# Exact solutions of Einstein and Einstein-Maxwell equations

## 0.1 Topics

- Exact solutions in General Relativity and Supergravity
- Cosmology and Astrophysics
- Quantum Fields
- Fundamental Relativity

## 0.2 Participants

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## 0.4 Research activity

### 0.4.1 Exact solutions in General Relativity and Supergravity

• In 2014 ICRANet started the new program "Exact solutions in the supersymmetric General Relativity" in collaboration with the group of Prof. Hermann Nicolai at Albert Einstein Institute at Potsdam (Germany). In 2017 in the framework of this program the new work has been done dedicated to the exactly integrable models in supergravity. It was continued the general way of extension of the pure gravity inverse scattering integration technique

to the case when fermions (introduced on the base of supersymmetry) are present. In this year the integrability technique for simple (N=1) supergravity in two space-time dimensions coupled to the matter fields taking values in the Lie algebra of E8(+8) group was developed. This theory contains matter living only in one Weyl representation of SO (16) and represents the reduction to two dimensions of the three-dimensional simple supergravity constructed earlier by H. Nishino and S. Rajpoot (2002). The proposed spectral linear problem use superspace and covers the complete set of principal bosonic and fermionic equations of motion. This linear system, as in pure gravity, contains only the first order poles with respect to the spectral parameter. The procedure of constructing the exact super-solitonic solutions is outlined. Reference (1).

• The classes of electrovacuum Einstein - Maxwell fields (with a cosmological constant), which metrics admit an Abelian two-dimensional isometry group G2 with non-null orbits are considered. For the fields with such symmetries, we describe the structures of the so called "non-dynamical degrees of freedom" which presence as well as the presence of a cosmological constant change (in a strikingly similar ways) the vacuum and electrovacuum dynamical equations and destroy their well known integrable structures. The modifications of the known reduced forms of Einstein - Maxwell equations – the Ernst equations and self-dual Kinnersley equations which take into account the presence of non-dynamical degrees of freedom are found and the subclasses of fields with different non-dynamical degrees of freedom are considered. These are: (I) vacuum metrics with cosmological constant, (II) space-time geometries in vacuum with isometry groups G2 which orbits do not admit the orthogonal 2-surfaces (none-orthogonally-transitive isometry groups) and (III) electrovacuum fields with more general structures of electromagnetic fields than in the known integrable cases. For each of these classes of fields, in the case of diagonal metrics, all field equations can be reduced to the only nonlinear equation of the fourth order for one real function which characterise the element of area on the orbits of the isometry group G2 . Simple examples of solutions for each of these classes are presented. It is pointed out that if for some two-dimensional reduction of Einstein's field equations in four or higher dimensions, the aforementioned function possess a "harmonic" structure, instead of being (together with other field variables) a solution of some nonlinear equations, this can be an indication of possible complete integrability of these reduced dynamical equations for the fields with vanishing of all non-dynamical degrees of freedom. Reference (2).

## 0.4.2 Cosmology and Astrophysics

### Cosmology

• During the year 2017 the book "The cosmological Singularity" (Cambridge University Press) by V.Belinski and M.Henneaux has been completed. The book have been published 26 October 2017 and now is available at ICRANet.

Abstract of the book. Written for researchers focusing on general relativity, supergravity, and cosmology, this is a self-contained exposition of the structure of the cosmological singularity in generic solutions of the Einstein equations, and an up-to-date mathematical derivation of the theory underlying the Belinski–Khalatnikov–Lifshitz (BKL) conjecture on this field. Part I provides a comprehensive review of the theory underlying the BKL conjecture. The generic asymptotic behavior near the cosmological singularity of the gravitational field, and fields describing other kinds of matter, is explained in detail. Part II focuses on the billiard reformulation of the BKL behavior. Taking a general approach, this section does not assume any simplifying symmetry conditions and applies to theories involving a range of matter fields and space-time dimensions, including supergravities. Overall, this book will equip theoretical and mathematical physicists with the theoretical fundamentals of the Big Bang, Big Crunch, Black Hole singularities, the billiard description, and emergent mathematical structures. See Reference (3).

