Generalizations of the Kerr-Newman solution

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2 Brief description

One of the most important metrics in general relativity is the Kerr-Newman solution that describes the gravitational and electromagnetic fields of a rotating charged mass. For astrophysical purposes, however, it is necessary to take into account the effects due to the moment of inertia of the object. To attack this problem, we investigate new exact solutions of Einstein-Maxwell equations which posses an infinite set of gravitational and electromagnetic multipole moments and contain the Kerr-Newman solution as special case.

We study the collapse of a thin dust shell from the point of view of the horizon dynamics. We identify the critical surfaces at which time and space coordinates interchange their roles and investigate their properties by using the formalism of trapped surfaces. We show the existence of marginally outer trapped surfaces that are associated with the presence of quasi-local horizons. A particular shell configuration that avoids the formation of horizons is interpreted as naked shell.

We investigate the interior Einstein's equations in the case of a static, axially symmetric, perfect fluid source. We present a particular line element that is specially suitable for the investigation of this type of interior gravitational fields. Assuming that the deviation from spherically symmetry is small, we linearize the corresponding line element and field equations and find several classes of vacuum and perfect fluid solutions. We find some particular approximate solutions by imposing appropriate matching conditions.

3 Introduction

It is hard to overemphasize the importance of the Kerr geometry not only for general relativity itself, but also for the very fundamentals of physics. It assumes this position as being the most physically relevant rotating generalization of the static Schwarzschild geometry. Its charged counterpart, the Kerr-Newman solution, representing the exterior gravitational and electromagnetic fields of a charged rotating object, is an exact solution of the Einstein-Maxwell equations.

Its line element in Boyer–Lindquist coordinates can be written as

$$\begin{aligned} ds^2 &= \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 \\ &\quad -\frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[(r^2 + a^2)d\varphi - adt]^2 \\ &\quad -\frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 \,, \end{aligned} \tag{3.0.1}$$

where M is the total mass of the object, $a = J/M$ is the specific angular momentum, and Q is the electric charge. In this particular coordinate system, the metric functions do not depend on the coordinates t and ϕ, indicating the existence of two Killing vector fields $\xi^I = \partial_t$ and $\xi^{II} = \partial_\varphi$ which represent the properties of stationarity and axial symmetry, respectively.

An important characteristic of this solution is that the source of gravity is surrounded by two horizons situated at a distance

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{3.0.2}$$

from the origin of coordinates. Inside the interior horizon, r_-, a ring singularity is present which, however, cannot be observed by any observer situated outside the exterior horizon. If the condition $M^2 < a^2 + Q^2$ is satisfied, no horizons are present and the Kerr–Newman spacetime represents the exterior field of a naked singularity.

Despite of its fundamental importance in general relativity, and its theoretical and mathematical interest, this solution has not been especially useful for describing astrophysical phenomena, first of all, because observed astrophysical objects do not possess an appreciable net electric charge. Furthermore, the limiting Kerr metric takes into account the mass and the rotation, but does not consider the moment of inertia of the object. For astrophysical applications it is, therefore, necessary to use more general solutions with higher multipole moments which are due not only to the rotation of the body but also to its shape. This means that even in the limiting case of a static spacetime, a solution is needed that takes into account possible deviations from spherically symmetry.

4 The general static vacuum solution

In general relativity, stationary axisymmetric solutions of Einstein's equations [58] play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of know exact solutions drastically increased after Ernst [2] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies lead finally to the development of solution generating techniques [58] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [4] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

4.1 Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindric coordinates (t, ρ, z, φ). Stationarity implies that t can be chosen as the time coordinate and the metric does not depend on time, i.e. $\partial g_{\mu\nu}/\partial t = 0$. Consequently, the corresponding timelike Killing vector has the components δ^μ_t. A second Killing vector field is associated to the axial symmetry with respect to the axis $\rho = 0$. Then, choosing φ as the azimuthal angle, the metric satisfies the conditions $\partial g_{\mu\nu}/\partial\varphi = 0$, and the components of the corresponding spacelike Killing vector are δ^μ_φ.

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form $g_{\mu\nu} = g_{\mu\nu}(\rho, z)$, it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [4, 5, 6]

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] , \qquad (4.1.1)$$

where f, ω and γ are functions of ρ and z, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$\rho\partial_\rho\Omega = f^2\partial_z\omega , \qquad \rho\partial_z\Omega = -f^2\partial_\rho\omega , \qquad (4.1.2)$$

the vacuum field equations $R_{\mu\nu} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \partial_z^2 f + \frac{1}{f}[(\partial_\rho\Omega)^2 + (\partial_z\Omega)^2 - (\partial_\rho f)^2 - (\partial_z f)^2] = 0 , \qquad (4.1.3)$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Omega) + \partial_z^2\Omega - \frac{2}{f}\left(\partial_\rho f\,\partial_\rho\Omega + \partial_z f\,\partial_z\Omega\right) = 0 , \qquad (4.1.4)$$

$$\partial_\rho\gamma = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right] , \qquad (4.1.5)$$

$$\partial_z\gamma = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right) . \qquad (4.1.6)$$

It is clear that the field equations for γ can be integrated by quadratures, once f and Ω are known. For this reason, the equations (4.1.3) and (4.1.4) for f and Ω are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only. It is interesting to mention that this set of equations can be geometrically interpreted in the context of nonlinear sigma models [7].

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation $\varphi \to -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (4.1.1) with $\omega = 0$, and the field equations can be written as

$$\partial_\rho^2 \psi + \frac{1}{\rho}\partial_\rho \psi + \partial_z^2 \psi = 0\,, \quad f = \exp(2\psi)\,, \tag{4.1.7}$$

$$\partial_\rho \gamma = \rho\left[(\partial_\rho\psi)^2 - (\partial_z\psi)^2\right]\,, \quad \partial_z\gamma = 2\rho\partial_\rho\psi\,\partial_z\psi\,. \tag{4.1.8}$$

We see that the main field equation (4.1.7) corresponds to the linear Laplace equation for the metric function ψ.

4.2 Static solution

The general solution of Laplace's equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as [4, 58]

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2+z^2)^{\frac{n+1}{2}}} P_n(\cos\theta)\,, \qquad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}\,, \tag{4.2.1}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The expression for the metric function γ can be calculated by quadratures by using the set of first order differential equations (4.1.8). Then

$$\gamma = -\sum_{n,m=0}^{\infty} \frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2+z^2)^{\frac{n+m+2}{2}}} \left(P_n P_m - P_{n+1}P_{m+1}\right)\,. \tag{4.2.2}$$

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one of the most interesting special solutions which is Schwarzschild's spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants a_n in such a way that the infinite sum (4.2.1) converges to the Schwarzschild solution in cylindric coordinates. But, or course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzchild's metric.

In fact, it turns out that to investigate the properties of solutions with multipole moments it is more convenient to use prolate spheroidal coordinates (t, x, y, φ) in which the line element can be written as

$$ds^2 = f dt^2 - \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right]$$

where

$$x = \frac{r_+ + r_-}{2\sigma}\,, \quad (x^2 \geq 1), \quad y = \frac{r_+ - r_-}{2\sigma}\,, \quad (y^2 \leq 1) \tag{4.2.3}$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2\,, \quad \sigma = const\,, \tag{4.2.4}$$

and the metric functions are f, ω, and γ depend on x and y, only. In this coordinate system, the general static solution which is also asymptotically flat can be expressed as

$$f = \exp(2\psi)\,, \quad \psi = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n(y) Q_n(x)\,, \quad q_n = const$$

where $P_n(y)$ are the Legendre polynomials, and $Q_n(x)$ are the Legendre functions of second kind. In particular,

$$P_0 = 1, \quad P_1 = y, \quad P_2 = \frac{1}{2}(3y^2 - 1)\,, ...$$

$$Q_0 = \frac{1}{2}\ln\frac{x+1}{x-1}\,, \quad Q_1 = \frac{1}{2}x\ln\frac{x+1}{x-1} - 1\,,$$

$$Q_2 = \frac{1}{2}(3x^2-1)\ln\frac{x+1}{x-1} - \frac{3}{2}x\,, ...$$

The corresponding function γ can be calculated by quadratures and its general expression has been explicitly derived in [8]. The most important special cases contained in this general solution are the Schwarzschild metric

$$\psi = -q_0 P_0(y) Q_0(x)\ , \quad \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}\ ,$$

and the Erez-Rosen metric [9]

$$\psi = -q_0 P_0(y) Q_0(x) - q_2 P_2(y) Q_2(x)\ , \quad \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2} + ...$$

In the last case, the constant parameter q_2 turns out to determine the quadrupole moment. In general, the constants q_n represent an infinite set of parameters that determines an infinite set of mass multipole moments.

5 Stationary generalization

The solution generating techniques [12] can be applied, in particular, to any static seed solution in order to obtain the corresponding stationary generalization. One of the most powerful techniques is the inverse method (ISM) developed by Belinski and Zakharov [13]. We used a particular case of the ISM, which is known as the Hoenselaers–Kinnersley-Xanthopoulos (HKX) transformation to derive the stationary generalization of the general static solution in prolate spheroidal coordinates.

5.1 Ernst representation

In the general stationary case $(\omega \neq 0)$ with line element

$$\begin{aligned} & ds^2 = f(dt - \omega d\varphi)^2 \\ - & \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right] \end{aligned}$$

it is useful to introduce the the Ernst potentials

$$E = f + i\Omega\,, \quad \xi = \frac{1-E}{1+E}\,,$$

where the function Ω is now determined by the equations

$$\sigma(x^2-1)\Omega_x = f^2\omega_y\,, \quad \sigma(1-y^2)\Omega_y = -f^2\omega_x\,.$$

Then, the main field equations can be represented in a compact and symmetric form:

$$(\xi\xi^* - 1)\left\{[(x^2-1)\xi_x]_x + [(1-y^2)\xi_y]_y\right\} = 2\xi^*[(x^2-1)\xi_x^2 + (1-y^2)\xi_y^2]\,.$$

This equation is invariant with respect to the transformation $x \leftrightarrow y$. Then, since the particular solution

$$\xi = \frac{1}{x} \to \Omega = 0 \to \omega = 0 \to \gamma = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2}$$

represents the Schwarzschild spacetime, the choice $\xi^{-1} = y$ is also an exact solution. Furthermore, if we take the linear combination $\xi^{-1} = c_1 x + c_2 y$ and introduce it into the field equation, we obtain the new solution

$$\xi^{-1} = \frac{\sigma}{M} x + i \frac{a}{M} y \ , \ \sigma = \sqrt{M^2 - a^2} \ ,$$

which corresponds to the Kerr metric in prolate spheroidal coordinates.

In the case of the Einstein-Maxwell theory, the main field equations can be expressed as

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2 \xi = 2(\xi^* \nabla \xi - \mathcal{F}^* \nabla \mathcal{F}) \nabla \xi \ ,$$

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2 \mathcal{F} = 2(\xi^* \nabla \xi - \mathcal{F}^* \nabla \mathcal{F}) \nabla \mathcal{F}$$

where ∇ represents the gradient operator in prolate spheroidal coordinates. Moreover, the gravitational potential ξ and the electromagnetic $\mathcal{F}$ Ernst potential are defined as

$$\xi = \frac{1 - f - i\Omega}{1 + f + i\Omega} , \quad \mathcal{F} = 2 \frac{\Phi}{1 + f + i\Omega} \ .$$

The potential Φ can be shown to be determined uniquely by the electromagnetic potentials A_t and A_φ One can show that if ξ_0 is a vacuum solution, then the new potential

$$\xi = \xi_0 \sqrt{1 - e^2}$$

represents a solution of the Einstein-Maxwell equations with effective electric charge e. This transformation is known in the literature as the Harrison transformation [10]. Accordingly, the Kerr–Newman solution in this representation acquires the simple form

$$\xi = \frac{\sqrt{1 - e^2}}{\frac{\sigma}{M} x + i \frac{a}{M} y} , \qquad e = \frac{Q}{M} , \qquad \sigma = \sqrt{M^2 - a^2 - Q^2} \ .$$

In this way, it is very easy to generalize any vacuum solution to include the case of electric charge. More general transformations of this type can be used in order to generate solutions with any desired set of gravitational and electromagnetic multipole moments [11].

5.2 Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds (M, γ) and (N, G) of dimension m and n, respectively. Let M be coordinatized by x^a, and N by X^μ, so that the metrics on M and N can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $G = G(X)$. A harmonic map is a smooth map $X : M \to N$, or in coordinates $X : x \longmapsto X$ so that X becomes a function of x, and the X's satisfy the motion equations following from the action [14]

$$S = \int d^m x \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \qquad (5.2.1)$$

which sometimes is called the "energy" of the harmonic map X. The straightforward variation of S with respect to X^μ leads to the motion equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\, \gamma^{ab}\, \partial_a X^\nu\, \partial_b X^\lambda = 0\,, \qquad (5.2.2)$$

where $\Gamma^\mu_{\ \nu\lambda}$ are the Christoffel symbols associated to the metric $G_{\mu\nu}$ of the target space N. If $G_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $G_{\mu\nu} = \eta_{\mu\nu} = diag(\pm1, ..., \pm1)$, the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (5.2.1) is usually referred to as the Polyakov action [16].

Consider now the case in which the base space M is a stationary axisymmetric spacetime. Then, γ^{ab}, $a, b = 0, ..., 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (4.1.1), i.e.

$$\gamma_{ab} = \begin{pmatrix} f & 0 & 0 & -f\omega \\ 0 & -f^{-1}e^{2k} & 0 & 0 \\ 0 & 0 & -f^{-1}e^{2k} & 0 \\ -f\omega & 0 & 0 & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}. \qquad (5.2.3)$$

Let the target space N be 2-dimensional with metric $G_{\mu\nu} = (1/2)f^{-2}\delta_{\mu\nu}$, $\mu, \nu = 1, 2$, and let the coordinates on N be $X^\mu = (f, \Omega)$. Then, it is straightforward to show that the action (5.2.1) becomes

$$S = \int \mathcal{L}\, dt d\varphi d\rho dz\,, \qquad \mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right]\,, \tag{5.2.4}$$

and the corresponding motion equations (5.2.2) are identical to the main field equations (4.1.3) and (4.1.4).

Notice that the field equations can also be obtained from (5.2.4) by a direct variation with respect to f and Ω. This interesting result was obtained originally by Ernst [2], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \to 2)-$nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2, R)/SO(2)$ [15], and the Lagrangian (5.2.4) can be written explicitly [17] in terms of the generators of the Lie group $SL(2, R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [14].

The form of the Lagrangian (5.2.4) with two gravitational field variables, f and Ω, depending on two coordinates, ρ and z, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (5.2.4). Indeed, if we consider γ^{ab} as a 2-dimensional metric that depends on the parameters ρ and z, the diagonal form of the Lagrangian (5.2.4) implies that $\sqrt{|\gamma|}\gamma^{ab} = \delta^{ab}$. Clearly, this choice is not compatible with the factor ρ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian (5.2.4) cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to "absorb" the unpleasant factor ρ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2,R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields [58]. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

5.3 Representation as a generalized harmonic map

Consider two (pseudo-)Riemannian manifolds (M,γ) and (N,G) of dimension m and n, respectively. Let x^a and X^μ be coordinates on M and N, respectively. This coordinatization implies that in general the metrics γ and G become functions of the corresponding coordinates. Let us assume that not only γ but also G can explicitly depend on the coordinates x^a, i.e. let $\gamma = \gamma(x)$ and $G = G(X,x)$. This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map $X : M \to N$ will be called an $(m \to n)-$generalized harmonic map if it satisfies the Euler-Lagrange equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\,\gamma^{ab}\,\partial_a X^\nu \partial_b X^\lambda + G^{\mu\lambda}\gamma^{ab}\,\partial_a X^\nu\,\partial_b G_{\lambda\nu} = 0\,, \tag{5.3.1}$$

which follow from the variation of the generalized action

$$S = \int d^m x \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.2}$$

with respect to the fields X^μ. Here the Christoffel symbols, determined by the metric $G_{\mu\nu}$, are calculated in the standard manner, without considering the explicit dependence on x. Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term $G_{\mu\nu}(X,x)$ in the Lagrangian density implies that we are taking into account the "interaction" between the base space M and the target space N. This interaction leads to an extra term in the motion equations, as can be seen in (5.3.1). It turns out that this interaction is the result of the effective presence of the gravitational field.

Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (5.3.1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^{\mu}_{\ \nu\lambda}\, \partial_b X^{\lambda} + G^{\mu\lambda}\, \partial_b G_{\lambda\nu})\partial_a X^{\nu} = 0\,, \qquad (5.3.3)$$

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma^{\mu}_{\ \nu\lambda} = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\lambda}\partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma^{\mu}_{\ \nu\lambda} \neq 0$ and (5.3.3) cannot be satisfied. In the general case of a curved target metric, conditions (5.3.3) represent a system of m first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

As we mentioned before, the generalized action (5.3.2) includes an interaction between the base space N and the target space M, reflected on the fact that $G_{\mu\nu}$ depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To see this explicitly we calculate the covariant derivative of the generalized Lagrangian density

$$\mathcal{L} = \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^{\mu} \partial_b X^{\nu} G_{\mu\nu}(X, x)\,, \qquad (5.3.4)$$

and replace in the result the corresponding motion equations (5.3.1). Then, the final result can be written as

$$\nabla_b \widetilde{T}_a^{\ b} = -\frac{\partial \mathcal{L}}{\partial x^a} \qquad (5.3.5)$$

where $\widetilde{T}_a{}^b$ represents the canonical energy-momentum tensor

$$\widetilde{T}_a{}^b = \frac{\partial\mathcal{L}}{\partial(\partial_b X^\mu)}(\partial_a X^\mu) - \delta_a^b\mathcal{L} = 2\sqrt{\gamma}G_{\mu\nu}\left(\gamma^{bc}\partial_a X^\mu\,\partial_c X^\nu - \frac{1}{2}\delta_a^b\gamma^{cd}\partial_c X^\mu\,\partial_d X^\nu\right). \tag{5.3.6}$$

The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space $\gamma_{ab} = \eta_{ab}$, the explicit dependence of the metric of the target space $G_{\mu\nu}(X, x)$ on x generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.

$$T_{ab} = \frac{\delta\mathcal{L}}{\delta\gamma^{ab}}\,. \tag{5.3.7}$$

A straightforward computation shows that for the action under consideration here we have that $\widetilde{T}_{ab} = 2T_{ab}$ so that the generalized conservation law (5.3.5) can be written as

$$\nabla_b T_a{}^b + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial x^a} = 0\,. \tag{5.3.8}$$

For a given metric on the base space, this represents in general a system of m differential equations for the "fields" X^μ which must be satisfied "on-shell".

If the base space is 2-dimensional, we can use a reparametrization of x to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, $T_a{}^a = 0$.

In Section 5.1 we described stationary, axially symmetric, gravitational fields as a $(4 \to 2)-$nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \to 2)-$generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space M, and $X^\mu = (f, \Omega)$ the coordinates on the target space N. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

$$\gamma_{ab} = \delta_{ab} \qquad \text{and} \qquad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \qquad (a,b=1,2;\ \mu,\nu=1,2). \tag{5.3.9}$$

A straightforward computation shows that the generalized Lagrangian (5.3.4) coincides with the Lagrangian (5.2.4) for stationary axisymetric fields, and that the equations of motion (5.3.1) generate the main field equations (4.1.3) and (4.1.4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$. Then

$$T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right], \tag{5.3.10}$$

$$T_{\rho z} = \frac{\rho}{2f^2}\left(\partial_\rho f\, \partial_z f + \partial_\rho\Omega\, \partial_z\Omega\right). \tag{5.3.11}$$

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (5.3.8) on-shell:

$$\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\rho} = 0\,, \tag{5.3.12}$$

$$\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0\,. \tag{5.3.13}$$

Incidentally, the last equation coincides with the integrability condition for the metric function k, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs.(4.1.5,4.1.6) and (5.3.10,5.3.11), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (5.3.13) becomes an identity. Although we have eliminated from the starting Lagrangian (5.2.4) the variable k by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [17] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about k at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)-$generalized harmonic map with metrics given as in (5.3.9). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space M is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on M. Then, M represents the world-sheet of a bosonic string in which τ is measures the time and z is the parameter along the string. The string is "embedded" in the target space N whose metric is conformally flat and explicitly depends on the time parameter τ. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates ρ and z are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a\to\infty} f = 1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \omega = c_1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \Omega = O\left(\frac{1}{x^a}\right) \tag{5.3.14}$$

where c_1 is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate φ. If we choose the domain of the spatial coordinates as $\rho \in [0, \infty)$ and $z \in (-\infty, +\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space N satisfy the boundary conditions

$$\dot{X}^\mu(\rho, -\infty) = 0 = \dot{X}^\mu(\rho, \infty) , \quad X'^\mu(\rho, -\infty) = 0 = X'^\mu(\rho, \infty) \tag{5.3.15}$$

where the dot stands for a derivative with respect to ρ and the prime represents derivation with respect to z. These relationships are known in string theory [16] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume ρ as a "time" parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D-$branes situated at plus and minus infinity in the $z-$direction.

5.4 Dimensional extension

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space N, and study the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an $(m \to D)-$generalized harmonic map. As before we denote by $\{x^a\}$ the coordinates on M. Let $\{X^\mu, X^\alpha\}$ with $\mu = 1, 2$ and $\alpha = 3, 4, ..., D$ be the coordinates on N. The metric structure on M is again $\gamma = \gamma(x)$, whereas the metric on N can in general depend on all coordinates of M and N, i.e. $G = G(X^\mu, X^\alpha, x^a)$. The general structure of the corresponding field equations is as given in (5.3.1). They can be divided into one set of equations for X^μ and one set of equations for X^α. According to the results of the last section, the class of gravitational fields under consideration can be represented as a $(2 \to 2)-$generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates X^μ of the target space. Then, the gravitational sector of the target space will be contained in the components $G_{\mu\nu}$ $(\mu, \nu = 1, 2)$ of the metric, whereas the components $G_{\alpha\beta}$ $(\alpha, \beta = 3, 4, ..., D)$ represent the sector of the dimensional extension.

Clearly, the set of differential equations for X^μ also contains the variables X^α and its derivatives $\partial_a X^\alpha$. For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing X^α and its derivatives in the equations for X^μ. It is easy to show that this can be achieved by imposing the conditions

$$G_{\mu\alpha} = 0 \, , \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0 \, , \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0 \, . \tag{5.4.1}$$

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., $G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a)$, $\gamma = 3, 4, ..., D$. Furthermore, the variables X^α must satisfy the differential equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\alpha \right) + \Gamma^\alpha_{\ \beta\gamma}\, \gamma^{ab}\, \partial_a X^\beta \partial_b X^\gamma + G^{\alpha\beta}\gamma^{ab}\, \partial_a X^\gamma\, \partial_b G_{\beta\gamma} = 0 \, . \tag{5.4.2}$$

This shows that any given $(2 \to 2)-$generalized map can be extended, without affecting the field equations, to a $(2 \to D)-$generalized harmonic map.

It is worth mentioning that the fact that the target space N becomes split in two separate parts implies that the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. $T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)$. The generalized conservation law as given in (5.3.8) is satisfied by the sum of both parts.

Consider the example of stationary axisymmetric fields given the metrics (5.3.9). Taking into account the conditions (5.4.1), after a dimensional extension the metric of the target space becomes

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\ 0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\ . & . & \cdots & \cdots & \cdots \\ 0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x) \end{pmatrix} . \tag{5.4.3}$$

Clearly, to avoid that this metric becomes degenerate we must demand that $\det(G_{\alpha\beta}) \neq 0$, a condition that can be satisfied in view of the arbitrariness of the components of the metric. With the extended metric, the Lagrangian density gets an additional term

$$\begin{aligned} \mathcal{L} = \quad & \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right] \\ & + \left(\partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta\right) G_{\alpha\beta} \, , \end{aligned} \tag{5.4.4}$$

which nevertheless does not affect the field equations for the gravitational variables f and Ω. On the other hand, the new fields must be solutions of the extra field equations

$$\left(\partial_\rho^2 + \partial_z^2\right) X^\alpha + \Gamma^\alpha{}_{\beta\gamma}\left(\partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma\right) \tag{5.4.5}$$

$$+G^{\alpha\gamma}\left(\partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma}\right) = 0 \, . \tag{5.4.6}$$

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice $G_{\alpha\beta} = \eta_{\alpha\beta}$ with additional fields X^α given as arbitrary harmonic functions. This choice opens the possibility of introducing a “time” coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section 4.1, these fields are obtained from the general stationary fields in the limiting case $\Omega = 0$ (or equivalently, $\omega = 0$). If we consider the representation as an $SL(2,R)/SO(2)$ nonlinear sigma model or as a $(2 \to 2)-$generalized harmonic map, we see immediately that the limit $\Omega = 0$ is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case $\Omega = 0$. In the most simple case of an extension with $G_{\alpha\beta} = \delta_{\alpha\beta}$, the resulting $(2 \to 2)-$generalized map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4.7}$$

where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable f. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string "living" in a D-dimensional target space N. The string world-sheet is parametrized by the coordinates ρ and z. The gravitational sector of the target space depends explicitly on the metric functions f and Ω and on the parameter ρ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)-$dimensional Minkowski spacetime with time parameter τ. Then, the string world-sheet is a 2-dimensional flat hypersurface which is "frozen" along the time τ.

5.5 The general solution

If we take as seed metric the general static solution, the application of two HXK transformations generates a stationary solution with an infinite number of gravitoelectric and gravitomagnetic multipole moments. The HKX method is applied at the level of the Ernst potential from which the metric functions can be calculated by using the definition of the Ernst potential E and the field equations for γ. The resulting expressions in the general case are quite cumbersome. We quote here only the special case in which only an arbitrary quadrupole parameter is present. In this case, the result can be written as

$$\begin{aligned} f &= \frac{R}{L} e^{-2qP_2Q_2} , \\ \omega &= -2a - 2\sigma \frac{\mathcal{M}}{R} e^{2qP_2Q_2} , \\ e^{2\gamma} &= \frac{1}{4}\left(1+\frac{M}{\sigma}\right)^2 \frac{R}{x^2-y^2} e^{2\hat{\gamma}} , \end{aligned} \tag{5.5.1}$$

where

$$\begin{aligned} R &= a_+a_- + b_+b_- , \qquad L = a_+^2 + b_+^2 , \\ \mathcal{M} &= \alpha x(1-y^2)(e^{2q\delta_+} + e^{2q\delta_-})a_+ + y(x^2-1)(1-\alpha^2 e^{2q(\delta_+ + \delta_-)})b_+ , \\ \hat{\gamma} &= \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 + q^2(1-P_2)\Big[(1+P_2)(Q_1^2-Q_2^2) \\ &\quad + \frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\Big] . \end{aligned} \tag{5.5.2}$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$\begin{aligned} a_\pm &= x(1-\alpha^2 e^{2q(\delta_+ + \delta_-)}) \pm (1+\alpha^2 e^{2q(\delta_+ + \delta_-)}) , \\ b_\pm &= \alpha y(e^{2q\delta_+} + e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+} - e^{2q\delta_-}) , \\ \delta_\pm &= \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2 \mp xy) + \frac{3}{4}[x(1-y^2) \mp y(x^2-1)]\ln\frac{x-1}{x+1} , \end{aligned}$$

the quantity α being a constant

$$\alpha = \frac{\sigma - M}{a} , \qquad \sigma = \sqrt{M^2 - a^2} . \tag{5.5.3}$$

The physical significance of the parameters entering this metric can be clarified by calculating the Geroch-Hansen [18, 19] multipole moments

$$M_{2k+1} = J_{2k} = 0 , \quad k = 0, 1, 2, ... \tag{5.5.4}$$

$$M_0 = M\,, \quad M_2 = -Ma^2 + \frac{2}{15}qM^3\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.5}$$

$$J_1 = Ma\,, \quad J_3 = -Ma^3 + \frac{4}{15}qM^3a\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.6}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the symmetry with respect to the equatorial plane. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q.

We analyzed the geometric and physical properties of the above solution. The special cases contained in the general solution suggest that it can be used to describe the exterior asymptotically flat gravitational field of rotating body with arbitrary quadrupole moment. This is confirmed by the analysis of the motion of particles on the equatorial plane. The quadrupole moment turns out to drastically change the geometric structure of spacetime as well as the motion of particles, especially near the gravitational source.

We investigated in detail the properties of the Quevedo-Mashhoon (QM) spacetime which is a generalization of Kerr spacetime, including an arbitrary quadrupole. Our results show [20] that a deviation from spherical symmetry, corresponding to a non-zero electric quadrupole, completely changes the structure of spacetime. A similar behavior has been found in the case of the Erez-Rosen spacetime. In fact, a naked singularity appears that affects the ergosphere and introduces regions where closed timelike curves are allowed. Whereas in the Kerr spacetime the ergosphere corresponds to the boundary of a simply-connected region of spacetime, in the present case the ergosphere is distorted by the presence of the quadrupole and can even become transformed into non simply-connected regions. All these changes occur near the naked singularity which is situated at $x = 1$, a value that corresponds to the radial distance $r = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates. In the limiting case $a/M > 1$, the multipole moments and the metric become complex, indicating that the physical description breaks down. Consequently, the extreme Kerr black hole represents the limit of applicability of the QM spacetime.

Since standard astrophysical objects satisfy the condition $a/M < 1$, we can conclude that the QM metric can be used to describe their exterior gravitational field. Two alternative situations are possible. If the characteristic radius of the body is greater than the critical distance $M + \sqrt{M^2 - a^2}$, i.e. $x > 1$, the exterior solution must be matched with an interior solution in order to describe the entire spacetime. If, however, the characteristic radius of the body is smaller than the critical distance $M + \sqrt{M^2 - a^2}$, the QM metric describes the field of a naked singularity.

6 Black shells and naked shells

6.1 Introduction

The gravitational collapse is one of the most interesting predictions of general relativity. It is associated with the formation of black holes and gravitational waves, which are expected to contain information about the end state of highly interacting compact objects and about the dynamics of the physical processes that occur during the collapse. To find out the details of the formation of black holes and gravitational waves in general relativity, it is necessary to consider the entire set of Einstein equations and apply several methods of numerical relativity to find numerical solutions. Numerical relativity is a research area by itself that implies the use of highly accurate computational tools [21].

An alternative approach consists in considering only the most essential aspects of the gravitational collapse by analyzing an idealized model that reduces the complexity of the problem. This is the case of the black shell scenario, a toy model in which a thin shell made of matter collapses under the influence of its own gravitational field [22, 23, 24]. In this case, the mathematical complexity of the problem reduces drastically and, as a consequence, we are allowed to apply mainly analytical methods. In the black shell model, we will assume that the contraction of a spherically symmetric shell starts at some radial distance and leads to a reduction of the shell radius with respect to a fiducial observer located at infinity. As the shell shrinks, the evolution is assumed to be described by an Oppenheimer-Snyder collapsing process [25].

In this work, to analyze the dynamics of the surface, where the thin shell is located, we consider the norm of the vector orthogonal to the surface and investigate the conditions under which critical surfaces appear, i.e., radii at which the norm of the vector vanishes and an interchange between time and space coordinates occurs. This method allows us to find all the critical surfaces that can appear during the evolution of the shell. For instance, this procedure predicts the existence of a horizon that appears as the shell radius equals its gravitational radius. Furthermore, we will see that, in general, there exists a second critical surface with a radius that is always greater than the gravitational radius of the shell. To investigate the properties of the critical surfaces, we use the formalism of trapped surfaces and quasi-local horizons. We show that once the shell reaches its gravitational radius, space and time coordinates interchange their roles and the corresponding surface is a null surface with zero expansion so that it can be interpreted as an apparent horizon. The second surface corresponds to a marginally outer trapped surface that can be associated with a quasi-local horizon.

We will see that there exists a particular case in which no horizons appear during the evolution of a shell, whose end state corresponds to a curvature singularity. We call this particular configuration a naked shell. Some properties of naked shells are also studied.

