Generalizations of the Kerr-Newman solution

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2 Brief description

One of the most important metrics in general relativity is the Kerr-Newman solution that describes the gravitational and electromagnetic fields of a rotating charged mass. For astrophysical purposes, however, it is necessary to take into account the effects due to the moment of inertia of the object. To attack this problem, we investigate new exact solutions of Einstein-Maxwell equations which posses an infinite set of gravitational and electromagnetic multipole moments and contain the Kerr-Newman solution as special case.

The link between exterior solutions to the Einstein gravitational field equations such as the exact Erez-Rosen metric and approximate Hartle-Thorne metric is established here for the static case in the limit of linear mass quadrupole moment and second order terms in total mass. To this end, the Geroch-Hansen multipole moments are calculated for the Erez-Rosen and Hartle-Thorne solutions in order to find the relationship among the parameters of both metrics. The coordinate transformations are sought in a general form with two unknown functions in the corresponding limit. By employing perturbation theory, the approximate Erez-Rosen metric is written in the same coordinates as the Hartle-Thorne metric. By equating the radial and azimuthal components of the metric tensor of both solutions the sought functions are found in a straightforward way. The results obtained here can be applied to different astrophysical situations.

We also consider various equations of state of neutron star matter, which include from the point of neutron drops formation to supra nuclear densities. Particular attention is paid to the nucleon–nucleon interaction since, in addition to the kinetic energies of the particles, the interactions among nucleons play a key role. Moreover, we investigate the properties of super-dense matter with various sets of particles. In order to achieve these goals, different potentials were considered, which are in a good agreement with experimental data. Furthermore, we find the energy of the system by using various multi-particle methods, including the interaction of nucleons. Thanks to this information, the thermodynamic parameters such as the pressure and energy density are calculated. In conclusion, the latest observational constraints on the equation of state are taken into account and we show that the observational data require that the equation of state be stiff.

3 Introduction

It is hard to overemphasize the importance of the Kerr geometry not only for general relativity itself, but also for the very fundamentals of physics. It assumes this position as being the most physically relevant rotating generalization of the static Schwarzschild geometry. Its charged counterpart, the Kerr-Newman solution, representing the exterior gravitational and electromagnetic fields of a charged rotating object, is an exact solution of the Einstein-Maxwell equations.

Its line element in Boyer–Lindquist coordinates can be written as

$$\begin{aligned} ds^2 &= \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 \\ &\quad - \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[(r^2 + a^2)d\varphi - adt]^2 \\ &\quad - \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 , \end{aligned} \tag{3.0.1}$$

where M is the total mass of the object, $a = J/M$ is the specific angular momentum, and Q is the electric charge. In this particular coordinate system, the metric functions do not depend on the coordinates t and ϕ, indicating the existence of two Killing vector fields $\xi^I = \partial_t$ and $\xi^{II} = \partial_\varphi$ which represent the properties of stationarity and axial symmetry, respectively.

An important characteristic of this solution is that the source of gravity is surrounded by two horizons situated at a distance

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{3.0.2}$$

from the origin of coordinates. Inside the interior horizon, r_-, a ring singularity is present which, however, cannot be observed by any observer situated outside the exterior horizon. If the condition $M^2 < a^2 + Q^2$ is satisfied, no horizons are present and the Kerr–Newman spacetime represents the exterior field of a naked singularity.

Despite of its fundamental importance in general relativity, and its theoretical and mathematical interest, this solution has not been especially useful for describing astrophysical phenomena, first of all, because observed astrophysical objects do not possess an appreciable net electric charge. Furthermore, the limiting Kerr metric takes into account the mass and the rotation, but does not consider the moment of inertia of the object. For astrophysical applications it is, therefore, necessary to use more general solutions with higher multipole moments which are due not only to the rotation of the body but also to its shape. This means that even in the limiting case of a static spacetime, a solution is needed that takes into account possible deviations from spherically symmetry.

4 The general static vacuum solution

In general relativity, stationary axisymmetric solutions of Einstein's equations [1] play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of know exact solutions drastically increased after Ernst [2] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies lead finally to the development of solution generating techniques [1] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [4] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

4.1 Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindric coordinates (t, ρ, z, φ). Stationarity implies that t can be chosen as the time coordinate and the metric does not depend on time, i.e. $\partial g_{\mu\nu}/\partial t = 0$. Consequently, the corresponding timelike Killing vector has the components δ^μ_t. A second Killing vector field is associated to the axial symmetry with respect to the axis $\rho = 0$. Then, choosing φ as the azimuthal angle, the metric satisfies the conditions $\partial g_{\mu\nu}/\partial\varphi = 0$, and the components of the corresponding spacelike Killing vector are δ^μ_φ.

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form $g_{\mu\nu} = g_{\mu\nu}(\rho, z)$, it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [4, 5, 6]

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] , \tag{4.1.1}$$

where f, ω and γ are functions of ρ and z, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$\rho\partial_\rho\Omega = f^2\partial_z\omega , \qquad \rho\partial_z\Omega = -f^2\partial_\rho\omega , \tag{4.1.2}$$

the vacuum field equations $R_{\mu\nu} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \partial_z^2 f + \frac{1}{f}[(\partial_\rho\Omega)^2 + (\partial_z\Omega)^2 - (\partial_\rho f)^2 - (\partial_z f)^2] = 0 , \tag{4.1.3}$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Omega) + \partial_z^2\Omega - \frac{2}{f}\left(\partial_\rho f\,\partial_\rho\Omega + \partial_z f\,\partial_z\Omega\right) = 0 , \tag{4.1.4}$$

$$\partial_\rho\gamma = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right] , \tag{4.1.5}$$

$$\partial_z\gamma = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right) . \tag{4.1.6}$$

It is clear that the field equations for γ can be integrated by quadratures, once f and Ω are known. For this reason, the equations (4.1.3) and (4.1.4) for f and Ω are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only. It is interesting to mention that this set of equations can be geometrically interpreted in the context of nonlinear sigma models [7].

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation $\varphi \to -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (6.2.1) with $\omega = 0$, and the field equations can be written as

$$\partial_\rho^2 \psi + \frac{1}{\rho}\partial_\rho \psi + \partial_z^2 \psi = 0 \,, \quad f = \exp(2\psi) \,, \tag{4.1.7}$$

$$\partial_\rho \gamma = \rho \left[(\partial_\rho \psi)^2 - (\partial_z \psi)^2 \right] \,, \quad \partial_z \gamma = 2\rho \partial_\rho \psi \, \partial_z \psi \,. \tag{4.1.8}$$

We see that the main field equation (4.1.7) corresponds to the linear Laplace equation for the metric function ψ.

4.2 Static solution

The general solution of Laplace's equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as [4, 1]

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2 + z^2)^{\frac{n+1}{2}}} P_n(\cos\theta) \,, \qquad \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}} \,, \tag{4.2.1}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The expression for the metric function γ can be calculated by quadratures by using the set of first order differential equations (4.1.8). Then

$$\gamma = - \sum_{n,m=0}^{\infty} \frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2 + z^2)^{\frac{n+m+2}{2}}} \left(P_n P_m - P_{n+1} P_{m+1} \right) \,. \tag{4.2.2}$$

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one of the most interesting special solutions which is Schwarzschild's spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants a_n in such a way that the infinite sum (4.2.1) converges to the Schwarzschild solution in cylindric coordinates. But, or course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzchild's metric.

In fact, it turns out that to investigate the properties of solutions with multipole moments it is more convenient to use prolate spheroidal coordinates (t, x, y, φ) in which the line element can be written as

$$ds^2 = f dt^2 - \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right]$$

where

$$x = \frac{r_+ + r_-}{2\sigma}\,, \quad (x^2 \geq 1), \quad y = \frac{r_+ - r_-}{2\sigma}\,, \quad (y^2 \leq 1) \tag{4.2.3}$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2\,, \quad \sigma = const\,, \tag{4.2.4}$$

and the metric functions are f, ω, and γ depend on x and y, only. In this coordinate system, the general static solution which is also asymptotically flat can be expressed as

$$f = \exp(2\psi)\,, \quad \psi = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n(y) Q_n(x)\,, \quad q_n = const$$

where $P_n(y)$ are the Legendre polynomials, and $Q_n(x)$ are the Legendre functions of second kind. In particular,

$$P_0 = 1, \quad P_1 = y, \quad P_2 = \frac{1}{2}(3y^2 - 1)\,, ...$$

$$Q_0 = \frac{1}{2}\ln\frac{x+1}{x-1}\,, \quad Q_1 = \frac{1}{2}x\ln\frac{x+1}{x-1} - 1\,,$$

$$Q_2 = \frac{1}{2}(3x^2-1)\ln\frac{x+1}{x-1} - \frac{3}{2}x\,, ...$$

The corresponding function γ can be calculated by quadratures and its general expression has been explicitly derived in [8]. The most important special cases contained in this general solution are the Schwarzschild metric

$$\psi = -q_0 P_0(y) Q_0(x) \, , \quad \gamma = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2} \, ,$$

and the Erez-Rosen metric [9]

$$\psi = -q_0 P_0(y) Q_0(x) - q_2 P_2(y) Q_2(x) \, , \quad \gamma = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2} + \ldots .$$

In the last case, the constant parameter q_2 turns out to determine the quadrupole moment. In general, the constants q_n represent an infinite set of parameters that determines an infinite set of mass multipole moments.

5 Stationary generalization

The solution generating techniques [12] can be applied, in particular, to any static seed solution in order to obtain the corresponding stationary generalization. One of the most powerful techniques is the inverse method (ISM) developed by Belinski and Zakharov [13]. We used a particular case of the ISM, which is known as the Hoenselaers–Kinnersley-Xanthopoulos (HKX) transformation to derive the stationary generalization of the general static solution in prolate spheroidal coordinates.

