Generalizations of the Kerr-Newman solution

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2 Brief description

One of the most important metrics in general relativity is the Kerr-Newman solution that describes the gravitational and electromagnetic fields of a rotating charged mass. For astrophysical purposes, however, it is necessary to take into account the effects due to the moment of inertia of the object. To attack this problem, we investigate new exact solutions of Einstein-Maxwell equations which posses an infinite set of gravitational and electromagnetic multipole moments and contain the Kerr-Newman solution as special case.

We investigate the interior Einstein's equations in the case of a static, axially symmetric, perfect fluid source. We present a particular line element that is specially suitable for the investigation of this type of interior gravitational fields. Assuming that the deviation from spherically symmetry is small, we linearize the corresponding line element and field equations and find several classes of vacuum and perfect fluid solutions. We find physically meaningful spacetimes by imposing appropriate matching conditions.

We propose a criterion for finding the minimum distance at which an interior solution of Einstein's equations can be matched with an exterior asymptotically flat solution.The location of the matching hypersurface is thus constrained by a criterion defined in terms of the eigenvalues of the Riemann curvature tensor by using repulsive gravity effects. To determine the location of the matching hypersurface, we use the first derivatives of the curvature eigenvalues, implying C^3 differentiability conditions. The matching itself is performed by demanding continuity of the curvature eigenvalues across the matching surface. We apply the C^3 matching approach to spherically symmetric perfect fluid spacetimes and obtain the physically meaningful condition that density and pressure should vanish on the matching surface. Several perfect fluid solutions in Newton and Einstein gravity are tested.

3 Introduction

It is hard to overemphasize the importance of the Kerr geometry not only for general relativity itself, but also for the very fundamentals of physics. It assumes this position as being the most physically relevant rotating generalization of the static Schwarzschild geometry. Its charged counterpart, the Kerr-Newman solution, representing the exterior gravitational and electromagnetic fields of a charged rotating object, is an exact solution of the Einstein-Maxwell equations.

Its line element in Boyer–Lindquist coordinates can be written as

$$
\begin{aligned}
ds^2 \;=\;& \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 \\
&- \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[(r^2 + a^2)d\varphi - adt]^2 \\
&- \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 \, , \qquad (3.0.1)
\end{aligned}
$$

where M is the total mass of the object, $a = J/M$ is the specific angular momentum, and Q is the electric charge. In this particular coordinate system, the metric functions do not depend on the coordinates t and ϕ, indicating the existence of two Killing vector fields $\xi^I = \partial_t$ and $\xi^{II} = \partial_\varphi$ which represent the properties of stationarity and axial symmetry, respectively.

An important characteristic of this solution is that the source of gravity is surrounded by two horizons situated at a distance

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \qquad (3.0.2)$$

from the origin of coordinates. Inside the interior horizon, r_-, a ring singularity is present which, however, cannot be observed by any observer situated outside the exterior horizon. If the condition $M^2 < a^2 + Q^2$ is satisfied, no horizons are present and the Kerr–Newman spacetime represents the exterior field of a naked singularity.

Despite of its fundamental importance in general relativity, and its theoretical and mathematical interest, this solution has not been especially useful for describing astrophysical phenomena, first of all, because observed astrophysical objects do not possess an appreciable net electric charge. Furthermore, the limiting Kerr metric takes into account the mass and the rotation, but does not consider the moment of inertia of the object. For astrophysical applications it is, therefore, necessary to use more general solutions with higher multipole moments which are due not only to the rotation of the body but also to its shape. This means that even in the limiting case of a static spacetime, a solution is needed that takes into account possible deviations from spherically symmetry.

4 The general static vacuum solution

In general relativity, stationary axisymmetric solutions of Einstein's equations [22] play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of know exact solutions drastically increased after Ernst [2] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies lead finally to the development of solution generating techniques [22] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [4] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

4.1 Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindric coordinates (t, ρ, z, φ). Stationarity implies that t can be chosen as the time coordinate and the metric does not depend on time, i.e. $\partial g_{\mu\nu}/\partial t = 0$. Consequently, the corresponding timelike Killing vector has the components δ^μ_t. A second Killing vector field is associated to the axial symmetry with respect to the axis $\rho = 0$. Then, choosing φ as the azimuthal angle, the metric satisfies the conditions $\partial g_{\mu\nu}/\partial\varphi = 0$, and the components of the corresponding spacelike Killing vector are δ^μ_φ.

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form $g_{\mu\nu} = g_{\mu\nu}(\rho, z)$, it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [4, 5, 6]

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] , \tag{4.1.1}$$

where f, ω and γ are functions of ρ and z, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$\rho\partial_\rho\Omega = f^2\partial_z\omega , \qquad \rho\partial_z\Omega = -f^2\partial_\rho\omega , \tag{4.1.2}$$

the vacuum field equations $R_{\mu\nu} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \partial_z^2 f + \frac{1}{f}[(\partial_\rho\Omega)^2 + (\partial_z\Omega)^2 - (\partial_\rho f)^2 - (\partial_z f)^2] = 0 , \tag{4.1.3}$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Omega) + \partial_z^2\Omega - \frac{2}{f}\left(\partial_\rho f\,\partial_\rho\Omega + \partial_z f\,\partial_z\Omega\right) = 0 , \tag{4.1.4}$$

$$\partial_\rho\gamma = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right] , \tag{4.1.5}$$

$$\partial_z\gamma = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right) . \tag{4.1.6}$$

It is clear that the field equations for γ can be integrated by quadratures, once f and Ω are known. For this reason, the equations (4.1.3) and (4.1.4) for f and Ω are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only. It is interesting to mention that this set of equations can be geometrically interpreted in the context of nonlinear sigma models [7].

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation $\varphi \to -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (4.1.1) with $\omega = 0$, and the field equations can be written as

$$\partial_\rho^2 \psi + \frac{1}{\rho}\partial_\rho \psi + \partial_z^2 \psi = 0 \, , \quad f = \exp(2\psi) \, , \tag{4.1.7}$$

$$\partial_\rho \gamma = \rho \left[(\partial_\rho \psi)^2 - (\partial_z \psi)^2 \right] \, , \quad \partial_z \gamma = 2\rho \partial_\rho \psi \, \partial_z \psi \, . \tag{4.1.8}$$

We see that the main field equation (4.1.7) corresponds to the linear Laplace equation for the metric function ψ.

4.2 Static solution

The general solution of Laplace's equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as [4, 22]

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2 + z^2)^{\frac{n+1}{2}}} P_n(\cos\theta) \, , \qquad \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}} \, , \tag{4.2.1}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The expression for the metric function γ can be calculated by quadratures by using the set of first order differential equations (4.1.8). Then

$$\gamma = -\sum_{n,m=0}^{\infty} \frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2 + z^2)^{\frac{n+m+2}{2}}} \left(P_n P_m - P_{n+1} P_{m+1} \right) \, . \tag{4.2.2}$$

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one of the most interesting special solutions which is Schwarzschild's spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants a_n in such a way that the infinite sum (4.2.1) converges to the Schwarzschild solution in cylindric coordinates. But, or course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzchild's metric.

In fact, it turns out that to investigate the properties of solutions with multipole moments it is more convenient to use prolate spheroidal coordinates (t, x, y, φ) in which the line element can be written as

$$ds^2 = f dt^2 - \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right]$$

where

$$x = \frac{r_+ + r_-}{2\sigma}\,, \quad (x^2 \geq 1), \quad y = \frac{r_+ - r_-}{2\sigma}\,, \quad (y^2 \leq 1) \tag{4.2.3}$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2\,, \quad \sigma = const\,, \tag{4.2.4}$$

and the metric functions are f, ω, and γ depend on x and y, only. In this coordinate system, the general static solution which is also asymptotically flat can be expressed as

$$f = \exp(2\psi)\,, \quad \psi = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n(y) Q_n(x)\,, \quad q_n = const$$

where $P_n(y)$ are the Legendre polynomials, and $Q_n(x)$ are the Legendre functions of second kind. In particular,

$$P_0 = 1, \quad P_1 = y, \quad P_2 = \frac{1}{2}(3y^2 - 1)\,, ...$$

$$Q_0 = \frac{1}{2}\ln\frac{x+1}{x-1}\,, \quad Q_1 = \frac{1}{2}x\ln\frac{x+1}{x-1} - 1\,,$$

$$Q_2 = \frac{1}{2}(3x^2-1)\ln\frac{x+1}{x-1} - \frac{3}{2}x\,, ...$$

The corresponding function γ can be calculated by quadratures and its general expression has been explicitly derived in [8]. The most important special cases contained in this general solution are the Schwarzschild metric

$$\psi = -q_0 P_0(y) Q_0(x) \,, \quad \gamma = \frac{1}{2} \ln \frac{x^2-1}{x^2-y^2} \,,$$

and the Erez-Rosen metric [9]

$$\psi = -q_0 P_0(y) Q_0(x) - q_2 P_2(y) Q_2(x) \,, \quad \gamma = \frac{1}{2} \ln \frac{x^2-1}{x^2-y^2} + ... $$

In the last case, the constant parameter q_2 turns out to determine the quadrupole moment. In general, the constants q_n represent an infinite set of parameters that determines an infinite set of mass multipole moments.

5 Stationary generalization

The solution generating techniques [12] can be applied, in particular, to any static seed solution in order to obtain the corresponding stationary generalization. One of the most powerful techniques is the inverse method (ISM) developed by Belinski and Zakharov [13]. We used a particular case of the ISM, which is known as the Hoenselaers–Kinnersley-Xanthopoulos (HKX) transformation to derive the stationary generalization of the general static solution in prolate spheroidal coordinates.

5.1 Ernst representation

In the general stationary case $(\omega \neq 0)$ with line element

$$\begin{aligned} & ds^2 = f(dt - \omega d\varphi)^2 \\ - & \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right] \end{aligned}$$

it is useful to introduce the the Ernst potentials

$$E = f + i\Omega\,, \quad \xi = \frac{1-E}{1+E}\,,$$

where the function Ω is now determined by the equations

$$\sigma(x^2-1)\Omega_x = f^2\omega_y\,, \quad \sigma(1-y^2)\Omega_y = -f^2\omega_x\,.$$

Then, the main field equations can be represented in a compact and symmetric form:

$$(\xi\xi^* - 1)\left\{[(x^2-1)\xi_x]_x + [(1-y^2)\xi_y]_y\right\} = 2\xi^*[(x^2-1)\xi_x^2 + (1-y^2)\xi_y^2]\,.$$

This equation is invariant with respect to the transformation $x \leftrightarrow y$. Then, since the particular solution

$$\xi = \frac{1}{x} \to \Omega = 0 \to \omega = 0 \to \gamma = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2}$$

represents the Schwarzschild spacetime, the choice $\xi^{-1} = y$ is also an exact solution. Furthermore, if we take the linear combination $\xi^{-1} = c_1 x + c_2 y$ and introduce it into the field equation, we obtain the new solution

$$\xi^{-1} = \frac{\sigma}{M} x + i \frac{a}{M} y \, , \; \sigma = \sqrt{M^2 - a^2} \, ,$$

which corresponds to the Kerr metric in prolate spheroidal coordinates.

In the case of the Einstein-Maxwell theory, the main field equations can be expressed as

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2 \xi = 2(\xi^* \nabla \xi - \mathcal{F}^* \nabla \mathcal{F}) \nabla \xi \, ,$$

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2 \mathcal{F} = 2(\xi^* \nabla \xi - \mathcal{F}^* \nabla \mathcal{F}) \nabla \mathcal{F}$$

where ∇ represents the gradient operator in prolate spheroidal coordinates. Moreover, the gravitational potential ξ and the electromagnetic $\mathcal{F}$ Ernst potential are defined as

$$\xi = \frac{1 - f - i\Omega}{1 + f + i\Omega} \, , \quad \mathcal{F} = 2 \frac{\Phi}{1 + f + i\Omega} \, .$$

The potential Φ can be shown to be determined uniquely by the electromagnetic potentials A_t and A_φ One can show that if ξ_0 is a vacuum solution, then the new potential

$$\xi = \xi_0 \sqrt{1 - e^2}$$

represents a solution of the Einstein-Maxwell equations with effective electric charge e. This transformation is known in the literature as the Harrison transformation [10]. Accordingly, the Kerr–Newman solution in this representation acquires the simple form

$$\xi = \frac{\sqrt{1 - e^2}}{\frac{\sigma}{M} x + i \frac{a}{M} y} \, , \qquad e = \frac{Q}{M} \, , \qquad \sigma = \sqrt{M^2 - a^2 - Q^2} \, .$$

In this way, it is very easy to generalize any vacuum solution to include the case of electric charge. More general transformations of this type can be used in order to generate solutions with any desired set of gravitational and electromagnetic multipole moments [11].

5.2 Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds (M, γ) and (N, G) of dimension m and n, respectively. Let M be coordinatized by x^a, and N by X^μ, so that the metrics on M and N can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $G = G(X)$. A harmonic map is a smooth map $X : M \to N$, or in coordinates $X : x \longmapsto X$ so that X becomes a function of x, and the X's satisfy the motion equations following from the action [14]

$$S = \int d^m x \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \tag{5.2.1}$$

which sometimes is called the "energy" of the harmonic map X. The straightforward variation of S with respect to X^μ leads to the motion equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^{\mu}_{\ \nu\lambda}\, \gamma^{ab}\, \partial_a X^\nu\, \partial_b X^\lambda = 0\,, \tag{5.2.2}$$

where $\Gamma^{\mu}_{\ \nu\lambda}$ are the Christoffel symbols associated to the metric $G_{\mu\nu}$ of the target space N. If $G_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $G_{\mu\nu} = \eta_{\mu\nu} = diag(\pm 1, ..., \pm 1)$, the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (5.2.1) is usually referred to as the Polyakov action [16].

Consider now the case in which the base space M is a stationary axisymmetric spacetime. Then, γ^{ab}, $a, b = 0, ..., 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (4.1.1), i.e.

$$\gamma_{ab} = \begin{pmatrix} f & 0 & 0 & -f\omega \\ 0 & -f^{-1}e^{2k} & 0 & 0 \\ 0 & 0 & -f^{-1}e^{2k} & 0 \\ -f\omega & 0 & 0 & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}\,. \tag{5.2.3}$$

Let the target space N be 2-dimensional with metric $G_{\mu\nu} = (1/2)f^{-2}\delta_{\mu\nu}$, $\mu, \nu = 1, 2$, and let the coordinates on N be $X^\mu = (f, \Omega)$. Then, it is straightforward to show that the action (5.2.1) becomes

$$S = \int \mathcal{L}\, dt d\varphi d\rho dz\ , \qquad \mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right]\ , \tag{5.2.4}$$

and the corresponding motion equations (5.2.2) are identical to the main field equations (4.1.3) and (4.1.4).

Notice that the field equations can also be obtained from (5.2.4) by a direct variation with respect to f and Ω. This interesting result was obtained originally by Ernst [2], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \to 2)-$nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2, R)/SO(2)$ [15], and the Lagrangian (5.2.4) can be written explicitly [17] in terms of the generators of the Lie group $SL(2, R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [14].

The form of the Lagrangian (5.2.4) with two gravitational field variables, f and Ω, depending on two coordinates, ρ and z, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (5.2.4). Indeed, if we consider γ^{ab} as a 2-dimensional metric that depends on the parameters ρ and z, the diagonal form of the Lagrangian (5.2.4) implies that $\sqrt{|\gamma|}\gamma^{ab} = \delta^{ab}$. Clearly, this choice is not compatible with the factor ρ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian (5.2.4) cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to "absorb" the unpleasant factor ρ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2,R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields [22]. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

5.3 Representation as a generalized harmonic map

Consider two (pseudo-)Riemannian manifolds (M,γ) and (N,G) of dimension m and n, respectively. Let x^a and X^μ be coordinates on M and N, respectively. This coordinatization implies that in general the metrics γ and G become functions of the corresponding coordinates. Let us assume that not only γ but also G can explicitly depend on the coordinates x^a, i.e. let $\gamma = \gamma(x)$ and $G = G(X,x)$. This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map $X : M \to N$ will be called an $(m \to n)-$generalized harmonic map if it satisfies the Euler-Lagrange equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\,\gamma^{ab}\,\partial_a X^\nu \partial_b X^\lambda + G^{\mu\lambda}\gamma^{ab}\,\partial_a X^\nu\,\partial_b G_{\lambda\nu} = 0\,, \tag{5.3.1}$$

which follow from the variation of the generalized action

$$S = \int d^m x \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a\, X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.2}$$

with respect to the fields X^μ. Here the Christoffel symbols, determined by the metric $G_{\mu\nu}$, are calculated in the standard manner, without considering the explicit dependence on x. Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term $G_{\mu\nu}(X,x)$ in the Lagrangian density implies that we are taking into account the "interaction" between the base space M and the target space N. This interaction leads to an extra term in the motion equations, as can be seen in (5.3.1). It turns out that this interaction is the result of the effective presence of the gravitational field.

Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (5.3.1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^{\mu}_{\ \nu\lambda}\,\partial_b X^\lambda + G^{\mu\lambda}\,\partial_b G_{\lambda\nu})\partial_a X^\nu = 0\,, \tag{5.3.3}$$

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma^{\mu}_{\ \nu\lambda} = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\lambda}\partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma^{\mu}_{\ \nu\lambda} \neq 0$ and (5.3.3) cannot be satisfied. In the general case of a curved target metric, conditions (5.3.3) represent a system of m first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

As we mentioned before, the generalized action (5.3.2) includes an interaction between the base space N and the target space M, reflected on the fact that $G_{\mu\nu}$ depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To see this explicitly we calculate the covariant derivative of the generalized Lagrangian density

$$\mathcal{L} = \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.4}$$

and replace in the result the corresponding motion equations (5.3.1). Then, the final result can be written as

$$\nabla_b \widetilde{T}_a^{\ b} = -\frac{\partial \mathcal{L}}{\partial x^a} \tag{5.3.5}$$

where $\widetilde{T}_a{}^b$ represents the canonical energy-momentum tensor

$$\widetilde{T}_a{}^b = \frac{\partial \mathcal{L}}{\partial(\partial_b X^\mu)}(\partial_a X^\mu) - \delta_a^b \mathcal{L} = 2\sqrt{\gamma} G_{\mu\nu} \left(\gamma^{bc} \partial_a X^\mu \, \partial_c X^\nu - \frac{1}{2} \delta_a^b \gamma^{cd} \partial_c X^\mu \, \partial_d X^\nu \right) . \tag{5.3.6}$$

The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space $\gamma_{ab} = \eta_{ab}$, the explicit dependence of the metric of the target space $G_{\mu\nu}(X, x)$ on x generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.

$$T_{ab} = \frac{\delta \mathcal{L}}{\delta \gamma^{ab}} . \tag{5.3.7}$$

A straightforward computation shows that for the action under consideration here we have that $\widetilde{T}_{ab} = 2T_{ab}$ so that the generalized conservation law (5.3.5) can be written as

$$\nabla_b T_a{}^b + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial x^a} = 0 . \tag{5.3.8}$$

For a given metric on the base space, this represents in general a system of m differential equations for the "fields" X^μ which must be satisfied "on-shell".

If the base space is 2-dimensional, we can use a reparametrization of x to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, $T_a{}^a = 0$.

In Section 5.1 we described stationary, axially symmetric, gravitational fields as a $(4 \to 2)-$nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \to 2)-$generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space M, and $X^\mu = (f, \Omega)$ the coordinates on the target space N. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

$$\gamma_{ab} = \delta_{ab} \qquad \text{and} \qquad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \qquad (a, b = 1, 2;\ \mu, \nu = 1, 2). \tag{5.3.9}$$

A straightforward computation shows that the generalized Lagrangian (5.3.4) coincides with the Lagrangian (5.2.4) for stationary axisymetric fields, and that the equations of motion (5.3.1) generate the main field equations (4.1.3) and (4.1.4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$. Then

$$T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho \Omega)^2 - (\partial_z f)^2 - (\partial_z \Omega)^2\right], \tag{5.3.10}$$

$$T_{\rho z} = \frac{\rho}{2f^2}\left(\partial_\rho f\, \partial_z f + \partial_\rho \Omega\, \partial_z \Omega\right). \tag{5.3.11}$$

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (5.3.8) on-shell:

$$\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\rho} = 0\,, \tag{5.3.12}$$

$$\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0\,. \tag{5.3.13}$$

Incidentally, the last equation coincides with the integrability condition for the metric function k, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs.(4.1.5,4.1.6) and (5.3.10,5.3.11), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (5.3.13) becomes an identity. Although we have eliminated from the starting Lagrangian (5.2.4) the variable k by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [17] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about k at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)-$generalized harmonic map with metrics given as in (5.3.9). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space M is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on M. Then, M represents the world-sheet of a bosonic string in which τ is measures the time and z is the parameter along the string. The string is "embedded" in the target space N whose metric is conformally flat and explicitly depends on the time parameter τ. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates ρ and z are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a\to\infty} f = 1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \omega = c_1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \Omega = O\left(\frac{1}{x^a}\right) \tag{5.3.14}$$

where c_1 is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate φ. If we choose the domain of the spatial coordinates as $\rho \in [0, \infty)$ and $z \in (-\infty, +\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space N satisfy the boundary conditions

$$\dot{X}^\mu(\rho, -\infty) = 0 = \dot{X}^\mu(\rho, \infty) , \quad X'^\mu(\rho, -\infty) = 0 = X'^\mu(\rho, \infty) \tag{5.3.15}$$

where the dot stands for a derivative with respect to ρ and the prime represents derivation with respect to z. These relationships are known in string theory [16] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume ρ as a "time" parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D-$branes situated at plus and minus infinity in the $z-$direction.

5.4 Dimensional extension

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space N, and study the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an $(m \to D)-$generalized harmonic map. As before we denote by $\{x^a\}$ the coordinates on M. Let $\{X^\mu, X^\alpha\}$ with $\mu = 1,2$ and $\alpha = 3,4,...,D$ be the coordinates on N. The metric structure on M is again $\gamma = \gamma(x)$, whereas the metric on N can in general depend on all coordinates of M and N, i.e. $G = G(X^\mu, X^\alpha, x^a)$. The general structure of the corresponding field equations is as given in (5.3.1). They can be divided into one set of equations for X^μ and one set of equations for X^α. According to the results of the last section, the class of gravitational fields under consideration can be represented as a $(2 \to 2)-$generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates X^μ of the target space. Then, the gravitational sector of the target space will be contained in the components $G_{\mu\nu}$ $(\mu, \nu = 1,2)$ of the metric, whereas the components $G_{\alpha\beta}$ $(\alpha, \beta = 3,4,...,D)$ represent the sector of the dimensional extension.

Clearly, the set of differential equations for X^μ also contains the variables X^α and its derivatives $\partial_a X^\alpha$. For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing X^α and its derivatives in the equations for X^μ. It is easy to show that this can be achieved by imposing the conditions

$$G_{\mu\alpha} = 0\,, \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0\,, \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0\,. \tag{5.4.1}$$

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., $G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a)$, $\gamma = 3,4,...,D$. Furthermore, the variables X^α must satisfy the differential equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\alpha\right) + \Gamma^\alpha_{\ \beta\gamma}\,\gamma^{ab}\,\partial_a X^\beta \partial_b X^\gamma + G^{\alpha\beta}\gamma^{ab}\,\partial_a X^\gamma\,\partial_b G_{\beta\gamma} = 0\,. \tag{5.4.2}$$

This shows that any given $(2 \to 2)-$generalized map can be extended, without affecting the field equations, to a $(2 \to D)-$generalized harmonic map.

It is worth mentioning that the fact that the target space N becomes split in two separate parts implies that the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. $T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)$. The generalized conservation law as given in (5.3.8) is satisfied by the sum of both parts.

Consider the example of stationary axisymmetric fields given the metrics (5.3.9). Taking into account the conditions (5.4.1), after a dimensional extension the metric of the target space becomes

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\ 0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\ . & . & \cdots & \cdots & \cdots \\ 0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x) \end{pmatrix} . \tag{5.4.3}$$

Clearly, to avoid that this metric becomes degenerate we must demand that $\det(G_{\alpha\beta}) \neq 0$, a condition that can be satisfied in view of the arbitrariness of the components of the metric. With the extended metric, the Lagrangian density gets an additional term

$$\begin{aligned} \mathcal{L} = \quad & \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho\Omega)^2 + (\partial_z\Omega)^2\right] \\ & + \left(\partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta\right) G_{\alpha\beta} \, , \end{aligned} \tag{5.4.4}$$

which nevertheless does not affect the field equations for the gravitational variables f and Ω. On the other hand, the new fields must be solutions of the extra field equations

$$\left(\partial_\rho^2 + \partial_z^2\right) X^\alpha + \Gamma^\alpha{}_{\beta\gamma}\left(\partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma\right) \tag{5.4.5}$$

$$+ G^{\alpha\gamma}\left(\partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma}\right) = 0 \, . \tag{5.4.6}$$

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice $G_{\alpha\beta} = \eta_{\alpha\beta}$ with additional fields X^α given as arbitrary harmonic functions. This choice opens the possibility of introducing a "time" coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section 4.1, these fields are obtained from the general stationary fields in the limiting case $\Omega = 0$ (or equivalently, $\omega = 0$). If we consider the representation as an $SL(2, R)/SO(2)$ nonlinear sigma model or as a $(2 \to 2)-$generalized harmonic map, we see immediately that the limit $\Omega = 0$ is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case $\Omega = 0$. In the most simple case of an extension with $G_{\alpha\beta} = \delta_{\alpha\beta}$, the resulting $(2 \to 2)-$generalized map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4.7}$$

where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable f. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string "living" in a D-dimensional target space N. The string world-sheet is parametrized by the coordinates ρ and z. The gravitational sector of the target space depends explicitly on the metric functions f and Ω and on the parameter ρ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)-$dimensional Minkowski spacetime with time parameter τ. Then, the string world-sheet is a 2-dimensional flat hypersurface which is "frozen" along the time τ.

5.5 The general solution

If we take as seed metric the general static solution, the application of two HXK transformations generates a stationary solution with an infinite number of gravitoelectric and gravitomagnetic multipole moments. The HKX method is applied at the level of the Ernst potential from which the metric functions can be calculated by using the definition of the Ernst potential E and the field equations for γ. The resulting expressions in the general case are quite cumbersome. We quote here only the special case in which only an arbitrary quadrupole parameter is present. In this case, the result can be written as

$$\begin{aligned} f &= \frac{R}{L} e^{-2qP_2Q_2} , \\ \omega &= -2a - 2\sigma \frac{\mathcal{M}}{R} e^{2qP_2Q_2} , \\ e^{2\gamma} &= \frac{1}{4}\left(1+\frac{M}{\sigma}\right)^2 \frac{R}{x^2-y^2} e^{2\hat{\gamma}} , \end{aligned} \tag{5.5.1}$$

where

$$\begin{aligned} R &= a_+a_- + b_+b_- , \qquad L = a_+^2 + b_+^2 , \\ \mathcal{M} &= \alpha x(1-y^2)(e^{2q\delta_+} + e^{2q\delta_-})a_+ + y(x^2-1)(1-\alpha^2 e^{2q(\delta_+ + \delta_-)})b_+ , \\ \hat{\gamma} &= \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 + q^2(1-P_2)\Big[(1+P_2)(Q_1^2-Q_2^2) \\ &\quad + \frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\Big] . \end{aligned} \tag{5.5.2}$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$\begin{aligned} a_\pm &= x(1-\alpha^2 e^{2q(\delta_+ + \delta_-)}) \pm (1+\alpha^2 e^{2q(\delta_+ + \delta_-)}) , \\ b_\pm &= \alpha y(e^{2q\delta_+} + e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+} - e^{2q\delta_-}) , \\ \delta_\pm &= \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2 \mp xy) + \frac{3}{4}[x(1-y^2) \mp y(x^2-1)] \ln\frac{x-1}{x+1} , \end{aligned}$$

the quantity α being a constant

$$\alpha = \frac{\sigma - M}{a} , \qquad \sigma = \sqrt{M^2 - a^2} . \tag{5.5.3}$$

The physical significance of the parameters entering this metric can be clarified by calculating the Geroch-Hansen [18, 19] multipole moments

$$M_{2k+1} = J_{2k} = 0 , \quad k = 0, 1, 2, ... \tag{5.5.4}$$

$$M_0 = M\,, \quad M_2 = -Ma^2 + \frac{2}{15}qM^3\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,,... \tag{5.5.5}$$

$$J_1 = Ma\,, \quad J_3 = -Ma^3 + \frac{4}{15}qM^3a\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,,.... \tag{5.5.6}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the symmetry with respect to the equatorial plane. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q.

We analyzed the geometric and physical properties of the above solution. The special cases contained in the general solution suggest that it can be used to describe the exterior asymptotically flat gravitational field of rotating body with arbitrary quadrupole moment. This is confirmed by the analysis of the motion of particles on the equatorial plane. The quadrupole moment turns out to drastically change the geometric structure of spacetime as well as the motion of particles, especially near the gravitational source.

We investigated in detail the properties of the Quevedo-Mashhoon (QM) spacetime which is a generalization of Kerr spacetime, including an arbitrary quadrupole. Our results show [20] that a deviation from spherical symmetry, corresponding to a non-zero electric quadrupole, completely changes the structure of spacetime. A similar behavior has been found in the case of the Erez-Rosen spacetime. In fact, a naked singularity appears that affects the ergosphere and introduces regions where closed timelike curves are allowed. Whereas in the Kerr spacetime the ergosphere corresponds to the boundary of a simply-connected region of spacetime, in the present case the ergosphere is distorted by the presence of the quadrupole and can even become transformed into non simply-connected regions. All these changes occur near the naked singularity which is situated at $x = 1$, a value that corresponds to the radial distance $r = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates. In the limiting case $a/M > 1$, the multipole moments and the metric become complex, indicating that the physical description breaks down. Consequently, the extreme Kerr black hole represents the limit of applicability of the QM spacetime.

Since standard astrophysical objects satisfy the condition $a/M < 1$, we can conclude that the QM metric can be used to describe their exterior gravitational field. Two alternative situations are possible. If the characteristic radius of the body is greater than the critical distance $M + \sqrt{M^2 - a^2}$, i.e. $x > 1$, the exterior solution must be matched with an interior solution in order to describe the entire spacetime. If, however, the characteristic radius of the body is smaller than the critical distance $M + \sqrt{M^2 - a^2}$, the QM metric describes the field of a naked singularity.

6 Approximate perfect fluid solutions with quadrupole moment

6.1 Introduction

Based on experimental evidence, general relativity is considered today as one of the best candidates to describe the gravitational field of compact astrophysical objects. As a theory for the gravitational field, it should be able to describe the field of all possible physical configurations, in which gravity is involved. All the information about the gravitational field should be contained in the metric tensor which must be a solution of Einstein's equations. Consider the case of a compact object like a star or a planet. From the point of view of the multipole structure of the source, to describe the field of a compact object, we need an interior and an exterior solution, both containing at least three independent physical parameters, namely, mass, angular momentum and quadrupole moment. Consider first the case of a source with only mass and angular momentum. The corresponding exterior solution is represented by the Kerr spacetime [21] for which no physically reasonable interior solution is known. This is a major problem in classical general relativity [22]. Many methods have been suggested to find a suitable interior Kerr solution, including exotic matter models and specially adapted equations of state, but none of them has lead to a definite answer. In view of this situation, we consider that alternative approaches should be considered. In particular, we believe that additional physical parameters can be taken into account that are relevant for the description of the gravitational field. The simplest of such additional parameters is the quadrupole moment which is responsible for the deformation of any realistic mass distribution. Indeed, if we add a quadrupole moment to a spherically symmetric object, we end up with an axisymmetric mass distribution, which implies new degrees of freedom at the level of the corresponding field equations. This is the main idea of the alternative approach we propose to attack the problem of finding interior solutions to describe the interior gravitational structure of compact objects. As a first step to develop such an approach, we will focus in this work on the case of a source with only mass and quadrupole, neglecting the contribution of the angular momentum.

In a recent work [23], it was proposed to use the Zipoy-Voorhees transformation [24, 25] to generate the quadrupolar metric ($q-$metric), which can be interpreted as the simplest generalization of the Schwarzschild metric, describing the gravitational field of a distribution of mass whose non-spherically symmetric shape is represented by an independent quadrupole parameter. In the literature, this metric is known as the Zipoy-Voorhees metric, delta-metric, gamma-metric and q-metric [26, 27]. Here, we will use the name q-metric to highlight the importance of the quadrupole parameter q. Consequently, this metric can be used to describe the exterior gravitational field of deformed distributions of mass in which the quadrupole moment is the main parameter that describes the deformation.

Circular and radial geodesics of the exterior gamma-metric ($\gamma = 1 + q$) have been compared with the spherically symmetric case to establish the sensitivity of the trajectories to the gamma parameter [28]. Moreover, it was shown that the properties of the accretion disks in the field of the gamma-metric can be drastically different from those of disks around black holes [29, 30, 31].