6.2 Dynamics

In this section, we will follow the Darmois-Israel formalism [22, 26, 27, 28, 29, 30] in which the starting point is a spherically symmetric thin shell described by the hypersurface Σ with coordinates $\xi^a = \{\tau, \theta, \varphi\}$. The corresponding line element on Σ is assumed to be of the form

$$ds^2_\Sigma = -d\tau^2 + R^2(\tau) d\Omega^2 \,. \tag{6.2.1}$$

Thus, Σ splits the spacetime into two regions V_-, inside Σ, and V_+, outside Σ. The thin shell is assumed to be described by an energy-momentum tensor S^{ab}.

To describe the spacetime, we assume that the inside region V_- corresponds to the Minkowski spacetime

$$ds^2_- = -dt^2 + dr^2 + r^2 d\Omega^2 \,. \tag{6.2.2}$$

On Σ, it is convenient to introduce new coordinates $T(\tau)$ and $R(\tau)$ such that

$$t = T(\tau) \,,\, dt = \dot{T} d\tau \,, \quad r = R(\tau) \,,\, dr = \dot{R} d\tau \,, \tag{6.2.3}$$

where τ is the proper time and a dot represents derivative with respect to τ.

Thus, the interior Minkowski metric on Σ becomes

$$ds^2_-|_\Sigma = -\left(\dot{T}^2 - \dot{R}^2\right)d\tau^2 + R^2(\tau)d\Omega^2\,. \tag{6.2.4}$$

Furthermore, we will assume that the outside region corresponds to the Schwarzschild spacetime

$$ds^2_+ = -fdt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,,\; f = 1 - \frac{2M}{r}, \tag{6.2.5}$$

which on Σ in coordinates (6.2.3) becomes

$$ds^2_+|_\Sigma = -\left(F\dot{T}^2 - \frac{\dot{R}^2}{F}\right)d\tau^2 + R^2(\tau)d\Omega^2,\; F = 1 - \frac{2M}{R}\,. \tag{6.2.6}$$

To guarantee that the entire spacetime is well defined as a differential manifold, one can impose the Darmois matching conditions

$$[h_{ab}] = h^+_{ab} - h^-_{ab} = 0\,, \quad [K_{ab}] = K^+_{ab} - K^-_{ab} = 0\,, \tag{6.2.7}$$

where $h^\pm_{ab}$ is the metric induced on Σ by the metric of $V_\pm$ and $K^\pm_{ab}$ is the corresponding extrinsic curvature, respectively. The first condition implies simply that $ds^2_+|_\Sigma = ds^2_-|_\Sigma$, i.e.,

$$\dot{T}^2 - \dot{R}^2 = F\dot{T}^2 - \frac{\dot{R}^2}{F}\,. \tag{6.2.8}$$

In general, it is quite difficult to satisfy the second condition of Eq.(6.2.7). A less strict version of this condition was proposed by Israel and consists in assuming that the jump in the extrinsic curvature, $[K_{ab}] \neq 0$, determines a thin shell with energy momentum tensor S_{ab}, which is defined as

$$S_{ab} = -\frac{1}{8\pi}([K_{ab}] - [K]h_{ab}) \tag{6.2.9}$$

where $K = K_{ab}h^{ab}$. For simplicity, let us consider the case of a dust shell $S_{ab} = \sigma u_a u_b$, where σ is the surface density of the dust and u_a is the 3-velocity of the shell. It is then straightforward to compute the extrinsic curvature of V_+ and V_- and the right-hand side of Eq.(6.2.9), which determines the behavior of the surface density σ. The final result can be expressed as

$$R(\sqrt{1+\dot{R}^2} - \sqrt{F+\dot{R}^2}) = m = 4\pi\sigma R^2 \tag{6.2.10}$$

where m is an integration constant. This equation can be interpreted as the motion equation of the shell. Indeed, a rearrangement of Eq.(6.2.10) leads to the expression

$$M = m\sqrt{1+\dot{R}^2} - \frac{m^2}{2R}\,, \tag{6.2.11}$$

which can be interpreted as representing the conservation of energy during the motion of the shell. In fact, the first term on the right-hand side represents a relativistic quantity, which includes the energy at rest and the kinetic energy. Then, the second term can be interpreted as the binding energy of the system. Consequently, M represents the gravitational mass of the shell and m its rest mass [23]. The equation of motion (6.2.11) can be rewritten as

$$\dot{R}^2 = \left(\frac{M}{m} + \frac{m}{2R} - 1\right)\left(\frac{M}{m} + \frac{m}{2R} + 1\right)\,. \tag{6.2.12}$$

Since the right-hand side of this equation must be positive, it follows that if $m \geq M$ the radius of the shell can take values only within the interval

$$R \in \left(0, \frac{m^2}{2m-2M}\right] \tag{6.2.13}$$

with boundaries

$$\begin{aligned} \text{for } m \to M \quad &\Rightarrow \quad R \in (0,\infty)\,, \\ \text{for } m = 2M \quad &\Rightarrow \quad R \in (0,2M]\,, \\ \text{for } m \to \infty \quad &\Rightarrow \quad R \in (0,\infty)\,. \end{aligned} \tag{6.2.14}$$

On the other hand, if $m < M$, during the evolution of the shell its radius can have any positive value, $R \in (0,\infty)$. We see that the value of the rest mass m is important for determining the motion of the shell. The lower limit $(R \to 0)$ follows from the interpretation of the function $R(\tau)$ as the radius of the shell and also, as we will show below, from the fact that it corresponds to a curvature singularity.

The result of integrating the motion equation (6.2.11) is shown in Fig. 7.1.

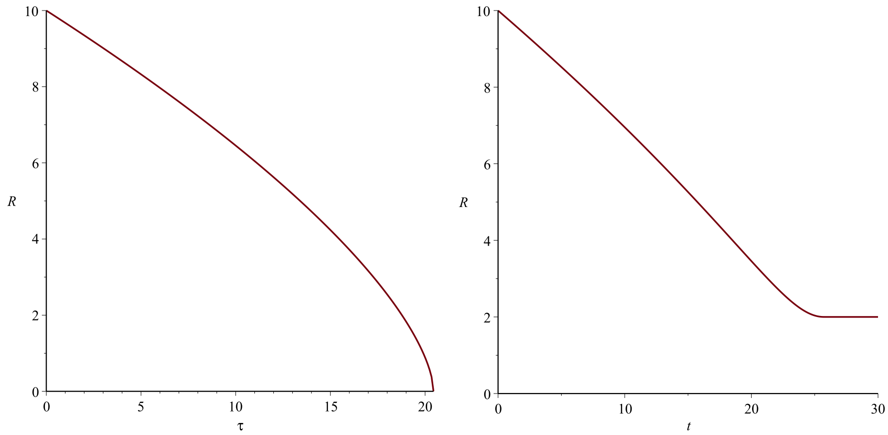

Figure 6.1: The radius of the shell in terms of the proper time τ and the coordinate time t for the particular masses $M = 1$ and $m = 1$.

We present the result in terms of the proper time τ and the coordinate time t. As expected, in terms of the proper time τ, the shell reaches the origin of coordinate in finite time, whereas for an observer at infinity the shell never reaches the radius $R = 2M$.

6.3 Critical surfaces and horizons

In a spacetime, horizons are usually defined in terms of Killing vectors. For instance, if the spacetime is static with a timelike Killing vector ξ^μ, the condition $\xi^\mu \xi_\mu = 0$ determines a hypersurface which is interpreted as the event horizon. In the case of the shell we are considering here, the corresponding spacetime has no timelike Killing vector and so it is not possible to use the above definition to search for horizons. In general, in the case of time-dependent spacetimes, horizons must be treated in a different manner. We will see this in the next section.

In this section, we will search for critical surfaces that resemble the properties of a horizon. To this end, let us consider a radially directed vector $U^\mu = (\dot{T}, \dot{R}, 0, 0)$ with norm

$$U^2 = -F\dot{T}^2 + \frac{\dot{R}^2}{F} \,. \tag{6.3.1}$$

The behavior of the norm U^2 along the radial coordinate contains information about the surfaces orthogonal to it. For instance, the locations at which the norm vanishes can be interpreted as indicating the presence of a null surface, which is one of the properties of horizons. To be more specific, let us consider the case in which the norm is timelike when the shell is at rest $\dot{R} = 0$, i.e., $F\dot{T}^2 = 1$. This is a pure coordinate condition that determines how the time coordinate T depends on the spatial coordinate R. For simplicity, we assume that this condition is valid all the way along the radial coordinate and so the norm becomes

$$U^2 = -1 + \frac{\dot{R}^2}{F}\,, \tag{6.3.2}$$

which can be expressed as

$$U^2 = \left[-2 + \frac{2M}{R} + \left(\frac{M}{m} + \frac{m}{2R}\right)^2\right]\left(1 - \frac{2M}{R}\right)^{-1}. \tag{6.3.3}$$

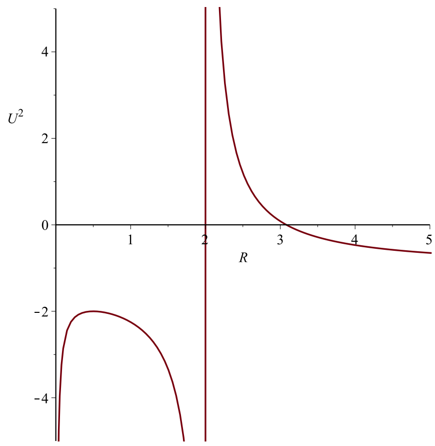

Figure 6.2: Norm of the observer's 4-velocity, according to Eq.(6.3.3) for the particular masses $M = 1$ and $m = 1$.

In Fig. 7.4, we illustrate the behavior of U^2. We see that there are two critical points, where the norm vanishes, namely $R = R_{gr} = 2M$ and $R = R_h$. The former value is obtained from the algebraic equation

$$4(2m^2 - M^2)R^2 - 12m^2MR - m^4 = 0\,, \tag{6.3.4}$$

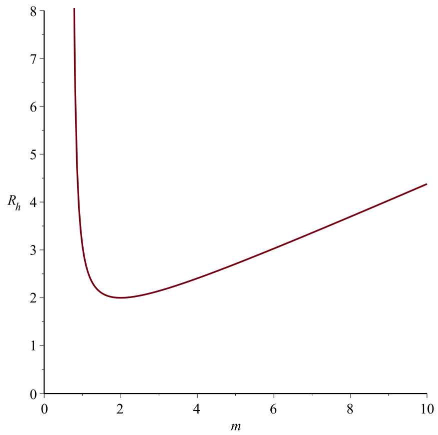

Figure 6.3: Location of the horizon radius R_h in terms of the mass m. Here we choose $M = 1$, which means that R_h and m are given in multiples of M.

for which we find the positive solution

$$R_h = \frac{m^2(3M + \mathcal{M})}{2(2m^2 - M^2)}, \quad \mathcal{M} = \sqrt{2m^2 + 8M^2}\,. \tag{6.3.5}$$

In both cases, the surface is determined by the equation $R = const$ so that the corresponding normal vector is $n_\mu = (0, 1, 0, 0)$ with norm $n^2 = g^{RR} = 1 - 2M/R$. It follows that the surface $R = R_{gr} = 2M$ is the only null surface which, according to the behavior of the norm U^2 around $R = 2M$, resembles the properties of an event horizon.

The radius R_h is given in Eq.(6.3.5) in terms of the gravitational mass M and the rest mass m. The behavior of this quantity is illustrated in Fig. 7.3. We can see two special points in this plot. First, for $m = M/\sqrt{2}$, the radius diverges, indicating that R_h exists only for values of $m > M/\sqrt{2}$. For rest masses with $m < M/\sqrt{2}$, Eq.(6.3.5) indicates that no radius exists ($R_h < 0$). The second point, $R_h = 2M$ with $m = 2M$, is a minimum value that corresponds to the Schwarzschild radius. In the particular case of a shell at rest at infinity ($\dot{R} = 0,\ R \to \infty$), from the equation of motion (6.2.11), it follows that the rest mass and the gravitational mass coincide, $m = M$, and then the equation for the radius R_h reduces to

$$R_h = \frac{M}{2}(3 + \sqrt{10}) \,. \tag{6.3.6}$$

Figure 7.4 shows the location of the radius R_h in accordance with Eq.(6.3.5). Furthermore, we see that at $R = R_h$ the norm changes its sign. This is an effect that is observed exactly on the horizon of black holes. Consequently, the surface with radius R_h denotes the particular location, where an interchange between the time and spatial coordinates takes place. However, this is not a null surface, which is a property of event horizons. Nevertheless, we will see below that it can be associated with the presence of a quasi-local horizon.

According to the above results, a black shell consists of a central singularity with one or two horizons, which are located as follows:

$$if\ m < \frac{M}{\sqrt{2}} \Rightarrow \text{an event horizon at } R = 2M \,, \tag{6.3.7}$$

$$if\ m \geq \frac{M}{\sqrt{2}}\ and\ m \neq 2M \Rightarrow \text{an event horizon at } R = 2M \tag{6.3.8}$$

$$\text{and a dynamic horizon at } R = R_h \,.$$

The inner horizon located at $R = 2M$ is always present, except in the case $m = 2M$ that we will consider below. The outer horizon located at $R = R_h > 2M$ is not always present; its existence and location depend on the value of the rest mass m. From Eq.(6.3.5) it follows that for the particular value $m = 2M$, the radius of the exterior dynamic horizon R_h reduces to its minimum value $R_h = 2M$, i.e., it coincides with the inner horizon located at the Schwarzschild radius $R_S = 2M$. This could be interpreted as a degenerate case in which the two horizons coincide. However, a detailed analysis shows that in this case, no horizon exists. This will be shown in the next section.

6.4 Naked shells

In this section, we will consider a particular configuration that can exist only for a very specific value of the rest and gravitational masses. From the expression for the dynamic horizon radius given in Eq.(6.3.5), it follows that for the particular value $m = 2M$, the radius R_h reduces to its minimum value

$R_h = 2M$, i.e., it coincides with the horizon located at the Schwarzschild radius $R_S = 2M$. This could be interpreted as the degenerate case in which the two horizons coincide.

However, a straightforward computation of the norm U^2 leads to the expression

$$U^2(m = 2M) = -\frac{2M + 7R}{4R} , \tag{6.4.1}$$

which has no zeros for any positive values of R. This means that during the evolution of a particular shell, in which the rest mass is twice the gravitational mass, no horizons are formed. Moreover, the end state of the shell evolution corresponds to a curvature singularity. Indeed, the computation of the Kretschmann scalar for the shell metric (6.2.1) leads to the expression

$$K = R_{abcd}R^{abcd} = 4\frac{1 + 2\dot{R}^2 + \dot{R}^4 + 2\ddot{R}R^2}{R^4} . \tag{6.4.2}$$

We see that the only singularity occurs when $R \to 0$, i.e., as the radius of the shell shrinks to its minimum value. No other singularities exist during the collapse of the shell as long as its velocity and acceleration remain finite.

The particular configuration described above in which a curvature singularity is formed as the end state of evolution of a thin shell, but no horizons appear during the evolution, will be called naked shell. It exists only for a very specific value of the rest mass. For any other value of the rest mass, the collapse of the shell is characterized by the appearance of horizons, implying that the corresponding configuration is a black shell.

From the equation of motion (6.2.12), it follows that in the case of a naked shell the dynamics is governed by the equation

$$\dot{R} = -\sqrt{\left(\frac{M}{R} - \frac{1}{2}\right)\left(\frac{M}{R} + \frac{3}{2}\right)} , \tag{6.4.3}$$

where the minus sign has been chosen in order for the equation to describe the motion of a collapsing shell. The motion is constrained within the interval $R \in (0, 2M]$. This means that a shell can start collapsing at any $R \leq 2M$, where $R = 2M$ is not a horizon and will reach the singularity in a finite proper time. Any observer within the radial distance $R \leq 2M$ can communicate with an observer located infinitesimally close to the central singularity.

