Self-gravitating Systems of Dark Matter Particles

Contents

1 Topics

The problem of the distribution of stars in globular clusters, and more general in galactic systems, has implied one of the results of most profound interest in classical astronomy. The pioneering works of Michie (1963) and King (1966) considered the effects of collisional relaxation and tidal cutoff by studying solutions of the Fokker-Planck equation. There, it was shown that stationary solutions are well described by pseudo-isothermal sphere models, based on simple Maxwell energy distribution functions with a constant subtracting term interpreted as an energy cutoff. An extension of this statistical analysis with thermodynamic considerations, which includes the effects of violent (collisionless) relaxation, was studied by Lynden-Bell (1967) with important implications to the problem of virialization of dark matter (DM) halos which are still of current interest.

Later on, in a series of works by R. Ruffini and collaborators (see, e.g., Ruffini and Stella, 1983 in Newtonian gravity and Gao et al., 1990 in GR), the emphasis changed from self-gravitating systems of classic stars (which verify Maxwellian distributions) to systems of fermionic particles, with the aim of describing galactic DM halos. In this line, an important contribution was given by Chavanis (2004), who studied generalized kinetic theories accounting for collisionless relaxation processes, obtaining a class of generalized Fokker-Planck equation for fermions with applications to DM halo formation. It was there explicitly shown the possibility to obtain, out of general thermodynamic principles, a generalized Fermi-Dirac distribution function including an energy cutoff, extending the former results by Michie and King to quantum particles.

Within this field of research, our group aims to contribute to the understanding of the DM nature. In particular, we mainly focus on a possible fermionic nature of the DM particles, and its consequences in astrophysics and cosmology.

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3 Highlighted Publications 2022

3.1 Fermionic dark matter and galactic structures

The extensive and continuous monitoring of the closest stars to the Galactic center has been producing over decades a large amount of high-quality data of their positions and velocities. The explanation of these data, especially the S2 star motion, reveals the presence of a compact source, Sagittarius A* (Sgr A*), whose mass must be about $4 \times 10^6\ M_\odot$. This result has been protagonist of the awarded Nobel Prize in Physics 2020 to Reinhard Genzel and Andrea Ghez "*for the discovery of a supermassive compact object at the centre of our galaxy*". Traditionally, the nature of Sgr A* has been attributed to a supermassive black hole (SMBH), even though a robust proof of its existence is absent. Further, recent data on the motion of the G2 cloud show that its post-peripassage velocity is lower than the expected one from a Keplerian orbit around the hypothesized SMBH. An attempt to overcome this difficulty has used a friction force, produced (arguably) by an accretion flow whose presence is also observationally unconfirmed.

We have advanced in Argüelles et al. (2018) and in Becerra-Vergara et al. (2020, 2021); Argüelles et al. (2021a, 2022) an alternative scenario that identifies the nature of the supermassive compact object in the Milky Way center, with a highly concentrated core of DM made of fermions (referred from now on as *darkinos*). The existence of a high dense core of DM at the center of galaxies had been demonstrated in Ruffini et al. (2015), where it was shown that *core-halo* profiles are obtained from the Ruffini-Argüelles-Rueda (RAR) fermionic DM model. In the RAR model, the DM galactic structure is calculated treating the *darkinos* as a self-gravitating system at finite temperatures, in thermodynamic equilibrium, in general relativity. It has been already shown that this model, for *darkinos* of 48–345 keV, successfully explains the observed halo rotation curves of the Milky Way (Argüelles et al., 2018) as well as the ones of other galaxy types (Argüelles et al., 2019).

Therefore, since 2020 we move forward by performing first in Becerra-

Vergara et al. (2020) and then in Becerra-Vergara et al. (2021); Argüelles et al. (2021a, 2022), observational tests of the theoretically predicted dense quantum core at the Galactic center within the DM-RAR model. Namely, to test whether the quantum core of *darkinos* could work as an alternative to the SMBH scenario for SgrA*. For this task, we have first shown in Becerra-Vergara et al. (2020) that the sole DM core, for 56 keV *darkinos*, explains the orbit of S2 (position and radial velocities) and G2 with even better accuracy than the traditional Scwarzchild BH scenario, and without the need to invoque a drag force nor other external agents acting on G2. More recently in Becerra-Vergara et al. (2021), we have extended the above results to all the best observationally resolved S-cluster stars, namely to the up-to-date astrometry data of the 17 S-stars orbiting Sgr A*, achieving to explain the dynamics of the S-stars with similar (and some cases better) accuracy compared to a central BH model.