The question arises whether it is possible to find an interior metric that can be matched to the exterior one in such a way that the entire spacetime is described as a whole. To this end, it is usually assumed that the interior mass distribution can be described by means of a perfect fluid with two physical parameters, namely, energy density and pressure. The energy-momentum tensor of the perfect fluid is then used in the Einstein equations as the source of the gravitational field. It turns out that the system of corresponding differential equations cannot be solved because the number of equations is less than the number of unknown functions. This problem is usually solved by imposing equations of state that relate the pressure and density of the fluid. In this work, however, we will explore a different approach that was first proposed by Synge [32]. To apply this method, one first uses general physical considerations to postulate the form of the interior metric and, then, one evaluates the energy-momentum tensor of the source by using Einstein's equations. In this manner, any interior metric can be considered as an exact solution of the Einstein equations for some energy-momentum tensor. However, the main point of the procedure is to impose physical conditions on the resulting matter source so that it corresponds to a physical reasonable configuration. In general, one can impose energy conditions, matching conditions with the exterior metric, and conditions on the behavior of the metric functions near the center of the source and on the boundary with the exterior field.

Hernandez [33] has shown how to modify *ad hoc* an interior spherically symmetric solution to obtain an approximate interior solution for the corresponding family of exterior Weyl metrics, provided the exterior metric contains the Schwarzschild metric as a particular case. The Hernandez approach has been generalized by Stewart et al. [34] to obtain an exact interior solution to the gamma-metric. They found two different interior solutions which match the exterior gamma-metric. In general, however, this *ad hoc* method does not lead to interior solutions corresponding to simple fluids. The matching between interior and exterior solutions, in general, requires the fulfillment of several mathematical conditions on the matching surface [35, 36].

This work is organized as follows. In Sec. 6.2, we consider the $q-$metric as describing the exterior gravitational field of a deformed source with mass and quadrupole moment. In Sec. 7.1, we present the exact field equations for a perfect fluid source. In Sec. 6.4, we construct the approximate line element with a quadrupole moment and in Sec. 6.5, we present some particular interior solutions. Finally, Sec. 7.7 contains discussions of our results.

6.2 Exterior $q-$metric

Zipoy [24] and Voorhees [25] investigated static, axisymmetric vacuum solutions of Einstein's equations and found a simple transformation, which allows one to generate new solutions from a known solution. If we start from the Schwarzschild solution and apply a Zipoy-Voorhees transformation, the new line element can be written as

$$ds^2 = \left(1-\frac{2m}{r}\right)^{1+q} dt^2$$
$$-\left(1-\frac{2m}{r}\right)^{-q}\left[\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)}\left(\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right)+r^2\sin^2\theta d\varphi^2\right]. \quad (6.2.1)$$

A detailed analysis of this metric shows that m and q are constant parameters that determine the total mass and the quadrupole moment of the gravitational source [23]. A stationary generalization of the $q-$metric, satisfying the main physical conditions of exterior spacetimes, has been obtained in [37]. The metric (6.2.1) has been interpreted as the simplest generalization of the Schwarzschild metric with a quadrupole.

Whereas the seed metric is the spherically symmetric Schwarzschild solution, which describes the gravitational field of a black hole, the generated $q-$metric is axially symmetric, and describes the exterior field of a naked singularity [23]. In fact, this can be shown explicitly by calculating the invariant Geroch multipoles [38, 39]. The lowest mass multipole moments M_n, $n = 0, 1, \ldots$ are given by

$$M_0 = (1+q)m \, , \quad M_2 = -\frac{m^3}{3} q(1+q)(2+q) \, , \tag{6.2.2}$$

whereas higher moments are proportional to mq and can be completely rewritten in terms of M_0 and M_2. Accordingly, the arbitrary parameters m and q determine the mass and quadrupole which are the only independent multipole moments of the solution. In the limiting case $q = 0$, only the monopole $M_0 = m$ survives, as in the Schwarzschild spacetime. In the limit $m = 0$, with $q \neq 0$, all moments vanish identically, implying that no mass distribution is present and the spacetime must be flat. The same is true in the limiting case $q \to -1$ which corresponds to the Minkowski metric. Moreover, notice that all odd multipole moments are zero because the solution possesses an additional reflection symmetry with respect to the equatorial plane $\theta = \pi/2$.

The deformation is described by the quadrupole moment M_2 which is positive for a prolate source and negative for an oblate source. This implies that the parameter q can be either positive or negative. Since the total mass M_0 of the source must be positive, we must assume that $q > -1$ for positive values of m, and $q < -1$ for negative values of m. We conclude that the above metric can be used to describe the exterior gravitational field of a static positive mass M_0 with a positive or negative quadrupole moment M_2.

A study of the curvature of the $q-$metric shows that the outermost singularity is located at $r = 2m$, a hypersurface which in all known compact objects is situated inside the surface of the body. This implies that in order to describe the entire gravitational field, it is necessary to cover this type of singularity with an interior solution.

6.3 Interior metric

As mentioned in the Introduction, in this work, we will concentrate on the case of static perfect fluid spacetimes. There are many forms to write down the corresponding line element and, in principle, all of them must be equivalent [22]. However, certain forms of the line element turn out to be convenient for investigating a particular problem. Our experience with numerical perfect fluid solutions [50] indicates that for the case under consideration the line element

$$ds^2 = f dt^2 - \frac{e^{2\gamma}}{f}\left(\frac{dr^2}{h} + d\theta^2\right) - \frac{\mu^2}{f} d\varphi^2 \,, \tag{6.3.1}$$

is particularly convenient. Here $f = f(r,\theta)$, $\gamma = \gamma(r,\theta)$, $\mu = \mu(r,\theta)$, and $h = h(r)$. A redefinition of the coordinate r leads to an equivalent line element which has been used to investigate anisotropic static fluids [41].

The Einstein equations for a perfect fluid with 4-velocity U_α, density ρ, and pressure p (we use geometric units with $G = c = 1$)

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi \left[(\rho + p) U_\alpha U_\beta - p g_{\alpha\beta}\right] \tag{6.3.2}$$

for the line element (6.3.1) can be represented as two second-order differential equations for μ and f

$$\mu_{,rr} = -\frac{1}{2h}\left(2\mu_{,\theta\theta} + h_{,r}\mu_{,r} - 32\pi p \frac{\mu e^{2\gamma}}{f}\right) \,, \tag{6.3.3}$$

$$f_{,rr} = \frac{f_{,r}^2}{f} - \left(\frac{h_{,r}}{2h} + \frac{\mu_{,r}}{\mu}\right) f_{,r} + \frac{f_{,\theta}^2}{hf} - \frac{\mu_{,\theta} f_{,\theta}}{\mu h} - \frac{f_{,\theta\theta}}{h} + 8\pi \frac{(3p+\rho)e^{2\gamma}}{h} \,, \tag{6.3.4}$$

where a subscript represents partial derivative. Moreover, the function γ is determined by a set of two partial differential equations

$$\gamma_{,r} = \frac{1}{h\mu_{,r}^2 + \mu_{,\theta}^2}\left\{\frac{\mu}{f^2}\left[\frac{\mu_{,r}}{4}\left(hf_{,r}^2 - f_{,\theta}^2\right) + \frac{1}{2}\mu_{,\theta} f_{,\theta} f_{,r} + 8\pi \mu_{,r} p f e^{2\gamma}\right] + \mu_{,\theta}\mu_{,r\theta} - \mu_{,r}\mu_{,\theta\theta}\right\} \,, \tag{6.3.5}$$

$$\gamma_{,\theta} = \frac{1}{h\mu_{,r}^2 + \mu_{,\theta}^2}\left\{\frac{\mu}{f^2}\left[\frac{\mu_{,\theta}}{4}\left(f_{,\theta}^2 - hf_r^2\right) + \frac{1}{2}h\mu_{,r} f_{,\theta} f_{,r} - 8\pi \mu_{,\theta} p f e^{2\gamma}\right] + h\mu_{,r}\mu_{,r\theta} + \mu_{,\theta}\mu_{,\theta\theta}\right\} \,, \tag{6.3.6}$$

which can be integrated by quadratures once f, μ, p, and h are known. The integrability condition of these partial differential equations turns out to be satisfied identically by virtue of the remaining field equations. Notice that there is no equation for the function $h(r)$. This means that it can be absorbed in the definition of the radial coordinate r. Nevertheless, one can also fix it arbitrarily; it turns out that this freedom is helpful when solving the equations and investigating the physical significance of the solutions.

The advantage of using the line element (6.3.1) is that the field equations are split into two sets. The main set consists of the equations (6.3.3) and (6.3.4) for μ and f which must be solved simultaneously. The second set consists of the first-order equations for γ which plays a secondary role in the sense that they can be integrated once the remaining functions are known. Notice also that the pressure p and the density ρ must be given *a priori* in order to solve the main set of differential equations for μ and f. As follows from Eq.(6.3.4), the equation of state $3p + \rho = 0$ reduces the complexity of this equation; nevertheless, this condition leads to negative pressures which, from a physical point of view, are not expected to be present inside astrophysical compact objects.

Finally, we mention that from the conservation law $T^{\alpha\beta}_{;\beta} = 0$, we obtain two first-order differential equations for the pressure

$$p_{,r} = -\frac{1}{2}(p+\rho)\frac{f_{,r}}{f} \,, \quad p_{,\theta} = -\frac{1}{2}(p+\rho)\frac{f_{,\theta}}{f} \,, \tag{6.3.7}$$

which can be integrated for any given functions $f(r,\theta)$ and $\rho(r,\theta)$.

It is very difficult to find physically reasonable solutions for the above field equations, because the underlying differential equations are highly nonlinear with very strong couplings between the metric functions. In [42], we presented a new method for generating perfect fluid solutions of the Einstein equations, starting from a given seed solution. The method is based upon the introduction of a new parameter at the level of the metric functions of the seed solution in such a way that the generated new solution is characterized by physical properties which are different from those of the seed solutions.

In this work, we will analyze approximate solutions which satisfy the conditions for being applicable in the case of astrophysical compact objects. We will see that it is then possible to perform a numerical integration by imposing appropriate initial conditions. In particular, if we demand that the metric functions and the pressure are finite at the axis, it is possible to find a class of numerical solutions which can be matched with the exterior $q-$metric with a pressure that vanishes at the matching surface.

6.4 Linearized quadrupolar metrics

Our general goal is to investigate how the quadrupole moment influences the structure of spacetimes that can be used to describe the gravitational field of compact deformed gravitational sources. In particular, we aim to find perfect fluid solutions that can be matched with the exterior $q-$metric (6.2.1). Our approach consists in postulating the interior line element and evaluating the energy-momentum tensor from the Einstein equations, a method which was first proposed by Synge and has been applied to find several approximate interior solutions [43, 44]. Once the components of the energy-momentum are calculated, a comparison with the perfect fluid tensor allows us to find explicit expressions for the the density and the pressure, on which standard physical conditions are imposed.

To find the corresponding interior line element, we proceed as follows. Consider the case of a slightly deformed mass. This means that the parameter q for the exterior $q-$metric can be considered as small and we can linearize the line element as

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left[1 + q\ln\left(1 - \frac{2m}{r}\right)\right]dt^2 - r^2\left[1 - q\ln\left(1 - \frac{2m}{r}\right)\right]\sin^2\theta d\varphi^2$$
$$- \left[1 + q\ln\left(1 - \frac{2m}{r}\right) - 2q\ln\left(1 - \frac{2m}{r} + \frac{m^2}{r^2}\sin^2\theta\right)\right]\left(\frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2\right) \ . \quad (6.4.1)$$

We will assume that the exterior gravitational field of the compact object is described to the first order in q by the line element (6.4.1), which represents a particular approximate solution to Einstein's equations in vacuum.

To construct the approximate interior line element, we start from the exact line element (6.3.1) and use the approximate solution (6.4.1) as a guide. Following this procedure, an appropriate interior line element can be expressed as

$$ds^2 = e^{2\nu}(1+qa)dt^2 - (1+qc+qb)\frac{dr^2}{1-\frac{2\tilde{m}}{r}}$$
$$-(1+qa+qb)r^2 d\theta^2 - (1-qa)r^2 \sin^2\theta d\varphi^2 \,, \tag{6.4.2}$$

where the functions $\nu = \nu(r)$, $a = a(r)$, $c = c(r)$, $\tilde{m} = \tilde{m}(r)$, and $b = b(r, \theta)$. Notice that we have introduced an additional auxiliary function $c(r)$ which plays a role similar to that of the auxiliary function $h(r)$ of the interior line element (6.3.1). Notice that this approximate line element contains also the approximate exterior $q-$metric (6.4.1) as a particular case. This implies that vacuum fields can also be investigated in this approximate approach.

6.4.1 General vacuum solution

To test the consistency of the linearized approach, we will derive explicitly the approximate vacuum $q-$metric (6.4.1). To this end, we compute the vacuum field equations and obtain

$$\tilde{m}_{,r} = 0 \quad \text{i.e.} \quad \tilde{m} = m = const. \,, \tag{6.4.3}$$

$$\nu_{,r} = \frac{m}{r\,(r-2m)} \,, \tag{6.4.4}$$

$$(r-m)(a_{,r} - c_{,r}) + (a-c) = 0 \,, \tag{6.4.5}$$

$$2r\,(r-2m)\,a_{,rr} + (3r-m)\,a_{,r} + (r-3m)\,c_{,r} - 2\,(a-c) = 0, \tag{6.4.6}$$

$$r\,(r-2m)\,b_{,rr} + b_{,\theta\theta} + (r-m)\,b_{,r} - 2\,(r-2m)\,c_{,r} + 2\,(a-c) = 0, \tag{6.4.7}$$

$$\left(r^2 - 2mr + m^2\sin^2\theta\right) b_{,\theta} + 2r\,(r-2m)\,(ma_{,r} - a + c)\sin\theta\cos\theta = 0, \tag{6.4.8}$$

$$\left(r^2 - 2mr + m^2\sin^2\theta\right) b_{,r} + 2\,(r-2m)\left(r - m\sin^2\theta\right) a_{,r}$$
$$+2\,(r-m)\,(a-c)\sin^2\theta = 0 \quad , \tag{6.4.9}$$

where for simplicity we have replaced the solution of the first equation $\tilde{m} = m = const.$ in the remaining equations.

Then, Eqs.(6.4.4) and (6.4.5) can be integrated and yield

$$\nu = \frac{1}{2}\ln\left(1-\frac{2m}{r}\right) + \alpha_1\,, \quad a - c = \frac{\alpha_2 m^2}{(r-m)^2}\,, \tag{6.4.10}$$

where α_1 and α_2 are dimensionless integration constants. The remaining system of partial differential equations can be integrated in general and yields

$$a = -\frac{\alpha_2 m}{r-m} + \frac{1}{2}(\alpha_3 - \alpha_2)\ln\left(1-\frac{2m}{r}\right) + \alpha_4\,, \tag{6.4.11}$$

$$c = -\frac{\alpha_2 m r}{(r-m)^2} + \frac{1}{2}(\alpha_3 - \alpha_2)\ln\left(1-\frac{2m}{r}\right) + \alpha_4\,, \tag{6.4.12}$$

$$b = \frac{2\alpha_2 m}{r-m} - (\alpha_3 - \alpha_2)\left[\ln 2 + \ln\left(1-\frac{2m}{r}+\frac{m^2\sin^2\theta}{r^2}\right)\right] + \alpha_5\,, \tag{6.4.13}$$

where α_3, α_4 and α_5 are dimensionless integration constants.

Thus, we see that the general approximate exterior solution with quadrupole moment is represented by a 5-parameter family of solutions. The particular case

$$\alpha_1 = 0\,, \quad \alpha_2 = 0\,, \quad \alpha_3 = 2\,, \quad \alpha_4 = 0\,, \quad \alpha_5 = 2\ln 2\,, \tag{6.4.14}$$

corresponds to the linearized $q-$metric as represented in Eq.(6.4.1). Another interesting particular case corresponds to the choice

$$\alpha_1 = 0\,, \quad \alpha_3 = \alpha_2\,, \quad \alpha_4 = 0\,, \quad \alpha_5 = 0\,, \tag{6.4.15}$$

which leads to the following line element

$$\begin{aligned} ds^2 \;=\; & \left(1-\frac{2m}{r}\right)\left(1-\frac{q\alpha_2 m}{r-m}\right)dt^2 - \left(1+\frac{q\alpha_2 m}{r-m}\right)r^2\sin^2\theta d\varphi^2 \\ & - \left[1+\frac{q\alpha_2 m(r-2m)}{(r-m)^2}\right]\frac{dr^2}{1-\frac{2m}{r}} - \left(1+\frac{q\alpha_2 m}{r-m}\right)r^2 d\theta^2\,. \end{aligned} \tag{6.4.16}$$

This is an asymptotically flat approximate solution with parameters m, q and α_2. The singularity structure can be found by analyzing the Kretschmann invariant $K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ which in this case reduces to

$$K = \frac{48m^2}{r^6}\left(1 + q\alpha_2\frac{r-4m}{r-m} + \mathcal{O}(q^2)\right) , \tag{6.4.17}$$

where the term proportional to q^2 has been neglected due to the approximate character of the solution. We see that there is a central singularity at $r = 0$ and a second one at $r = m$. We conclude that the solution (6.4.16) describes the exterior field of two naked singularities of mass m and quadrupole q. The parameter α_2 can be absorbed by redefining the constant q and so it has no special physical meaning. In the general solution (6.4.11)-(6.4.13), the additive constants α_4 and α_5 can be chosen such that at infinity the solution describes the Minkowski spacetime in spherical coordinates. This means that non asymptotically flat solutions are also contained in the 5−parameter family (6.4.11) – (6.4.13). This is the most general vacuum solution which is linear in the quadrupole moment. To our knowledge, this general solution is new.