As far as we know, the naked shell configuration described above as a dynamical process is not known in the literature. However, the concept of shell-like null singularities has been introduced previously in a completely different situation to describe a particular 5-dimensional non-dynamical braneworld model [31]. On the other hand, the naked shell configuration is a realistic theoretical prediction of the approach presented here. Indeed, the singularity at $r = 0$ is a genuine prediction of our approach since it is based upon the analysis of a coordinate-independent curvature scalar. As such it corresponds to an observable gravitational object. Furthermore, the property that this true singularity is not surrounded by a horizon is fulfilled for the particular case $m = 2M$, i.e., for a shell with a large binding energy, which is a physically realizable condition. In this sense, the prediction of the existence of naked shells is in agreement with the physical expectations from a theoretical point of view. In addition, the formation of a naked shell is dynamically allowed since the equation of motion in this case leads only to a restriction on the spatial interval, where the naked shell can move, namely, $R \in (0, 2M]$. This means that the interior metric can be used to describe the region of spacetime inside a sphere of radius $2M$. Outside this sphere, the spacetime is described by the Schwarzschild metric.

6.5 Quasi-local horizons

As shown in Sec. 6.3, during the collapse of a shell special surfaces can appear around which the time and spatial coordinates interchange their roles. The interpretation of these surfaces as event horizons is problematic because they are defined as global properties of spacetime. The physical evolution of a global horizon is at least problematic, if not impossible. For this reason, an alternative approach is necessary in which local properties of spacetime are invoked. This is the formalism of quasi-local horizons which is based upon the concept of trapped surfaces, dating back to the formulation of the singularity theorems proved by Penrose [32] and Hawking and Ellis [33]. In turn, trapped surfaces are defined in terms of the behavior of the expansion of null vectors orthogonal to the surfaces.

In the case under consideration, all the surfaces to be investigated are spherically symmetric, which allows us to perform all the calculations explicitly. Indeed, we are interested in the behavior of the critical surfaces described in Sec. 6.3, which correspond to 2-spheres with $r = const$. To be more specific, consider the null vectors l^α and N^α with $l_\alpha N^\alpha = -1$, which are orthogonal to the surface Σ of the shell. Then, the expansions of l and N are defined as [36]

$$\Theta_{(l)} = q^{\alpha\beta}\nabla_\alpha l_\beta \,, \quad \Theta_{(N)} = q^{\alpha\beta}\nabla_\alpha N_\beta \,, \tag{6.5.1}$$

where the components of the metric $q^{\alpha\beta}$ are defined as

$$q^{\alpha\beta} = g^{\alpha\beta} + l^\alpha N^\beta + N^\alpha l^\beta \,. \tag{6.5.2}$$

To determine the null vectors l and N, we introduce the auxiliary orthonormal vectors n^α and r^α, which are obtained by representing the spacetime metric $g^{\alpha\beta}$ in terms of the induced metric h^{ab} as [23]

$$g^{\alpha\beta} = n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b \,, \tag{6.5.3}$$

with

$$n_\alpha n^\alpha = 1 \,, \quad e^\alpha_a = \frac{\partial x^\alpha}{\partial \xi^a} \tag{6.5.4}$$

where the induced metric h_{ab} has been used to determine the line element on the surface of the shell Σ, i.e.,

$$ds^2|_\Sigma = g_{\alpha\beta} dx^\alpha dx^\beta|_\Sigma = g_{\alpha\beta} e^\alpha_a e^\beta_b d\xi^d \xi^b = h_{ab} d\xi^a d\xi^b \,. \tag{6.5.5}$$

A straightforward computation by using the exterior metric (6.2.5) leads to the following expression for the components of the vector n

$$n^\alpha = \frac{1}{(F\dot{T}^2 - F^{-1}\dot{R}^2)^{1/2}} \left(\frac{\dot{R}}{F}, F\dot{T}, 0, 0 \right) \,. \tag{6.5.6}$$

Furthermore, we introduce a vector r^α such that $r^\alpha r_\alpha = -1$ and $n_\alpha r^\alpha = 0$. Then, we obtain

$$r^\alpha = \frac{1}{(F\dot{T}^2 - F^{-1}\dot{R}^2)^{1/2}} \left(\dot{T}, \dot{R}, 0, 0 \right) \,. \tag{6.5.7}$$

Finally, the null vectors are defined as

$$l^\alpha = \frac{1}{\sqrt{2}}(r^\alpha + n^\alpha) \,, \quad N^\alpha = \frac{1}{\sqrt{2}}(r^\alpha - n^\alpha) \,, \tag{6.5.8}$$

and can be written explicitly as

$$l^\alpha = \frac{1}{\sqrt{2}(F\dot{T}^2 - F^{-1}\dot{R}^2)^{1/2}} \left(\frac{\dot{R}}{F} + \dot{T}, F\dot{T} + \dot{R}, 0, 0 \right) , \tag{6.5.9}$$

$$N^\alpha = \frac{1}{\sqrt{2}(F\dot{T}^2 - F^{-1}\dot{R}^2)^{1/2}} \left(\dot{T} - \frac{\dot{R}}{F}, \dot{R} - F\dot{T}, 0, 0 \right) , \tag{6.5.10}$$

which satsisfy the conditions $N_\alpha N^\alpha = l_\alpha l^\alpha = 0$ and $l_\alpha N^\alpha = -1$. These expressions can now be introduced into Eq.(6.5.2) for the components of the metric q to obtain $q_{\alpha\beta} = diag(0, 0, r^2, r^2 \sin^2\theta)$. Then, the expansions (6.5.1) can be computed and we obtain

$$\Theta_{(l)} = 0\,, \quad \Theta_{(N)} = 0\,, \tag{6.5.11}$$

indicating that the surfaces with $r = const.$ are marginally outer trapped surfaces [34]. We notice that a similar computation by using the inner metric leads to the same result.

We can now identify the critical surfaces obtained in Sec. 6.3. The surface $r = 2M$ is thus a null surface with zero expansion and so it corresponds to an apparent horizon [35]. On the other hand, the surface $r = R_h$, which is not null but is accompanied by an interchange between the time and spatial coordinates, is a marginally outer trapped surface, which can be interpreted as corresponding to a quasi-local horizon.

6.6 Conclusions

In this work, we analyzed the collapse of a spherically symmetric thin shell made of pure dust. To describe the dynamics of the shell, we employ the Darmois-Israel formalism, according to which the complete spacetime is split into three different parts that must satisfy the matching conditions. In our case, the interior part corresponds to a flat Minkowski spacetime, the exterior one is described by the spherically symmetric Schwarzschild metric and the boundary between them is described by an induced metric that satisfies the matching conditions and can be interpreted as corresponding to a thin shell of dust. As a result of demanding compatibility between the three spacetime metrics, we obtain a differential equation that governs the motion of the shell and depends on the gravitational mass M and on an additional integration constant m, which is interpreted as the rest mass of the shell.

By using a particular vector oriented along with the motion of the shell, we search for critical surfaces around which an interchange between space and time coordinates occurs, which resembles the behavior around a horizon. We found two critical surfaces. The first one appears when the radius of the shell equals its gravitational radius ($R = 2M$), i.e., it corresponds to the Schwarzschild horizon R_S of the exterior spacetime. The radius R_h of the second critical surface is always greater than the gravitational radius of the shell and its explicit value depends on the values of the gravitational and rest masses. The interpretation of these critical surfaces as horizons is problematic due to the dynamic behavior of the shell. The standard definition of an event horizon is associated with global properties of the spacetime, which makes it difficult or even impossible to analyze its evolution. Therefore, we use the alternative formalism of trapped surfaces, which is based on local properties of the spacetime. The idea is to analyze the behavior of null vectors orthogonal to the surface of the shell. It turns that all the surfaces with constant radial distance are characterized by a vanishing expansion. Consequently, the two critical surfaces located at R_S and R_h correspond to marginally outer trapped surfaces, which are associated with quasi-local horizons. In addition, since the critical surface located at the gravitational radius $R = 2M$ is a null surface, it can be interpreted as an apparent horizon.

Furthermore, for the particular case of a thin shell, whose rest mass is twice the gravitational mass ($m = 2M$), no horizon exists and the end state of the shell evolution corresponds to a curvature singularity, which appears as the radius of the shell tends to zero. We thus denote the resulting configuration as a naked shell. This is a very peculiar configuration that appears only because of the existence of the second critical surface R_h. Indeed, whereas the vector that we use to detect critical surfaces, for $m \neq 2M$ predicts the existence of two locations with zero norm at R_S and R_h, in the limiting case with $m = 2M$ shows no zeros at all. This means that during the collapse of a shell with $m = 2M$ no interchange between space and time occurs and so no horizon appears that could hide the curvature singularity from the outer spacetime.

We have shown that the critical surfaces that appear during the collapse of a thin shell can be interpreted as quasi-local horizons, which imply the existence of a sophisticated structure from the point of view of dynamics. In a related context, the idea of interpreting black holes as macroscopic quantum objects was proposed in [30]. The present work can be generalized to include another kind of thin shells and spacetimes. For instance, one could consider the case of thin shells with internal pressure or additional gravitational charges. We expect to study such generalized configurations in future works.

7 PERFECT FLUID SOLUTIONS WITH QUADRUPOLE MOMENT

7.1 Introduction

Based on experimental evidence, general relativity is considered today as one of the best candidates to describe the gravitational field of compact astrophysical objects. As a theory for the gravitational field, it should be able to describe the field of all possible physical configurations, in which gravity is involved. All the information about the gravitational field should be contained in the metric tensor which must be a solution of Einstein's equations. Consider the case of a compact object like a star or a planet. From the point of view of the multipole structure of the source, to describe the field of a compact object, we need an interior and an exterior solution, both containing at least three independent physical parameters, namely, mass, angular momentum and quadrupole moment. Consider first the case of a source with only mass and angular momentum. The corresponding exterior solution is represented by the Kerr spacetime [57] for which no physically reasonable interior solution is known. This is a major problem in classical general relativity [58]. Many methods have been suggested to find a suitable interior Kerr solution, including exotic matter models and specially adapted equations of state, but none of them has lead to a definite answer; in this regard, see [97] for a recent promising proposal. In view of this situation, we consider that alternative approaches should be considered. In particular, we believe that additional physical parameters can be taken into account that are relevant for the description of the gravitational field. The simplest of such additional parameters is the quadrupole moment which is responsible for the deformation of any realistic mass distribution. Indeed, if we add a quadrupole moment to a spherically symmetric object, we end up with an axisymmetric mass distribution, which implies new degrees of freedom at the level of the corresponding field equations. This is the main idea of the alternative approach we propose to attack the problem of finding interior solutions to describe the interior gravitational structure of compact objects. As a first step to develop such an approach, we will focus in this work on the case of a source with only mass and quadrupole, neglecting the contribution of the angular momentum. In a recent work [59], it was proposed to use the Zipoy-Voorhees transformation [79, 96] to generate the quadrupolar metric ($q-$metric), which can be interpreted as the simplest generalization of the Schwarzschild metric, describing the gravitational field of a distribution of mass whose non-spherically symmetric shape is represented by an independent quadrupole parameter. In the literature, this metric is known as the Zipoy-Voorhees metric, delta-metric, gamma-metric and q-metric [86, 81]. Here, we will use the name q-metric to highlight the importance of the quadrupole parameter q. Indeed, several studies show the physical importance of the quadrupole parameter. Circular and radial geodesics of the exterior gamma-metric ($\gamma = 1 + q$) have been compared with the spherically symmetric case to establish the sensitivity of the trajectories to the gamma parameter [82]. Moreover, it was shown that the properties of the accretion disks in the field of the gamma-metric can be drastically different from those of disks around black holes [80, 56]. These studies show that the $q-$metric can be used to describe the exterior gravitational field of deformed distributions of mass in which the quadrupole moment is the main parameter that describes the deformation.

The question arises whether it is possible to find an interior metric that can be matched to the exterior one in such a way that the entire spacetime is described as a whole. To this end, it is usually assumed that the interior mass distribution can be described by means of a perfect fluid with two physical parameters, namely, energy density and pressure. The energy-momentum tensor of the perfect fluid is then used in the Einstein equations as the source of the gravitational field. It turns out that the system of corresponding differential equations cannot be solved because the number of equations is less than the number of unknown functions. This problem is usually solved by imposing equations of state that relate the pressure and density of the fluid. Moreover, one should impose energy conditions, matching conditions with the exterior metric, and conditions on the behavior of the metric functions near the center of the source and on the boundary with the exterior field.

Hernandez [85] has shown how to modify *a*d hoc an interior spherically symmetric solution to obtain an approximate interior solution for the corresponding family of exterior Weyl metrics, provided the exterior metric contains the Schwarzschild metric as a particular case. The Hernandez approach has been generalized to obtain an exact interior solution to the gamma-metric. They found two different interior solutions which match the exterior gamma-metric. In general, however, this *a*d hoc method does not lead to interior solutions corresponding to simple fluids. The matching between interior and exterior solutions, in general, requires the fulfillment of several mathematical conditions on the matching surface [84, 83].

The above discussion shows that the search for physically relevant interior solutions is not an easy task. The difficulties increase once we consider the non-uniqueness of the solutions. Indeed, whereas the Birkhoff theorem guarantees that the Schwarzschild metric is the only spherically symmetric vacuum solution of Einstein's equations, there exist many spherically symmetric interior solutions that can be matched with the Schwarzschild metric. In the case of axial symmetry, the situation is even more complicated. In a recent work [98, 99], several exterior solutions with quadrupole were compared. It was found that they are all represented by diverse analytical expressions and are characterized by different sets of multipole moments. In this sense, they are all physically different from each other. One can, therefore, expect that there will be many interior metrics that can be matched with each one of the exterior solutions. One example of this situation is given by a recently proposed interior solution for the $q-$metric [61] and the solutions that we will analyze in this work. In fact, to obtain the solution found in [61] the authors have chosen a particular line element which does not contain the line element used in this work. For the sake of completeness, we show this result in 7.8.

7.2 Exterior $q-$metric

Zipoy [79] and Voorhees [96] investigated static, axisymmetric vacuum solutions of Einstein's equations and found a simple transformation, which allows one to generate new solutions from a known solution. If we start from the Schwarzschild solution and apply a Zipoy-Voorhees transformation, the new line element can be written as [59]

$$
\begin{aligned}
ds^2 = \quad & \left(1-\frac{2m}{r}\right)^{1+q} dt^2 - \left(1-\frac{2m}{r}\right)^{-q} \\
& \times \left[\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)} \left(\frac{dr^2}{1-\frac{2m}{r}} + r^2 d\theta^2\right) \right. \\
& \left. + r^2\sin^2\theta d\varphi^2 \right] . \qquad (7.2.1)
\end{aligned}
$$

This metric is axially symmetric and, consequently, it is physically different from the seed Schwarzschild metric. A detailed analysis shows that m and q are constant parameters that determine the total mass and the quadrupole moment of the gravitational source [59]. A stationary generalization of the $q-$metric, satisfying the main physical conditions of exterior spacetimes, has been obtained in [94, 37]. The metric (7.2.1) has been interpreted as the simplest generalization of the Schwarzschild metric with a quadrupole.

As mentioned above, whereas the seed metric is the spherically symmetric Schwarzschild solution, which describes the gravitational field of a black hole, the generated $q-$metric is axially symmetric, and describes the exterior field of a naked singularity [59]. In fact, this can be shown explicitly by calculating the invariant Geroch multipoles [93, 92]. The lowest mass multipole moments M_n, $n = 0, 1, \ldots$ are given by

$$
M_0 = (1+q)m\,, \quad M_2 = -\frac{m^3}{3}q(1+q)(2+q)\,, \qquad (7.2.2)
$$

whereas higher moments are proportional to mq and can be completely rewritten in terms of M_0 and M_2. Accordingly, the arbitrary parameters m and q determine the mass and quadrupole which are the only independent multipole moments of the solution. In the limiting case $q = 0$, only the monopole $M_0 = m$ survives, as in the Schwarzschild spacetime. In the limit $m = 0$, with $q \neq 0$, all moments vanish identically, implying that no mass distribution is present and the spacetime must be flat. The same is true in the limiting case $q \to -1$ which corresponds to the Minkowski metric. Moreover, notice that all odd multipole moments are zero because the solution possesses an additional reflection symmetry with respect to the equatorial plane $\theta = \pi/2$.