5.1 Ernst representation

In the general stationary case $(\omega \neq 0)$ with line element

$$\begin{aligned} ds^2 &= f(dt - \omega d\varphi)^2 \\ &- \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right] \end{aligned}$$

it is useful to introduce the the Ernst potentials

$$E = f + i\Omega\,, \quad \xi = \frac{1-E}{1+E}\,,$$

where the function Ω is now determined by the equations

$$\sigma(x^2-1)\Omega_x = f^2\omega_y\,, \quad \sigma(1-y^2)\Omega_y = -f^2\omega_x\,.$$

Then, the main field equations can be represented in a compact and symmetric form:

$$(\xi\xi^* - 1)\left\{[(x^2-1)\xi_x]_x + [(1-y^2)\xi_y]_y\right\} = 2\xi^*[(x^2-1)\xi_x^2 + (1-y^2)\xi_y^2]\,.$$

This equation is invariant with respect to the transformation $x \leftrightarrow y$. Then, since the particular solution

$$\xi = \frac{1}{x} \to \Omega = 0 \to \omega = 0 \to \gamma = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2}$$

represents the Schwarzschild spacetime, the choice $\xi^{-1} = y$ is also an exact solution. Furthermore, if we take the linear combination $\xi^{-1} = c_1 x + c_2 y$ and introduce it into the field equation, we obtain the new solution

$$\xi^{-1} = \frac{\sigma}{M} x + i \frac{a}{M} y \, , \; \sigma = \sqrt{M^2 - a^2} \, ,$$

which corresponds to the Kerr metric in prolate spheroidal coordinates.

In the case of the Einstein-Maxwell theory, the main field equations can be expressed as

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\xi \, ,$$

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\mathcal{F} = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\mathcal{F}$$

where ∇ represents the gradient operator in prolate spheroidal coordinates. Moreover, the gravitational potential ξ and the electromagnetic $\mathcal{F}$ Ernst potential are defined as

$$\xi = \frac{1 - f - i\Omega}{1 + f + i\Omega} \, , \quad \mathcal{F} = 2\frac{\Phi}{1 + f + i\Omega} \, .$$

The potential Φ can be shown to be determined uniquely by the electromagnetic potentials A_t and A_φ One can show that if ξ_0 is a vacuum solution, then the new potential

$$\xi = \xi_0\sqrt{1 - e^2}$$

represents a solution of the Einstein-Maxwell equations with effective electric charge e. This transformation is known in the literature as the Harrison transformation [10]. Accordingly, the Kerr–Newman solution in this representation acquires the simple form

$$\xi = \frac{\sqrt{1 - e^2}}{\frac{\sigma}{M} x + i \frac{a}{M} y} \, , \qquad e = \frac{Q}{M} \, , \qquad \sigma = \sqrt{M^2 - a^2 - Q^2} \, .$$

In this way, it is very easy to generalize any vacuum solution to include the case of electric charge. More general transformations of this type can be used in order to generate solutions with any desired set of gravitational and electromagnetic multipole moments [11].

5.2 Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds (M, γ) and (N, G) of dimension m and n, respectively. Let M be coordinatized by x^a, and N by X^μ, so that the metrics on M and N can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $G = G(X)$. A harmonic map is a smooth map $X : M \to N$, or in coordinates $X : x \longmapsto X$ so that X becomes a function of x, and the X's satisfy the motion equations following from the action [14]

$$S = \int d^m x \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \tag{5.2.1}$$

which sometimes is called the "energy" of the harmonic map X. The straightforward variation of S with respect to X^μ leads to the motion equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\, \gamma^{ab}\, \partial_a X^\nu\, \partial_b X^\lambda = 0\,, \tag{5.2.2}$$

where $\Gamma^\mu_{\ \nu\lambda}$ are the Christoffel symbols associated to the metric $G_{\mu\nu}$ of the target space N. If $G_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $G_{\mu\nu} = \eta_{\mu\nu} = diag(\pm 1, ..., \pm 1)$, the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (5.2.1) is usually referred to as the Polyakov action [16].

Consider now the case in which the base space M is a stationary axisymmetric spacetime. Then, γ^{ab}, $a, b = 0, ..., 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (6.2.1), i.e.

$$\gamma_{ab} = \begin{pmatrix} f & 0 & 0 & -f\omega \\ 0 & -f^{-1}e^{2k} & 0 & 0 \\ 0 & 0 & -f^{-1}e^{2k} & 0 \\ -f\omega & 0 & 0 & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}. \tag{5.2.3}$$

Let the target space N be 2-dimensional with metric $G_{\mu\nu} = (1/2)f^{-2}\delta_{\mu\nu}$, $\mu, \nu = 1, 2$, and let the coordinates on N be $X^\mu = (f, \Omega)$. Then, it is straightforward to show that the action (5.2.1) becomes

$$S = \int \mathcal{L}\, dtd\varphi d\rho dz \,, \qquad \mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right] \,, \tag{5.2.4}$$

and the corresponding motion equations (5.2.2) are identical to the main field equations (4.1.3) and (4.1.4).

Notice that the field equations can also be obtained from (5.2.4) by a direct variation with respect to f and Ω. This interesting result was obtained originally by Ernst [2], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \to 2)-$nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2,R)/SO(2)$ [15], and the Lagrangian (5.2.4) can be written explicitly [17] in terms of the generators of the Lie group $SL(2,R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [14].

The form of the Lagrangian (5.2.4) with two gravitational field variables, f and Ω, depending on two coordinates, ρ and z, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (5.2.4). Indeed, if we consider γ^{ab} as a 2-dimensional metric that depends on the parameters ρ and z, the diagonal form of the Lagrangian (5.2.4) implies that $\sqrt{|\gamma|}\gamma^{ab} = \delta^{ab}$. Clearly, this choice is not compatible with the factor ρ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian (5.2.4) cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to "absorb" the unpleasant factor ρ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2,R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields [1]. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

5.3 Representation as a generalized harmonic map

Consider two (pseudo-)Riemannian manifolds (M,γ) and (N,G) of dimension m and n, respectively. Let x^a and X^μ be coordinates on M and N, respectively. This coordinatization implies that in general the metrics γ and G become functions of the corresponding coordinates. Let us assume that not only γ but also G can explicitly depend on the coordinates x^a, i.e. let $\gamma=\gamma(x)$ and $G=G(X,x)$. This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map $X: M\to N$ will be called an $(m\to n)-$generalized harmonic map if it satisfies the Euler-Lagrange equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right)+\Gamma^{\mu}_{\ \nu\lambda}\,\gamma^{ab}\,\partial_a X^\nu\partial_b X^\lambda+G^{\mu\lambda}\gamma^{ab}\,\partial_a X^\nu\,\partial_b G_{\lambda\nu}=0\,, \tag{5.3.1}$$

which follow from the variation of the generalized action

$$S=\int d^m x\sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a\,X^\mu\partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.2}$$

with respect to the fields X^μ. Here the Christoffel symbols, determined by the metric $G_{\mu\nu}$, are calculated in the standard manner, without considering the explicit dependence on x. Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term $G_{\mu\nu}(X,x)$ in the Lagrangian density implies that we are taking into account the "interaction" between the base space M and the target space N. This interaction leads to an extra term in the motion equations, as can be seen in (5.3.1). It turns out that this interaction is the result of the effective presence of the gravitational field.

Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (5.3.1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^{\mu}_{\ \nu\lambda}\, \partial_b X^{\lambda} + G^{\mu\lambda}\, \partial_b G_{\lambda\nu})\partial_a X^{\nu} = 0\,, \tag{5.3.3}$$

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma^{\mu}_{\ \nu\lambda} = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\lambda}\partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma^{\mu}_{\ \nu\lambda} \neq 0$ and (5.3.3) cannot be satisfied. In the general case of a curved target metric, conditions (5.3.3) represent a system of m first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

As we mentioned before, the generalized action (5.3.2) includes an interaction between the base space N and the target space M, reflected on the fact that $G_{\mu\nu}$ depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To see this explicitly we calculate the covariant derivative of the generalized Lagrangian density

$$\mathcal{L} = \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a X^{\mu}\partial_b X^{\nu} G_{\mu\nu}(X,x)\,, \tag{5.3.4}$$

and replace in the result the corresponding motion equations (5.3.1). Then, the final result can be written as

$$\nabla_b \widetilde{T}_a^{\ b} = -\frac{\partial \mathcal{L}}{\partial x^a} \tag{5.3.5}$$

where $\widetilde{T}_a{}^b$ represents the canonical energy-momentum tensor

$$\widetilde{T}_a{}^b = \frac{\partial \mathcal{L}}{\partial(\partial_b X^\mu)}(\partial_a X^\mu) - \delta_a^b \mathcal{L} = 2\sqrt{\gamma} G_{\mu\nu} \left(\gamma^{bc} \partial_a X^\mu \, \partial_c X^\nu - \frac{1}{2} \delta_a^b \gamma^{cd} \partial_c X^\mu \, \partial_d X^\nu \right) . \tag{5.3.6}$$

The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space $\gamma_{ab} = \eta_{ab}$, the explicit dependence of the metric of the target space $G_{\mu\nu}(X, x)$ on x generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.

$$T_{ab} = \frac{\delta \mathcal{L}}{\delta \gamma^{ab}} \, . \tag{5.3.7}$$

A straightforward computation shows that for the action under consideration here we have that $\widetilde{T}_{ab} = 2T_{ab}$ so that the generalized conservation law (5.3.5) can be written as

$$\nabla_b T_a{}^b + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial x^a} = 0 \, . \tag{5.3.8}$$

For a given metric on the base space, this represents in general a system of m differential equations for the "fields" X^μ which must be satisfied "on-shell".

If the base space is 2-dimensional, we can use a reparametrization of x to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, $T_a{}^a = 0$.