6.4.2 Perfect fluid solutions

We now apply the approximate line element (6.4.2) to the study of perfect fluid solutions. First, we note that in this case the conservation law (6.3.7) reduces to

$$p_{,r} = -(\rho + p)\nu_{,r} , \qquad p_{,\theta} = 0 . \tag{6.4.18}$$

Calculating the second derivative $p_{,r\theta} = 0$, the above conservation laws lead to

$$\rho_{,\theta} = 0 , \tag{6.4.19}$$

implying that the perfect fluid variables can depend on the coordinate r only. Notice that this does not imply that the source is spherically symmetric. In fact, due to the presence of the quadrupole parameter q in the line element (6.4.2), the coordinate r is no longer a radial coordinate and the equation r =constant represents, in general, a non-spherically symmetric deformed surface [45].

The corresponding linearized Einstein equations can be represented as

$$\overset{(0)}{G^{\nu}_{\mu}} + q\,\overset{(q)}{G^{\nu}_{\mu}} = 8\pi\left(\overset{(0)}{T^{\nu}_{\mu}} + q\,\overset{(q)}{T^{\nu}_{\mu}}\right) , \tag{6.4.20}$$

where the $(0)-$terms correspond to the limiting case of spherical symmetry. As for the energy-momentum tensor, we assume that density and pressure can also be linearized as

$$p(r) = p_0(r) + q\, p_1(r), \;\; \rho(r) = \rho_0(r) + q\rho_1(r), \tag{6.4.21}$$

in accordance with the conservation law conditions (6.4.18) and (6.4.19). Here, $p_0(r)$ and $\rho_0(r)$ are the pressure and density of the background spherically symmetric solution. If we now compute the linearized field equations (6.4.20) for the line element (6.4.2), we arrive at a set of nine differential equations for the functions ν, $\tilde{m}$, a, b, c, and p_1. After lengthy computations, it is then possible to isolate an equation that relates $p_1(r)$ and $b(r,\theta)$ from which it follows that

$$b_{,\theta} = 0\,. \tag{6.4.22}$$

This means that for the particular approximate line element (6.4.2), the field equations for a perfect fluid do not allow the metric functions to explicitly depend on the angular coordinate θ. To search for concrete solutions which can be matched with an exterior metric with quadrupole, it is necessary to modify the exterior metric accordingly. Therefore, we will now consider a modification of the approximate $q-$metric (6.4.1), which can be expressed as

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left[1 + q\ln\left(1 - \frac{2m}{r}\right)\right] dt^2 - r^2\left[1 - q\ln\left(1 - \frac{2m}{r}\right)\right]\sin^2\theta d\varphi^2 \\ - \left[1 - q\ln\left(1 - \frac{2m}{r}\right)\right]\left(\frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2\right). \tag{6.4.23}$$

We will see that this approximate exterior solution can be used together with the interior line element (6.4.2) to search for approximate solutions with a perfect fluid source. Taking into account that the conservation laws and the approximate field equations for a perfect fluid imply that the physical quantities p and ρ and the metric function b depend only on the spatial coordinate r, the remaining field equations can be represented explicitly as given in Appendix 6.6.

6.4.3 The background solution

For the zeroth component of the linearized field equations, we will consider a spherically symmetric spacetime. If we set $q = 0$ in the line element (6.4.2), only the metric functions ν and $\tilde{m}$ remain for which we obtain the field equations

$$\tilde{m}_{,r} = 4\pi r^2 \rho_0, \tag{6.4.24}$$

$$\nu_{,r} = \frac{\tilde{m} + 4\pi r^3 p_0}{r\,(r - 2\tilde{m})}. \tag{6.4.25}$$

If we assume that the density is constant, ρ_0 =const., we obtain a particular solution that can be represented as

$$e^{\nu} = e^{\nu_0} = \frac{3}{2} f_0(R) - \frac{1}{2} f_0(r) \quad p_0 = \rho_0 \frac{f_0(r) - f_0(R)}{3f_0(R) - f_0(r)}\,, \quad \tilde{m} = \frac{4\pi}{3}\rho_0 r^3, \tag{6.4.26}$$

with

$$f_0(r) = \sqrt{1 - \frac{2mr^2}{R^3}}\,, \tag{6.4.27}$$

where the integration constants have been chosen such that at the surface radius $r = R$, the Schwarzschild exterior metric is obtained. Solution (6.4.26) is the simplest spherically symmetric perfect fluid solution and is known as the interior Schwarzschild solution. In this work, we will use it as the zeroth approximation of the interior quadrupolar solutions to be obtained below.

6.4.4 Matching conditions

The importance of writing the approximate line elements as given above is that the matching between the interior and the exterior metrics can be performed in a relatively easy manner. Indeed, let us consider the boundary conditions at the matching surface $r = R$ by comparing the above interior metric (6.4.2) with the $q-$metric (6.4.1) to first order in q. Then, we obtain the matching conditions

$$\tilde{m}(R) = m\,, \quad a(R) = c(R) = 2\nu(R) = -\frac{1}{2}b(R) = \ln\left(1 - \frac{2m}{R}\right)\,. \tag{6.4.28}$$

In addition, we can impose the physically meaningful condition that the total pressure vanishes at the matching surface, i.e.,

$$p(R) = 0\,. \tag{6.4.29}$$

From the point of view of a numerical integration, the above matching conditions can be used as boundary values for the integration of the corresponding differential equations.

Notice that we reach the desired matching by fixing only the spatial coordinate as $r = R$; however, this does not mean that the matching surface is a sphere. Indeed, the shape of the matching surface is determined by the conditions $t = const$ and $r = R$ which, according to Eq.(6.4.1), determine a surface with explicit $\theta-$dependence. The coordinate r is therefore not a radial coordinate. This has been previously observed in the case of a different metric with quadrupole moment [45].

6.5 Particular solutions

Our goal is to find interior solutions to the linearized system of differential equations which take into account the contribution of the quadrupole parameter q only up to the first order. The explicit form of the corresponding field equations is given in Appendix 6.6. One can see that it can be split into two sets that can be treated separately. The first set relates only the functions $\tilde{m}(r)$, $\nu(r)$ and $\rho_0(r)$, which must satisfy the equations

$$\tilde{m}_{,r} = 4\pi\rho_0 r^2\,, \tag{6.5.1}$$

$$\nu_{,r} = \frac{4\pi p_0 r^3 + \tilde{m}}{r\,(r - 2\tilde{m})}\,, \tag{6.5.2}$$

and

$$p_{0,r} = -\frac{\left(4p_0\pi r^3 + \tilde{m}\right)(p_0 + \rho_0)}{r\,(r - 2\tilde{m})}\,. \tag{6.5.3}$$

This set of equations can be integrated immediately once the value of the density ρ_0 is known. In particular, for a constant ρ_0, we obtain the interior Schwarzschild metric (6.4.26), which is the zeroth order solution we will use in the following sections to integrate the field equations.

In addition, for the remaining functions $a(r)$, $b(r)$ and $c(r)$, we obtain a set of two second-order and three first-order partial differential equations which are presented explicitly in Appendix 6.6. In the next sections, we will analyze this set of equations and derive several particular solutions.

6.5.1 Solutions determined by constants

To find an interior counterpart for the above approximate exterior metric, we consider first the simplest case in which a, b, c and ρ_1 are constants. The explicit form of the remaining field equations (see Appendix 6.6) suggests the relationship

$$a = c \, , \tag{6.5.4}$$

which reduces considerably the complexity of the equations. Indeed, the only non-trivial equations in this case are

$$\rho_1 = -(b+c)\,\rho_0 \, , \quad p_1 = -(b+c)\,p_0, \tag{6.5.5}$$

so that the total pressure and density are

$$p\,(r) = p_0(r)\,[1 - q\,(b+c)] \, , \quad \rho = \rho_0\,[1 - q\,(b+c)] \; , \tag{6.5.6}$$

where $p_0(r)$ is given in Eq.(6.4.26). The values of the constants a, b and c can be determined from the matching conditions with the exterior metric on the surface $r = R$. We obtain

$$\tilde{m}(R) = m \, , \quad a = c = 2\nu(R) = \ln\left(1 - \frac{2m}{R}\right) , \; b = -2\ln\left(1 - \frac{2m}{R}\right) . \tag{6.5.7}$$

This is a simple approximate interior solution in which the presence of the quadrupole parameter essentially leads to a modification of the pressure of the body. For instance, for the particular choice

$$\rho_0 = \frac{3m}{4\pi R^3}, \; R = 1, \; m = 0.435, \; q = \frac{1}{100} \, , \tag{6.5.8}$$

we obtain the pressure and the metric functions depicted in Fig. 6.1. We conclude that this interior solution is singularity free and can be matched continuously across the matching surface $r = R$ with the approximate exterior $q-$metric (6.4.23).

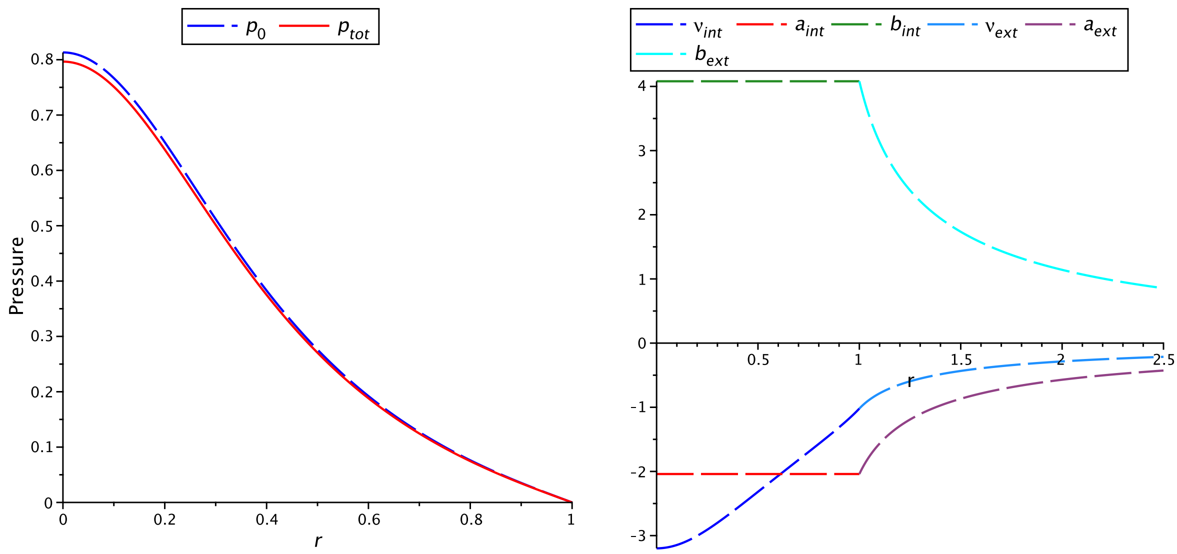

Figure 6.1: Behavior of the pressure and the metric functions in terms of the spatial coordinate r in units of m.

The corresponding interior line element can be expressed as

$$ds^2 = e^{2\nu}(1+qa)dt^2 - (1-qa)\left(\frac{dr^2}{1-\frac{2\tilde{m}}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) , \quad (6.5.9)$$

with

$$a = \ln\left(1 - \frac{2m}{R}\right) . \quad (6.5.10)$$

In the limiting case $q \to 0$, we turn back to the interior Schwarzschild solution.

6.5.2 Solutions with radial dependence

We now assume that the functions a, b and c depend on the radial coordinate. As before, the interior Schwarzschild solution (6.4.26) is taken as the zeroth approximation. An analysis of the field equations shows that the following cases need to be considered.

1) Let $b(r) = 0$ and $a(r) = c(r)$. The field equations allow only one solution, namely, $a = const$. However, this is a trivial case that is equivalent to multiplying the density, pressure and some metric components by a constant quantity.

2) Let $b(r) = 0$ and $a(r) \neq c(r)$. In this case, the boundary conditions (6.4.28), which imply that $a = c = a(R)$, are not consistent with the remaining field equations. No solutions are found in this case.

3) Let $b(r) \neq 0$ and $a(r) \neq c(r)$. From Eqs.(6.6.8) and (6.6.9), we obtain that

$$b(r) = -2a(r) . \tag{6.5.11}$$

This relationship simplifies the remaining equations. Nevertheless, we were unable to find analytical solutions. Therefore, we perform a numerical integration of the remaining equations for the functions $a(r)$, $c(r)$, and $p_1(r)$. We take as a particular example the values specified in (6.5.8) for the mass m, the matching radius R and the quadrupole parameter q. Then, from the boundary conditions (6.4.28), we obtain

$$a(R) = c(R) = -2.0402, \quad a_{,r}\,|_{r=R} = 13.6 . \tag{6.5.12}$$

Moreover, to perform the numerical integration, it is necessary to specify the profile of the density function $\rho_1(r)$, which we take as

$$\rho_1(r) = \rho_1(0) - r - \frac{1}{2}r^2 - \frac{1}{6}r^3 , \tag{6.5.13}$$

where the constant $\rho_1(0)$ must be chosen such that the total density $\rho = \rho_0 + q\rho_1(r)$ vanishes at the surface and is finite at the center $r = 0$. For the particular parameter values (6.5.8), we obtain $\rho_1(0) = -8.718$ and the behavior of the total density is illustrated in Fig. 6.2.

With this density function, the numerical integration can be performed explicitly, leading to a solution for the pressure which is presented in Fig. 6.2. Moreover, the result of the numerical integration of the metric functions $a(r), b(r)$ and $c(r)$ is represented in Fig. 6.3. We see that all the boundary conditions for the variables of the perfect fluid and the metric functions are satisfied and that all the quantities show a regular behavior. We conclude that in this particular case it is possible to obtain physically meaningful solutions.

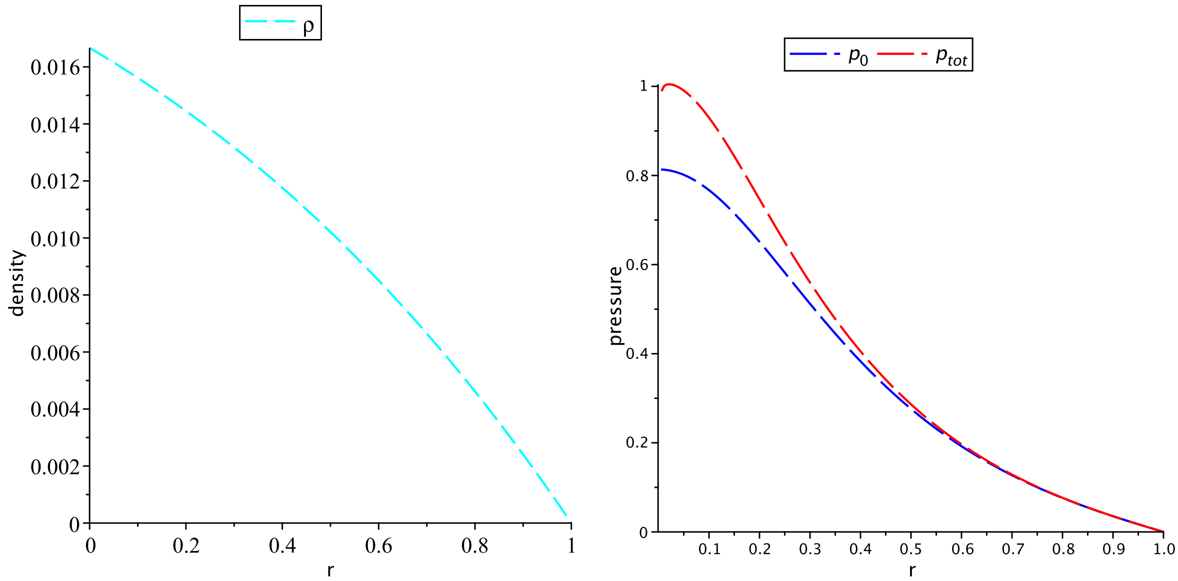

Figure 6.2: Behavior of the density and pressure as functions of the radial coordinate.

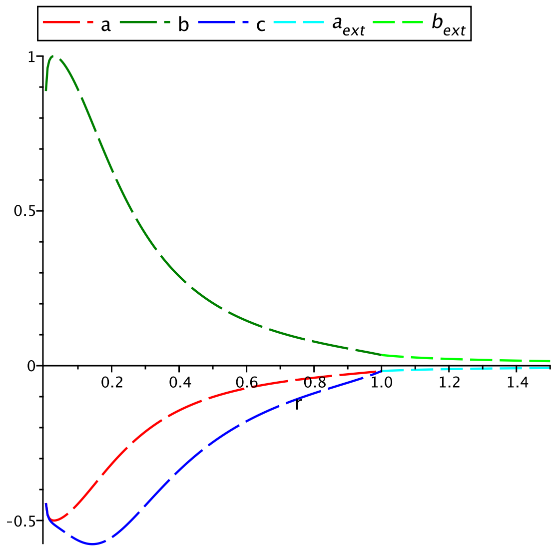

Figure 6.3: Behavior and matching of the metric functions.