The deformation is described by the quadrupole moment M_2 which is positive for a prolate source and negative for an oblate source. This implies that the parameter q can be either positive or negative. Since the total mass M_0 of the source must be positive, we must assume that $q > -1$ for positive values of m, and $q < -1$ for negative values of m. We conclude that the above metric can be used to describe the exterior gravitational field of a static positive mass M_0 with a positive or negative quadrupole moment M_2.

A study of the curvature of the $q-$metric shows that the outermost singularity is located at $r = 2m$, a hypersurface which in all known compact objects is situated inside the surface of the body. This implies that in order to describe the entire gravitational field, it is necessary to cover this type of singularity with an interior solution.

7.3 Interior metric

As mentioned in the Introduction, in this work, we will concentrate on the case of static perfect fluid spacetimes. There are many forms to write down the corresponding line element and, in principle, all of them could lead to different particular solutions [58]. Therefore, the choice of the line element is important for obtaining particular families of solutions. Certain forms of the line element turn out to be convenient for investigating a particular problem. Our experience with numerical perfect fluid solutions [91] indicates that for the case under consideration the line element

$$ds^2 = f dt^2 - \frac{e^{2\gamma}}{f}\left(\frac{dr^2}{h} + d\theta^2\right) - \frac{\mu^2}{f} d\varphi^2 \,, \tag{7.3.1}$$

is particularly convenient. Here $f = f(r,\theta)$, $\gamma = \gamma(r,\theta)$, $\mu = \mu(r,\theta)$, and $h = h(r)$. A redefinition of the coordinate r leads to an equivalent line element which has been used to investigate anisotropic static fluids [89].

The Einstein equations for a perfect fluid with 4-velocity U_α, density ρ, and pressure p (we use geometric units with $G = c = 1$)

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi \left[(\rho + p) U_\alpha U_\beta - p g_{\alpha\beta}\right] \tag{7.3.2}$$

for the line element (7.3.1) can be represented as two second-order differential equations for μ and f

$$\mu_{,rr} = -\frac{1}{2h}\left(2\mu_{,\theta\theta} + h_{,r}\mu_{,r} - 32\pi p\frac{\mu e^{2\gamma}}{f}\right) , \tag{7.3.3}$$

$$\begin{aligned} f_{,rr} = \quad & \frac{f_{,r}^2}{f} - \left(\frac{h_{,r}}{2h} + \frac{\mu_{,r}}{\mu}\right) f_{,r} + \frac{f_{,\theta}^2}{hf} - \frac{\mu_{,\theta} f_{,\theta}}{\mu h} - \frac{f_{,\theta\theta}}{h} \\ & + 8\pi\frac{(3p+\rho)e^{2\gamma}}{h} , \end{aligned} \tag{7.3.4}$$

where a subscript represents partial derivative. Moreover, the function γ is determined by a set of two partial differential equations

$$\begin{aligned} \gamma_{,r} = \quad & \frac{1}{h\mu_{,r}^2 + \mu_{,\theta}^2}\left\{\frac{\mu}{f^2}\left[\frac{\mu_{,r}}{4}\left(hf_{,r}^2 - f_{,\theta}^2\right) + \frac{1}{2}\mu_{,\theta} f_{,\theta} f_{,r}\right.\right. \\ & \left.\left. + 8\pi\mu_{,r} p f e^{2\gamma}\right] + \mu_{,\theta}\mu_{,r\theta} - \mu_{,r}\mu_{,\theta\theta}\right\}, \end{aligned} \tag{7.3.5}$$

$$\begin{aligned} \gamma_{,\theta} = \quad & \frac{1}{h\mu_{,r}^2 + \mu_{,\theta}^2}\left\{\frac{\mu}{f^2}\left[\frac{\mu_{,\theta}}{4}\left(f_{,\theta}^2 - hf_{r}^2\right) + \frac{1}{2}h\mu_{,r} f_{,\theta} f_{,r}\right.\right. \\ & \left.\left. - 8\pi\mu_{,\theta} p f e^{2\gamma}\right] + h\mu_{,r}\mu_{,r\theta} + \mu_{,\theta}\mu_{,\theta\theta}\right\} , \end{aligned} \tag{7.3.6}$$

which can be integrated by quadratures once f, μ, p, and h are known. The integrability condition of these partial differential equations turns out to be satisfied identically by virtue of the remaining field equations. Notice that there is no equation for the function $h(r)$. This means that it can be absorbed in the definition of the radial coordinate r. Nevertheless, one can also fix it arbitrarily; it turns out that this freedom is helpful when solving the equations and investigating the physical significance of the solutions.

The advantage of using the line element (7.3.1) is that the field equations are split into two sets. The main set consists of the equations (7.3.3) and (7.3.4) for μ and f which must be solved simultaneously. The second set consists of the first-order equations for γ which plays a secondary role in the sense that they can be integrated once the remaining functions are known. Notice also that the pressure p and the density ρ must be given *a* priori in order to solve the main set of differential equations for μ and f. As follows from Eq.(7.3.4), the equation of state $3p + \rho = 0$ reduces the complexity of this equation; nevertheless, this condition leads to negative pressures which, from a physical point of view, are not expected to be present inside astrophysical compact objects.

Finally, we mention that from the conservation law $T^{\alpha\beta}_{;\beta} = 0$, we obtain two first-order differential equations for the pressure

$$p_{,r} = -\frac{1}{2}(p+\rho)\frac{f_{,r}}{f} , \quad p_{,\theta} = -\frac{1}{2}(p+\rho)\frac{f_{,\theta}}{f} , \tag{7.3.7}$$

that can be integrated for any given functions $f(r,\theta)$ and $\rho(r,\theta)$, which satisfy Einstein's equations.

It is very difficult to find physically reasonable solutions for the above field equations, because the underlying differential equations are highly nonlinear with very strong couplings between the metric functions. In [90], some of us presented a new method for generating perfect fluid solutions of the Einstein equations, starting from a given seed solution. The method is based upon the introduction of a new parameter at the level of the metric functions of the seed solution in such a way that the generated new solution is characterized by physical properties which are different from those of the seed solutions.

In this work, we will analyze approximate solutions which satisfy the conditions for being applicable in the case of astrophysical compact objects. We will see that it is then possible to perform a numerical integration by imposing appropriate initial conditions. In particular, if we demand that the metric functions and the pressure are finite at the axis, it is possible to find a class of numerical solutions which can be matched with the exterior $q-$metric with a pressure that vanishes at the matching surface.

7.4 Linearized quadrupolar metrics

Our general goal is to investigate how the quadrupole moment influences the structure of spacetimes that can be used to describe the gravitational field of compact deformed gravitational sources. In particular, we aim to find perfect fluid solutions that can be matched with the exterior $q-$metric (7.2.1).

To find the corresponding interior line element, we proceed as follows. Consider the case of a slightly deformed mass. This means that the parameter q for the exterior $q-$metric can be considered as small and we can linearize the line element as

$$
\begin{aligned}
ds^2 = \quad & \left(1-\frac{2m}{r}\right)\left[1+q\ln\left(1-\frac{2m}{r}\right)\right]dt^2 \\
& -r^2\left[1-q\ln\left(1-\frac{2m}{r}\right)\right]\sin^2\theta d\varphi^2 \\
& -\left[1+q\ln\left(1-\frac{2m}{r}\right)-2q\ln\left(1-\frac{2m}{r}\right.\right. \\
& \left.\left.+\frac{m^2}{r^2}\sin^2\theta\right)\right]\left(\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right) \qquad (7.4.1)
\end{aligned}
$$

We will assume that the exterior gravitational field of the compact object is described to the first order in q by the line element (7.4.1), which represents a particular approximate solution to Einstein's equations in vacuum.

To construct the approximate interior line element, we start from the exact line element (7.3.1) and use the approximate solution (7.4.1) as a guide. Following this procedure, an appropriate interior line element can be expressed as

$$
\begin{aligned}
ds^2 = \quad & e^{2\nu}(1+qa)dt^2-(1+qc+qb)\frac{dr^2}{1-\frac{2\tilde{m}}{r}} \\
& -(1+qa+qb)r^2d\theta^2-(1-qa)r^2\sin^2\theta d\varphi^2\ , \qquad (7.4.2)
\end{aligned}
$$

where the functions $\nu=\nu(r)$, $a=a(r)$, $c=c(r)$, $\tilde{m}=\tilde{m}(r)$, and $b=b(r,\theta)$. Notice that we have introduced an additional auxiliary function $c(r)$ which plays a role similar to that of the auxiliary function $h(r)$ of the interior line element (7.3.1). Notice that the approximate line element (7.4.2) contains also the approximate exterior $q-$metric (7.4.1) as a particular case. This implies that vacuum fields can also be investigated by using this approximate line element.

7.4.1 General vacuum solution

To test the consistency of the linearized approach, we will derive explicitly the approximate vacuum $q-$metric (7.4.1). To this end, we compute the vacuum field equations from the line element (7.4.2) and obtain

$$\tilde{m}_{,r} = 0 \quad \text{i.}e. \quad \tilde{m} = m = const.\,, \tag{7.4.3}$$

$$\nu_{,r} = \frac{m}{r\,(r-2m)}\,, \tag{7.4.4}$$

$$(r-m)(a_{,r} - c_{,r}) + (a-c) = 0\,, \tag{7.4.5}$$

$$\begin{aligned} 2 \qquad & r\,(r-2m)\,a_{,rr} + (3r-m)\,a_{,r} + (r-3m)\,c_{,r} \\ & -2\,(a-c) = 0, \end{aligned} \tag{7.4.6}$$

$$\begin{aligned} r \qquad & (r-2m)\,b_{,rr} + b_{,\theta\theta} + (r-m)\,b_{,r} - 2\,(r-2m)\,c_{,r} \\ & +2\,(a-c) = 0, \end{aligned} \tag{7.4.7}$$

$$\begin{aligned} & \left(r^2 - 2mr + m^2\sin^2\theta\right) b_{,\theta} + 2r\,(r-2m) \\ & \times (ma_{,r} - a + c)\sin\theta\cos\theta = 0, \end{aligned} \tag{7.4.8}$$

$$\begin{aligned} & \left(r^2 - 2mr + m^2\sin^2\theta\right) b_{,r} + 2\,(r-2m) \\ & \times \left(r - m\sin^2\theta\right) a_{,r} + 2\,(r-m)\,(a-c)\sin^2\theta = 0, \end{aligned} \tag{7.4.9}$$

where for simplicity we have replaced the solution of the first equation $\tilde{m} = m = const.$ in the remaining equations.

Then, Eqs.(7.4.4) and (7.4.5) can be integrated and yield

$$\nu = \frac{1}{2} \ln \left(1 - \frac{2m}{r}\right) + \alpha_1 \,, \quad a - c = \frac{\alpha_2 m^2}{(r-m)^2} \,, \tag{7.4.10}$$

where α_1 and α_2 are dimensionless integration constants. The remaining system of partial differential equations can be integrated in general and yields

$$a = -\frac{\alpha_2 m}{r-m} + \frac{1}{2}(\alpha_3 - \alpha_2) \ln \left(1 - \frac{2m}{r}\right) + \alpha_4 \,, \tag{7.4.11}$$

$$c = -\frac{\alpha_2 m r}{(r-m)^2} + \frac{1}{2}(\alpha_3 - \alpha_2) \ln \left(1 - \frac{2m}{r}\right) + \alpha_4 \,, \tag{7.4.12}$$

$$\begin{aligned} b = \; & \frac{2\alpha_2 m}{r-m} - (\alpha_3 - \alpha_2) \Bigg[\ln 2 \\ & + \ln \left(1 - \frac{2m}{r} + \frac{m^2 \sin^2\theta}{r^2}\right) \Bigg] + \alpha_5 \,, \end{aligned} \tag{7.4.13}$$

where α_3, α_4 and α_5 are dimensionless integration constants.

Thus, we see that the general approximate exterior solution with quadrupole moment is represented by a 5-parameter family of solutions. The particular case

$$\begin{aligned} & \alpha_1 = 0 \,, \quad \alpha_2 = 0 \,, \quad \alpha_3 = 2 \,, \quad \alpha_4 = 0 \,, \\ & \quad \alpha_5 = 2 \ln 2 \,, \end{aligned} \tag{7.4.14}$$

corresponds to the linearized $q-$metric as represented in Eq.(7.4.1). Another interesting particular case corresponds to the choice

$$\alpha_1 = 0 \,, \quad \alpha_3 = \alpha_2 \,, \quad \alpha_4 = 0 \,, \quad \alpha_5 = 0 \,, \tag{7.4.15}$$

which leads to the following line element

$$\begin{aligned} ds^2 &= \left(1-\frac{2m}{r}\right)\left(1-\frac{q\alpha_2 m}{r-m}\right)dt^2 \\ &- \left(1+\frac{q\alpha_2 m}{r-m}\right) r^2\sin^2\theta d\varphi^2 \\ &- \left[1+\frac{q\alpha_2 m(r-2m)}{(r-m)^2}\right]\frac{dr^2}{1-\frac{2m}{r}} \\ &- \left(1+\frac{q\alpha_2 m}{r-m}\right) r^2 d\theta^2 \,. \end{aligned} \tag{7.4.16}$$

This is an asymptotically flat approximate solution with parameters m, q and α_2. The singularity structure can be found by analyzing the Kretschmann invariant $K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ which in this case reduces to

$$K = \frac{48m^2}{r^6}\left(1 + q\alpha_2\frac{r-4m}{r-m} + \mathcal{O}(q^2)\right) , \tag{7.4.17}$$

where the term proportional to q^2 has been neglected due to the approximate character of the solution. We see that there is a central singularity at $r = 0$ and a second one at $r = m$. We conclude that the solution (7.4.16) describes the exterior field of two naked singularities of mass m and quadrupole q. The parameter α_2 can be absorbed by redefining the constant q and so it has no special physical meaning. In the general solution (7.4.11)-(7.4.13), the additive constants α_4 and α_5 can be chosen such that at infinity the solution describes the Minkowski spacetime in spherical coordinates. This means that non asymptotically flat solutions are also contained in the $5-$parameter family (7.4.11)-(7.4.13). This is the most general vacuum solution which is linear in the quadrupole moment. To our knowledge, this general solution is new.

7.4.2 Newtonian limit

To further investigate the physical meaning of the solution (7.4.16), let us consider the coordinate transformations [81]

$$r = R\left[1 - q\frac{m}{R}\left(1 + \frac{m}{R}\left(\beta_1 + \sin^2\vartheta\right) + \frac{m^2}{R^2}\left(\beta_2 - \sin^2\vartheta\right) + ..\right)\sin^2\vartheta\right], \tag{7.4.18}$$

and

$$\theta = \vartheta - q\frac{m^2}{R^2}\left(1 + 2\frac{m}{R} + ...\right)\sin\vartheta\cos\vartheta, \tag{7.4.19}$$

where the β_1 and β_2 are constants and we have neglected terms older than m^3/R^3. Inserting the above coordinates into the metric (7.4.16), we obtain the approximate line element

$$ds^2 = (1 + 2\Phi)\,dt^2 - \frac{dR^2}{1 + 2\Phi} - U\,(R, \vartheta)\,R^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{7.4.20}$$

with

$$\Phi = -\frac{GM}{R} + \frac{GQ}{R^3}P_2\,(\cos\vartheta)\,, \tag{7.4.21}$$

$$U\,(R, \vartheta) = 1 - 2\frac{GM}{R^3}P_2(\cos\vartheta), \tag{7.4.22}$$

where $P_2(\cos\vartheta)$ is the Legendre polynomial of degree 2, and we have chosen the free constants as $\alpha_2 = 2$, $\beta_1 = 1/3$ and $\beta_2 = 5/3$.

We recognize the metric (7.4.20) as the Newtonian limit of general relativity, where Φ represents the Newtonian potential. Moreover, the constants

$$M = (1 + q)\,m, \quad Q = \frac{2}{3}qm^3, \tag{7.4.23}$$

can be interpreted as the Newtonian mass and quadrupole moment of the corresponding mass distribution.

We conclude that the metric (7.4.16) represents the exterior gravitational field of a slightly deformed mass. We will use this exterior approximate metric to match the interior solutions we will investigate in the following section.

Notice that in order to obtain the Newtonian limit (7.4.20), it is necessary to apply the coordinate transformation (7.4.18), which relates r with R and ϑ. This shows that r cannot interpreted as a radial coordinate and surfaces with

$r = const$ do not correspond to spheres.