More recently in Argüelles et al. (2022), we aimed the focus to the study of the periapsis precession of the S2 star orbit. We have there quantified for the first time within the RAR-DM model, the effects on the S2-star periapsis (precession) shift due to an extended DM mass filling the S2 orbit, in contrast with the vacuum solution of the traditional Schwarzschild BH scenario. The main result can be summarized as follows: while the Schwarzschild BH scenario predicts a unique prograde precession for S2, in the DM scenario it can be either retrograde or prograde, depending on the amount of DM mass enclosed within the S2 orbit, which in turn is a function of the *darkino* mass (see Fig. 3.1, together with Table below).

We further show in Fig. 3.2 the relativistic precession of S2 projected orbit, in a *right ascension - declination* plot. It can be there seen that while the positions in the plane of the sky nearly coincide about the last pericentre passage in the three models, they can be differentiated close to next apocentre. Specifically, the upper right panel evidences the difference at apocentre between the prograde case (as for the BH and RAR model with $m = 58$ keV), and the retrograde case (i.e. RAR model with $m = 56$ keV).

Unfortunately, as can be already evidenced from Fig. 3.2, all the current and publicly available data of S2 can not discriminate between the two models, but upcoming S2 astrometry close to next apocentre passage could potentially establish if Sgr A* is governed by a classical BH or by a quantum DM system.

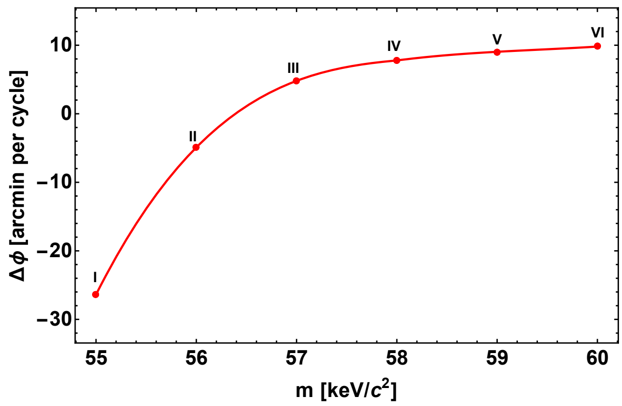

Figure 3.1: Relativistic periapsis precession $\Delta\phi$ per orbit as a function of the *darkino* mass as predicted by the RAR DM models for the S2-star. The precession is retrograde for $m < 56.4$ keV$/c^2$ while it becomes prograde for $m > 56.4$ keV$/c^2$ (see also Table below).

Table 3.1: Comparison of the BH and RAR DM models that best fit of all the publicly available data of the S2 orbit. The $2nd$ column shows the central object mass, M_{CO}. For the Schwarzschild BH model, $M_{CO} = M_{BH}$, while for the RAR model, $M_{CO} = M_c$, with M_c the DM core mass. The $3rd$ column shows the radius of the central object, r_c. For the Schwarzschild BH model, r_c is given by the event horizon radius, $R_{Sch} = 2GM_{BH}/c^2$. The $4th$ column shows the DM mass enclosed within the S2 orbit, $\Delta M_{DM}/M_{CO}$. The best fitting pericentre and apocentre radii of the S2 orbit are given, respectively, in the $5th$ and $6th$ column. The values of the average reduced-χ^2 of the best fits, defined as in Becerra-Vergara et al., 2020, are given in the $7th$ column. The last two columns shows, respectively, the model predictions of the periapsis precession of the real orbit, $\Delta\phi$, and of the sky-projected orbit, $\Delta\phi_{sky}$.

	Model	M_{CO} $[10^6 M_\odot]$	r_c $[mpc]$	$\Delta M_{DM}/M_{CO}$	r_p $[as]$	r_a $[as]$	$\langle\bar{\chi}^2\rangle$	$\Delta\phi$ $[arcmin]$	$\Delta\phi_{sky}$ $[arcmin]$
I	RAR ($m = 55$ keV$/c^2$)	3.55	0.446	1.39×10^{-2}	0.01417	0.23723	2.9719	−26.3845	−32.1116
II	RAR ($m = 56$ keV$/c^2$)	3.50	0.427	5.99×10^{-3}	0.01418	0.23618	3.0725	−4.9064	−5.9421
III	RAR ($m = 57$ keV$/c^2$)	3.50	0.407	2.21×10^{-3}	0.01417	0.23617	3.2766	4.8063	5.8236
IV	RAR ($m = 58$ keV$/c^2$)	3.50	0.389	7.13×10^{-4}	0.01424	0.23609	3.2814	7.7800	9.4243
V	RAR ($m = 59$ keV$/c^2$)	3.50	0.371	2.93×10^{-4}	0.01418	0.23613	3.3356	9.0456	10.9613
VI	RAR ($m = 60$ keV$/c^2$)	3.50	0.355	1.08×10^{-4}	0.01423	0.23610	3.3343	9.8052	11.8764
	BH	4.07	3.89×10^{-4}	0	0.01427	0.23623	3.3586	11.9501	14.4947