The line element for this perfect fluid solution can be written as

$$ds^2 = e^{2\nu}(1+qa)dt^2 - (1+qb+qc)\frac{dr^2}{1-\frac{2\tilde{m}}{r}} + (1-qa)(d\theta^2 + r^2\sin^2\theta d\varphi^2) \tag{6.5.14}$$

and leads to the interior Schwarzschild spacetime for vanishing quadrupole parameter.

6.6 Linearized field equations

In general, the linearized field equations which follow from the line element

$$\begin{aligned} ds^2 &= e^{2\nu}(1+qa)dt^2 - (1+qc+qb)\frac{dr^2}{1-\frac{2\tilde{m}}{r}} \\ &\quad -(1+qa+qb)r^2 d\theta^2 - (1-qa)r^2\sin^2\theta d\varphi^2 \,, \end{aligned} \tag{6.6.1}$$

where all the metric functions depend on r only, can be written as

$$\tilde{m}_{,r} = 4\pi\rho_0 r^2 \tag{6.6.2}$$

$$\nu_{,r} = \frac{4\pi p_0 r^3 + \tilde{m}}{r\,(r-2\tilde{m})} \tag{6.6.3}$$

$$p_{0,r} = -\frac{\left(4p_0\pi r^3 + \tilde{m}\right)(p_0+\rho_0)}{r\,(r-2\tilde{m})}, \tag{6.6.4}$$

$$\begin{aligned} &2r\,(r-2\tilde{m})\,a_{,rr} + \left[(3p_0 - 2\rho_0)\,4\pi r^3 + 3r - \tilde{m}\right]a_{,r} + \left(r - 3\tilde{m} - 4\pi p_0 r^3\right)c_{,r} \\ &-16\pi r^2\left[(b+c)\,(\rho_0+p_0) + \rho_1 + p_1\right] - 2\,(a-c) = 0\,, \end{aligned} \tag{6.6.5}$$

$$\begin{aligned} &r\,(r-2\tilde{m})\,b_{,rr} + \left(r - \tilde{m} - 4\pi\rho_0 r^3\right)b_{,r} - 2\,(r-2\tilde{m})\,c_{,r} \\ &+16\pi r^2\left[(c+b)\,\rho_0 + \rho_1\right] + 2\,(a-c) = 0\,, \end{aligned} \tag{6.6.6}$$

$$\left(4\pi p_0 r^3 + r - \tilde{m}\right)(a_{,r} - c_{,r}) - 32\pi r^2[(c+b)\,p_0 + p_1] + 2\,(a-c) = 0\,, \tag{6.6.7}$$

$$2\left(4\pi p_0 r^3+\tilde{m}\right)a_{,r}+\pi r^2\left(\left(c+b\right)p_0+p_1\right)-2\left(a-c\right)=0\,, \quad (6.6.8)$$

$$\begin{aligned}&[4\left(4\pi p_0 r^3+r-\tilde{m}\right)^2\sin^2\theta+r\left(r-2\tilde{m}\right)\cos^2\theta]b_{,r}\\&+2[\left(4\pi p_0 r^3-\tilde{m}\right)\sin^2\theta+r]\left(r-2\tilde{m}\right)a_{,r}\\&-2\sin^2\theta\left(4\pi p_0 r^3+r-\tilde{m}\right)\{8\pi r^2[\left(c+b\right)p_0+p_1]-a+c\}=0\end{aligned} \quad (6.6.9)$$

6.7 Conclusions

In this work, we have investigated approximate interior solutions of Einstein's equations in the case of static and axially symmetric perfect fluid spacetimes, which can be matched smoothly with an exterior spacetime, characterized by an arbitrary mass and a small quadrupole moment.

We present the general form of the field equations for a perfect fluid source by using a specially adapted line element. Due to the complexity of the corresponding set of differential equations, we reduce the problem to the study of the particular case in which the quadrupole parameter can be considered as a small quantity, which is then used to linearize the line element and the field equations. The interior linearized line element is chosen such that it can easily be compared with the exterior approximate $q-$metric. We analyze the corresponding linearized field equations and derive several classes of new vacuum and perfect fluid solutions.

We limit ourselves to the study of interior solutions that can be matched with the exterior $q-$metric, satisfy the energy conditions for the density and the pressure and are free of singularities in the entire spacetime. This implies that our solutions can be used to describe the gravitational field of realistic compact objects in which the quadrupole moment is small.

The results obtained here can be considered as a first step towards the determination of an exact solution of Einstein's equations which describes correctly the gravitational field of a rotating deformed source. The important feature of our approach is that we consider explicitly the influence of the quadrupole on the structure of spacetime and the corresponding field equations. In the present work, we only considered the simple and idealized case of a static mass distribution with a small quadrupole and obtained compatible and physically reasonable results. To study more realistic configurations, it is necessary to take into account the rotation the exact quadrupole of the mass. We expect to investigate these problems in future works.

7 C^3 matching for asymptotically flat spacetimes

7.1 Introduction

General relativity is a theory of the gravitational interaction and, in particular, should describe the gravitational field of relativistic compact objects. In this case, the spacetime can be split into two different parts, namely, the interior region described by an exact solution $g^-_{\mu\nu}$ of Einstein's equations with a physically reasonable energy-momentum tensor and the exterior region, which corresponds to an exact vacuum solution $g_{\mu\nu}$. This implies that the spacetime M can be considered as split into two regions $(M^-, g^-_{\mu\nu})$ and $(M^+, g^+_{\mu\nu})$ with a special hypersurface Σ at which the two regions should be matched. In the case of compact objects, Σ should be identified with the surface of the object, i.e., it is a time-like hypersurface. It then follows that at Σ certain matching conditions should be imposed in order for the spacetime to be well defined.

Two sets of matching conditions are commonly used in the literature. The Darmois conditions [46] demand that the first and second fundamental forms (the intrinsic metric and the extrinsic curvature) be continuous across Σ. The Lichnerowicz conditions state that the metric and all its first derivatives must be continuous across Σ in "admissible" coordinates that traverse Σ. The Darmois conditions are expressed in terms of tensorial quantities and hence they can be considered as a covariant formulation of the matching problem. In the case of the Lichnerowicz conditions, the term "admissible" coordinates is used, indicating that the choice of a coordinate system is essential. The equivalence between Darmois and Lichnerowicz conditions can be proved by using Gaussian normal coordinates. This proof allows one also to precise the concept of "admissible" coordinates which are then defined as coordinates related to Gaussian normal coordinates by means of (C^2_Σ, C^4) transformations [47]. However, as pointed out by Israel in [48], the explicit form of the matching conditions are of limited utility, since "admissible" coordinates usually are not the most convenient for handling the matching problem in practice.

An alternative approach in which the extrinsic curvature is not necessarily continuous across Σ was proposed by Israel [48]. An effective energy-momentum tensor which determines a thin shell is defined in terms of the difference of the extrinsic curvature evaluated inside and outside the hypersurface Σ. This means that Σ can now be interpreted as a thin shell that separates M^+ and M^-, is part of the entire spacetime M and as such plays an important role in the determination of the spacetime dynamics.

In all the C^2 matching conditions described above, it is important to know a priori the location of Σ. Although in the case of compact objects, we can identify Σ with the surface of the body, in general, it is not easy to find the equation that determines the surface and, if possible, it often is not given in the "admissible" coordinates that are essential for treating the matching problem. This is probably the reason why the matching conditions have been applied so far only in cases characterized by very high symmetries.

In this work, we propose to use a C^3 criterion to find information about the location of the hypersurface Σ. It is defined in terms of the eigenvalues of the Riemann curvature tensor which are invariant quantities. The idea is simple. Since the curvature tensor is a measure of the gravitational interaction, the curvature eigenvalues provides us with an invariant measure of the gravitational interaction. This invariance should be understood as independence with respect to the choice of coordinates and reference frames. This is due to the fact that the curvature eigenvalues can be treated as scalars under coordinate transformations. Whereas single components of the curvature tensor are not scalars in general and, consequently, cannot be used to define gravitational interaction in an invariant way, the eigenvalues with their scalar property represent the gravitational interaction independently of the choice of coordinates. An example of the application of the curvature eigenvalues is in the formulation of the invariant Petrov classification of the curvature tensor and, correspondingly, of the gravitational fields [49].

Since for a compact object, one expects the spacetime to be asymptotically flat, the curvature eigenvalues should vanish at spatial infinite and the behavior of the eigenvalues approaching the gravitational source could give some information about its boundaries. Here, we implement this simple idea in an invariant manner and show its applicability in the case of several exact solutions of Einstein's equations. The original idea of using curvature eigenvalues to formulate C^3 matching conditions was first presented by one of us in [50]. It has been used also to formulate an invariant definition of repulsive gravity [51, 52] and to construct cosmological models [53].

This paper is organized as follows. In Sec. 7.2, we use Cartan's formalism to investigate the general form of a curvature tensor that satisfies Einstein equations with a perfect-fluid source. Moreover, we find the general form of the curvature eigenvalues and derive some identities relating them. In Sec. 7.3, we review the definition of repulsive gravity in terms of the curvature eigenvalues. This definition is then used in Sec. 7.4 to propose the C^3 matching approach, whose objective is to perform the matching in such a way that the effects of repulsive gravity cannot be detected. We also apply the method to spherically symmetric solutions in Newton gravity and obtain the general result that the mass density should vanish at the matching surface. In Einstein gravity, in addition, the pressure should also vanish. Then, in the Appendix, we present all the metrics of the Tolman class of interior solutions and calculate the corresponding curvature eigenvalues to show that none of them satisfies the C^3 matching conditions. Finally, in Sec. 7.7, we discuss our results and propose some tasks for future investigation.

7.2 Curvature eigenvalues and Einstein equations

Our approach is based upon the analysis of the behavior of the curvature eigenvalues. There are different ways to determine these eigenvalues [49]. Our strategy is to use local tetrads and differential forms. From the physical point of view, a local orthonormal tetrad is the simplest and most natural choice for an observer in order to perform local measurements of time, space, and gravity. Moreover, once a local orthonormal tetrad is chosen, all the quantities related to this frame are invariant with respect to coordinate transformations. The only freedom remaining in the choice of this local frame is a Lorentz transformation. So, let us choose the orthonormal tetrad as

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \vartheta^a \otimes \vartheta^b \ , \tag{7.2.1}$$

with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, and $\vartheta^a = e^a{}_\mu dx^\mu$. The first

$$d\vartheta^a = -\omega^a{}_b \wedge \vartheta^b \ , \tag{7.2.2}$$

and second Cartan equations

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} \vartheta^c \wedge \vartheta^d \tag{7.2.3}$$

allow us to compute the components of the Riemann curvature tensor in the local orthonormal frame.

It is possible to represent the curvature tensor as a (6×6)-matrix by introducing the bivector indices $A, B, \ldots$ which encode the information of two different tetrad indices, i.e., $ab \to A$. We follow the convention proposed in [54] which establishes the following correspondence between tetrad and bivector indices

$$01 \to 1\,, \quad 02 \to 2\,, \quad 03 \to 3\,, \quad 23 \to 4\,, \quad 31 \to 5\,, \quad 12 \to 6\,. \tag{7.2.4}$$

Then, the Riemann tensor can be represented by the symmetric matrix $\mathbf{R}_{AB} = \mathbf{R}_{BA}$ with 21 components. The first Bianchi identity $R_{a[bcd]} = 0$, which in bivector representation reads

$$\mathbf{R}_{14} + \mathbf{R}_{25} + \mathbf{R}_{36} = 0\,, \tag{7.2.5}$$

reduces the number of independent components to 20.

Einstein's equations with cosmological constant

$$R_{ab} - \frac{1}{2} R \eta_{ab} + \Lambda \eta_{ab} = \kappa T_{ab}\,, \quad R_{ab} = R^c{}_{acb}\,, \tag{7.2.6}$$

can be written explicitly in terms of the components of the curvature tensor in the bivector representation, resulting in a set of ten algebraic equations that relate the components $\mathbf{R}_{AB}$. This means that only ten components $\mathbf{R}_{AB}$ are algebraic independent which can be arranged in the 6×6 curvature matrix

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{7.2.7}$$

where

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} - \kappa T_{03} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} + \kappa T_{02} & \mathbf{R}_{26} - \kappa T_{01} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix},$$

and $\mathbf{M}_1$ and $\mathbf{M}_2$ are 3×3 symmetric matrices

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} - \Lambda + \kappa\left(\frac{T}{2} + T_{00}\right) \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} -\mathbf{R}_{11} + \kappa\left(\frac{T}{2} + T_{00} - T_{11}\right) & -\mathbf{R}_{12} - \kappa T_{12} & -\mathbf{R}_{13} - \kappa T_{13} \\ -\mathbf{R}_{12} - \kappa T_{12} & -\mathbf{R}_{22} + \kappa\left(\frac{T}{2} + T_{00} - T_{22}\right) & -\mathbf{R}_{23} - \kappa T_{23} \\ -\mathbf{R}_{13} - \kappa T_{13} & -\mathbf{R}_{23} - \kappa T_{23} & \mathbf{R}_{11} + \mathbf{R}_{22} + \Lambda - \kappa T_{33} \end{pmatrix},$$

with $T = \eta^{ab} T_{ab}$. This is the most general form of a curvature tensor that satisfies Einstein's equations with cosmological constant and arbitrary energy-momentum tensor. The traces of the matrices entering the final form of the curvature turn out to be of particular importance. First, the matrix $\mathbf{L}$ is traceless by virtue of the Bianchi identities, as shown above. Moreover, for the remaining matrices we obtain

$$\mathrm{Tr}(\mathbf{M_1}) = -\Lambda + \kappa\left(\frac{T}{2} + T_{00}\right), \quad \mathrm{Tr}(\mathbf{M_2}) = +\Lambda + \kappa T_{00}, \tag{7.2.8}$$

so that

$$\mathrm{Tr}(\mathbf{R_{AB}}) = \kappa\left(\frac{T}{2} + 2T_{00}\right). \tag{7.2.9}$$

We see that these traces depend on the components of the energy-momentum tensor only.

7.2.1 Vacuum spacetimes

In the particular case of vanishing cosmological constant ($\Lambda = 0$) and vacuum fields ($R_{ab} = 0$), the curvature matrix reduces to

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M} & \mathbf{L} \\ \mathbf{L} & -\mathbf{M} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & \mathbf{R}_{33} \end{pmatrix}, \tag{7.2.10}$$

and the 3×3 matrices L and M are symmetric and trace free,

$$\mathrm{Tr}(\mathbf{L}) = 0, \qquad \mathrm{Tr}(\mathbf{M}) = 0. \tag{7.2.11}$$

Notice that the relation $\mathrm{Tr}(\mathbf{L}) = 0$ is valid in general as a consequence of the Bianchi identities, whereas $\mathrm{Tr}(\mathbf{M}) = 0$ holds only in the limiting vacuum case.

7.2.2 Conformally flat spacetimes

In the case of a conformally flat spacetime

$$g_{ab} = e^{\Psi}\eta_{ab} \tag{7.2.12}$$

where Ψ is an arbitrary C^2 function, we have that

$$R_{abcd} = \frac{1}{6}R(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}) + \frac{1}{2}(\eta_{ac}R_{bd} - \eta_{ad}R_{bc} - \eta_{bc}R_{ad} + \eta_{bd}R_{ac}), \tag{7.2.13}$$

where

$$R_{ab} = \Psi_{,ab} - \frac{1}{2}\Psi_{,a}\Psi_{,b} + \frac{1}{2}\eta_{ab}(\Box\Psi + \chi)\,, \tag{7.2.14}$$

and

$$R = 3e^{-\psi}\left(\Box\Psi + \frac{1}{2}\chi\right)\,, \tag{7.2.15}$$

with $\Box\Psi \equiv \eta^{ab}\Psi_{,ab}$ and $\chi \equiv \eta^{ab}\Psi_{,a}\Psi_{,b}$.