7.5 Perfect fluid solutions

We now apply the approximate line element (7.4.2) to the study of perfect fluid solutions. First, we note that in this case the conservation law (7.3.7) reduces to

$$p_{,r} = -(\rho + p)\nu_{,r} \,, \qquad p_{,\theta} = 0 \,. \tag{7.5.1}$$

Calculating the second derivative $p_{,r\theta} = 0$, the above conservation laws lead to

$$\rho_{,\theta} = 0 \,, \tag{7.5.2}$$

implying that the perfect fluid variables can depend on the coordinate r only. As mentioned in the previous section, this does not imply that the source is spherically symmetric. In fact, due to the presence of the quadrupole parameter q in the line element (7.4.2), the coordinate r is no longer a radial coordinate and the equation $r =$constant represents, in general, a non-spherically symmetric deformed surface [74].

The corresponding linearized Einstein equations can be represented as

$$\overset{(0)}{G^{\nu}_{\mu}} + q \overset{(q)}{G^{\nu}_{\mu}} = 8\pi \left(\overset{(0)}{T^{\nu}_{\mu}} + q \overset{(q)}{T^{\nu}_{\mu}} \right) , \tag{7.5.3}$$

where the $(0)-$terms correspond to the limiting case of spherical symmetry. As for the energy-momentum tensor, we assume that density and pressure can also be linearized as

$$p(r) = p_0(r) + q p_1(r), \;\; \rho(r) = \rho_0(r) + q\rho_1(r), \tag{7.5.4}$$

in accordance with the conservation law conditions (7.5.1) and (7.5.2). Here, $p_0(r)$ and $\rho_0(r)$ are the pressure and density of the background spherically symmetric solution, respectively. If we now compute the linearized field equations (7.5.3) for the line element (7.4.2), we arrive at a set of nine differential equations for the functions ν, $\tilde{m}$, a, b, c, ρ_1 and p_1. After lengthy computations, it is then possible to isolate an equation that relates $p_1(r)$ and

$b(r,\theta)$ from which it follows that

$$b_{,\theta} = 0 \,. \tag{7.5.5}$$

This means that for the particular approximate line element (7.4.2), the field equations for a perfect fluid do not allow the metric functions to explicitly depend on the angular coordinate θ. To search for concrete solutions which can be matched with an exterior metric with quadrupole, it is necessary to modify the exterior metric accordingly. Therefore, we will now consider the alternative approximate metric (7.4.16), with $\alpha_2 = 2$, which can be expressed as

$$\begin{aligned} ds^2 \;=\; & \left(1-\frac{2m}{r}\right)\left(1-\frac{2qm}{r-m}\right)dt^2 \\ & -\left(1+\frac{2qm}{r-m}\right)r^2\sin^2\theta d\varphi^2 \\ & -\left[1+\frac{2qm(r-2m)}{(r-m)^2}\right]\frac{dr^2}{1-\frac{2m}{r}} \\ & -\left(1+\frac{2qm}{r-m}\right)r^2 d\theta^2 \,. \end{aligned} \tag{7.5.6}$$

As shown in the previous section, this approximate solution leads to the Newtonian limit (7.4.20) and can be used to describe the exterior field of a slightly deformed mass.

We will see that this approximate exterior solution can be used together with the interior line element (7.4.2) to search for approximate solutions with a perfect fluid source. Taking into account that the conservation laws and the approximate field equations for a perfect fluid imply that the physical quantities p and ρ and the metric function b depend only on the spatial coordinate r, the remaining field equations can be represented explicitly as given in 7.9.

7.5.1 The background solution

For the zeroth component of the linearized field equations, we will consider a spherically symmetric spacetime. If we set $q = 0$ in the line element (7.4.2), only the metric functions ν and $\tilde{m}$ remain for which we obtain the field equations

$$\tilde{m}_{,r} = 4\pi r^2 \rho_0, \tag{7.5.7}$$

$$\nu_{,r} = \frac{\tilde{m} + 4\pi r^3 p_0}{r\,(r - 2\tilde{m})}. \tag{7.5.8}$$

If we assume that the density is constant, $\rho_0 = $ const., we obtain a particular solution that can be represented as

$$\begin{aligned} e^{\nu} &= e^{\nu_0} = \frac{3}{2} f_0(R) - \frac{1}{2} f_0(r),\ \tilde{m} = \frac{4\pi}{3}\rho_0 r^3, \\ p_0 &= \rho_0 \frac{f_0(r) - f_0(R)}{3f_0(R) - f_0(r)}, \end{aligned} \tag{7.5.9}$$

with

$$f_0(r) = \sqrt{1 - \frac{2mr^2}{R^3}}, \tag{7.5.10}$$

where the integration constants have been chosen such that at the surface radius $r = R$ the exterior Schwarzschild metric is obtained. The resulting line element

$$\begin{aligned} ds^2 = \quad & \frac{1}{4}[3f_0(R) - f_0(r)]^2 dt^2 - \frac{dr^2}{1 - \frac{8\pi}{3}\rho_0 r^2} \\ & - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned} \tag{7.5.11}$$

represents the simplest spherically symmetric perfect fluid solution and is known as the interior Schwarzschild metric. In this work, we will use it as the zeroth approximation of the interior quadrupolar solutions to be obtained below.

7.5.2 Matching conditions

The importance of writing the approximate line elements as given above is that the matching between the interior and the exterior metrics can be performed in a relatively easy manner. Indeed, let us consider the boundary conditions at the matching surface $r = r_\Sigma$ by comparing the above interior metric (7.4.2) with the $q-$metric (7.5.6) to first order in q. Then, we obtain the matching conditions

$$a(r_\Sigma) = -\frac{2m}{r_\Sigma - m}, \quad c(r_\Sigma) = -\frac{2m}{(r_\Sigma - m)^2} \quad b(r_\Sigma) = \frac{4m}{r_\Sigma - m} + \alpha_5,$$
$$\nu(r_\Sigma) = \frac{1}{2}\ln\left(1 - \frac{2m}{r_\Sigma}\right) \quad \tilde{m}(r_\Sigma) = m. \tag{7.5.12}$$

In addition, we can impose the physically meaningful condition that the total pressure vanishes at the matching surface, i.e.,

$$p(r_\Sigma) = 0\,. \tag{7.5.13}$$

From the point of view of a numerical integration, the above matching conditions can be used as boundary values for the integration of the corresponding differential equations.

Notice that we reach the desired matching by fixing only the spatial coordinate as $r = r_\Sigma$; however, as mentioned in the previous section, this does not mean that the matching surface is a sphere. Indeed, the shape of the matching surface is determined by the conditions $t = const$ and $r = r_\Sigma$ which, according to Eq.(7.4.20), determine a surface with explicit $\theta-$dependence. The coordinate r is therefore not a radial coordinate. This has been previously observed in the case of a different metric with quadrupole moment [74].

7.6 Particular interior solutions

Our goal is to find interior solutions to the linearized system of differential equations which take into account the contribution of the quadrupole parameter q only up to the first order. The explicit form of the corresponding field equations is given in 7.9. One can see that they can be split into two sets that can be treated separately. The first set relates only the functions $\tilde{m}(r)$, $\nu(r)$ and $\rho_0(r)$, which must satisfy the equations

$$\tilde{m}_{,r} = 4\pi\rho_0 r^2\,, \tag{7.6.1}$$

$$\nu_{,r} = \frac{4\pi p_0 r^3 + \tilde{m}}{r\,(r - 2\tilde{m})}\,, \tag{7.6.2}$$

and

$$p_{0,r} = -\frac{\left(4p_0\pi r^3 + \tilde{m}\right)\left(p_0 + \rho_0\right)}{r\left(r - 2\tilde{m}\right)} . \tag{7.6.3}$$

This set of equations can be integrated immediately once the value of the density ρ_0 is known. In particular, for a constant ρ_0, we obtain the interior Schwarzschild metric (7.5.9), which is the zeroth order solution we will use in the following sections to integrate the field equations.

In addition, for the remaining functions $a(r)$, $b(r)$ and $c(r)$, we obtain a set of two second-order and three first-order partial differential equations which are presented explicitly in 7.9. In the next sections, we will analyze this set of equations and derive several particular solutions.

7.6.1 Solutions determined by constants

To find an interior counterpart for the approximate exterior $q-$metric, we consider first the simplest case in which a, b, c and ρ_1 are constants. The explicit form of the remaining field equations (see 7.9) suggests the relationship

$$b = -2a + C_{ab} , \quad C_{ab} = const. , \tag{7.6.4}$$

which reduces considerably the complexity of the equations. Indeed, the only non-trivial equations in this case are

$$\rho_1 = -\left(b + c\right)\rho_0 , \;\; p_1 = -\left(b + c\right) p_0, \tag{7.6.5}$$

so that the total pressure and density are

$$\begin{aligned} p\left(r\right) &= p_0(r)\left[1 - q\left(b + c\right)\right] , \\ \rho &= \rho_0\left[1 - q\left(b + c\right)\right] \; , \end{aligned} \tag{7.6.6}$$

where $p_0(r)$ is given in Eq.(7.5.9). The values of the constants a, b and c can be determined from the matching conditions with the exterior metric on the surface $r = r_\Sigma$. We obtain

$$\begin{aligned} &\nu\left(r_\Sigma\right) = -1.020, \tilde{m}\left(r_\Sigma\right) = 0.435 , \quad a\left(r_\Sigma\right) = -1.540, \;\; c\left(r_\Sigma\right) = -2.725, \\ &b\left(r_\Sigma\right) = 3.080, \; C_{ab} = 0.01. \end{aligned} \tag{7.6.7}$$

This is a simple approximate interior solution in which the presence of the

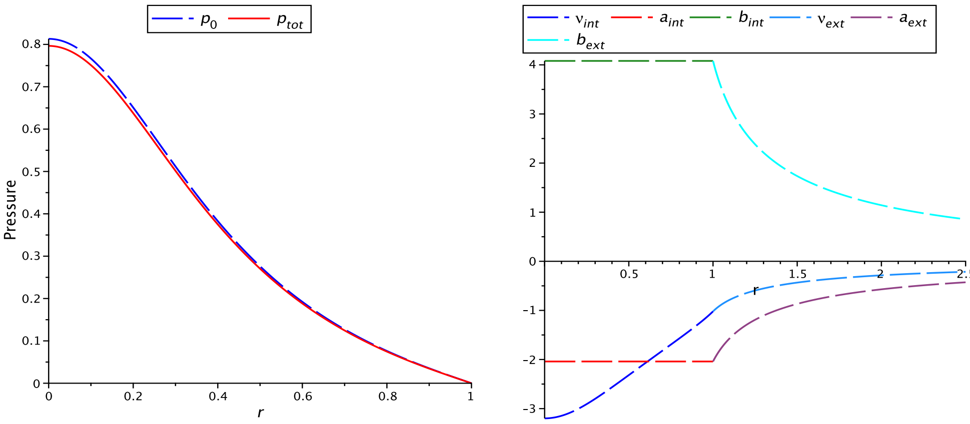

Figure 7.1: Behavior of the pressure and the metric functions in terms of the spatial coordinate r in units of m.

quadrupole parameter essentially leads to a modification of the pressure of the body. For instance, for the particular choice

$$\rho_0 = \frac{3m}{4\pi r_\Sigma^3}, \ R = 1, \ m = 0.435, \ \ q = \frac{1}{100}\,, \tag{7.6.8}$$

we obtain the pressure and the metric functions depicted in Fig.7.1. We conclude that this interior solution is singularity free and can be matched continuously across the matching surface $r = r_\Sigma$ with the approximate exterior metric (7.5.6).

Notice, however, the discontinuity in the derivatives of the metric functions. This means that the simple Ansatz of a constant interior metric is indeed compatible with the field equations, but leads to non physical solutions. In the next subsections, we will see that this problem can be solved by considering more general metric functions.

We conclude that in this case the corresponding interior line element can be expressed as

$$\begin{aligned} ds^2 = e^{2\nu}(1+qa)dt^2 - (1-qa)\Bigg(\frac{dr^2}{1-\frac{2\tilde{m}}{r}} \\ +r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\Bigg)\,, \end{aligned} \tag{7.6.9}$$

with

$$a = \ln\left(1 - \frac{2m}{R}\right) \, . \tag{7.6.10}$$

In the limiting case $q \to 0$, we turn back to the interior Schwarzschild solution.

7.6.2 Solutions with spatial dependence

We now assume that the functions a, b and c depend on the radial coordinate. As before, the interior Schwarzschild solution (7.5.9) is taken as the zeroth approximation. An analysis of the field equations shows that the following cases need to be considered.

1) Let $b(r) = 0$ and $a(r) = c(r)$. The field equations allow only one solution, namely, $a = const$. However, this is a trivial case that is equivalent to multiplying the density, pressure and some metric components by a constant quantity.

2) Let $b(r) = 0$ and $a(r) \neq c(r)$. In this case, the boundary conditions (7.5.12), which imply that $a = c = a\,(R)$, are not consistent with the remaining field equations. No solutions are found in this case.

3) Let $b(r) \neq 0$ and $a(r) \neq c(r)$. From Eqs. (7.9.8) and (7.9.9), we obtain that

$$b(r) = -2a(r) + C_{ab}. \tag{7.6.11}$$

where the C_{ab} is a constant. This relationship simplifies the remaining equations. Nevertheless, we were unable to find analytical solutions. Therefore, we perform a numerical integration of the remaining equations for the functions $a(r)$, $c(r)$, and $p_1(r)$. We take as a particular example the values specified in (7.6.8) for the mass m, the matching radius r_Σ and the quadrupole parameter q. Then, from the boundary conditions (7.5.12), we obtain

$$a\,(r_\Sigma) = -1.529, \; c\,(r_\Sigma) = -2.715, b\,(r_\Sigma) = 3.0696 \tag{7.6.12}$$

$$a_{,r}\,|_{r=r_\Sigma} = 25\;, C_{ab} = 0.01. \tag{7.6.13}$$

Moreover, to perform the numerical integration, it is necessary to specify

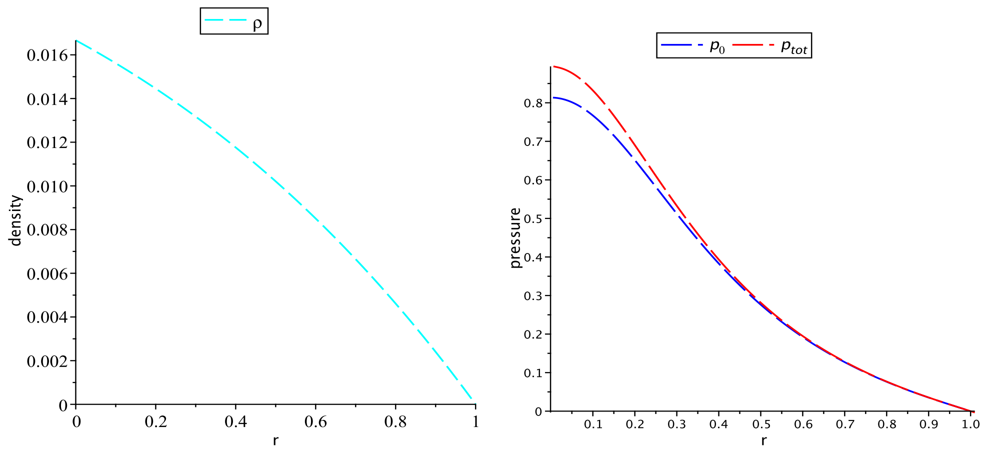

Figure 7.2: Behavior of the density and pressure as functions of the radial coordinate.

the profile of the density function $\rho_1(r)$, which we take as

$$\rho_1(r) = \rho_1(0) - r - \frac{1}{2}r^2 - \frac{1}{6}r^3 \,, \tag{7.6.14}$$

where the constant $\rho_1(0)$ must be chosen such that the total density $\rho = \rho_0 + q\rho_1(r)$ vanishes at the surface and is finite at the center $r = 0$. For the particular parameter values (7.6.8), we obtain $\rho_1(0) = -8.718$ and the behavior of the total density is illustrated in Fig. 7.2. With this density function, the numerical integration can be performed explicitly, leading to a solution for the pressure which is presented in Fig. 7.2.