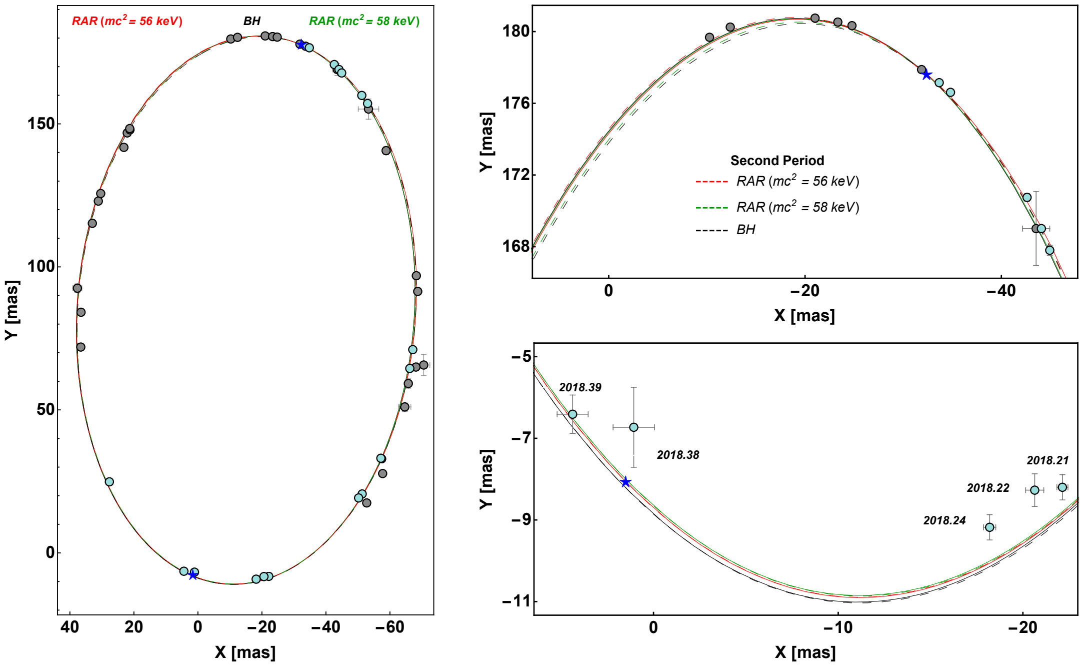

Figure 3.2: Relativistic precession of S2 in the projected orbit on the plane of the sky as predicted in the BH and RAR DM models. While it is prograde for the BH and RAR (m = 58 keV) (in dashed black and green respectively), it is retrograde for the RAR DM model (m = 56 keV) (in dashed red). The solid (theoretical) curves and gray (data) points correspond to the first period (≈ 1994–2010) while the dashed (theoretical) curves and cyan (data) points to the second period (≈ 2010–2026). *Right panels*: zoom of the region around apocentre *(top panel)* and pericentre *(bottom panel)*. The astrometric measurements are taken from Do et al. (2019)

A further interesting consequence of this scenario is that, a core made of these *darkinos*, becomes unstable against gravitational collapse into a BH for a threshold mass of $\sim 10^8\ M_{\odot}$. That is, collapsing DM cores can provide the BH seeds for the formation of SMBHs in active galaxies (such as M87), without the need of prior star formation, or other BH seed mechanisms involving super-Eddington accretion rates as demonstrated in Argüelles et al. (2021b) from thermodynamic arguments. This topic is of major interest within the group, and further consequences and ramifications are being currently studied and will be published in a list of papers in the coming years, including:

- to propose a new paradigm for SMBH formation and growth in a cosmological framework, which is neither based on baryonic matter nor on early Universe physics;
- to study the problem of disk-accretion around such DM-cores starting with the generalization of the Zakura & Suntaev disk equations in the presence of a high concentration of regular matter (i.e. instead of a singularity)
- to use fully relativistic ray-tracing techniques to predict the corresponding shadow-like images around these fermion cores, and compare with the shape and sizes of the ones obtained bt the EHT.

