerr-Taub-NUT (Newman - Unti - Tamburino) spacetime representing a source with a mass M, gravitomagnetic monopole moment $-\ell$, and gravitomagnetic dipole moment (angular momentum) per unit mass a. This equation can be separated into its radial and angular parts. The behavior of the radial functions at infinity and near the horizon is studied and used to examine the influence of l on the phenomenon of superradiance, while the angular equation leads to spin-weighted spheroidal harmonic solutions generalizing those of the Kerr spacetime. Finally, the coupling between the spin of the perturbing field and the gravitomagnetic monopole moment is discussed.

In [69] instead Bini and Cherubini investigate the algebraically special frequencies of Taub-NUT black holes in detail in comparison with known results concerning the Schwarzschild case. The periodicity of the time coordinate, required for regularity of the solution, prevents algebraically special frequencies to be physically acceptable. In the more involved Kerr-Taub-NUT case, the relevant equations governing the problem are obtained. The formalism is applied to the C-metric, and physical speculations are presented concerning the spin-acceleration coupling.

In [70] Bini, Cherubini and Mashhoon study the vacuum C metric and its physical interpretation in terms of the exterior spacetime of a uniformly accelerating spherically-symmetric gravitational source. Wave phenomena on the linearized C metric background are investigated. It is shown that the scalar perturbations of the linearized C metric correspond to the gravitational Stark effect. This effect is studied in connection with the Pioneer anomaly.

In [71] instead Bini, Cherubini and Mashhoon analysed the massless field perturbations of the accelerating Minkowski and Schwarzschild spacetimes. The results are extended to the propagation of the Proca field in Rindler spacetime. They examine critically the possibility of existence of a general spinacceleration coupling in complete analogy with the well-known spinrotation coupling. They argue that such a direct coupling between spin and linear acceleration does not exist.

In [72] Cherubini, Bini, Bruni and Perjes consider vacuum Kasner spacetimes, focusing on those that can be parametrized as linear perturbations of the special Petrov type D case. In particular they analyze in detail the perturbations which map the Petrov type D Kasner spacetime into another Kasner spacetime of Petrov type I. For these 'quasi-D' Kasner models they first investigate the modification to some curvature invariants and the principal null directions, both related to the Petrov classification of the spacetime. This simple Kasner example allows one to clarify the fact that perturbed spacetimes do not retain in general the speciality character of the background. In fact, there are four distinct principal null directions, although they are not necessarily first-order perturbations of the background principal null directions. Then in the Kasner type D background they derive a Teukolsky master equation, a classical tool for studying black-hole perturbations of any spin. This further step allows one to control totally general cosmologies around such a background as well as to show, from a completely new point of view, the well-known absence of gravitational waves in Kasner spacetimes.

Recent progress in black hole perturbations was obtained in 2013 by Bini and Damour [77] who showed how to explicitly compute in Post-Newtonian sense (as well as to re-sum and convert in the Effective One-Body formalism) gravitational perturbations due a massive particle in a circular orbit around a Schwarzschild black hole. This was made possible thanks to the use of certain Gravitational Self-Force tools. Since then, the same approach (with small modifications) has allowed to discuss perturbations in a Schwarzschild and Kerr spacetime, due to massive and spinning particles in bounded (eccentric) motion on the equatorial plane of these spacetimes. All results have been framed in a gauge-invariant form and have provided an impressive number of analytic information for several companion formalism (Post-Newtonian, Post-Minkowskian, Gravitational Self-Force, Effective Field Theories, Scattering Amplitudes, etc.). For the most recent results see e.g., [78, 79, 80, 81].

D. Cosmology

D.1. Mixmaster universe and the spectral index

The Bianchi type IX spatially homogeneous vacuum spacetime also known as the Mixmaster universe has served as a theoretical playground for many ideas in general relativity, one of which is the question of the nature of the chaotic behavior exhibited in some solutions of the vacuum Einstein equations and another is the question of whether or not one can interpret the spacetime as a closed gravitational wave. In particular, to describe the mathematical approach to an initial cosmological singularity, the exact Bianchi type IX dynamics leads to the BLK approximation involving the discrete BLK map which acts as the transition between phases of approximately Bianchi type I evolution. The parameters of this map are not so easily extracted from the numerical evolution of the metric variables. However, recently it has been realized that these parameters are directly related to transitions in the scale-free part of the Weyl tensor. In fact this leads to a whole new interpretation of what the BLK dynamics represents.

For a given foliation of any spacetime, one can always introduce the scale free part of the extrinsic curvature when its trace is nonzero by dividing by that trace. In the expansion-normalized approach to spatially homogeneous dynamics, this corresponds to the expansion-normalized gravitational velocity variables. This scale free extrinsic curvature tensor can be characterized by its eigenvalues, whose sum is 1 by definition: these define three functions of the time parametrizing the foliation which generalize the Kasner indices of Bianchi type I vacuum spacetimes. A phase of velocity-dominated evolution is loosely defined as an interval of time during which the spatial curvature terms in the spacetime curvature are negligible compared to the extrinsic curvature terms. Under these conditions the vacuum Einstein equations can be approximated by ordinary differential equations in the time. These lead to a simple scaling of the eigenvectors of the extrinsic curvature during which the generalized Kasner indices remain approximately constant and simulate the

Bianchi type I Kasner evolution.