Moreover, the result of the numerical integration of the metric functions $a(r), b(r)$ and $c(r)$ is represented in Fig. 7.3. We see that all the boundary conditions for the variables of the perfect fluid and the metric functions are satisfied and that all the quantities show a regular behavior. Moreover, to further analyze the physical significance of this solution, we can determine the corresponding equation of state from the value of the total density and pressure (see Fig. 7.2). The result is shown in Fig. 7.4. We notice a realistic behavior as the pressure increases with the density.

We conclude that in this particular case it is possible to obtain physically meaningful solutions. For completeness, we mention that the line element

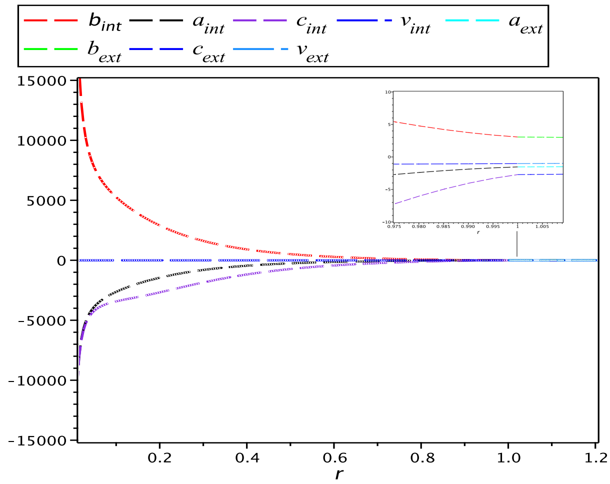

Figure 7.3: Behavior and matching of the metric functions.

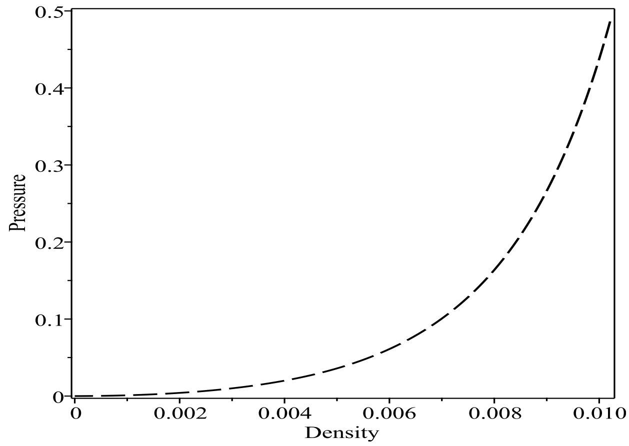

Figure 7.4: Equation of state of the solution represented in Fig. 7.2 (P versus ρ in geometrized units).

for this perfect fluid solution can be written as

$$\begin{aligned} ds^2 &= e^{2\nu}(1+qa)dt^2 - (1+qb+qc)\frac{dr^2}{1-\frac{2\tilde{m}}{r}} \\ &\quad -(1-qa)(d\theta^2 + r^2\sin^2\theta d\varphi^2) \end{aligned} \tag{7.6.15}$$

which, as expected, leads to the interior Schwarzschild spacetime for vanishing quadrupole parameter.

7.6.3 Barotropic solutions

A further method to obtain approximate interior solutions is to specify *a* priori an EoS that relates density and pressure. An interesting class of fluids are the barotropic fluids [75], which obey the EoS $p = p(e)$, where e is the total energy density. One of the interesting cases of barotropic fluids is provided by matter at zero temperature. In this case, the barotropic EoS is given as $p = p(\rho)$ and $e = e(\rho)$. One of the simplest cases is represented by the barotropic relation

$$p = w\rho(r) \ , \tag{7.6.16}$$

where w is the barotropic constant factor. The corresponding field equations can be obtained by replacing the above equation in the equations presented in 7.9. To proceed with the numerical integration of the field equations in this case, we choose the free parameters as

$$\begin{aligned} &\rho_0 = 0.7,\ r_\Sigma = 1,\ m = 0.23626,\ q = \frac{1}{100}, C_{ab} = 0.01, \\ &\nu(r_\Sigma) = -0.3198,\ a(r_\Sigma) = -0.6186,\ c(r_\Sigma) = -0.8101. \end{aligned} \tag{7.6.17}$$

Moreover, we assume the boundary conditions (7.5.12) for the metric functions and the vanishing of the pressure (7.5.13) on the matching surface. The boundary conditions are then used as initial values for the numerical integration. The behavior of the resulting total density is represented in Fig.7.5 for $w = 0.3$. To test our approach, we calculate numerically the value of the total pressure and obtain a behavior similar to that of the total density, which confirms the validity of the barotropic EoS as shown in Fig. 7.6. Furthermore, the numerical integration of the field equations indicates that the metric functions are regular inside the source and can be matched with the

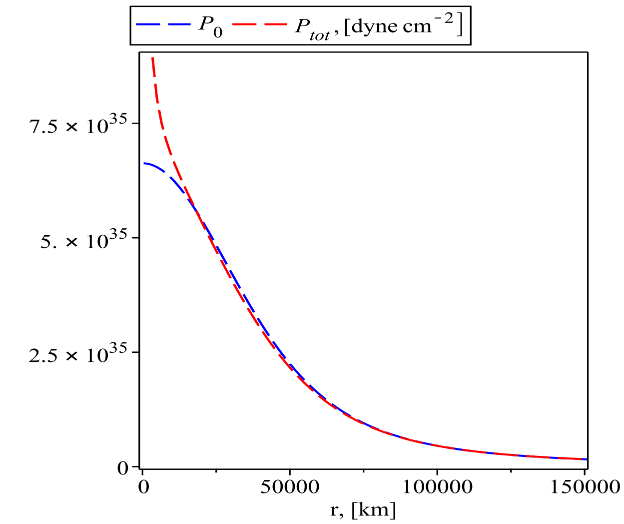

Figure 7.5: Behavior of the total density in cgs units. The central density is $\rho_0 = 10^6\ g/cm^3$

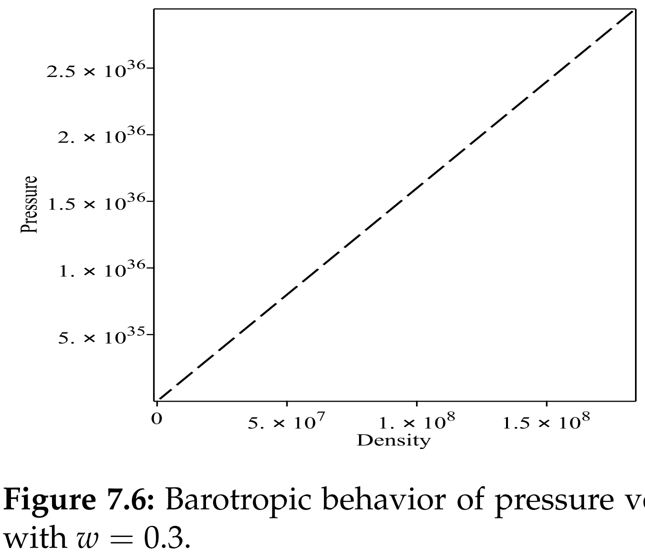

Figure 7.6: Barotropic behavior of pressure versus density $P(\rho)$ in cgs units with $w = 0.3$.

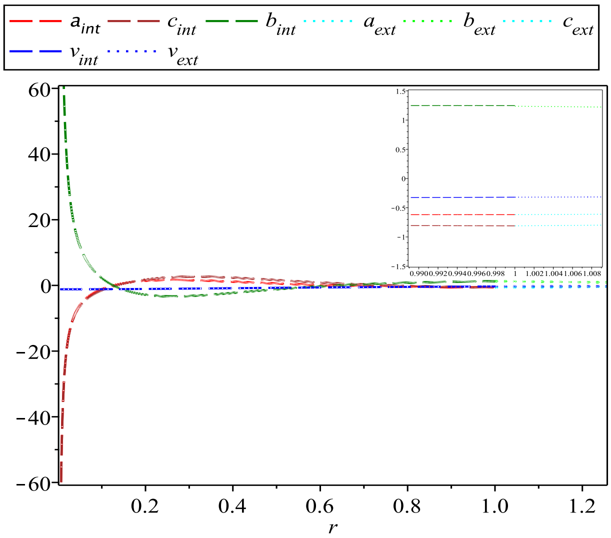

Figure 7.7: Metric functions inside and outside the matching surface in geometrized units.

metric functions of the exterior $q-$metric. This is shown explicitly in Fig. 7.7.

We conclude that it is possible to obtain approximate interior solutions with a barotropic EoS, which can be consistently matched with the exterior metric functions and are characterized by well-behaved density and pressure functions.

7.6.4 Polytropic solutions

A polytrope refers in the case considered here to a perfect fluid in which the pressure depends upon the density in the form

$$p = k\rho^{\gamma} , \tag{7.6.18}$$

where k is a proportionality constant and γ is the polytropic index. The corresponding field equations can be integrated numerically by assuming the values (7.6.17) for the free parameters.

Using the boundary conditions (7.5.12) and assuming the vanishing of the pressure on the matching surface, we obtain for the pressure the behavior depicted in Fig.7.8, whereas the metric functions that satisfy the field equations

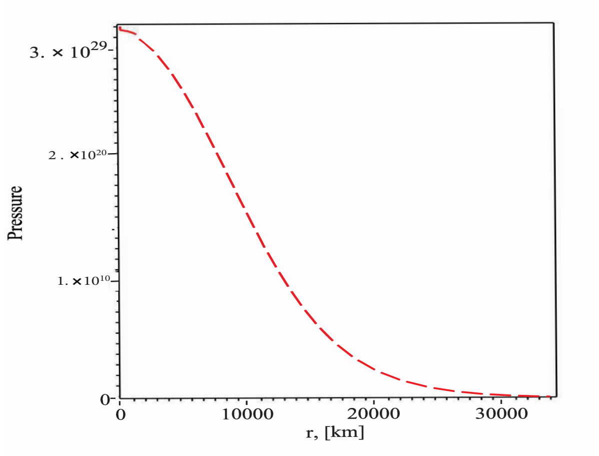

Figure 7.8: Behavior of the total pressure in cgs units for a polytropic perfect fluid with $k = 0.43$ and $\gamma = 3/2$.

and the matching conditions can be represented as in Fig.7.9.

Thus, we see that in the case of a polytropic fluid our approach allows us to obtain approximate interior solutions that can be matched with the exterior $q-$metric. The particular values of the proportionality constant $k = 0.43$ and the polytropic index $\gamma = 3/2$ have been chosen such that they correspond effectively to a realistic physical system, as will be shown in the next section.

7.7 On the physical significance of the solutions

To derive interior solutions with quadrupole moment that can be matched with the exterior approximate q-metric, first we have made a series of assumptions that reduce the mathematical complexity of the underlying field equations. As a result, we obtained numerical solutions that show the mathematical consistency of the proposed method. However, the physical significance of the solutions and their applicability to describe realistic compact objects seem to be drastically reduced by the assumption of the before mentioned mathematical conditions. Indeed, the first solution we obtained in Sec. 7.6.1 is characterized by a constant value of the energy density and constant values of some interior metric functions which, as mentioned before, lead to discontinuities in the derivatives of the metric functions and, consequently, to nonphysical solutions.

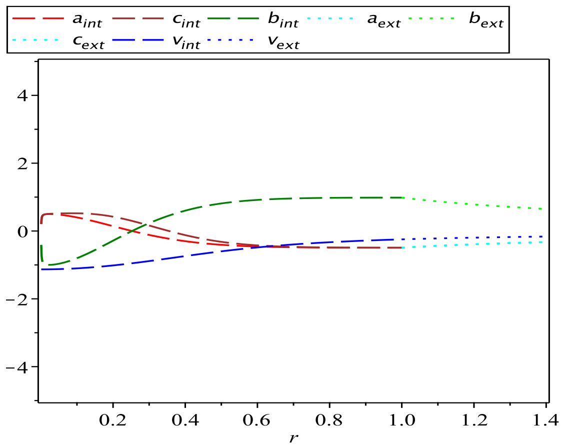

Figure 7.9: Behavior of the metric functions for a polytropic perfect fluid with $k = 0.43$ and $\gamma = 3/2$.

In Sec. 7.6.2, we applied a different procedure in which the profile of the density is given *a* priori as a function of the spatial coordinate. In this case, we obtain numerical solutions for the pressure and metric functions with a more realistic physical behavior, as shown in Figs. 7.2 and 7.3. Indeed, the density and pressure are maximal at the center of the source, then their value diminish continuously until they vanish at the surface of the source.

Then, in Sections 7.6.3 and 7.6.4, we apply a different method that consists in specifying *a* priori an equation of state, which allows us to integrate the corresponding field equations. We assume barotropic and a polytropic equations of state and find numerical solutions for the pressure and the metric functions that are well behaved from a physical point of view.

We now investigate whether it is possible to apply our results to describe the gravitational field of realistic compact objects, in which several properties of the internal structure of the gravitational sources are taken into account. Indeed, the most recent surveys with observational data show that white dwarfs [47, 46, 48] and neutron stars [42, 44, 43, 45] are realistic compact objects, where relativistic effects are expected to play a non-negligible role. Accordingly, we will compare the physical properties of our solutions, as expressed at the level of the EoS, with those of realistic compact objects. Consider, for instance, a white dwarf whose interior is described by the Chandrasekhar EoS in parametric form (geometrized units) [74, 78, 72, 77]

$$\rho_{Ch} = \frac{32}{3}\left(\frac{m_e}{m_n}\right)^3 K_n \left(\frac{\overline{A}}{Z}\right) y\left(x\right)^3 , \tag{7.7.1}$$

$$\begin{aligned} p_{Ch} &= \frac{4}{3}\left(\frac{m_e}{m_n}\right)^4 K_n \Big[y\left(x\right)\left(2y\left(x\right)^2 - 3\right)\sqrt{1+y\left(x\right)^2} \\ &\quad +3\ln\left(y\left(x\right)+\sqrt{1+y\left(x\right)^2}\right) , \Big] \end{aligned} \tag{7.7.2}$$

with

$$K_n = \frac{m_n^4}{32\pi^2}, \tag{7.7.3}$$

where $\overline{A}$ and Z are the average atomic weight and atomic number of the corresponding nuclei; $y\left(x\right) = p_e\left(x\right)/m_e$, with $p_e\left(x\right)$, m_e and m_n are the Fermi momentum, the mass of the electron and the mass of the nucleon, respectively. Here, we consider the particular case for the average molecular weight $\overline{A}/Z = 2$.

The Chandrasekhar EoS is considered to be the simplest, but also the most relevant and basic EoS for the description of white dwarf matter. It should be noted that there are plenty of more sophisticated EoS, which are used to describe the interior of white dwarfs and outer crusts of neutron stars [73, 71, 64, 70, 69]. These equations take into account the electron-electron, electron-ion and ion-ion Coulomb interactions, nuclear composition, Thomas-Fermi corrections, finite temperature effects, phase transitions, magnetic fields, etc. [68, 67, 50, 66, 65, 64, 63, 60]. However, for our purposes we will restrict ourselves to the Chandrasekhar EoS for simplicity and clarity.

In the case of neutron stars, the internal structure can be described by the EoS of a pure degenerate neutron gas, which in parametric form can be written as [74, 77, 69]

$$\begin{aligned}
\rho_{NS} &= \frac{\epsilon_0}{8}\Bigg[\left(2y\left(x\right)^3 + y\left(x\right)\right)\sqrt{1+y\left(x\right)^2} \\
&\quad - \ln\left(y\left(x\right) + \sqrt{1+y\left(x\right)^2}\right)\Bigg], \qquad (7.7.4)\\
p_{NS} &= \frac{\epsilon_0}{24}\Bigg[\left(2y\left(x\right)^3 - 3y\left(x\right)\right)\sqrt{1+y\left(x\right)^2} \\
&\quad + 3\ln\left(y\left(x\right) + \sqrt{1+y\left(x\right)^2}\right)\Bigg], \qquad (7.7.5)
\end{aligned}$$

where $\epsilon_0 = m_n^4 c^5/\pi^2$ is the energy density.