Then, taking into account Einstein equations, the curvature matrix reduces to

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{7.2.16}$$

with

$$\mathbf{L} = \frac{k}{2}\begin{pmatrix} 0 & T_{03} & -T_{02} \\ -T_{03} & 0 & T_{01} \\ T_{02} & -T_{01} & 0 \end{pmatrix}\,, \tag{7.2.17}$$

$$\mathbf{M_1} = \begin{pmatrix} -\frac{1}{3}\Lambda + \frac{1}{3}kT + \frac{1}{2}k(T_{00} - T_{11}) & -\frac{1}{2}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & -\frac{1}{3}\Lambda + \frac{1}{3}kT + \frac{1}{2}k(T_{00} - T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & -\frac{1}{3}\Lambda + \frac{1}{3}kT + \frac{1}{2}k(T_{00} - T_{33}) \end{pmatrix} \quad (7.2.18)$$

and

$$\mathbf{M_2} = \begin{pmatrix} \frac{1}{3}\Lambda + \frac{1}{6}kT + \frac{1}{2}k(T_{00} - T_{11}) & -\frac{1}{3}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & \frac{1}{3}\Lambda + \frac{1}{6}kT + \frac{1}{2}k(T_{00} - T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & \frac{1}{3}\Lambda + \frac{1}{6}kT + \frac{1}{2}k(T_{00} - T_{33}) \end{pmatrix} . \quad (7.2.19)$$

As before, the matrix **L** is traceless by virtue of the Bianchi identities. Moreover, for the remaining matrices we obtain

$$\mathrm{Tr}(\mathbf{M_1}) = -\Lambda + \kappa\left(\frac{T}{2} + T_{00}\right) , \quad \mathrm{Tr}(\mathbf{M_2}) = +\Lambda + \kappa T_{00} , \quad (7.2.20)$$

so that

$$\mathrm{Tr}(\mathbf{R_{AB}}) = \kappa\left(\frac{T}{2} + 2T_{00}\right) . \quad (7.2.21)$$

An additional reduction is obtained if conformal invariance with $T = 0$ is demanded:

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix} , \quad (7.2.22)$$

with

$$\mathbf{L} = \frac{k}{2}\begin{pmatrix} 0 & T_{03} & -T_{02} \\ -T_{03} & 0 & T_{01} \\ T_{02} & -T_{01} & 0 \end{pmatrix} , \quad (7.2.23)$$

$$\mathbf{M_1} = \begin{pmatrix} -\frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{11}) & -\frac{1}{2}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & -\frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & -\frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{33}) \end{pmatrix} \qquad (7.2.24)$$

and

$$\mathbf{M_2} = \begin{pmatrix} \frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{11}) & -\frac{1}{3}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & \frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & \frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{33}) \end{pmatrix} . \qquad (7.2.25)$$

Again, the matrix **L** is traceless by virtue of the Bianchi identities. Moreover, for the remaining matrices we obtain

$$\text{Tr}(\mathbf{M_1}) = -\Lambda + \kappa T_{00} \,, \quad \text{Tr}(\mathbf{M_2}) = +\Lambda + \kappa T_{00} \,, \qquad (7.2.26)$$

so that

$$\text{Tr}(\mathbf{R_{AB}}) = 2\kappa T_{00} \,. \qquad (7.2.27)$$

7.2.3 Perfect fluid spacetimes

For later use, we also consider the case of a perfect fluid energy-momentum tensor with density ρ and pressure p

$$T_{ab} = (\rho + p)u_a u_b + p\eta_{ab} \,, \qquad (7.2.28)$$

where u_a is the four-velocity of the fluid which for simplicity can always be chosen as the comoving velocity $u^a = (-1, 0, 0, 0)$. Then,

$$T_{ab} = \text{diag}(\rho, p, p, p) \,. \qquad (7.2.29)$$

The curvature matrix for a perfect fluid is then given by Eq.(7.2.7) with

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} & \mathbf{R}_{26} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix},$$

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} - \Lambda + \frac{\kappa}{2}\left(3p + \rho\right) \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} -\mathbf{R}_{11} + \frac{\kappa}{2}(\rho + p) & -\mathbf{R}_{12} & -\mathbf{R}_{13} \\ -\mathbf{R}_{12} & -\mathbf{R}_{22} + \frac{\kappa}{2}(\rho + p) & -\mathbf{R}_{23} \\ -\mathbf{R}_{13} & -\mathbf{R}_{23} & \mathbf{R}_{11} + \mathbf{R}_{22} + \Lambda - \kappa p \end{pmatrix}.$$

The eigenvalues of the curvature tensor correspond to the eigenvalues of the matrix $\mathbf{R}_{AB}$. In general, they are functions λ_i, with $i = 1, 2, ..., 6$,which depend on the parameters and coordinates entering the tetrads ϑ^a. As shown above, in the case of a vacuum solution the curvature matrix is traceless and hence the eigenvalues must satisfy the condition

$$\sum_{i=1}^{6} \lambda_i = 0 \,. \tag{7.2.30}$$

In the case of a perfect fluid solution, the curvature eigenvalues are related by

$$\sum_{i=1}^{6} \lambda_i = \frac{3\kappa}{2}(\rho + p) \,. \tag{7.2.31}$$

These identities are a consequence of applying Einstein's equations to the general form of the curvature matrix $\mathbf{R_{AB}}$ and, consequently, they should contain information about the behavior of the gravitational field. We will verify these statements in the examples to be presented below.

7.3 Repulsive gravity

The effects of repulsive gravity have been found in several contexts and different gravitational fields (see, for instance, [55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65] and the references cited therein.)In particular, repulsive effects have been identified in the gravitational field of naked singularities and near black holes [66]. In the literature, there are several intuitive definitions of repulsive gravity, but only recently an invariant definition was proposed in [52] by using the eigenvalues of the curvature tensor. The idea consists in using the eigenvalues to detect the regions of the gravitational field of compact objects, where repulsive effects are of importance. Indeed, since the gravitational field of compact objects is asymptotically flat, the eigenvalues should vanish at infinity. When approaching the object, the eigenvalues can either increase exponentially until they diverge at the singularity or they change their sign at some point, indicating the character of gravity has changed. This behavior is schematically illustrated in Fig. 7.1. The dotted curve corresponds to an

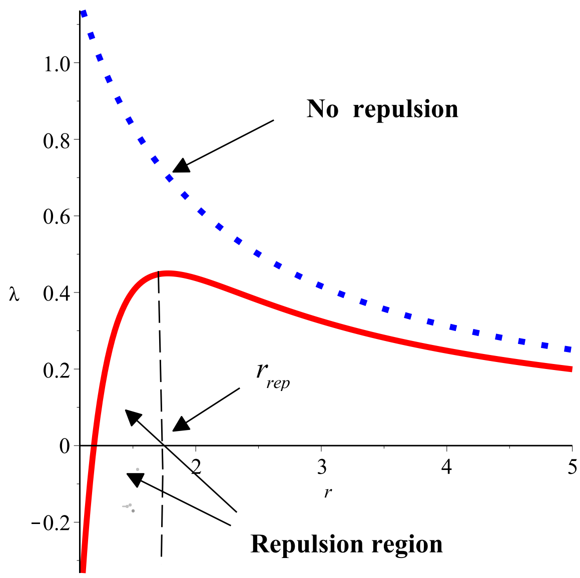

Figure 7.1: Schematic representation of the behavior of the curvature eigenvalue. Here, r represents the distance from the source. The repulsive region $r < r_{rep}$ includes positive values of the eigenvalue, where repulsive effects can be detected, and negative values, where repulsion becomes dominant.

eigenvalue with no change in the character of gravity whereas the solid curve shows a maximum at a distance $r = r_{rep}$, indicating the presence of repulsive gravity. Within the region $r < r_{rep}$, repulsive gravity exists and even becomes dominant once the eigenvalue changes its sign.

The above intuitive description of repulsive gravity can be formalized as follows. Let $\{\lambda_i^+\}$ $(i = 1, ..., 6)$ represents the set of eigenvalues of an exterior spacetime. As explained above, the presence of an extremum in an eigenvalue is an indication of the existence of repulsive gravity. Then, let $\{r_l\}, l = 1, 2, ...$ with $0 < r_l < \infty$ represents the set of solutions of the equation

$$\frac{\partial \lambda_i}{\partial r}\bigg|_{r=r_l} = 0\,, \quad \text{with} \quad r_{rep} = \max\{r_l\}\,, \tag{7.3.1}$$

i.e., r_{rep} is the largest extremum of the eigenvalues and is called repulsion radius. Notice that at $r_{rep} = \max\{r_l\}$ the corresponding curvature eigenvalue should show a true change in the behavior of the gravitational interaction, i.e., the set up of repulsive gravity should be clear. For this reason, one should also impose the condition that the second derivative of the eigenvalue at the extremum be different from zero to avoid the presence of saddle points. In all the examples we will consider below, this condition is satisfied.

This definition is obviously related to the existence of eigenvalue extrema. It could happen that no extrema exist at all. This would correspond to the case illustrated in Fig. 7.1 with a dotted curve, i.e., the non-existence of extrema implies that no repulsion radius exists and, consequently, no repulsive effects can be observed in such a gravitational field. We will see below that this is the case of the Schwarzschild spacetime. If there is only one extremum, the repulsion radius coincides with the value of the radial coordinate at the extremum. On the other hand, if several extrema exist, the region contained between two extrema would show the effects corresponding to a transition between, say, a local maximum attraction and a local maximum repulsion. We will show in the next section that in the Kerr, Reissner-Nordström, and Kerr-Newman spacetimes there are several extrema. This seems to indicate that a general condition for the existence of a repulsion radius is related to the existence of multipole moments other than the mass monopole.

Since the curvature eigenvalues characterize in an invariant way the gravitational interaction, the above definition represents a invariant method to derive the repulsion region of asymptotically flat spacetimes, which describe the gravitational field of compact objects.

7.3.1 Repulsive gravity around black holes

In this work, we will use the above definition of repulsive radius to establish the minimum radius at which an exterior spacetime can be matched with an interior spacetime. In particular, we are interested in matching asymptotically flat exterior spacetimes with interior spacetimes that are free of singularities inside a certain particular region. It is therefore important to determine the location of the repulsion radius of known spacetimes that describe the gravitational field of compact objects. Let us thus consider the Kerr-Newman black hole spacetime [49]:

$$\begin{aligned} ds^2 &= -\frac{r^2-2Mr+a^2+Q^2}{r^2+a^2\cos^2\theta}(dt-a\sin^2\theta d\varphi)^2 \\ &\quad +\frac{\sin^2\theta}{r^2+a^2\cos^2\theta}[(r^2+a^2)d\varphi-adt]^2 \\ &\quad +\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2+Q^2}dr^2+(r^2+a^2\cos^2\theta)d\theta^2\,, \end{aligned} \tag{7.3.2}$$

where M is the mass of the rotating central object, $a = J/M$ is the specific angular momentum, and Q represents the total electric charge. The orthonormal tetrad can be chosen as

$$\begin{aligned} \vartheta^0 &= \left(\frac{r^2-2Mr+a^2+Q^2}{r^2+a^2\cos^2\theta}\right)^{1/2}(dt-a\sin^2\theta d\varphi)\,, \\ \vartheta^1 &= \frac{\sin\theta}{(r^2+a^2\cos^2\theta)^{1/2}}[(r^2+a^2)d\varphi-adt]\,, \\ \vartheta^2 &= (r^2+a^2\cos^2\theta)^{1/2}d\theta\,, \\ \vartheta^3 &= \left(\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2+Q^2}\right)^{1/2}dr\,. \end{aligned} \tag{7.3.3}$$

Following the approach presented in the last section, lengthy computations lead to the following eigenvalues [52]

$$\lambda_1^+ + i\lambda_4^+ = \lambda_2^+ + i\lambda_5^+ = l\,, \quad \lambda_3^+ + i\lambda_6^+ = -2l+k\,, \tag{7.3.4}$$

where

$$l = -\left[M - \frac{Q^2\,(r+ia\cos\theta)}{r^2+a^2\cos^2\theta^2}\right]\left(\frac{r-ia\cos\theta}{r^2+a^2\cos^2\theta}\right)^3\,, \tag{7.3.5}$$

$$k = -\frac{Q^2}{(r^2 + a^2\cos^2\theta)^2} \, . \tag{7.3.6}$$

A straightforward analysis shows that all the eigenvalues have extrema located at different values of r, but the largest one is associated with the equation $\frac{\partial \lambda_3^+}{\partial r} = 0$, the roots of which are determined by the equation

$$r^3(Mr - 2Q^2) + a^2\cos^2\theta[2Q^2 r + M(a^2\cos^2\theta - 6r^2)] = 0 \, . \tag{7.3.7}$$

The analytical solution to this quartic equation can be represented as

$$r_{rep}^{KN} = \frac{1}{2M}\left[Q^2 + A\left(1 + \sqrt{2 + \frac{2Q^2}{A}}\right)\right] , \quad A = \sqrt{Q^4 + 4a^2M^2\cos^2\theta}, \tag{7.3.8}$$

which in the limiting case $a = 0$ reduces to the Reissner-Nordström repulsion radius

$$r_{rep}^{RN} = \frac{2Q^2}{M} \, , \tag{7.3.9}$$

for $Q = 0$ leads to repulsion radius of the Kerr source

$$r_{rep}^{K} = (1 + \sqrt{2})a\cos\theta \, , \tag{7.3.10}$$

and in the Schwarzschild limiting case yields no repulsion radius. We see that the repulsion region is located very closed to the source. In the case of the Reissner-Nordström spacetime, the repulsion radius is twice the classical radius of a particle with mass M and charge Q, implying that repulsion is a pure classical effect. In the case of elementary particles, like the electron, the value of the classical radius would imply that the surrounding Reissner-Nordström spacetime is a naked singularity. If we also take into account the rotational properties of the particle and use the Kerr-Newman metric, the resulting spacetime corresponds to a naked singularity as well (see [67] and the references cited therein). In the Kerr spacetime, the repulsion radius depends on the polar angle, vanishing on the equatorial plane and reaching its maximum value at the poles. This result is frame independent in the sense that it is derived from a curvature eigenvalue, which can be considered as a scalar. The angular dependence of the Kerr repulsion radius is also in agreement with the fact that on the equator the curvature of the Kerr metric coincides with that of the Schwarzschild metric, which has no repulsion radius. In Fig.

7.2, we illustrate the shape of the repulsion radii of black holes and compare them with the location of the horizon and ergosphere. Indeed, we see that the repulsion region is located in the region of spacetime occupied by the horizon and ergosphere, which is closed to the gravitational source. This agrees with the regions where repulsive effects have been identified by using a completely different approach based on the analysis of the motion of test particles along circular orbits around black holes and naked singularities [68, 69, 66].

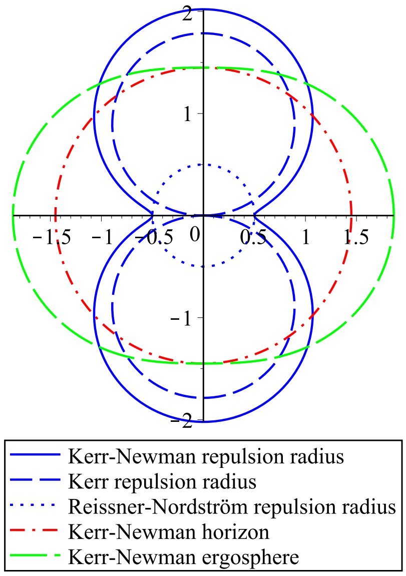

Figure 7.2: Repulsion radius of black holes with $M = 1$, $Q = 0.5$ and $a = 0.74$. For comparison, the horizon and ergosphere of the Kerr-Newman black hole are also plotted.

7.4 C^3 matching

The exterior field of compact objects is usually described by vacuum exact solutions characterized by singularities in a region closed to the source of gravity. To describe the entire spacetime and to avoid the presence of singularities, we usually say that the exterior spacetime must be matched with an interior spacetime which "covers" the region with singularities. To do this in concrete examples, we usually apply physical intuition to determine the matching surface and anyone of the C^2 methods mentioned in Sec. 7.1 to carry out the matching. The goal of the C^3 matching approach is to determine in an invariant manner where and how to "cover" the singular spacetime region.

To be more specific, consider an exterior spacetime $(M^+, g^+_{\mu\nu})$ and an interior spacetime $(M^-, g^-_{\mu\nu})$, which are characterized by the curvature eigenvalues $\{\lambda_i^+\}$ and $\{\lambda_i^-\}$, respectively. Then, the C^3 matching procedure consists in (i) establishing the matching surface Σ as determined by the matching radius [50]

$$r_{match} \in [r_{rep}, \infty) \,, \quad \text{with} \quad r_{rep} = \max\{r_l\} \,, \quad \left.\frac{\partial \lambda_i^+}{\partial r}\right|_{r=r_l} = 0 \,, \tag{7.4.1}$$

and (ii) performing the matching of the spacetimes $(M^+, g^+_{\mu\nu})$ and $(M^-, g^-_{\mu\nu})$ at Σ by imposing the conditions

$$\left.\lambda_i^+\right|_\Sigma = \left.\lambda_i^-\right|_\Sigma \quad \forall i \,. \tag{7.4.2}$$

The idea of defining the matching surface in terms of the matching radius was first proposed in [50]. Continuing with the formalization of the C^3 matching procedure, in the present work, we introduce the matching conditions (7.4.2) that are C^2 conditions similar to those assumed in other matching formalisms.