In Section 5.1 we described stationary, axially symmetric, gravitational fields as a $(4 \to 2)-$nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \to 2)-$generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space M, and $X^\mu = (f, \Omega)$ the coordinates on the target space N. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

$$\gamma_{ab} = \delta_{ab} \qquad \text{and} \qquad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \qquad (a,b = 1,2;\ \mu,\nu = 1,2). \tag{5.3.9}$$

A straightforward computation shows that the generalized Lagrangian (5.3.4) coincides with the Lagrangian (5.2.4) for stationary axisymetric fields, and that the equations of motion (5.3.1) generate the main field equations (4.1.3) and (4.1.4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$. Then

$$T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right], \tag{5.3.10}$$

$$T_{\rho z} = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right). \tag{5.3.11}$$

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (5.3.8) on-shell:

$$\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\rho} = 0\,, \tag{5.3.12}$$

$$\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0\,. \tag{5.3.13}$$

Incidentally, the last equation coincides with the integrability condition for the metric function k, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs.(4.1.5,4.1.6) and (5.3.10,5.3.11), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (5.3.13) becomes an identity. Although we have eliminated from the starting Lagrangian (5.2.4) the variable k by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [17] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about k at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)-$generalized harmonic map with metrics given as in (5.3.9). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space M is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on M. Then, M represents the world-sheet of a bosonic string in which τ is measures the time and z is the parameter along the string. The string is "embedded" in the target space N whose metric is conformally flat and explicitly depends on the time parameter τ. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates ρ and z are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a\to\infty} f = 1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \omega = c_1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \Omega = O\left(\frac{1}{x^a}\right) \tag{5.3.14}$$

where c_1 is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate φ. If we choose the domain of the spatial coordinates as $\rho \in [0, \infty)$ and $z \in (-\infty, +\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space N satisfy the boundary conditions

$$\dot{X}^\mu(\rho, -\infty) = 0 = \dot{X}^\mu(\rho, \infty) , \quad X'^\mu(\rho, -\infty) = 0 = X'^\mu(\rho, \infty) \tag{5.3.15}$$

where the dot stands for a derivative with respect to ρ and the prime represents derivation with respect to z. These relationships are known in string theory [16] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume ρ as a "time" parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D-$branes situated at plus and minus infinity in the $z-$direction.

5.4 Dimensional extension

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space N, and study the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an $(m \to D)-$generalized harmonic map. As before we denote by $\{x^a\}$ the coordinates on M. Let $\{X^\mu, X^\alpha\}$ with $\mu = 1,2$ and $\alpha = 3,4,...,D$ be the coordinates on N. The metric structure on M is again $\gamma = \gamma(x)$, whereas the metric on N can in general depend on all coordinates of M and N, i.e. $G = G(X^\mu, X^\alpha, x^a)$. The general structure of the corresponding field equations is as given in (5.3.1). They can be divided into one set of equations for X^μ and one set of equations for X^α. According to the results of the last section, the class of gravitational fields under consideration can be represented as a $(2 \to 2)-$generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates X^μ of the target space. Then, the gravitational sector of the target space will be contained in the components $G_{\mu\nu}$ $(\mu, \nu = 1,2)$ of the metric, whereas the components $G_{\alpha\beta}$ $(\alpha, \beta = 3,4,...,D)$ represent the sector of the dimensional extension.

Clearly, the set of differential equations for X^μ also contains the variables X^α and its derivatives $\partial_a X^\alpha$. For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing X^α and its derivatives in the equations for X^μ. It is easy to show that this can be achieved by imposing the conditions

$$G_{\mu\alpha} = 0\,, \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0\,, \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0\,. \tag{5.4.1}$$

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., $G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a)$, $\gamma = 3,4,...,D$. Furthermore, the variables X^α must satisfy the differential equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\alpha\right) + \Gamma^\alpha_{\ \beta\gamma}\,\gamma^{ab}\,\partial_a X^\beta \partial_b X^\gamma + G^{\alpha\beta}\gamma^{ab}\,\partial_a X^\gamma\,\partial_b G_{\beta\gamma} = 0\,. \tag{5.4.2}$$

This shows that any given $(2 \to 2)-$generalized map can be extended, without affecting the field equations, to a $(2 \to D)-$generalized harmonic map.

It is worth mentioning that the fact that the target space N becomes split in two separate parts implies that the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. $T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)$. The generalized conservation law as given in (5.3.8) is satisfied by the sum of both parts.

Consider the example of stationary axisymmetric fields given the metrics (5.3.9). Taking into account the conditions (5.4.1), after a dimensional extension the metric of the target space becomes

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\ 0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\ . & . & \cdots & \cdots & \cdots \\ 0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x) \end{pmatrix} . \tag{5.4.3}$$

Clearly, to avoid that this metric becomes degenerate we must demand that $\det(G_{\alpha\beta}) \neq 0$, a condition that can be satisfied in view of the arbitrariness of the components of the metric. With the extended metric, the Lagrangian density gets an additional term

$$\begin{aligned} \mathcal{L} = \quad & \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right] \\ & + \left(\partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta\right) G_{\alpha\beta} \,, \end{aligned} \tag{5.4.4}$$

which nevertheless does not affect the field equations for the gravitational variables f and Ω. On the other hand, the new fields must be solutions of the extra field equations

$$\left(\partial_\rho^2 + \partial_z^2\right) X^\alpha + \Gamma^\alpha{}_{\beta\gamma}\left(\partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma\right) \tag{5.4.5}$$

$$+ G^{\alpha\gamma}\left(\partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma}\right) = 0 \,. \tag{5.4.6}$$

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice $G_{\alpha\beta} = \eta_{\alpha\beta}$ with additional fields X^α given as arbitrary harmonic functions. This choice opens the possibility of introducing a "time" coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section 4.1, these fields are obtained from the general stationary fields in the limiting case $\Omega = 0$ (or equivalently, $\omega = 0$). If we consider the representation as an $SL(2,R)/SO(2)$ nonlinear sigma model or as a $(2 \to 2)-$generalized harmonic map, we see immediately that the limit $\Omega = 0$ is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case $\Omega = 0$. In the most simple case of an extension with $G_{\alpha\beta} = \delta_{\alpha\beta}$, the resulting $(2 \to 2)-$generalized map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4.7}$$

where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable f. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string "living" in a D-dimensional target space N. The string world-sheet is parametrized by the coordinates ρ and z. The gravitational sector of the target space depends explicitly on the metric functions f and Ω and on the parameter ρ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)-$dimensional Minkowski spacetime with time parameter τ. Then, the string world-sheet is a 2-dimensional flat hypersurface which is "frozen" along the time τ.

5.5 The general solution

If we take as seed metric the general static solution, the application of two HXK transformations generates a stationary solution with an infinite number of gravitoelectric and gravitomagnetic multipole moments. The HKX method is applied at the level of the Ernst potential from which the metric functions can be calculated by using the definition of the Ernst potential E and the field equations for γ. The resulting expressions in the general case are quite cumbersome. We quote here only the special case in which only an arbitrary quadrupole parameter is present. In this case, the result can be written as

$$\begin{aligned} f &= \frac{R}{L}e^{-2qP_2Q_2} , \\ \omega &= -2a - 2\sigma\frac{\mathcal{M}}{R}e^{2qP_2Q_2} , \\ e^{2\gamma} &= \frac{1}{4}\left(1+\frac{M}{\sigma}\right)^2 \frac{R}{x^2-y^2}e^{2\hat{\gamma}} , \end{aligned} \tag{5.5.1}$$

where

$$\begin{aligned} R &= a_+a_- + b_+b_- , \qquad L = a_+^2 + b_+^2 , \\ \mathcal{M} &= \alpha x(1-y^2)(e^{2q\delta_+} + e^{2q\delta_-})a_+ + y(x^2-1)(1-\alpha^2 e^{2q(\delta_+ + \delta_-)})b_+ , \\ \hat{\gamma} &= \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 + q^2(1-P_2)\Big[(1+P_2)(Q_1^2-Q_2^2) \\ &\quad + \frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\Big] . \end{aligned} \tag{5.5.2}$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$\begin{aligned} a_\pm &= x(1-\alpha^2 e^{2q(\delta_+ + \delta_-)}) \pm (1+\alpha^2 e^{2q(\delta_+ + \delta_-)}) , \\ b_\pm &= \alpha y(e^{2q\delta_+} + e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+} - e^{2q\delta_-}) , \\ \delta_\pm &= \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2 \mp xy) + \frac{3}{4}[x(1-y^2) \mp y(x^2-1)]\ln\frac{x-1}{x+1} , \end{aligned}$$

the quantity α being a constant

$$\alpha = \frac{\sigma - M}{a} , \qquad \sigma = \sqrt{M^2 - a^2} . \tag{5.5.3}$$

The physical significance of the parameters entering this metric can be clarified by calculating the Geroch-Hansen [18, 19] multipole moments

$$M_{2k+1} = J_{2k} = 0 , \quad k = 0, 1, 2, ... \tag{5.5.4}$$

$$M_0 = M\,, \quad M_2 = -Ma^2 + \frac{2}{15}qM^3\left(1-\frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.5}$$

$$J_1 = Ma\,, \quad J_3 = -Ma^3 + \frac{4}{15}qM^3a\left(1-\frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.6}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the symmetry with respect to the equatorial plane. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q.

We analyzed the geometric and physical properties of the above solution. The special cases contained in the general solution suggest that it can be used to describe the exterior asymptotically flat gravitational field of rotating body with arbitrary quadrupole moment. This is confirmed by the analysis of the motion of particles on the equatorial plane. The quadrupole moment turns out to drastically change the geometric structure of spacetime as well as the motion of particles, especially near the gravitational source.

We investigated in detail the properties of the Quevedo-Mashhoon (QM) spacetime which is a generalization of Kerr spacetime, including an arbitrary quadrupole. Our results show [20] that a deviation from spherical symmetry, corresponding to a non-zero electric quadrupole, completely changes the structure of spacetime. A similar behavior has been found in the case of the Erez-Rosen spacetime. In fact, a naked singularity appears that affects the ergosphere and introduces regions where closed timelike curves are allowed. Whereas in the Kerr spacetime the ergosphere corresponds to the boundary of a simply-connected region of spacetime, in the present case the ergosphere is distorted by the presence of the quadrupole and can even become transformed into non simply-connected regions. All these changes occur near the naked singularity which is situated at $x = 1$, a value that corresponds to the radial distance $r = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates. In the limiting case $a/M > 1$, the multipole moments and the metric become complex, indicating that the physical description breaks down. Consequently, the extreme Kerr black hole represents the limit of applicability of the QM spacetime.

Since standard astrophysical objects satisfy the condition $a/M < 1$, we can conclude that the QM metric can be used to describe their exterior gravitational field. Two alternative situations are possible. If the characteristic radius of the body is greater than the critical distance $M + \sqrt{M^2 - a^2}$, i.e. $x > 1$, the exterior solution must be matched with an interior solution in order to describe the entire spacetime. If, however, the characteristic radius of the body is smaller than the critical distance $M + \sqrt{M^2 - a^2}$, the QM metric describes the field of a naked singularity.