It should be noted that the pure degenerate neutron gas represents the simplest EoS for neutron stars, though there is a plenty of other more complicated EoSs in the literature, accounting for the nucleon-nucleon interactions, contribution of various particles-carriers of interaction, etc. [69, 55]. The most recent and realistic EoSs are tested via measuring X-ray emissions, tidal deformation and gravitational wave events [54, 49, 50, 51, 52, 53].

To prove that the interior metrics investigated in this work can be used to describe the internal gravitational field of white dwarfs and neutron stars, we should integrate the above EoSs together with the field equations given in 7.9. A preliminary computation shows that this is possible, but requires a detailed analysis of the parameters and constants that enter the EoSs. This we expect to investigate in future works. For the purposes of the present work, we propose to use a different method, which consists in finding an effective EoS from the above parametric EoS. To this end, we compute the parametric pressure and density and plot the result as an effective EoS of pressure vs. density. The result is shown in Fig. 7.10 for white dwarfs and in Fig. 7.11 for neutron stars, respectively. It is then easy to show that in both cases the effective EoS can be approximated by a polytrope, $p = k\rho^\gamma$, with $k = 5.510^{-6}$ and $\gamma = 3/2$ for white dwarfs and $k = 7.1310^{19}$ and $\gamma = 1.365$ for neutron stars, respectively. On the other hand, in Sec. 7.6.4, we have shown that it is possible to obtain interior solutions with a polytropic EoS. We thus conclude that the method presented in this work can be used to describe the interior gravitational field of white dwarfs and neutron stars with an effective polytropic EoS.

However, notice that to integrate the field equations in our approach we

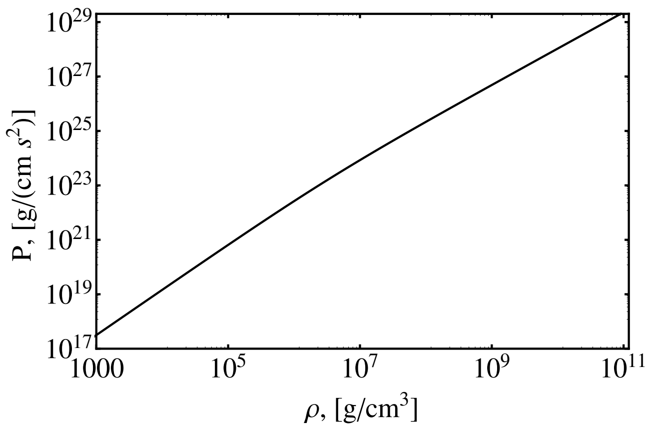

Figure 7.10: Effective EoS (log scale) for white dwarfs obtained from the parametric Chandrasekhar EoS.

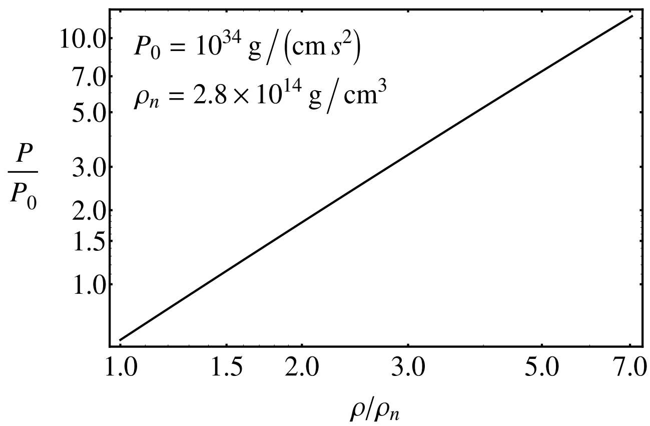

Figure 7.11: Effective EoS (log scale) for neutron stars described by a pure degenerate neutron gas. Here ρ_n is the average nuclear density.

must fix *a* priori the values of the mass and the matching radius so that we cannot establish any limits on these quantities, which are very important for the understanding of the physical properties of astrophysical compact objects like white dwarfs and neutron stars. Our approach does not allow us to investigate this problem. To this end, it is necessary to take into account the equilibrium conditions for the interior solution. In the case of spherically symmetric interior solutions, this is done by considering the Tolman-Oppenheimer-Volkoff (TOV) equation. Consequently, to perform a more physical analysis of the inner structure of astrophysical objects with quadrupole moment, it is necessary to generalize the TOV procedure to include axially symmetric sources. We expect to investigate this problem in future works.

7.8 The HPHM metric

Hernández-Pastora, Herrera and Marti (HPHM) presented recently in [61] a general procedure to find static and axially symmetric interior solutions to the Einstein equations by focusing on the matching of the interior spacetime to a given exterior solution. To this end, a particular line element was chosen that can be expressed as

$$ds^2 = -e^{2\hat{a}} Z\left(r\right)^2 dt^2 = \frac{e^{2\hat{g}-2\hat{a}}}{A\left(r\right)} dr^2 + e^{2\hat{g}-2\hat{a}} r^2 d\theta^2 + e^{2\hat{a}} r^2 \sin^2\theta d\varphi^2 \,, \tag{7.8.1}$$

where $\hat{a}$ and $\hat{g}$ are functions of r and θ, in general.

To find the relationship between the interior solution found in [61] for the exterior $q-$metric and the interior solutions presented in the present work, we compare (7.8.1) with the line element proposed here in Eq.(7.4.2), i.e.,

$$\begin{aligned} ds^2 = e^{2\nu}(1+qa)dt^2 - (1+qc+qb)\frac{dr^2}{1-\frac{2\tilde{m}}{r}} \\ -(1+qa+qb)r^2 d\theta^2 - (1-qa)r^2\sin^2\theta d\varphi^2 \,. \end{aligned} \tag{7.8.2}$$

Since the set of coordinates is the same in both line elements, we can perform a direct comparison term by term of the metric functions and obtain the set of equations

$$e^{2\hat{a}} Z^2 = e^{2\nu}(1+qa) \,, \tag{7.8.3}$$

$$e^{2\hat{g}-2\hat{a}}A^{-1} = (1+qc+qb)\left(1-\frac{2\tilde{m}}{r}\right)^{-1} , \tag{7.8.4}$$

$$e^{2\hat{g}-2\hat{a}} = 1+qa+qb , \tag{7.8.5}$$

$$e^{-2\hat{a}} = 1-qa . \tag{7.8.6}$$

Obviously, the compatibility between Eqs.(7.8.3) and (7.8.6) is guaranteed. On the other hand, since the functions A and $\tilde{m}$ depend on r only and both of them can be absorbed by an appropriate reparametrization of the coordinate r, without loss of generality we can set

$$A = 1-\frac{2\tilde{m}}{r} . \tag{7.8.7}$$

Then, from Eqs.(7.8.4) and (7.8.5), we obtain

$$e^{2\hat{g}-2\hat{a}} = 1+qc+qb = 1+qa+qb , \tag{7.8.8}$$

which implies that

$$a = c . \tag{7.8.9}$$

All the physically relevant interior solutions that we find in Sec. 7.6 are characterized by the condition $a \neq c$. It follows that the interior metrics derived in the present work cannot be obtained as particular solutions of the HPHM solution.

7.9 Linearized field equations

In general, up to the first order in q, the field equations which follow from the line element

$$\begin{aligned} ds^2 = {} & e^{2\nu}(1+qa)dt^2 - (1+qc+qb)\frac{dr^2}{1-\frac{2\tilde{m}}{r}} \\ & -(1+qa+qb)r^2 d\theta^2 - (1-qa)r^2\sin^2\theta d\varphi^2 , \end{aligned} \tag{7.9.1}$$

where all the metric functions depend on r only, can be written as

$$\tilde{m}_{,r} = 4\pi\rho_0 r^2, \tag{7.9.2}$$

$$\nu_{,r} = \frac{4\pi p_0 r^3 + \tilde{m}}{r\left(r - 2\tilde{m}\right)}, \tag{7.9.3}$$

,

$$p_{0,r} = -\frac{\left(4p_0\pi r^3 + \tilde{m}\right)\left(p_0 + \rho_0\right)}{r\left(r - 2\tilde{m}\right)}, \tag{7.9.4}$$

$$\begin{aligned} &2r\left(r - 2\tilde{m}\right)a_{,rr} + \left[\left(3p_0 - 2\rho_0\right)4\pi r^3 + 3r - \tilde{m}\right]a_{,r} \\ &+\left(r - 3\tilde{m} - 4\pi p_0 r^3\right)c_{,r} - 16\pi r^2\left[\left(b + c\right)\left(\rho_0 + p_0\right)\right. \\ &\left.+\rho_1 + p_1\right] - 2\left(a - c\right) = 0, \end{aligned} \tag{7.9.5}$$

$$\begin{aligned} &r\left(r - 2\tilde{m}\right)b_{,rr} + \left(r - \tilde{m} - 4\pi\rho_0 r^3\right)b_{,r} \\ &-2\left(r - 2\tilde{m}\right)c_{,r} + 16\pi r^2\left[\left(c + b\right)\rho_0 + \rho_1\right] \\ &+2\left(a - c\right) = 0, \end{aligned} \tag{7.9.6}$$

$$\begin{aligned} &\left(4\pi p_0 r^3 + r - \tilde{m}\right)\left(a_{,r} - c_{,r}\right) - 32\pi r^2\left[\left(c + b\right)p_0 + p_1\right] \\ &+2\left(a - c\right) = 0, \end{aligned} \tag{7.9.7}$$

$$\begin{aligned} &2\left(4\pi p_0 r^3 + \tilde{m}\right)a_{,r} + \pi r^2\left[\left(c + b\right)p_0\right. \\ &\left.+p_1\right] - 2\left(a - c\right) = 0, \end{aligned} \tag{7.9.8}$$

$$\begin{aligned} &\left[4\left(4\pi p_0 r^3 + r - \tilde{m}\right)^2\sin^2\theta + r\left(r - 2\tilde{m}\right)\cos^2\theta\right]b_{,r} \\ &+2\left[\left(4\pi p_0 r^3 - \tilde{m}\right)\sin^2\theta + r\right]\left(r - 2\tilde{m}\right)a_{,r} \\ &-2\sin^2\theta\left(4\pi p_0 r^3 + r - \tilde{m}\right)\left\{8\pi r^2\left[\left(c + b\right)p_0 + p_1\right]\right. \\ &\left.-a + c\right\} = 0. \end{aligned} \tag{7.9.9}$$

The first three equations constitute a separated set of differential equations that can be integrated for any given value background density ρ_0. From this set of equations we can derive solutions for the metric functions ν and $\tilde{m}$ and for the background pressure p_0. These solutions can then be used in the second set of equations (7.9.5)-(7.9.9) to find solutions for the metric functions a, b and c and for the pressure function p_1.

7.10 Conclusions and remarks

In this work, we have investigated interior solutions of Einstein's equations in the case of static and axially symmetric perfect fluid spacetimes. We impose the physical condition that the interior spacetime can be matched smoothly with the exterior $q-$metric characterized by two parameters, which determine the mass and quadrupole moment of the source.

We assume in this work that the interior counterpart of the exterior $q-$metric is described by an isotropic perfect fluid, for the sake of simplicity. Indeed, as has been shown in [89], the most general interior solution, which is compatible with an exterior static and axisymmetric gravitational source, accepts up to four stresses. Thus, we are assuming here that three of them are negligible small so that we end up with a perfect fluid with only one isotropic stress. Moreover, the isotropic perfect fluid is a very idealized model since it has been shown that static and isotropic perfect fluid sources must be spherical, at least in the case of an incompressible equation of state [100, 89]. This is related to the fact that even small pressure anisotropies can generate breaks in the fluid distribution, drastically moving away from the idealized isotropic configuration [101]. Furthermore, analyzing the stability of the isotropy condition, it has been established recently in [102] that the realistic physical processes that occur during the stellar evolution inevitably lead to the appearance of pressure anisotropies, which cannot disappear during the dynamic evolution of the mass distribution. Consequently, several results indicate that the final equilibrium configuration of a realistic stellar evolution is characterized by the presence of pressure anisotropies. Nevertheless, in this work, we assume the isotropy condition to simplify the mathematical complexity of the resulting field equations. Our results show that it is possible to find approximate perfect fluid solutions with quadrupole, which are consistent from the mathematical point of view, in the sense that Einstein's equations are satisfied for certain equations of state.

In general, it is difficult to find interior solutions for a given exterior solution by using Einstein's equations. Due to the complexity of the corresponding set of differential equations, we reduce the problem to the particular case in which the quadrupole parameter can be considered as a small quantity, which is then used to linearize the exterior $q-$metric. As for the interior metric, we use the particular linearized Ansatz (7.4.2), leading to a simplified set of differential equations in which, as a consequence of the conservation law (7.5.1) and (7.5.2), the parameters of the perfect fluid do not depend on the angular coordinate θ. We then analyze the corresponding linearized field equations and derive several classes of new vacuum and perfect fluid solutions, which depend on the spatial coordinate r, only.

We search for interior solutions that can be matched with the approximate exterior $q-$metric, satisfy the energy conditions for the density and the pressure and are free of singularities in the entire spacetime. Our results show that it is possible to find such solutions, implying that the approximate approach is compatible with the imposed physical conditions. Indeed, we first found a particular analytical solution for which pressure and density are well-behaved functions, but the derivatives of the metric functions present discontinuities that are considered as nonphysical. However, when we impose a particular functional dependence for the density, a barotropic or a polytropic EoS, we obtain numerical solutions that satisfy the matching conditions for the metric functions and the energy conditions for the fluid. In addition, we analyzed the behavior of the pressure and density as given at the level of the EoS. To verify if these particular solutions can be applied to describe the exterior and interior gravitational field of realistic compact objects, we analyzed the Chandrasekhar EoS for white dwarfs and the pure degenerate neutron gas EoS for neutron stars. We found that the inner properties of these compact objects can be represented effectively by a polytropic EoS, which is essentially the behavior observed in the numerical solutions found in Sec. 7.6.4. We interpret this result as an indication that our approximate solutions with arbitrary mass and small quadrupole moment can be used to study the exterior and interior gravitational field of relativistic compact objects. Nevertheless, we also noticed that the model developed in this work does not impose any limits on the value of the mass of compact objects, a result which is well known from observations and more sophisticated theoretical studies. This is a disadvantage of our model that can be improved by considering the equilibrium conditions for the interior solutions. To this end, it would be necessary to consider a generalization of the TOV procedure to include axisymmetric gravitational fields. We will consider this problem in future works.

In the literature, there are several methods that deal with approximate solutions of Einstein's equations [41, 40, 85, 39, 38] Usually, they intend to find numerical solutions, starting from approximate metrics, field equations, numerical EoS and sometimes even approximate coordinates. Sometimes those methods do not care about the matching conditions at the level of the derivatives of the metric. With the same initial conditions and physical assumptions, the method we use here should be able to reproduce the results obtained by other methods. However, the intention of our method is to go beyond approximate numerical solutions. For instance, in the case of the quadrupole, other methods do not care about the form of the corresponding exact metric. In our case, this is very important because we are using the simplest known exact generalization of the Schwarzschild solution with quadrupole [98], for which we expect to find a physically reasonable interior counterpart. The approximate solutions found in this work show that our method is self-consistent and can deliver physically meaningful solutions. This first approximate approach is necessary as a first proof of the capability of the method.

The obtained solutions are mathematically consistent, but are physically restricted because they depend on only one spatial coordinate. A more realistic case would necessarily imply to consider solutions with an additional angular dependence, which corresponds to taking into account the rotation of the gravitational source. To this end, it will be binding to investigate generalizations of the particular Ansatz (7.4.2). This is a task of a future work.

The results obtained here can be considered as a first step towards the determination of an exact solution of Einstein's equations which describes correctly the gravitational field of a rotating deformed source. The important feature of our approach is that we consider explicitly the influence of the quadrupole on the structure of spacetime and the corresponding field equations. In the present work, we only considered the simple and idealized case of a static mass distribution with a small quadrupole and obtained compatible and physically reasonable results. To study more realistic configurations, it is necessary to take into account the rotation and the exact quadrupole of the mass. We expect to investigate these problems in future works.

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