Thus, the C^3 matching demands that the curvature eigenvalues be continuous across the matching surface Σ which should be located at any radius between the repulsion radius and infinity. Notice that the repulsion radius is determined by the eigenvalues of the exterior curvature tensor. Accordingly, the minimum matching radius coincides with the repulsion radius. Physically, this means that the C^3 matching is intended to avoid the presence of repulsive gravity in the case of gravitational compact objects. The motivation for this requirement is that so far no repulsive gravity has been detected in the gravitational field of realistic compact objects like white dwarfs, neutron stars and other kinds of stars and planets. We can, therefore, consider repulsive gravity as an unphysical phenomenon at the level of compact objects and we propose to use the C^3 matching procedure to avoid such unphysical situation by covering the repulsive region by an physical interior solution.

7.4.1 Newtonian gravity

It is well known that Newtonian gravity is contained in the Einstein gravity theory as a special case. In some sense, we could also say that Newtonian gravity is the simplest non-trivial special case of Einstein gravity. It is, therefore, reasonable to test the C^3 matching in this simple special case. If it turns out that this procedure does not lead to physically meaningful results in the Newtonian limiting case, it would imply a failure of the method. Then, the possibility of being successful in the general Einstein theory would be very small.

Let us consider the line element for the nearly Newtonian metric in spherical coordinates $x^\alpha = (t, r, \theta, \varphi)$ (see [54], page 470)

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \qquad (7.4.3)$$

where $\Phi << 1$ is the Newtonian potential. In this work, we limit ourselves to the study of spherically symmetric gravitational configurations and so we assume that Φ depends on r only. The components of the orthonormal tetrad are then

$$\vartheta^0 = \sqrt{1+2\Phi}dt\,, \quad \vartheta^1 = \sqrt{1-2\Phi}dr\,, \qquad (7.4.4)$$

$$\vartheta^2 = \sqrt{1-2\Phi}\, r\, d\theta\,, \quad \vartheta^3 = \sqrt{1-2\Phi}\, r \sin\theta\, d\varphi\,, \qquad (7.4.5)$$

which in the first-order approximation lead to the connection 1-form

$$\omega^0{}_1 = \Phi_r \vartheta^0, \quad \omega^2{}_3 = -\frac{1}{r}(1+\Phi)\cot\theta\vartheta^3, \qquad (7.4.6)$$

$$\omega^1{}_2 = -\frac{1}{r}(1+\Phi - r\Phi_r)\vartheta^2, \quad \omega^1{}_3 = -\frac{1}{r}(1+\Phi - r\Phi_r)\vartheta^3\,, \qquad (7.4.7)$$

where Φ_r denotes the radial derivative of Φ. Moreover, the only non-vanishing components of the curvature 2-form can be expressed up to the first order in Φ as

$$\Omega^0{}_1 = -\Phi_{rr}\,\vartheta^0 \wedge \vartheta^1, \ \Omega^0{}_2 = -\tfrac{1}{r}\Phi_r\,\vartheta^0 \wedge \vartheta^2, \qquad (7.4.8)$$

$$\Omega^0{}_3 = -\tfrac{1}{r}\Phi_r\,\vartheta^0 \wedge \vartheta^3, \ \Omega^2{}_3 = \tfrac{2}{r}\Phi_r\vartheta^2 \wedge \vartheta^3, \qquad (7.4.9)$$

$$\Omega^3{}_1 = (\Phi_{rr} + \tfrac{1}{r}\Phi_r)\vartheta^2 \wedge \vartheta^3, \ \Omega^1{}_2 = (\Phi_{rr} + \tfrac{1}{r}\Phi_r)\vartheta^1 \wedge \vartheta^2\,. \qquad (7.4.10)$$

It then follows that the only non-zero components of the curvature tensor are

$$R_{0101} = \mathbf{R}_{11} = \Phi_{rr}\,, \;\; R_{0202} = \mathbf{R}_{22} = R_{0303} = \mathbf{R}_{33} = \tfrac{1}{r}\Phi_r\,, \qquad (7.4.11)$$

$$R_{2323} = \mathbf{R}_{44} = \tfrac{2}{r}\Phi_r\,, \;\; R_{3131} = \mathbf{R}_{55} = R_{1212} = \mathbf{R}_{66} = \Phi_{rr} + \tfrac{1}{r}\Phi_r\,. \qquad (7.4.12)$$

Consequently, the curvature matrix $\mathbf{R_{AB}}$ (7.2.7) is diagonal with eigenvalues

$$\lambda_1 = \Phi_{rr}\,, \;\; \lambda_2 = \lambda_3 = \tfrac{1}{r}\Phi_r\,, \qquad (7.4.13)$$

$$\lambda_4 = \tfrac{2}{r}\Phi_r\,, \;\; \lambda_5 = \lambda_6 = \Phi_{rr} + \tfrac{1}{r}\Phi_r\,, \qquad (7.4.14)$$

which satisfy the relationship

$$\sum_{i=1}^{6} \lambda_i = 3\left(\Phi_{rr} + \frac{2}{r}\Phi_r\right) = 3\,\nabla^2\Phi\,. \qquad (7.4.15)$$

We then conclude that in Newtonian gravity the eigenvalue identity (7.2.31) is equivalent to the Poisson equation, i.e.,

$$\sum_{i=1}^{6} \lambda_i = \frac{3\kappa}{2}\rho \quad \Leftrightarrow \quad \nabla^2\Phi = \frac{\kappa}{2}\rho\,. \qquad (7.4.16)$$

7.4.2 C^3 matching in Newtonian gravity

To illustrate the C^3 matching approach in Newtonian gravity, let us consider a spherically symmetric solution. Then, the corresponding exterior field should correspond to that of a sphere described by a solution of the Laplace equation with

$$\rho_{ext} = 0\,, \qquad \Phi_{ext} = -\frac{M}{r}, \qquad (7.4.17)$$

where M is a constant. It is then straightforward to calculate the curvature eigenvalues which turn out to be

$$\lambda_1^+ = -\lambda_4^+ = -\frac{2M}{r^3}, \;\; \lambda_2^+ = \lambda_3^+ = -\lambda_5^+ = -\lambda_6^+ = \frac{M}{r^3}. \qquad (7.4.18)$$

The condition

$$\left.\frac{d\lambda_i^+}{dr}\right|_{r_{match}} = 0 \qquad (7.4.19)$$

does not offer any positive finite root for the minimum matching radius, indicating that the matching can be performed at any radius $r_{match} \in (0, \infty)$. We proceed now to the second step and demand the equality of the eigenvalues across the matching surface, i.e.,

$$\lambda_i^- \Big|_{r_{match}} = \lambda_i^+ \Big|_{r_{match}} . \tag{7.4.20}$$

Using the expressions (7.4.13) for the eigenvalues of the interior solution, we obtain that at the matching radius the following conditions must be satisfied

$$\lambda_1^- = \frac{\kappa}{2}\rho - \frac{2}{r}\Phi_r = -\frac{2M}{r^3} , \quad \lambda_2^- = \frac{1}{r}\Phi_r = \frac{M}{r^3} , \quad \lambda_3^- = \frac{1}{r}\Phi_r = \frac{M}{r^3} , \tag{7.4.21}$$

$$\lambda_4^- = \frac{2}{r}\Phi_r = \frac{2M}{r^3} , \ \lambda_5^- = \frac{\kappa}{2}\rho - \frac{1}{r}\Phi_r = -\frac{M}{r^3} , \ \lambda_6^- = \frac{\kappa}{2}\rho - \frac{1}{r}\Phi_r = -\frac{M}{r^3} , \tag{7.4.22}$$

where we have used Poisson's equation (7.4.16) to replace the second derivative of Φ. The above equations represent a system of three independent algebraic equations from which we obtain that the only compatible solution is

$$\rho = 0 \quad \text{at} \quad r = r_{match} . \tag{7.4.23}$$

This condition is in agreement with our physical intuition as we expect that the density vanishes at the matching surface. Notice that to obtain this result, we used, on the one hand, Poisson's equation for the internal potential, without specifying any particular equation of state for the matter density ρ, i.e, no particular solution of Poisson's equation was involved in the proof. Consequently, this result is independent of the equation of state of the spherical mass distribution. On the other hand, for the matching with the external metric, we used the Newtonian potential of a sphere, which is described by a unique solution (7.4.17) of the Laplace equation. We conclude, therefore, that the C^3 matching condition (7.4.23) is valid, in general, for any spherically symmetric field in Newtonian gravity.

7.4.3 Particular solutions in Newtonian gravity

We now consider some solutions of the Poisson equation which determine interior Newtonian fields. Remember that a solution of the Poisson equation $\nabla^2\Phi = \frac{\kappa}{2}\rho$ depends on the specific form of the density function ρ. In general, the density is given *a priori* and the potential Φ is determined from the Poisson differential equation. Several particular solutions are known. For concreteness, let us consider the following solutions [70]

$$\rho_{HS} = \rho_0, \qquad \Phi_{HS} = -\frac{k\rho_0}{4}\left(a^2 - \frac{r^2}{3}\right), \tag{7.4.24}$$

$$\rho_P = \frac{6Mb^2}{k(r^2+b^2)^{5/2}}, \quad \Phi_P = -\frac{M}{(r^2+b^2)^{1/2}}, \tag{7.4.25}$$

$$\rho_{IP} = \frac{2Mb\left[3b^2 + 3b(b^2+r^2)^{1/2} + 2r^2\right]}{k(b^2+r^2)^{3/2}\left[b + (b^2+r^2)^{1/2}\right]^3}, \quad \Phi_{IP} = -\frac{M}{b + (b^2+r^2)^{1/2}}, \tag{7.4.26}$$

where ρ_0, a and b are constants. These solutions are known as the homogeneous sphere, Plummer model and isochrone potential, respectively. As we can see, in general, none of these solutions satisfies the matching condition $\rho = 0$. This means that strictly speaking none of them can be matched with the exterior solution of a sphere (7.4.17). However, to illustrate the validity of the matching procedure, consider, for instance, the eigenvalues of the Plummer model, which can be written as

$$\lambda_1^- = \frac{M(b^2-2r^2)}{(b^2+r^2)^{5/2}}, \tag{7.4.27}$$

$$\lambda_2^- = \lambda_3^- = \lambda_4^-/2 = \frac{M}{(b^2+r^2)^{3/2}}, \tag{7.4.28}$$

$$\lambda_5^- = \lambda_6^- = \frac{M(2b^2-r^2)}{(b^2+r^2)^{5/2}}.$$

A straightforward computation of the conditions $\lambda_i^+ = \lambda_i^-$ shows that the only possible solution is $b = 0$, which coincides with the matching condition $\rho_P = 0$. Moreover, from the expressions for the eigenvalues we see that

$$\lim_{r\to\infty} \lambda_i^- = \lambda_{Ei}^+, \tag{7.4.29}$$

indicating that the matching can be performed only at infinity. An analysis of the interior solutions for the homogeneous sphere and the isochrone potential leads to similar results. This corroborates the validity of the C^3 matching conditions in Newtonian gravity.

7.5 Spherically symmetric relativistic fields

For the investigation of relativistic fields, we consider the general spherically symmetric line element

$$ds^2 = -e^{\nu} dt^2 + e^{\phi} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{7.5.1}$$

where ν and ϕ depend on r only. It then follows that the corresponding orthonormal tetrad can be chosen as

$$\vartheta^0 = e^{\nu/2} dt \,, \quad \vartheta^1 = e^{\phi/2} dr \,, \quad \vartheta^2 = r d\theta \,, \quad \vartheta^3 = r \sin\theta d\varphi \,. \tag{7.5.2}$$

It is straightforward to compute the connection 1-form

$$\begin{aligned} \omega^1{}_2 &= -\frac{1}{r} e^{-\phi/2} \vartheta^2 \,, \quad \omega^1{}_3 = -\frac{1}{r} e^{-\phi/2} \vartheta^3 \,, \\ \omega^2{}_3 &= -\frac{1}{r} \cot\theta \vartheta^3 \,, \quad \omega^1{}_0 = -\frac{\nu_r}{2r} e^{-\phi/2} \vartheta^4 \,, \end{aligned} \tag{7.5.3}$$

and from here the curvature 2-form and the components of the curvature tensor, which can be expressed as

$$\begin{aligned} R_{0101} = \mathbf{R}_{11} = -\frac{1}{4}(\phi_r \nu_r - \nu_r^2 - 2\nu_{rr}) e^{-\phi} \,, \quad R_{0202} = \mathbf{R}_{22} = \frac{1}{2r} \nu_r e^{-\phi} \,, \\ R_{0303} = \mathbf{R}_{33} = \frac{1}{2r} \nu_r e^{-\phi} \,, \quad R_{2323} = \mathbf{R}_{44} = \frac{1}{r^2}(1 - e^{-\phi}) \,, \\ R_{3131} = \mathbf{R}_{55} = \frac{1}{2r} \phi_r e^{-\phi} \,, \quad R_{1212} = \mathbf{R}_{66} = \frac{1}{2r} \phi_r e^{-\phi} \,. \end{aligned} \tag{7.5.4}$$

We then obtain the following eigenvalues for the curvature tensor of an interior perfect fluid solution

$$\lambda_1^- = \mathbf{R}_{11}, \; \lambda_2^- = \mathbf{R}_{22}, \; \lambda_3^- = \mathbf{R}_{33}, \tag{7.5.5}$$

$$\lambda_4^- = -\lambda_1^- + \frac{k(\rho+p)}{2},\ \lambda_5^- = -\lambda_2^- + \frac{k(\rho+p)}{2},\ \lambda_6^- = -\lambda_3^- + \frac{k(\rho+p)}{2}. \quad (7.5.6)$$

Moreover, Einstein's equations can be expressed as

$$\nu_{rr} + \frac{1}{2}\nu_r^2 - \frac{\nu_r}{2r}(2 + r\phi_r) - \frac{\phi_r}{r} - \frac{2}{r^2}(1 - e^{\phi}) = 0\,, \quad (7.5.7)$$

$$\kappa\rho = \frac{1}{r^2}[1 + e^{-\phi}(r\phi_r - 1)],\ \ \kappa p = -\frac{1}{r^2}[1 - e^{-\phi}(1 + r\nu_r)]. \quad (7.5.8)$$

7.5.1 C^3 matching in general relativity

To proceed with the matching, we consider the Schwarzschild solution

$$\rho_S = p_S = 0\,,\ \ \phi_S = -\nu_S = -\ln(1 - 2M/r). \quad (7.5.9)$$

as the only available spherically symmetric vacuum solution. The eigenvalues are as follows:

$$\lambda_1^+ = -\lambda_4^+ = -2M/r^3,\ \lambda_2^+ = \lambda_3^+ = -\lambda_5^+ = -\lambda_6^+ = M/r^3. \quad (7.5.10)$$

The C^3 matching condition $d\lambda_i^+/dr = 0$ does not lead to any repulsion radius and so the matching can be carried out within the interval $r_{match} \in (0, \infty)$, resembling the situation in the case of Newtonian gravity. If, in addition, we demand that the exterior (7.5.9) and interior eigenvalues (7.5.5) coincide on the matching surface, we obtain that the only solution is

$$\rho = 0,\ p = 0 \quad \text{at} \quad r = r_{match}\,. \quad (7.5.11)$$

Again, this result is very consistent and corroborates in an invariant way our physical expectation of vanishing pressure and density on the matching surface. In the Newtonian limit, which at the level of the energy-momentum tensor is obtained by neglecting the pressure, we corroborate the result obtained above for Newtonian gravity that the density should vanish at the matching surface.

Notice that these conditions should be imposed on the equation of state, which determines the internal structure of the spherically symmetric mass distribution. This means that they are physical conditions to be satisfied by the interior solution. Usually, it is difficult to find analytic solutions to the

Einstein equations. If, in addition, we demand that the solutions be physical, the number of known solutions reduces dramatically. This seems to be the case of spherically symmetric solutions. Indeed, the physical conditions (7.5.11), which follow from the C^3 matching procedure, seem to discard a large number of known analytic solutions. In the Appendix, we include a series of spherically symmetric interior solutions which are known as Tolman I-$VIII$. For all the solutions of this class we computed the interior eigenvalues, which are also included in the Appendix. It is straightforward to show that none of these solutions can be matched with the exterior Schwarzschild metric.

7.6 Spherically symmetric interior solutions

In 1939, Tolman [71] carried out an exhaustive analysis of the Einstein field equations in the presence of a perfect fluid with spherical symmetry. The method consisted in giving *a priori* one of the unknowns, the density ρ for instance, and finding the remaining unknowns from the field equations. As a result, Tolman derived eight different solutions, which are known as the Tolman class. This class of exact analytic solutions has many features that make it interesting to describing physical cosmology and for modeling high density astronomical objects. In particular, the solutions IV, VI and VII have received great attention and have been generalized in different contexts of the general relativity (see [72, 73]).