6 The Erez-Rosen and the Hartle-Thorne solutions in the limiting case $\sim Q$ and $\sim M^2$.

6.1 Introduction

There are a plenty of exact and approximate solutions to the Einstein field equations (EFE) in the literature [1]. Most of the solutions are pure mathematical and only some of them are physical and usually used in a various realistic astrophysical context. We here focus on the exterior exact Erez-Rosen (ER) [9] and approximate Hartle-Thorne (HT) [48, 49] solutions, which are very well established and widely exploited. The Erez-Rosen solution mainly involved in the description of the exterior gravitational field of a deformed astrophysical object. Instead the Hartle-Thorne solution is used to study both interior and exterior fields of slowly rotating and slightly deformed astrophysical objects in the strong field regime. In connection with this, it is interesting to show how to find the relationship between these solutions in the limiting static case with a small deformation.

The metric of a nonrotating mass with a quadrupole moment has been obtained by Erez and Rosen in 1959 [9] with a use of Weyl method [4]. This metric was also analyzed by applying the spheroidal coordinates, which are adapted to characterize the gravitational field of non-spherically symmetric bodies. Later corrected for several numerical coefficients by Doroshkevich (1966) [36], Winicour et al. (1968) [74] and Young and Coulter (1969) [77]. The physical properties of the ER metric were investigated by Zeldovich and Novikov [78] and later by Quevedo and Parkes [61]. More general solutions involving multipole moments were obtained by Quevedo, Quevedo and Mashhoon (QM) [59, 8, 56, 60].

In the article dated back 1967 Hartle put forward his approach for studying physical properties of slowly rotating relativistic stars [48]. Physical quantities that describe the equilibrium configurations of rotating stars such as the change in mass, gravitational potential, eccentricity, binding energy, change in moment of inertia, quadrupole moment, etc. were proportional to the square of the star's angular velocity Ω^2. Hartle and Thorne tested the formalism for different equations of state of relativistic objects [49]. From that moment this solution is widely known as the Hartle-Thorne (HT) solution. Unlike other solutions of the Einstein equations, the Hartle-Thorne solution has an internal counterpart [48, 68], which makes it more practical for investigation the equilibrium structure and physical characteristics of relativistic compact objects such as white dwarfs, neutron stars and hypothetical quark stars [29, 24, 73, 70, 79]. Relatively recently this solution was extended up to Ω^4 approximation [75].

The main objective of this work is to find the relationship between Hartle-Thorne solution and Erez-Rosen solution and show their equivalence in the limiting static case with a small deformation. We adopted the signature of the line elements for this article as (+ - - -) and used geometrical units ($G = c = 1$).

It should be emphasized that the relationship between the ER and HT solutions has been established by Mashhoon and Theiss in 1991 [53], involving the Zipoy-Voorhees transformation in the limiting static case for small deformation. In addition Frutos-Alfaro and Soffel has shown that in the limit of $\sim Q$ and $\sim M^2$ for static case one can find the relationship between the two metrics without involving the Zipoy-Voorhees transformation [41]. In [32] we revisited the derivation by Mashhoon and Theiss providing all technical details in an instructive way. However, in this work we revisit (revise, reproduce) the results of Frutos-Alfaro and Soffel [41], justifying physical significance, providing technical details. The paper pursues pure scientific and academic purposes.

In this work, we review the main properties of the ER solution in section 6.2. The main physical characteristics of the exterior Hartle-Thorne solution are discussed in section 6.3. Section 6.4 is devoted to the computation of the multipole structure of the solutions. The linearized, up to the first order in mass quadrupole moment Q and to the second order in total mass M the Erez-Rosen solution is considered in section 6.5. The Hartle-Thorne solution in the limit of $\sim Q$ and $\sim M^2$ is considered in section 6.6. Using the perturbation method, the coordinate transformations are sought in section 6.7. Finally, we summarize our conclusions and discuss about future prospects.

6.2 The Erez-Rosen metric

The Erez-Rosen metric is an exact exterior solution with mass (m) and quadrupole (q) parameters that describes the gravitational field of static deformed objects in the strong field regime [62]. It belongs to the Weyl class of static axisymmetric vacuum solutions in prolate spheroidal coordinates (t, x, y, φ), with $x \geq 1$ and $-1 \leq y \leq 1$:

$$ds^2 = e^{2\psi} dt^2 - m^2 e^{-2\psi} \left[e^{2\gamma}(x^2 - y^2) \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) d\varphi^2 \right] , \tag{6.2.1}$$

where the metric functions ψ and γ depend on the spatial coordinates x and y, only, and m represents the mass parameter.

The solution found by Erez and Rosen has the following form [58]

$$\psi = \frac{1}{2} \ln \left(\frac{x-1}{x+1} \right) + \frac{1}{2} q (3y^2 - 1) \left[\frac{1}{4}(3x^2 - 1) \ln \left(\frac{x-1}{x+1} \right) + \frac{3}{2} x \right] \tag{6.2.2}$$

and

$$\begin{aligned} \gamma = & \frac{1}{2}(1+q)^2 \ln \left(\frac{x^2-1}{x^2-y^2} \right) - \frac{3}{2} q (1-y^2) \left[x \ln \left(\frac{x-1}{x+1} \right) + 2 \right] \\ + & \frac{9}{16} q^2 (1-y^2) \left[x^2 + 4y^2 - 9x^2y^2 - \frac{4}{3} + x \left(x^2 + 7y^2 - 9x^2y^2 - \frac{5}{3} \right) \ln \left(\frac{x-1}{x+1} \right) \right. \\ + & \left. \frac{1}{4}(x^2-1)(x^2+y^2-9x^2y^2-1) \ln^2 \left(\frac{x-1}{x+1} \right) \right] , \end{aligned} \tag{6.2.3}$$

where q is the quadrupole parameter.

6.3 The exterior Hartle-Thorne solution

The general form of the exterior approximate HT metric [48, 49] in spherical (t, R, Θ, ϕ) coordinates is given by:

$$\begin{aligned} ds^2 = &\left(1-\frac{2M}{R}\right)\left[1+2k_1P_2(\cos\Theta)-2\left(1-\frac{2M}{R}\right)^{-1}\frac{J^2}{R^4}(2\cos^2\Theta-1)\right]dt^2 \\ &-\left(1-\frac{2M}{R}\right)^{-1}\left[1-2\left(k_1-\frac{6J^2}{R^4}\right)P_2(\cos\Theta)-2\left(1-\frac{2M}{R}\right)^{-1}\frac{J^2}{R^4}\right]dR^2 \\ &-R^2[1-2k_2P_2(\cos\Theta)](d\Theta^2+\sin^2\Theta d\phi^2)+\frac{4J}{R}\sin^2\Theta dt d\phi \end{aligned} \tag{6.3.1}$$

where

$$k_1 = \frac{J^2}{MR^3}\left(1+\frac{M}{R}\right)+\frac{5}{8}\frac{Q-J^2/M}{M^3}Q_2^2(x)\,, \tag{6.3.2}$$

$$k_2 = k_1+\frac{J^2}{R^4}+\frac{5}{4}\frac{Q-J^2/M}{M^2R}\left(1-\frac{2M}{R}\right)^{-1/2}Q_2^1(x)\,, \tag{6.3.3}$$

and

$$Q_2^1(x) = (x^2-1)^{1/2}\left[\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-2}{x^2-1}\right],$$

$$Q_2^2(x) = (x^2-1)\left[\frac{3}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^3-5x}{(x^2-1)^2}\right],$$

are the associated Legendre functions of the second kind, being $P_2(\cos\Theta) = (1/2)(3\cos^2\Theta - 1)$ the Legendre polynomial, and $x = R/M - 1$. The constants M, J and Q are the total mass, angular momentum and quadrupole moment of a rotating object, respectively. Note, that according to Hartle $Q > 0$ for oblate and $Q < 0$ for prolate objects.

In order to obtain the HT solution for static objects, we set $J = 0$ in a general form (6.3.1) and obtain

$$\begin{aligned} ds^2 = &\left(1-\frac{2M}{R}\right)\left[1+2k_1P_2(\cos\Theta)\right]dt^2-\left(1-\frac{2M}{R}\right)^{-1}\left[1-2k_1P_2(\cos\Theta)\right]dR^2 \\ &-R^2[1-2k_2P_2(\cos\Theta)](d\Theta^2+\sin^2\Theta d\phi^2) \end{aligned} \tag{6.3.4}$$

where

$$k_1 = \frac{5}{8}\frac{Q}{M^3}Q_2^2(x) , \tag{6.3.5}$$

$$k_2 = k_1 + \frac{5}{4}\frac{Q}{M^2R}\left(1-\frac{2M}{R}\right)^{-1/2} Q_2^1(x) , \tag{6.3.6}$$

Hence we can find the function $\psi(x,y)$ for the static HT metric from a simple relation

$$g_{tt} = e^{2\psi} \tag{6.3.7}$$

The solution found for ψ has the following form

$$\psi = \frac{1}{2}\ln\left[\left(\frac{x-1}{x+1}\right)\left(1-\frac{5Q\left(3y^2-1\right)\left(10x-6x^3+3\left(x^2-1\right)^2\ln\left[\frac{x+1}{x-1}\right]\right)}{16M^3\left(1-x^2\right)}\right)\right] \tag{6.3.8}$$

where $y = \cos\Theta$. Finally, for the function γ it is not possible to find an explicit expression, unless we consider the approximate version of the ER metric. This will be done below.

6.4 Geroch-Hansen multipole moments

Using the original definition formulated by Geroch [46], the calculation of multipole moments is quite laborious. Fodor et al. [38] found a relation between the Ernst potential [2, 37] and the multipole moments which facilitates the computation. In the case of static axisymmetric space–times, the Ernst potential is defined as

$$\xi(x,y) = \frac{1-e^{2\psi}}{1+e^{2\psi}} , \tag{6.4.1}$$

The idea is that the multipole moments can be obtained explicitly from the values of the Ernst potential on the axis by using the following procedure. On the axis of symmetry $y = 1$, we can introduce the inverse of the Weyl coordinate z as

$$\tilde{z} = \frac{1}{z} = \frac{1}{mx} \tag{6.4.2}$$

If we introduce the inverse potential as

$$\tilde{\xi}(\tilde{z},1) = \frac{1}{z}\xi(\tilde{z},1) \tag{6.4.3}$$

The multipole moments can be calculated as

$$M_n = M_n + d_n, \quad m_n = \frac{1}{n!}\frac{d^n\tilde{\xi}(\tilde{z},1)}{d\tilde{z}^n} \tag{6.4.4}$$

where d_n must be determined from the original Geroch definition (e.g. Refs. [60]). For the Erez–Rosen metric, the Geroch-Hansen multipole moments read

$$M_0 = m\,, \qquad M_2 = \frac{2}{15}qm^3 \tag{6.4.5}$$

where M_0 is the monopole moment and M_2 is the quadrupole moment.