3.2 Interactions in Warm Dark Matter: A View from Cosmological Perturbation Theory

The traditional ΛCDM paradigm of cosmology is in remarkable agreement with large scale cosmological observations and galaxy properties. However, there is an increasing number of tensions of the ΛCDM with observations on smaller scales, such as the so-called "missing" DM sub-halo problem and the "core-cusp" discrepancy.

Specifically, high-resolution cosmological simulations of average-sized halos in ΛCDM predict an overproduction of small-scale structures, significantly larger that the observed number of small satellite galaxies in the Local Group. Moreover, N-body simulations of CDM predict a cuspy density profile for virialized halos, while observations show dwarf spheroidal galaxies (dSphs) having flattened smooth density profiles in their central regions.

A possible alternative to alleviate or try to resolve such tensions, is to consider *warm* dark matter particles (WDM), meaning that they are semi-relativistic during the earliest stages of structure formation with non-negligible free-streaming particle length.

WDM models feature an intermediate velocity dispersion between HDM and CDM that results in a suppression of structures at small scales due to free-streaming. If this free streaming scale today is smaller than the size of galaxy clusters, it can provide a solution to the missing satellites problem. However, thermally produced WDM suffers from the so called *catch-22* problem when studied within N-body simulations (Shao et al., 2013). Such WDM-only simulations either show unrealistic core-sizes for particle masses above the keV range, or they acquire the right halo sizes though for sub-keV masses, in direct conflict with phase-space and Lyman-α constraints.

Another compelling alternative to collisionless CDM, apart from WDM, is to consider interactions in CDM. This consideration relaxes the assumption that CDM interacts only gravitationally after early decoupling, and includes interactions either between DM and SM particles or additional hidden particles, or among DM particles themselves. These later models are denominated as "self-interacting" DM models (SIDM).

With the aim of shedding light into this matter, recently in Yunis et al. (2020) we have provided a general framework for self-interacting WDM in cosmological perturbations, by deriving from first principles a Boltzmann hierarchy which retains certain independence from a particular interaction Lagrangian. There, elastic interactions among the massive particles were considered, to obtain a hierarchy which is more general than the ones usually obtained for non-relativistic (as for cold DM) or for ultra-relativistic (as for neutrinos) approximations. The more general momentum-dependent kernel integrals in the Boltzmann collision terms are explicitly calculated for different field-mediator models, including a scalar field or a massive vector field.

In particular, if the Self Interactions maintain the DM fluid in kinetic equilibrium all the way until the fluid becomes non relativistic, the background distribution function at that moment will switch into a non relativistic form, constituting the scenario known as Non Relativistic Self Decoupling (a.k.a. late kinetic decoupling). The consequences of this scenario are poorly explored in the literature, and only some preliminary results have been recently obtained within simplified DM fluid approximations in Egana-Ugrinovic et al. (2021). However, more recently in Yunis et al. (2022) this late kinetic decoupling physics was fully explored, but this time with self-interactions treated from first principles interaction Lagrangians (i.e. superseeding the fluid-approximation), following the formalism developed in Yunis et al. (2020). There, it was found that, if one imposes continuity of the limiting expressions for the energy density, the non relativistic distribution function can be found in an analytic expression (see Yunis et al. (2021, 2022) for details).

Figure 3.3 illustrates the effects of self-interactions in the matter power spectrum for the case of a massive scalar field-mediator, while including for the late kintetic decoupling case. We have used an extended version of the CLASS code incorporating our results for SI-WDM with particle masses in the $\sim$ keV range. There, we see some of the particular features of the models. We see that the inclusion of Self Interactions, for models with Non Relativistic Self Decoupling (i.e. late kinetic decoupling), the resulting power spectra may differ significantly from its relativistic counterpart. Indeed, we find that in this regime the models are "colder" (i. e. as if they correspond to a higher particle mass), and show even at smaller k values a distinctive oscillatory pattern (see e.g. dot-dashed curves in Fig. 3.3. This indeed has the effect of increasing the amount of small structure formed for these models, implying that the few keV (traditional) WDM models which were excluded from phase-space arguments, can now be back to agreement with observations in this new SI-WDM scenarios.