The Weyl tensor can be also be repackaged as a second rank but complex spatial tensor with respect to the foliation and its scale free part is determined by a single complex scalar function of its eigenvalues, a number of particular representations for which are useful. In particular the so called speciality index is the natural choice for this variable which is independent of the permutations of the spatial axes used to order the eigenvalues, and so is a natural 4-dimensional tracker of the evolving gravitational field quotienting out all 3-dimensional gauge-dependent quantities. During a phase of velocity-dominated ("Kasner") evolution, the Weyl tensor is approximately determined by the extrinsic curvature alone, and hence the scalefree invariant part of the Weyl tensor is locked to the generalized Kasner indices exactly as in a Kasner spacetime. Of course during transitions between velocity-dominated evolution where the spatial curvature terms are important, the generalized Kasner indices and the Weyl tensor are uncoupled in their evolution, but the transition between one set of generalized Kasner indices and the next is locked to a transition in the scalefree Weyl tensor. This idealized mapping, approximated by the BKL map between Kasner triplets, can be reinterpreted as a continuous transition in the Weyl tensor whose scale invariant part can be followed through the transition directly. For spatially homogeneous vacuum spacetimes, the BLK transition is a consequence of a Bianchi type II phase of the dynamics which can be interpreted as a single bounce with a curvature wall in the Hamiltonian approach to the problem. One can in fact follow this transition in the Weyl tensor directly with an additional first order differential equation which is easily extracted from the Newman-Penrose equations expressed in a frame adapted both to the foliation and the Petrov type of the Weyl tensor.

This type of Weyl transition in the Mixmaster dynamics can be followed approximately using the Bianchi type II approximation to a curvature bounce, leading to a temporal spike in the real and imaginary parts of the speciality index which represents a circuit in the complex plane between the two real asymptotic Kasner points (a "pulse"). The graph of the speciality index versus time thus serves as a sort of electrocardiogram of the "heart" of the Mixmaster dynamics, stripping away all the gauge and frame dependent details of its evolution except for the choice of time parametrization, which is a recent nice result of our investigation.

D.2. Cosmological fluids obeying a non-ideal equation of state

Current improvements in cosmological measurements strongly favor the standard model of the universe being spatially flat, homogeneous and isotropic on large scales and dominated by dark energy consistently with the effect of a cosmological constant and cold dark matter. In the literature such a concordance model is referred to as Λ-Cold Dark Matter (ΛCDM) model: the universe is well described by a Friedmann-Robertson-Walker (FRW) metric, whose gravity source is a mixture of non-interacting perfect fluids including a cosmological constant. At early times the universe was radiation-dominated, but the present contribution of radiation is negligibly small. The dominant contribution to the mass-energy budget of the universe today is due to dark energy, obeying an equation of state $p_{\rm de} \simeq -\rho_{\rm de}$. The cosmological constant thus acts as an effective negative pressure, allowing the total energy density of the universe to remain constant even though the universe expands.

However, no theoretical model determining the nature of dark energy is available as yet, leaving its existence still unexplained. A big effort has been spent in recent years in formulating cosmological models with modified equation of state [142]. These models include, for instance, a decaying scalar field (quintessence) minimally coupled to gravity, similar to the one assumed by inflation [143], scalar field models with nonstandard kinetic terms ($k-$essence) [144], the Chaplygin gas [145], braneworld models and cosmological models from scalar-tensor theories of gravity (see, e.g., Refs. [146, 147] and references therein).

Following the same line of thinking, we have recently proposed in Ref. [148] a cosmological model with a fluid source obeying a non-ideal equation of state with "asymptotic freedom," first introduced by Shan and Chen (SC) in the context of lattice kinetic theory [149]. Such an equation of state supports a phase transition between low and high density regimes, both characterized by an ideal gas behavior, i.e., pressure and density change in linear proportion to each other. Similarly to the case of lattice kinetic theory, in which the stabilizing effect of hard-core repulsion is replaced by an asymptotic-free attraction, the repulsive effect of the cosmological constant is here replaced by a scalar field with asymptotic-free attraction. We have used these properties to model the growth of the dark matter-energy component of the universe, showing that a cosmological FRW fluid obeying a SC-like equation of state naturally evolves from an ordinary energy density component towards a present-day universe with a suitable dark energy component, with no need of invoking any cosmological constant. We have also provided some observational tests in support to our model. More precisely, we have drawn the Hubble diagram (distance modulus vs redshift) as well as the expansion history of the universe (Hubble parameter vs redshift), showing that they are consistent with current astronomical data.

We have then investigated in Ref. [150] the possibility that a SC-like equation of state may also be used to describe an early inflationary universe. Inflation is an epoch of accelerated expansion, which was originally assumed as a mechanism to solve several puzzles of the standard Big Bang scenario, e.g., the flatness and horizon problems [151, 152, 153, 154, 155, 156]. It also provides plausible scenarios for the origin of the large scale structure of the universe, as well as the formation of anisotropies in the cosmic microwave background radiation. Many different kinds of inflationary models have been developed so far, including recent attempts to construct consistent models of inflation based on superstring or supergravity models.

In the context of inflation, we have represented a SC fluid in a flat FRW universe filled by a scalar field in an external potential, whose energy density and pressure are identified (and fixed by) with the SC corresponding quantities. Therefore, the potential is completely determined and we have analyzed in detail its role in the slow-roll approximation of inflation. We have found that simple choices of the free parameters of the SC model are consistent with current Planck and WMAP data, i.e., the minimal viability request for any model. Furthermore, the equation of state undergoes a transition between $p/\rho < 0$ (exotic matter) during inflation to $p/\rho > 0$ (ordinary matter) for late times, thus providing a natural exit mechanism. As a result, a SC-like equation of state for the fluid source of the early universe may play a role also in the context of inflation.

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