For the Hartle–Thorne metric, we obtain

$$M_0 = M\,, \qquad M_2 = -Q. \tag{6.4.6}$$

As one can see from the Geroch-Hansen definition of multipole moments the quadrupole moment of the HT metric has an opposite sign, which is due to the use of a different convention.

6.5 The approximated Erez-Rosen solution in the limit of $\sim Q$ and $\sim M^2$

In order to obtain the approximated ER metric we express m and q in terms of M and Q by equating relations (6.4.5) and (6.4.6), respectively as follows

$$m = M\,, \qquad q = -\frac{15}{2}\frac{Q}{M^3}\,. \tag{6.5.1}$$

and find its limit in $\sim Q$ and $\sim M^2$ by expanding in Taylor series keeping only Q and M^2 and neglecting QM^2 terms. Taking into account $x = r/m - 1$

and $y = \cos\theta$, the final result is written in spherical-like coordinates (t, r, θ, ϕ)

$$
\begin{aligned}
ds^2 = & \left(1 - \frac{2M}{r} + \frac{2QP_2(\cos\theta)}{r^3} + \frac{2MQP_2(\cos\theta)}{r^4}\right) dt^2 \\
& - \left(1 + \frac{2M}{r} + \frac{4M^2}{r^2} - \frac{2QP_2(\cos\theta)}{r^3} - \frac{2MQ(5P_2^2(\cos\theta) + 11P_2(\cos\theta) - 1)}{3r^4}\right) dr^2 \\
& - r^2 \left(1 - \frac{2QP_2(\cos\theta)}{r^3} - \frac{2MQ(-1 + 5P_2(\cos\theta) + 5P_2^2(\cos\theta))}{3r^4}\right) d\theta^2 \\
& - r^2 \sin^2\theta \left(1 - \frac{2QP_2(\cos\theta)}{r^3} - \frac{6MQP_2(\cos\theta)}{r^4}\right) d\phi^2 \qquad (6.5.2)
\end{aligned}
$$

6.6 The Hartle-Thorne solution in the limit of $\sim Q$ and $\sim M^2$

In order to obtain the HT metric for static objects we set $J = 0$ and find its limit in $\sim Q$ and $\sim M^2$, taking into account $x = R/M - 1$ and $y = \cos\Theta$. So the HT metric in standard spherical coordinates (t, R, Θ, ϕ) reads

$$
\begin{aligned}
ds^2 = & \left(1 - \frac{2M}{R} + \frac{2QP_2(\cos\Theta)}{R^3} + \frac{2MQP_2(\cos\Theta)}{R^4}\right) dt^2 \\
& - \left(1 + \frac{2M}{R} + \frac{4M^2}{R^2} - \frac{2QP_2(\cos\Theta)}{R^3} - \frac{10MQP_2(\cos\Theta)}{R^4}\right) dR^2 \\
& - R^2 \left(1 - \frac{2QP_2(\cos\Theta)}{R^3} - \frac{5MQP_2(\cos\Theta)}{R^4}\right) (d\Theta^2 + \sin^2\Theta d\phi^2) \qquad (6.6.1)
\end{aligned}
$$

6.7 Transformation from the Erez-Rosen metric to the Hartle-Thorne metric

To obtain the correspondence between the ER solution, with coordinates (t, r, θ, ϕ), and the HT solution, with coordinates (t, R, Θ, ϕ), both solutions must be written in the same coordinates. Therefore, we search for a coordinate transformation of the following form:

$$r \to R + \frac{MQ}{R^3} f_1(\Theta) \qquad \theta \to \Theta + \frac{MQ}{R^4} f_2(\Theta) \tag{6.7.1}$$

where $f_1(\Theta)$ and $f_2(\Theta)$ are the sought unknown functions. In view of $\sim Q$ and $\sim M^2$ approximation the functions f_1 and f_2 depend only on Θ. The total differentials of the coordinates are given by:

$$dr = \frac{\partial r}{\partial R} dR + \frac{\partial r}{\partial \Theta} d\Theta = \left(1 - \frac{3MQ}{R^4} f_1(\Theta)\right) dR + \frac{MQ}{R^3}\left(\frac{\partial f_1(\Theta)}{\partial \Theta}\right) d\Theta, \tag{6.7.2}$$

$$d\theta = \frac{\partial \theta}{\partial R} dR + \frac{\partial \theta}{\partial \Theta} d\Theta = \left(-\frac{4MQ}{R^5} f_2(\Theta)\right) dR + \left(1 + \frac{MQ}{R^4}\frac{\partial f_2(\Theta)}{\partial \Theta}\right) d\Theta. \tag{6.7.3}$$

These expressions should be plugged in the approximated ER solution (6.5.2). Then, only terms $\sim Q$ and $\sim M^2$ must be retained. We thus obtain the approximated ER metric in the same coordinates as the HT solution with the same parameters

$$\begin{aligned}
ds^2 = &\left(1 - \frac{2M}{R} + \frac{2QP_2(\cos\Theta)}{R^3} + \frac{2MQP_2(\cos\Theta)}{R^4}\right) dt^2 \\
&- \left(1 + \frac{2M}{R} + \frac{4M^2}{R^2} - \frac{2QP_2(\cos\Theta)}{R^3} - \frac{2MQ(5P_2^2(\cos\Theta) + 11P_2(\cos\Theta) - 9f_1(\Theta) - 1)}{3R^4}\right) dR^2 \\
&- R^2\left(1 - \frac{2QP_2(\cos\Theta)}{R^3} - \frac{2MQ(5P_2(\cos\Theta)(1 + P_2(\cos\Theta)) + 3(f_1(\Theta) - f_2'(\Theta)) - 1)}{3R^4}\right) d\Theta^2 \\
&+ \left(\frac{2MQ(f_1'(\Theta) - 4f_2(\Theta))}{R^3}\right) dR d\Theta \\
&- R^2\sin^2\Theta\left(1 - \frac{2QP_2(\cos\Theta)}{R^3} - \frac{2MQ(3P_2(\cos\Theta) + f_1(\Theta) + f_2(\Theta)\cot\Theta)}{R^4}\right) d\phi^2
\end{aligned} \tag{6.7.4}$$

Furthermore, by equating the corresponding g_{RR} components of the metric tensor of both approximated ER (6.7.4) and HT (6.6.1) solutions, written in

(t, R, Θ, ϕ) coordinates, we find the expression for function $f_1(\Theta)$ as follows

$$f_1(\Theta) = \frac{5P_2^2(\cos\Theta) - 4P_2(\cos\Theta) - 1}{9} \quad (6.7.5)$$

Analogously, by comparing only the azimuthal components of the metric tensor $g_{\phi\phi}$ of both solutions, we find the function $f_2(\Theta)$ as

$$f_2(\Theta) = \frac{1}{6}\left(2 - 5P_2(\cos\Theta)\right)\cos\Theta\sin\Theta \quad (6.7.6)$$

To this end, if we plug these functions into the mixed component of the metric tensor $g_{R\Theta}$ of the approximated ER solution (6.7.4), written in (R, Θ) coordinates, $g_{R\Theta}$ vanishes, as we expected.

6.8 Conclusion

We explored the Erez-Rosen and Hartle–Thorne metrics (in the absence of rotation) in the limit $\sim Q$ and $\sim M^2$ by using the perturbation method. The approximation that we used throughout the paper is physical and convenient to solve most problems of celestial mechanics in the post-Newtonian Physics. We showed that the approximate Erez–Rosen line element coincides with the Hartle–Thorne solution in the considered limit.

The use of Geroch and Hansen invariant definition of the multipole moments helped us to calculate the corresponding mass monopole and quadrupole moments and establish the interconnection among the parameters of both solutions .

In addition, we have showed that the explicit form of the coordinate transformations in the limit $\sim Q$ and $\sim M^2$ do not require the use of the Zipoy-Voorhees transformation in view of Ref. [40] (see also Ref. [32] for details).

Due to the recent results [30, 27, 21, 22], it will be interesting to find the connection between the Erez–Rosen and Zipoy-Voorhees (q-metric) solutions. This will be the issue of future studies.

7 Equations of state for neutron stars

7.1 Introduction

First theoretical calculations on the properties of neutron stars were carried out at the beginning of the 20th century by Tolman and Oppenheimer. It was shown that with increasing mass of a star, the electron pressure is no longer able to oppose the gravitational compression. Whereas the electrostatic corrections do not make a large contribution to the pressure of the star in the descriptions of white dwarf stars and were often neglected, the interactions between nucleons become crucial as the matter density becomes close to the nuclear density. The attraction between the nucleons reduces the pressure, but the repulsion caused by vector carriers increases the pressure, thereby preventing the collapse of the star [66]. If the Fermi energy is large, then in addition to neutrons other particles are formed in the neutron star. The Baym equation of state, which includes neutrons, protons and electrons, describes well neutron star matter at densities below the formation of neutron drops $\rho_{drip} = 4.3 \times 10^{11}$ g/cm^3 then it goes quite smoothly into the Baym-Bem-Petrik equation of state [52], which in turn describes the state of matter within $\rho_{drip} < \rho < \rho_{nuc}$, where $\rho_{nuc} = 2.8 \times 10^{14}$ g/cm^3 is the nuclear density. Further, the results vary widely within the densities above the density of nuclear matter. This is the range we will investigate in this work.

The structure of this work is organized as follows: In sections 7.2 and 7.3, neutron stars are considered at zero temperatures for the equation of state of a degenerate neutron gas. In sections 7.4, 7.5 and 7.6 different equations of state are compared, taking into account the interactions between particles. In section 7.7, the mass-radius relations are considered for various equations of state and compared with current observational data. In section 7.8, the main results are discussed and the corresponding conclusions are summarized.