Most of the tensions inherent to νMSM WDM models come from structure formation, namely MW satellite counts and Lyman-α observations. These are related to the fact that the preferred parameter ranges may underproduce small structure and almost rule out the available parameter space. So, the inclusion of Self Interactions can significantly relax the existing bounds on this family of models. The evaluation of the predictions of these models for the number of MW satellites as well for the observations of the Lyman-α forest were published in last year in Yunis et al. (2022), (see e.g. Fig. 3.4).

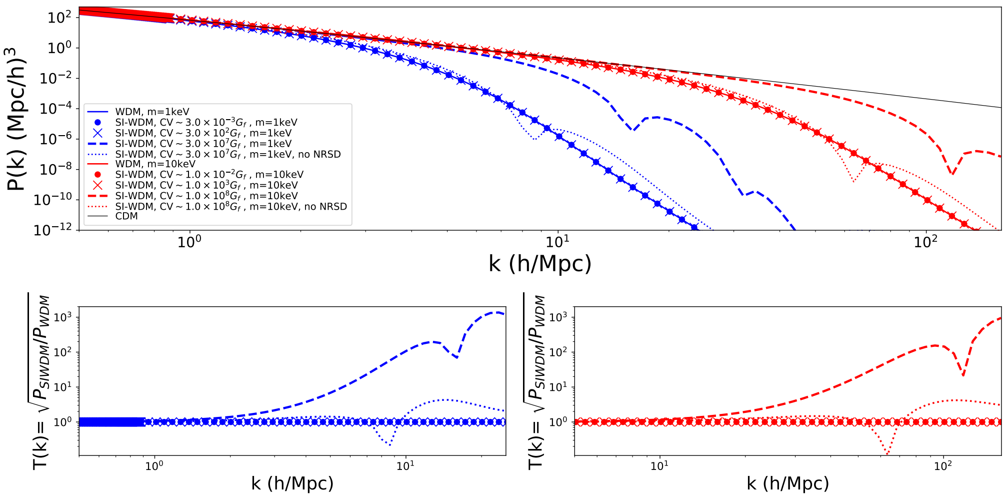

Figure 3.3: Power Spectrum (*top panel*) and Transfer Functions with respect to standard WDM (*bottom panels*) for a vector field SI-WDM model, simulated using the modification to CLASS described in Yunis et al. (2022) and considering the effects of non-relativistic self decoupling when appropriate. We assume the relaxation time approximation (see Yunis et al. (2022) for details), and consider two values of the DM particle mass: 1 and 10 keV. Also plotted are the power spectra of a CDM model and of WDM models with DM mass of 1 and 10 keV. All WDM and SI-WDM models consider a nonresonant production scenario (Dodelson-Widrow mechanism with $T \sim (4/11)^{1/3} T_\gamma$. We also show, for comparison, cosmologies where the interaction constant is high enough to be in the non-relativistic self decoupled regime but these effects are ignored, instead using a relativistic DF (dotted lines).