The Tolman solutions can be represented by the metric functions $\nu(r)$ and $\phi(r)$ of the line element

$$ds^2 = -e^{\nu}dt^2 + e^{\phi}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{7.6.1}$$

and by the density ρ and pressure p of the perfect fluid. In this Appendix, we present the explicit form of all the eight interior solutions and calculate the corresponding curvature eigenvalues λ_i. The solutions and eigenvalues are denoted by a subscript in front of all the relevant quantities.

Tolman I: The Einstein universe

Solution:

$$\begin{gathered}\phi_{EU} = -\ln\left(1 - r^2/\kappa^2\right), \quad \nu_{EU} = 2\ln(c), \\ \rho_{EU} = \frac{3}{k\kappa^2}, \quad p_{EU} = -\frac{\rho_{EU}}{3},\end{gathered} \tag{7.6.2}$$

where c, k and κ are constants.

Eigenvalues:

$$\begin{aligned} \lambda_{EU_1} &= \lambda_{EU_2} = \lambda_{EU_3} = 0, \\ \lambda_{EU_4} &= \lambda_{EU_5} = \lambda_{EU_6} = 1/\kappa^2. \end{aligned} \tag{7.6.3}$$

Tolman II: The Schwarzschild-de Sitter solution

Solution:

$$\begin{aligned} \phi_{SdS} &= -\ln\left(1 - 2M/r - r^2/\kappa^2\right), \\ \nu_{SdS} &= \ln\left[c^2\left(1 - 2M/r - r^2/\kappa^2\right)\right], \\ \rho_{SdS} &= -p_{SdS} = \tfrac{3}{k\kappa^2}, \end{aligned} \tag{7.6.4}$$

where c, k and κ are constants.

Eigenvalues:

$$\lambda_{SdS_1} = -\lambda_{SdS_4} = -\tfrac{2\kappa^2 M + r^3}{\kappa^2 r^3}, \tag{7.6.5}$$

$$\lambda_{SdS_2} = \lambda_{SdS_3} = -\lambda_{SdS_5} = -\lambda_{SdS_6} = \tfrac{\kappa^2 M - r^3}{\kappa^2 r^3}. \tag{7.6.6}$$

Tolman III: Schwarzschild interior solution Solution:

$$\begin{aligned} \phi_{SI} &= -\ln\left[1 - 2Mr^2/\kappa^3\right], \\ \nu_{SI} &= 2\ln\left[\tfrac{3(1-2M/\kappa)^{1/2}}{2} - \tfrac{\left(1-2Mr^2/\kappa^3\right)^{1/2}}{2}\right], \\ \rho_{SI} &= \tfrac{6M}{k\kappa^3}, \quad p_{SI} = \tfrac{6M\left[(1-2Mr^2/\kappa^3)^{1/2} - (1-2M/\kappa)^{1/2}\right]}{k\kappa^3\left[3(1-2M/\kappa)^{1/2} - (1-2Mr^2/\kappa^3)^{1/2}\right]}, \end{aligned} \tag{7.6.7}$$

where k, κ and M are constants.

Eigenvalues:

$$\begin{aligned} \lambda_{SI_1} &= \lambda_{SI_2} = \lambda_{SI_3} = -\tfrac{2M(2Mr^2-k^3)^{1/2}}{\kappa^3\left[(2Mr^2-k^3)^{1/2} - 3\kappa(2M-\kappa)^{1/2}\right]}, \\ \lambda_{SI_4} &= \lambda_{SI_5} = \lambda_{SI_6} = 2M/\kappa^3. \end{aligned} \tag{7.6.8}$$

Tolman IV

Solution:

$$\phi_{IV} = \ln\left[\frac{1+2r^2/A^2}{(1-r^2/\kappa^2)(1+r^2/A^2)}\right], \ \nu_{IV} = \ln\left[B^2(1+r^2/A^2)\right], \tag{7.6.9}$$
$$\rho_{IV} = \frac{3A^2(A^2+\kappa^2)+(7A^2+2\kappa^2)r^2+6r^4}{k\kappa^2(A^2+2r^2)^2}, \ p_{IV} = \frac{\kappa^2-A^2-3r^2}{k\kappa^2(A^2+2r^2)}.$$

Eigenvalues:

$$\begin{aligned}
\lambda_{IV_1} &= \frac{(\kappa^2-2r^2)A^2-2r^4}{\kappa^2(A^2+2r^2)^2}, \\
\lambda_{IV_2} &= \lambda_{IV_3} = \frac{\kappa^2-r^2}{\kappa^2(A^2+2r^2)}, \\
\lambda_{IV_4} &= \frac{\kappa^2+A^2+r^2}{\kappa^2(A^2+2r^2)}, \\
\lambda_{IV_5} &= \lambda_{IV_6} = \frac{A^4+(\kappa^2+2r^2)A^2+2r^4}{\kappa^2(A^2+2r^2)^2}.
\end{aligned} \tag{7.6.10}$$

Tolman V

Solution:

$$\begin{aligned}
\phi_V &= \ln\left[\frac{1+2K-K^2}{1-(1+2K-K^2)(r/\kappa)^N}\right], \quad \nu_V = \ln\left(B^2r^{2K}\right), \\
\rho_V &= \frac{2K-K^2}{k(1+2K-K^2)r^2} + \frac{3+5K-2K^2}{k(1+K)\kappa^2}\left(\frac{r}{\kappa}\right)^M, \\
p_V &= \frac{K^2}{k(1+2K-K^2)r^2} - \frac{1+2K}{k\kappa^2}\left(\frac{r}{\kappa}\right)^M,
\end{aligned} \tag{7.6.11}$$

where $N = 2(1+2K-K^2)/(1+K)$ and $M = 2K(1-K)/(1+K)$.

Eigenvalues:

$$\begin{aligned}
\lambda_{V_1} &= -\frac{K\left[2K(K^2-2K-1)(r/\kappa)^N+K^2-1\right]}{(K+1)(K^2-2K-1)r^2}, \\
\lambda_{V_2} &= \lambda_{V_3} = -\frac{K\left[1+(K^2-2K-1)(r/\kappa)^N\right]}{(K^2-2K-1)r^2}, \\
\lambda_{V_4} &= \frac{(K^2-2K-1)(r/\kappa)^N+K^2-2K}{(K^2-2K-1)r^2}, \\
\lambda_{V_5} &= \lambda_{V_6} = \frac{N}{2r^2}\left(\frac{r}{\kappa}\right)^N.
\end{aligned} \tag{7.6.12}$$

Tolman VI

Solution:

$$\begin{array}{c} \phi_{VI} = \ln\left(2-K^2\right), \quad \nu_{VI} = 2\ln\left(Ar^{1-K} - Br^{1+K}\right), \\ \rho_{VI} = \frac{1-K^2}{8\pi(2-K^2)r^2}, \\ p_{VI} = \frac{(1-K)^2A-(1+K)^2Br^{2K}}{8\pi(2-K^2)(A-Br^{2K})r^2} - \frac{1+2K}{8\pi\kappa^2}\left(\frac{r}{\kappa}\right)^M, \end{array} \tag{7.6.13}$$

where A, B and K are arbitrary constants.

Eigenvalues:

$$\begin{aligned} \lambda_{VI_1} &= \frac{K\left[A(K-1)-B(K+1)r^{2K}\right]}{(K^2-2)(A-Br^{2K})r^2}, \\ \lambda_{VI_2} = \lambda_{VI_3} &= \frac{-A(K-1)-B(K+1)r^{2K}}{(K^2-2)(A-Br^{2K})r^2}, \\ \lambda_{VI_4} &= \frac{K^2-1}{(K^2-1)r^2}, \\ \lambda_{VI_5} &= \lambda_{VI_6} = 0\,. \end{aligned} \tag{7.6.14}$$

Tolman VII

Solution:

$$\begin{aligned} \phi_{VII} &= -\ln\left(1-r^2/\kappa^2+4r^4/A^4\right), \nu_{VII} = 2\ln\left[B\sin\left(\ln T^{1/2}\right)\right], \\ \rho_{VII} &= \frac{3A^2-20\kappa^2r^2}{8\pi\kappa^2A^2} \end{aligned} \tag{7.6.15}$$

$$p_{VII} = \frac{B\cot\left(\ln T^{1/2}\right)-A^2(A^4-4\kappa^2r^2)\left(1-r^2/\kappa^2+4r^4/A^4\right)^{1/2}}{8\pi\kappa^2A^6\left(1-r^2/\kappa^2+4r^4/A^4\right)^{1/2}} \tag{7.6.16}$$

where $B \equiv \left[4\kappa^2A^4 - 4r^2(A^4-4\kappa^2r^2)\right]$, T is given by

$$c\,T \equiv \left(1-r^2/\kappa^2+4r^4/A^4\right)^{1/2} + 2r^2/A^2 - A^2/(4\kappa^2)$$

and c, κ and A are arbitrary constants.

Eigenvalues:

$$
\begin{aligned}
4\lambda_{VII_1} &= (1 - r^2/\kappa^2 + 4r^4/A^4) \\
&\quad \times \left[\tfrac{\cot(\ln T^{1/2})}{T}(FT_r + 2T_{rr}) - \tfrac{T_r^2}{T^2}\left(2\cot(\ln T^{1/2}) + 1\right)\right] \;, \quad (7.6.17) \\
F &\equiv \tfrac{-2r/\kappa^2 + 16r^3/A^4}{1 - r^2/\kappa^2 + 4r^4/A^4}, \\
\lambda_{VII_2} &= \lambda_{VII_3} = -(1 - r^2/\kappa^2 + 4r^4/A^4)\tfrac{T_r \cot(\ln T^{1/2})}{2rT} \;, \\
\lambda_{VII_4} &= 1/\kappa^2 - 4r^2/A^4 \;, \\
\lambda_{VII_5} &= \lambda_{VII_6} = 1/\kappa^2 - 8r^2/A^4 \;.
\end{aligned}
$$

Here T_r indicates derivative with respect to r.

Tolman VIII Solution:

$$
\begin{aligned}
\phi_{VIII} &= -\ln\left[\tfrac{2}{(a-b)(a+2b-1)} - \left(\tfrac{2M}{r}\right)^{a+2b-1} - \left(\tfrac{r}{\kappa}\right)^{a-b}\right] \;, \quad (7.6.18) \\
\nu_{VIII} &= \ln\left(B^2 r^{2b}\right) - \phi_{VIII} \;, \\
8\pi r^2 \rho_{VIII} &= 1 - \tfrac{2}{(a-b)(a+2b-1)} - (a + 2b - 2)\left(\tfrac{2M}{r}\right)^{a+2b-1} \\
&\quad + (a - b + 1)\left(\tfrac{r}{\kappa}\right)^{a-b} \;,
\end{aligned}
$$

$$
\begin{aligned}
p_{VIII} &= \Big[(a-2)(a+2b-1)(a-b)\left(\tfrac{2M}{r}\right)^{a+2b-1} \\
&\quad -(a+b+1)(a+2b-1)(a-b)\left(\tfrac{r}{\kappa}\right)^{a-b} \\
&\quad -a^2 + (1-b)a + 2b^2 + 3b + 2\Big/\Big[8\pi(a-b)(a+2b-1)r^2\Big] \;,
\end{aligned}
$$

where a, b and M are constants and $b \equiv (a^2 - a - 2)/(3 - a)$

Eigenvalues:

$$\lambda_{VIII_1}/D + 4b^2 + 4b = (a+2b-1)(a^2-b^2)(a-1) \times \left[\left(\frac{2M}{r}\right)^{a+2b-1} + \left(\frac{r}{\kappa}\right)^{a-b} \right] , \tag{7.6.19}$$

$$\lambda_{VIII_2}/D + 4b = (a+2b-1)(a-b) \times \left[(1-a)\left(\frac{2M}{r}\right)^{a+2b-1} + (a+b)\left(\frac{r}{\kappa}\right)^{a-b} \right] ,$$

$$\lambda_{VIII_3} = \lambda_{VIII_2} ,$$

$$\lambda_{VIII_4}/2D + 2 = (a+2b-1)(a-b) \times \left[\left(\frac{2M}{r}\right)^{a+2b-1} \left(\frac{r}{\kappa}\right)^{a-b} + 1 \right] ,$$

$$\lambda_{VIII_5/D} = \lambda_{VIII_6} = (b-a)\left(\frac{2M}{r}\right)^{a+2b-1} + (a+2b-1)\left(\frac{r}{\kappa}\right)^{a-b} ,$$

where

$$D \equiv \frac{1}{2(a-b)(a+2b-1)r^2} .$$

7.7 Conclusions

In this work, we propose a new method for matching two spacetimes in general relativity. We demand that the curvature eigenvalues of the interior and exterior solutions be continuous across the matching surface. To determine the matching surface, we assume that the exterior spacetime is asymptotically flat and consider the behavior of the corresponding eigenvalues as the source is approached. A monotonous growth of the eigenvalues is interpreted as corresponding to the presence of attractive gravitational interaction throughout the entire space. On the contrary, if an eigenvalue shows local extrema and even changes its sign as the source is approached, we interpret this behavior as due to the presence of repulsive gravity. The repulsion radius is defined by the location of the first extremum (C^3 condition), which appears as the source is approached from spatial infinity. In turn, the repulsion radius is defined as the minimum radius, where the matching can be carried out, i.e., the matching surface can be located anywhere between the repulsion radius and infinity. This means that the goal of fixing a minimum radius for the matching surface is to avoid the presence of repulsive gravity because it has not been detected at least in the gravitational field of compact objects.

We tested the C^3 matching procedure in the case of spherically symmetric perfect fluid spacetimes in Newtonian gravity and in general relativity. Remarkably, our method leads to completely general results, independently of any particular solution of the field equations. In the case of Newtonian gravity, we obtain that the matter density ρ of the gravitational source should vanish on the matching surface. This result is valid in general because to obtain it, we used, on the one hand, Poisson's equation for the internal potential, without specifying any particular form for the matter density ρ. Consequently, this result is independent of the equation of state of the spherical mass distribution. On the other hand, for the matching with the external metric, we used the Newtonian potential of a sphere, which is described by a unique solution of the Laplace equation. This means that the C^3 matching condition $\rho = 0$ on the matching surface is valid for any spherically symmetric field in Newtonian gravity. In the case of general relativity, we applied a similar procedure, in which only the interior Einstein field equations and the exterior Schwarzschild metric are involved in the analysis of the C^3 matching conditions, and obtained that the matter density and the pressure should vanish on the matching surface. Again, this result is independent of the equation of state, which determines the internal structure of the spherical mass distribution.These conditions are very plausible from a physical point of view and, therefore, establish the validity of the C^3 matching. We analyzed several particular examples of well-known spherically symmetric perfect fluid solutions and found out that, in general, it is not possible to satisfy the C^3 matching conditions. This result indicates that to obtain physically meaningful interior solutions, it would be convenient to start from metrics, satisfying the matching conditions *a priori* and containing arbitrary functions that are then determined by the field equations.

In this work, we limited ourselves to the study of spherically symmetric solutions so that the matching surface is easily identified as a sphere. However, it is possible to apply the C^3 matching method to the case of axially symmetric spacetimes, which are more realistic as models for describing the gravitational field of astrophysical compact objects. Preliminary results show that in the case of metrics with quadrupolar moment, the repulsion radius depends on the angular coordinate so that the matching surface is different from an ideal sphere. In this case, the C^3 matching implies a detailed numerical analysis of the curvature eigenvalues. This work is in progress and will be presented elsewhere.

As presented here, the C^3 matching method has been specially adapted for the study of asymptotically flat spacetimes, which can be used to describe the gravitational field of astrophysical compact objects. However, it is also possible to consider other physical situations in which, for instance, cosmological models or collapsing shells are to be matched. We expect to investigate this type of configurations in future works.

The generality of the results obtained in this work points out to the possibility of applying the C^3 matching procedure in other theories and scenarios. For instance, matching conditions are important in the context of local strings, domail walls, braneworld scenarios, etc. We will analyze these problems in other works.

The matching problem in general relativity and other theories can also be investigated by using the distributional calculus techniques. In particular, Israel's matching conditions of general relativity can be formulated in a very elegant way by using these techniques (see, for instance, [74]). Israel's conditions can be understood as a C^2 matching procedure because they involve only second order derivatives of the metric. The disadvantage of this method is that it can be applied only once the matching surface is known. On the other hand, the C^3 method we propose here can be used to find the matching surface by imposing the condition that no repulsive gravity exists in the gravitational field of compact objects. However, our approach is different and cannot be formulated with the elegant method of distributional calculus. Indeed, we impose that the curvature eigenvalues of the interior and exterior metrics coincide on the matching surface and, therefore, in the case of no coincidence our method does not apply. This is a disadvantage of our method. However, in the case of no coincidence it seems reasonable to try to apply distributional calculus to handle correctly the jump in the eigenvalues. We expect to consider this problem in future works.

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