7.2 Pure neutron gas

The pressure of a degenerate neutron gas is calculated in the so-called phase space. With an increase in the density of a neutron star, the uncertainty principle greatly increases the momentum phase space and the radius of the star decreases. Further, due to the gravitational instability, it will decrease to the gravitational radius $r_g = 2GM/c^2$, where the inner structure of the star is destroyed and a black hole is formed. Here G is the gravitational constant, c is the speed of light in vacuum and M is the total mass of the star.

We write the relativistic hydrostatic equilibrium equation, Tolman-Oppenheimer-Volkoff (TOV) equation, for a perfect fluid as:

$$\begin{cases} \frac{dm(r)}{dr} = \frac{4\pi r^2}{c^2}\varepsilon(r) \\ \frac{dp(r)}{dr} = -\varepsilon(r)\frac{Gm(r)}{c^2r^2}\left[1+\frac{p(r)}{\varepsilon(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{c^2 m(r)}\right]\left[1-\frac{2Gm(r)}{c^2 r}\right]^{-1} \end{cases} \tag{7.2.1}$$

where $\varepsilon(r) = c^2\rho(r)$ is the energy density of hadron matter, $p(r)$ is the pressure, $\rho(r)$ is the density, and $m(r)$ is the mass of matter inside a sphere enclosed within radius r. This is a system of first-order ordinary differential equations, the solutions of which should represent equilibrium configurations of a perfect fluid with density $\varepsilon(r)$ and pressure $p(r)$. The TOV equation cannot be solved in its present form because it is an open system of differential equations. To close it, we must add an equation, which is usually given in the form of an equation of state (EoS). Clearly, not every EoS can be used to generate physically reasonable solutions of the TOV equation. In fact, not every EoS can lead to an equilibrium configuration. In the case of a neutron star an appropriate EoS can be written in a parametric form as:

$$\begin{cases} \varepsilon(r) = \frac{\varepsilon_0}{8}\left[(2y(r)^3+y(r))\sqrt{1+y(r)^2} - \ln\left(y(r)+\sqrt{1+y(r)^2}\right)\right] \\ p(r) = \frac{\varepsilon_0}{24}\left[(2y(r)^3-3y(r))\sqrt{1+y(r)^2} + 3\ln\left(y(r)+\sqrt{1+y(r)^2}\right)\right] \end{cases} \tag{7.2.2}$$

where, $y(r) = k_n(r)/(m_n c)$, $k_n(r)$ is the dimensionless Fermi momentum of a neutron, $\varepsilon_0 = m_n^4 c^5/(\pi^2\hbar^3)$ is a constant having the dimension of energy density [65]. We introduce dimensionless quantities for the energy density and pressure in the following form $\varepsilon = \bar{\varepsilon}c^4/(Gb^2)$, $p = \bar{p}c^4/(Gb^2)$, $\rho = \bar{\rho}c^2/(Gb^2)$, where $\bar{\rho}$ is the dimensionless density, $\bar{p}$ is the dimensionless pressure, and $\bar{\varepsilon}$ is the dimensionless energy density. Furthermore, to make the systems of equations (7.2.1) and (7.2.2) dimensionless, we introduce the quantities $r = bx$, and $m = \overline{m}c^2 b/G$, where $b = \pi\sqrt{\hbar^3/(Gcm_n^4)}$ is a parameter with dimension of length, satisfying the equality $\varepsilon_0 = c^4/(Gb^2)$ [65], x is the dimensionless radial coordinate, and $\overline{m}$ is the dimensionless mass. Then, the EoS (7.2.2) reduces to

$$\begin{cases} \overline{\varepsilon}(x) = \frac{1}{8}\left[(2y(x)^3 + y(x))\sqrt{1+y(x)^2} - \ln\left(y(x) + \sqrt{1+y(x)^2}\right)\right] \\ \overline{p}(x) = \frac{1}{24}\left[(2y(x)^3 - 3y(x))\sqrt{1+y(x)^2} + 3\ln\left(y(x) + \sqrt{1+y(x)^2}\right)\right] \end{cases} \tag{7.2.3}$$

and the structure equations (7.2.1) become

$$\begin{cases} \frac{d\overline{m}(x)}{dx} = 4\pi x^2 \overline{\varepsilon}(x) \\ \frac{d\overline{p}(x)}{dx} = -\overline{\varepsilon}(x)\frac{\overline{m}(x)}{x^2}\left[1 + \frac{\overline{p}(x)}{\overline{\varepsilon}(x)}\right]\left[1 + \frac{4\pi x^3 \overline{p}(x)}{\overline{m}(x)}\right]\left[1 - \frac{2\overline{m}(x)}{x}\right]^{-1} \end{cases} \tag{7.2.4}$$

Equations (7.2.3)-(7.2.4) describe the behavior of matter at densities below the formation of neutron droplets and above nuclear matter.

7.3 Neutron stars with protons and electrons

The pressure caused by protons and electrons in a neutron star is small, but it is still present and softens the equation of state by slightly reducing the maximum mass. In order to achieve equilibrium, the electroneutrality condition $n_e = n_p$, where n is the particle number density [63], along with a balance between reactions $n \to p + e + \overline{\nu}_e$ and $p + e \to n + \nu_e$ must be fulfilled within the star. Hence, the equation of state is constructed in the same way as for degenerate noninteracting neutrons and has the following form [67]:

$$\begin{cases} \varepsilon = \pi \sum_{i=n,p,e} \frac{m_i c^2}{\Lambda_i^3}\left[(2y_i(x)^3 + y_i(x))\sqrt{1+y_i(x)^2} - \ln\left(y_i(x) + \sqrt{1+y_i(x)^2}\right)\right] \\ p = \frac{\pi}{3} \sum_{i=n,p,e} \frac{m_i c^2}{\Lambda_i^3}\left[(2y_i(x)^3 - 3y_i(x))\sqrt{1+y_i(x)^2} + 3\ln\left(y_i(x) + \sqrt{1+y_i(x)^2}\right)\right] \end{cases} \tag{7.3.1}$$

where the Compton wavelengths of the particles are $\Lambda_n = 1.319$ fm, $\Lambda_p = 1.321$ fm and $\Lambda_e = 2.42 \times 10^3$ fm, the rest masses are $m_e c^2 = 0.511$ MeV, $m_p c^2 = 938.272$ MeV, and the dimensionless Fermi momenta for protons and electrons are given by

$$
\begin{aligned}
y_p(x) &= \frac{\sqrt{m_e^4 + \left(m_p^2 - m_n^2(1+y_n(x))\right)^2 - 2m_e^2\left(m_p^2 + m_n^2(1+y_n(x)^2)\right)}}{2m_n m_p\sqrt{1+y_n(x)^2}} \\
y_e(x) &= y_p(x)
\end{aligned}
\tag{7.3.2}
$$

where for convenience we have denoted $m_i c^2 = m_i$. The Fermi momentum can be calculated from the condition that the lower threshold for the neutron formation at $y_n(x) = 0$ is equal to $y_p(x) = 0.001265$, $y_e(x) = y_p(x)$, and the central density is $\rho = 1.186 \times 10^7$ g/cm^3.

By solving the TOV equations using (7.3.1) we get the mass-radius relation for a neutron star depicted in Fig. 7.2. Note that in the $M - R$ diagram (right panel) the configurations to the left of M_{max} are unstable and collapse into a black hole. A similar situation also occurs in the $M - \rho$ diagram (left panel), only here unstable configurations are to the right of M_{max}. It can be seen that the contribution of protons and electrons to the pressure or density and correspondingly to the mass is not significant.

Another feature of the equation of state with non-interacting protons, neutrons and electrons is that with increasing Fermi energies, electrons can decay into muons [69] and muons, in turn, decay into electrons with the emission of neutrinos or anti-neutrinos. This is one of the few ways for cooling of a neutron star at high temperatures [66]. The minimum density at which muons are formed is $\rho = 8.21 \times 10^{14}$ g/cm^3. At densities $\rho = 1.36 \times 10^{15}$ g/cm^3, $T < T_c = 2.2 \times 10^{11}$ K pion condensates form [55]. If we take into account that during the formation of a neutron star, the temperature still reaches a value $T > T_c$, but due to the neutrino cooling the temperature decreases from 10^{11} until 10^9 K in a month, then the formation of pion condensation in the core of stars is inevitable. It means that, in realistic models and in all equations of state, the curves should strongly move down after $\rho \approx 10^{15}$ g/cm^3, since pion condensations soften the equation of state by decreasing the maximum mass and are highly dependent on the model, though they have no contribution to the pressure. As for other pions, for example, neutral pions π^0 in 98% cases decay into two gamma quanta $\pi^0 \to 2\gamma$, that is, there is an equilibrium state between $\pi_0 \leftrightarrows 2\gamma$, but the equation for the chemical potential gives $\mu_{\pi^0} = 0$, and for $\mu_{\pi^+} = -\mu_e < 0$; therefore, in both cases, the distribution function for

$T = 0\,K$ tends to $f = \frac{1}{e^{(E-\mu)/kT}-1} \rightarrow 0.$ 5 The formation of such particles is not expected in superdense substances, at least not at low temperatures; for the same reason positrons, anti-baryons and other mesons must be absent in an ideal gas [63].

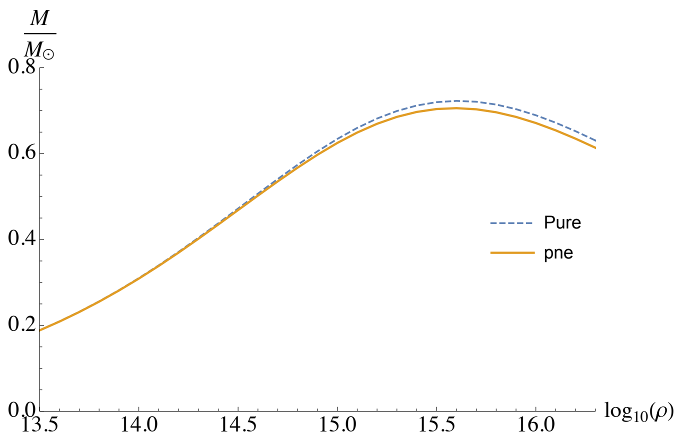

Figure 7.1: Mass-central density ($M - \rho$) relation for the equation of state of pure neutron matter, as well as for the degenerate matter with protons, neutrons and electrons.