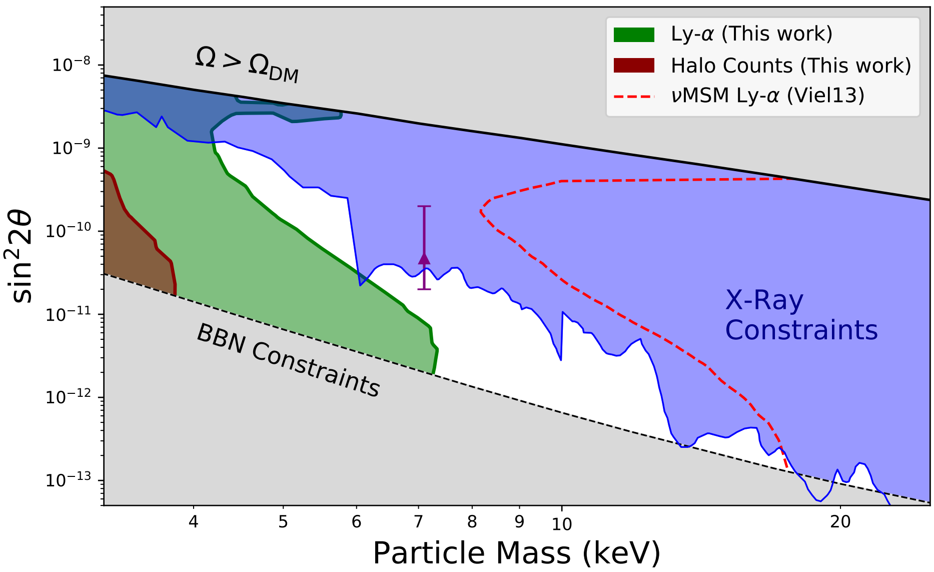

Figure 3.4: Parameter space constraints for νMSM, where MW satellite halo counts and Lyman-α forest bounds are analyzed under a self interacting model as outlined above. For each point (θ, m) in the parameter space we consider a self interacting model under a vector field mediator, with its interaction constant given by $\sigma/m \sim 0.144C_v^2/m^3 = 0.1\text{cm}^2/g$, the upper limit given by Bullet Cluster constraints (see Yunis et al. (2022) for details). For comparison, we plot the Lyman-alpha bounds for the non interacting case for a comparable analysis, plus other bounds to the νMSM parameter space for informative purposes, namely X-Ray indirect detection bounds (in blue) and sterile neutrino production bounds (in grey). We plot as well the sterile neutrino model compatible with a tentative 3.5 keV DM signal, subject of debate in recent years, as a purple triangle. The complete list of references of all such bounds can be found in Yunis et al. (2022).

4 Publications 2022

1. Argüelles, C. R.; Mestre, M. F.; Becerra-Vergara, E. A.; Crespi, V.; Krut, A.; Rueda, J. A.; Ruffini, R., "What does lie at the Milky Way centre? Insights from the S2 star orbit precession", Monthly Notices of the Royal Astronomical Society 511 (2022), issue 1, pp L35-L39.

   It has been recently demonstrated that both, a classical Schwarzschild black hole (BH), and a dense concentration of self-gravitating fermionic dark matter (DM) placed at the Galaxy centre, can explain the precise astrometric data (positions and radial velocities) of the S-stars orbiting Sgr A*. This result encompasses the 17 best resolved S-stars, and includes the test of general relativistic effects such as the gravitational redshift in the S2-star. In addition, the DM model features another remarkable result: the dense core of fermions is the central region of a continuous density distribution of DM whose diluted halo explains the Galactic rotation curve. In this Letter, we complement the above findings by analyzing in both models the relativistic periapsis precession of the S2-star orbit. While the Schwarzschild BH scenario predicts a unique prograde precession for S2, in the DM scenario it can be either retrograde or prograde, depending on the amount of DM mass enclosed within the S2 orbit, which in turn is a function of the DM fermion mass. We show that all the current and publicly available data of S2 can not discriminate between the two models, but upcoming S2 astrometry close to next apocentre passage could potentially establish if Sgr A* is governed by a classical BH or by a quantum DM system.

2. Yunis, R. I; Argüelles, C. R., Scóccola, C. G.; Nacir, D. L.; Giordano, G."Self-interacting dark matter in cosmology: accurate numerical implementation and observational constraints ", Journal of Cosmology and Astroparticle Physics (2022), Issue 02, id.024, pp. 48.

   This paper presents a systematic and accurate treatment of the evolution of cosmological perturbations in self-interacting dark matter models, for particles which decoupled from the primordial plasma while relativistic. We provide a numerical implementation of the Boltzmann hierarchies developed in a previous paper [JCAP, 09 (2020) 041] in a publicly available Boltzmann code and show how it can be applied to realistic DM candidates such as sterile neutrinos either under resonant or non-resonant production mechanisms, and for different field mediators. At difference with traditional fluid approximations - also known as a c eff-c vis parametrizations - our approach follows the evolution of phase-space perturbations under elastic DM interactions for a wide range of interaction models, including the effects of late kinetic decoupling. Finally, we analyze the imprints left by different self interacting models on linear structure formation, which can be constrained using Lyman- forest and satellite counts. We find new lower bounds on the particle mass that are less restrictive than previous constraints.

3. Argüelles, C. R.; Becerra-Vergara, E. A.; Krut, A.; Yunis. R.; Rueda, J. A.; Ruffini, R., "Reshaping our understanding on structure formation with the quantum nature of the dark matter ", International Journal of Moden Physics D 31 (2022), issue 2, id. 2230002.

   We study the nonlinear structure formation in cosmology accounting for the quantum nature of the dark matter (DM) particles in the initial conditions at decoupling, as well as in the relaxation and stability of the DM halos. Different from cosmological N-body simulations, we use a thermodynamic approach for collisionless systems of self-gravitating fermions in general relativity, in which the halos reach the steady state by maximizing a coarse-grained entropy. We show the ability of this approach to provide answers to crucial open problems in cosmology, among others: the mass and nature of the DM particle, the formation and nature of supermassive black holes in the early Universe, the nature of the intermediate mass black holes in small halos, and the core-cusp problem.

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