7.4 Bethe-Johnson nuclear interaction

Almost all equations of state of neutron stars are based on finding the energies of nucleon-nucleon interactions [23]. Nuclear scientists and theoreticians have spent plenty of time and effort studying nucleon-nucleon interactions. From interaction theories it is known that the attraction between nucleons is due to the exchange of scalar fields of pions, and repulsion by vector particles, in particular ρ, ω, φ (where ρ, ω, φ are the particle carriers of nuclear matter). Bethe and Johnson only considered repulsion from ω (783 MeV), since it is the particle with nucleons that possesses the strongest coupling constant and is approximately estimated in $g_\omega^2/\hbar c = 12.8680$ [64] and, most importantly, it describes well the experimental data of the binding energy of nucleons and the quadrupole moment of the deuteron. So, using the Bethe and Jones potential, taking into account the variational method and considering the agreement with experiments, we obtain an EoS that relates density

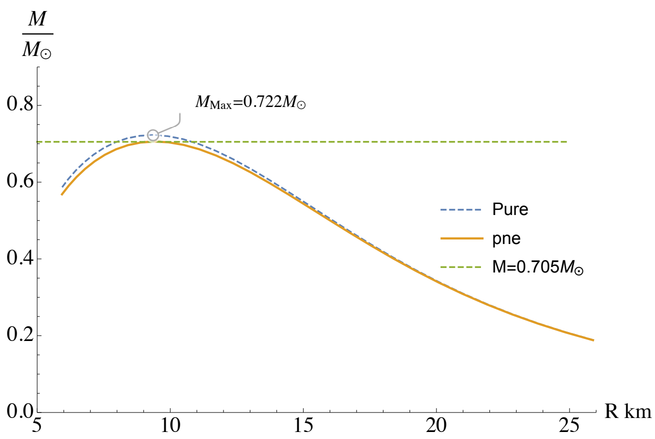

Figure 7.2: Mass-radius ($M - R$) relation. The dotted curve corresponds to the degenerate neutron gas and the solid curve to the equation of state which includes protons and electrons.

and pressure through the parametric equations [66]:

$$\begin{cases} \varepsilon = m_n n(x) + \frac{3(3\pi^2)^{2/3}(\hbar c)^2}{10 m_n} n(x)^{5/3} + 236 n(x)^{a+1} \\ p = n(x)^2 \frac{d(\varepsilon/n)}{dn} = \frac{(3\pi^2)^{2/3}(\hbar c)^2}{5 m_n} n(x)^{5/3} + 363 n(x)^{a+1} \end{cases} \tag{7.4.1}$$

where $a = 1.54$, $m_n = m_n c^2 = 939$ Mev and $\hbar c = 197.327$ MeV $\times$ fm. To make equation (7.4.1) dimensionless, we multiplied by $Gb^2/c^4 = 1.007 \times 10^{-4}$ fm^3/MeV. The speed of sound on the surface of a star is equal to

$$\left(\frac{c_s}{c}\right)^2 = \frac{dp}{d\varepsilon} = \frac{0.143 n^{2/3} + 0.649 n^a}{0.214 n^{2/3} + 1.01 + n^a} \tag{7.4.2}$$

We see that the speed of sound is always $c_s < c$ for any densities in the Bethe-Jones equation of state, as well as in the case of a degenerate neutron gas or mixed with protons and electrons. Numerical calculations for the total mass as a function of the total radius of a neutron star are given in Fig. 7.4.

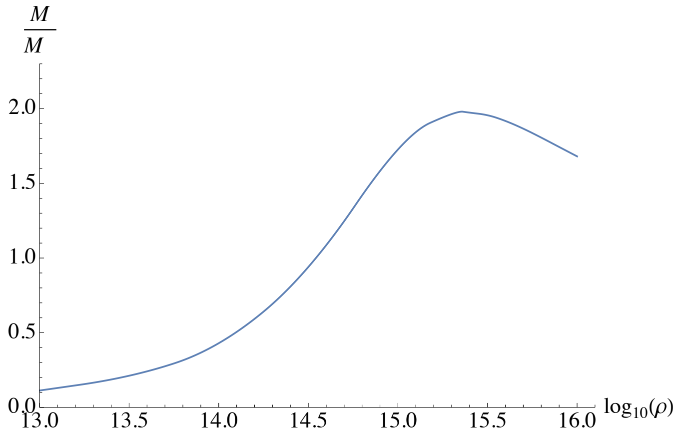

Figure 7.3: The mass-central density relation

7.5 Empirical nuclear-nuclear interaction

No matter how we choose the interaction potential between nucleons, it must be in agreement with the experimental data, that is, it must lead to the saturation of nuclear matter having satisfied 3 conditions: 1) the number density at which the saturation is achieved $n_0 = 0.16\ \mathrm{fm}^{-3}$, 2) the binding energy at saturation $BE = \frac{E}{A}|_{n=n_0} = -16$ MeV, where BE is the binding energy, 3) the compressibility of the nuclear matter $K(n_0) = 9\frac{dp}{dn} = 400$ MeV. It is known from the experiment that the compressibility value changes depending on the potentials. In our case, a test function for the equation of state of symmetric matter $n_p = n_n$ has the following form:

$$\begin{cases} \varepsilon/n = m_n + < E_0 > u^{2/3} + \frac{A}{2}u + \frac{B}{\sigma+1}u^{\sigma} \\ p/n_0 = \frac{2}{3} < E_0 > u^{5/3} + \frac{A}{2}u^2 + \frac{B\sigma}{\sigma+1}u^{\sigma+1} \end{cases} \tag{7.5.1}$$

After substituting in the conditions for saturation of the nuclear matter, the constants are fixed as $A = -118.2$ MeV, $B = 65.39$ MeV, $\sigma = 2.112$, $< E_0 >= 22.1$ MeV. A model describing only neutron matter $n = n_n$ is given by

$$\begin{cases} \varepsilon/n = m_n + 2^{2/3} < E_0 > u^{2/3} + \left(\frac{A}{2} - (2^{2/3}-1) < E_0 > +S_0\right)u + \frac{B}{\sigma+1}u^{\sigma}, \\ p/n_0 = \frac{2^{5/3}<E_0>}{3}u^{5/3} + \left(\frac{A}{2} - (2^{2/3}-1) < E_0 > +S_0\right)u^2 + \frac{B\sigma}{\sigma+1}u^{\sigma+1} \end{cases} \tag{7.5.2}$$

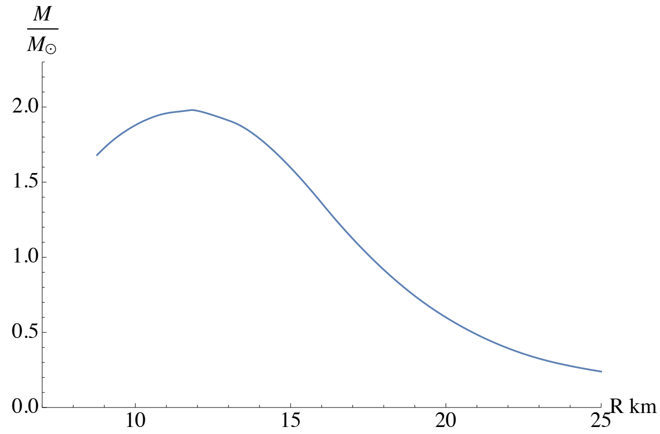

Figure 7.4: The mass-radius ratio for the Bethe and Jones equation of state. Maximum mass $M = 1.973 M_\odot$

where S_0 is the volume symmetry coefficient (degree of deviation from symmetry), $u = n/n_0$. The range of applicability of the equation is from the order of the nuclear density to $\rho = 1.135 \times 10^{15}$ g/cm^3 since beyond this value it will violate the principle of causality.

7.6 Skyrme interaction

In this section, we select a potential that gives a repulsive effect at high concentrations, since at high densities, which is equivalent to the reduction of the distance between nucleons, the repulsive force due to the exchange of vector particles dominates. Based on these arguments, we choose the potential in the form of $V(x-y) = \delta(x-y)\left(\frac{1}{6}t_3 n - t_0\right)$, where t_0 is the parameter that characterizes repulsion owing to the exchange of scalar particles between two nucleons, t_3 is the parameter describing the repulsion at high densities. Next, we find the energy with the help of the Hartree-Fock method, if the spin of particles is not considered (the calculation is performed by using the method of Hartree), we find that the energy increases twice. This is due to the fact that taking into account the spin of the particles reduces the total energy practically twice because in the first approximation only those pairs interact which

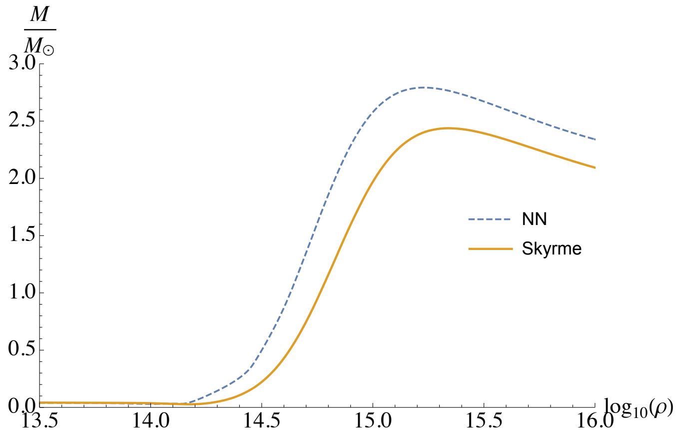

Figure 7.5: The mass-central density relations of Neutron Stars. For comparison, the dotted curve is for the empirical potential and the solid curve is for the skyrme potential

have antiparallel spins. The equation of state with spin reads as follows

$$\begin{cases} \varepsilon = m_n n(x) + \frac{3(3\pi^2)^{2/3}(\hbar c)^2}{10 m_n} n(x)^{5/3} - \frac{t_3}{24} n(x)^3 - \frac{t_0}{4} n(x)^2, \\ p = \frac{(3\pi^2)^{2/3}(\hbar c)^2}{5 m_n} n(x)^{5/3} - \frac{t_3}{12} n(x)^3 - \frac{t_0}{4} n(x)^2, \end{cases} \tag{7.6.1}$$

where the constants t_3, t_0 are found from the saturation condition for nuclear matter as well as for nucleon-nucleon interactions. After simple calculations one finds $t_3 = 14600.8$ MeV, $t_0 = 1024.1$ MeV. The domain of applicability of the equation of state is 2.707×10^{14} g/cm^3 $< \rho < 1.55 \times 10^{15}$ g/cm^3 [65].

7.7 Observational constraints

The physical constraints imposed on the equation of state are known. This is primarily a restriction on the speed of sound, since the speed of sound in the core and on the surface of a star must fulfill the condition $c_s/c < 1$. In our cases, these conditions are not met for the skyrme equation of state when $\rho > 1.55 \times 10^{15}$ g/cm^3 and for the empirical nucleon- nucleon interaction at $\rho > 1.135 \times 10^{15}$ g/cm^3. In Fig. 7.5, it can be seen that the maximum mass for the skyrme potential and for the empirical equation of state must not exist from the physical point of view, since it lies above the plane $c_s/c = 1$. In Fig.

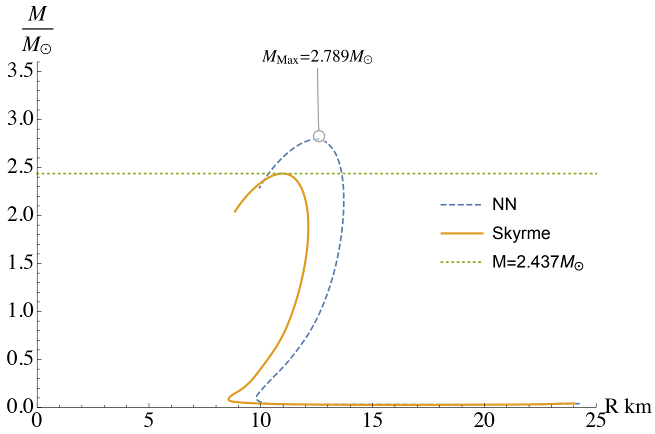

Figure 7.6: Equation of state for the skyrme potential with the limiting mass $M = 2.437M_\odot$ at $R = 10.937$ km. The dashed curve is the empirical equation of state.

7.8, the matter energy density is shown as a function of the pressure. As one can see, the behavior of each equation of state is different.

As for observational constraints, we learned from collecting data on neutron stars from 2019-2020 year [34] that with 95% confidence the mass of the most massive pulsar PSR J0740+6620, located at a distance of 1500 light years from earth, lies in the range $M = 2.14^{+0.20}_{-0.18}M_\odot$ with a rotation period of 2.89 ms. The lowest radius has been observed in the RX J1856-3754 [43]; for an observer at infinity the radius is $R_\infty = R\sqrt{1 - \frac{2GM}{c^2R}}$ so that for $R_{min} = 16.8$ km [72] the conditions read $2GM/c^2R > R - R^3/(R_\infty^{min})^2$ [33]. In Fig. 7.9, this is indicated by a dotted curve and a dotted line. Furthermore, the largest surface gravity of a neutron star is for $M = 1.4M_\odot$. On the other hand, the observed radius with 90% confidence is $15.64\,\text{km} < R_\infty < 18.86\,\text{km}$; in Fig. 7.9, we mark the lower limit of the surface gravity with two dotted curves of the form $2GM/c^2R > R - R^3/(R_\infty)^2$ [33]. The most famous and fastest rotating neutron star PSR J1748-2246 [50] has the highest rotation frequency 716 Hz and the equation for the frequency is determined as $\nu_{max} = 1045(M/M_\odot)^{1/2}(10\,\text{km}/R)^{3/2}$. Therefore, the constraint on the rotation is $M \geq 0.47(R/10\,\text{km})^3M_\odot$, which is shown in the figure with a dashed curve. Finally, the upper limit for the surface gravity of a neutron star is determined by the pulsar XTE J1814-338 and the constraint is $M/M_\odot < 2.4 \times 10^5c^2R/GM_\odot$, shown in the figure with a solid black line [31].

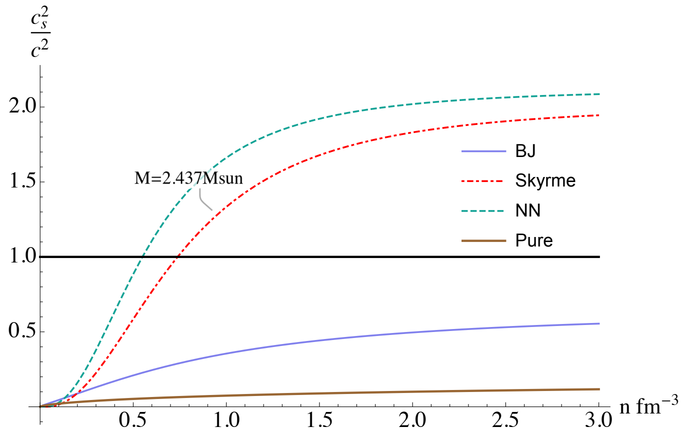

Figure 7.7: Dependence of the speed of sound on the number density nuclear matters n fm^{-3}.

Realistic mass-radius relations must pass through the region surrounded by the observational constraints. Therefore, as one can see, the pure degenerate neutron gas equation of state is not realistic.

7.8 Conclusions

In this paper, we obtained the dependence of mass on the radius and on the central density for different equations of states of Neutron Stars. It can be seen in Fig. 7.2 that if we do not take into account the nucleon-nucleon interaction, the equation of states will be extremely soft to describe the dependence of the mass on the radius, which does not correspond to the observations. In section 7.3, we discussed the appearance of pion condensation, but in realistic models, the formation of condensation must be prevented by many factors such as the pion-nucleon interaction [55]. In Ref. [25], the equation of state with protons, neutrons and electrons was studied, but instead of the local electroneutrality condition $n_p - n_e = 0$, the global electroneutrality condition was used in the form $\int \rho_{ch} d^3 r = \int e[n_p(r) - n_e(r)] = 0$. At the same time, the Lagrangian density $\mathcal{L}$ takes into account the repulsive force between nucleons from vector bosons ρ_μ and ω_μ an also the electromagnetic 4-potential A_μ or any other Lagrangian types [54].

As can be seen from the plots 7.4 and 7.6, the empirical equation of state has the most rigid dependence of mass on radius, but the plots quickly fall

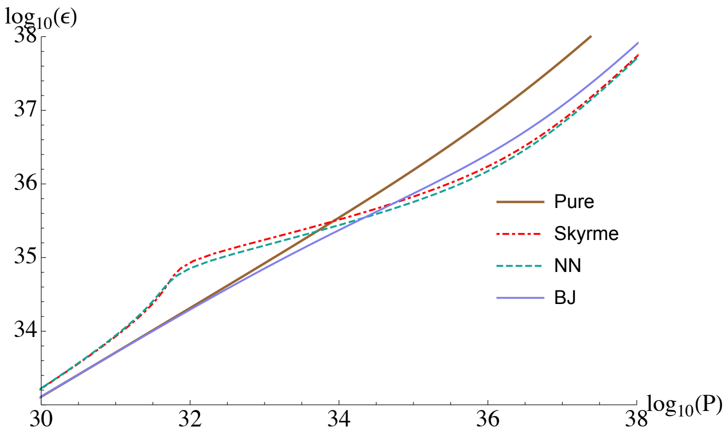

Figure 7.8: Energy density versus pressure

down with the inclusion of protons. The equation of Bethe-Jones is in good agreement with the experimental data, but at high densities it is not enough to describe the repulsive force with a single ω meson. Nevertheless, this equation of state does not contradict the latest observational and theoretical constraints, which is certainly a big advantage. The equation of state for the skyrme potential is just a good description of the matter at high densities up to $\rho < 5.5\rho_{nuc}$, which corresponds to a star with mass $M = 2.37M_{\odot}$.

The final results are shown in Fig. 7.9, as expected, the lower limit of observational constraints lies in the interval $1.3M_{\odot} < M < 1.4M_{\odot}$ that is close to the Chandrasekhar limit, which in turn lies in the range of $1.38 - 1.44$ solar masses. According to some estimates, the maximum observed mass of a neutron star PSR J0740+6620 is $2.17M_{\odot}$, which is slightly more than the Oppenheimer-Volkoff limit, whose estimate according to modern data is 2.01-2.16 solar masses. The existence of such an object is explained by the fact that a rapidly rotating neutron star increases the maximum mass by almost 15%. Apparently, in the past, PSR J0740+6620 absorbed a significant part of the substance of its companion — most likely, when it was still at the stage of the red giant [35].

According to 2015 data, about 2500 neutron stars are known. Out of them, only 10% have companions. Many massive neutron stars also have an inner core. The radius of the inner core can reach up to several kilometers and the density in the center of the nucleus can exceed the density of atomic nuclei by 7-8 times. The composition and equation of state of the substance of the inner

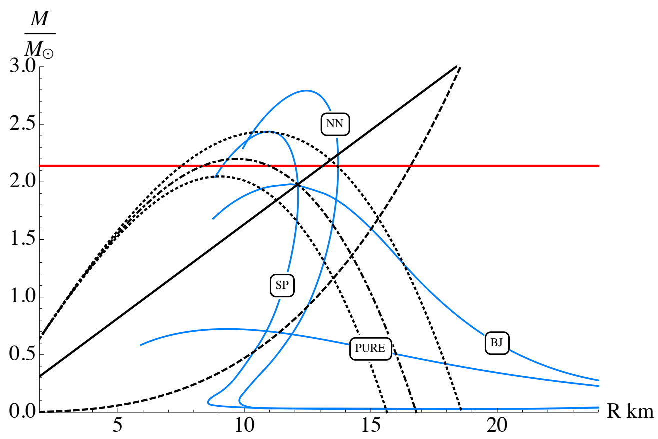

Figure 7.9: The $M - R$ relations for various models compared with observational constraints. NN is the empirical equation of state, SP is for the skyrme potential, Pure is the degenerate neutron gas and BJ is the Bethe-Jones equation of state.

core are not known for certain. At such densities, neutrons can give way to hyperons, three-quark particles that include at least one strange quark, or even consist of free quarks and gluons [76].

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