• It was proposed a new alternative (with respect to the accelerated universe paradigm) explanation of the discrepancy between values of the distances to the far galaxies following from the observations and from the standard Friedmann model. Observations show that these distances are a little bit larger in comparison with what is predicted by the usual Friedmann cosmology. However, this standard theory does not take into account traces the strong gravitational waves of cosmological origin leave in space. We show that such traces can be a cause for the aforementioned discrepancy. The sources of cosmological waves are inhomogeneities of a solitonic type of the gravitational field near the Big Bang. Due to expansion of the universe these inhomogeneities decay but each of them expels solitonic gravitational waves which also decay in course of propagation through the expanding space transfering, however, their energies to the Friedmann background making the distances different compared with those which would be observed without such waves. This effect has been described earlier (V. Belin-

ski, 1979) by example of single-soliton cylindrical wave propagating on the Friedmann background. Now the same phenomenon has been confirmed for double-soliton waves both for cylindrical and planar symmetries. Reference (4).

• Usingthe formalism of geometrothermodynamics to derive a fundamental thermodynamic equation, we construct a cosmological model in the framework of relativistic cosmology. In a first step, we describe a system without thermodynamic interaction, and show it to be equivalent to the standard LambdaCDM paradigm. The second step includes thermodynamic interaction and produces a model consistent with the main features of inflation. With the proposed fundamental equation we are thus able to describe all the known epochs in the evolution of our Universe, starting from the inflationary phase. Reference (9).

• It was developed a simple model for a study of classical fields in the background of a fluctuating spacetime volume. It is applied for a formulation of the Einstein equations with a perfect-fluid source. We investigate the particular case of a Friedmann-LemaItre-Robertson-Walker cosmology and show that the resulting field equations can lead to solutions which avoid the initial Big Bang singularity. By interpreting the fluctuations as a result of the presence of quantum spacetime, we conclude that classical singularities can be avoided even within a semiclassical model that includes quantum effects in a very simple manner. Reference (10).

### Astrophysics

• It was investigated the equilibrium configurations of uniformly rotating white dwarfs, using Chandrasekhar and Salpeter equations of state in the framework of Newtonian physics. The Hartle formalism is applied to integrate the field equation together with the hydrostatic equilibrium condition. We consider the equations of structure up to the second order in the angular velocity, and compute all basic parameters of rotating white dwarfs to test the so-called moment of inertia, rotational Love number, and quadrupole moment (I-Love-Q) relations. We found that the I-Love-Q relations are also valid for white dwarfs regardless of the equation of state and nuclear composition. In addition, we show that the moment of inertia, quadrupole moment, and eccentricity (I-Q-e) relations are valid as well. Reference (6).

• It was investigated charged particles circular motion in the gravitational field of a charged mass distribution described by the Reissner-Nordstrom

spacetime. We introduce a set of independent parameters completely characterizing the different spatial regions in which circular motion is allowed. We provide a most complete classification of circular orbits for different sets of particle and source charge-to-mass ratios. We study both black holes and naked singularities and show that the behavior of charged particles depend drastically on the type of source. Our analysis shows in an alternative manner that the behavior of circular orbits can in principle be used to distinguish between black holes and naked singularities. These results play an important role in the study of the coupled electromagnetic and gravitational interactions, and the investigation of the role of the charge in the gravitational collapse of compact objects. Reference (8).

• A novel approach for extracting gauge-invariant information about spin-orbit coupling in gravitationally interacting binary systems was introduced. This approach is based on the "scattering holonomy", i.e. the integration (from the infinite past to the infinite future) of the differential spin evolution along the two worldlines of a binary system in hyperbolic-like motion. We apply this approach to the computation, at the first post-Minkowskian approximation (i.e. first order in G and all orders in $v/c$), of the values of the two gyrogravitomagnetic ratios describing spin-orbit coupling in the Effective One-Body formalism. These gyrogravitomagnetic ratios are found to tend to zero in the ultrarelativistic limit. References (11).

• It was computed the (center-of-mass frame) scattering angle of hyperbolic-like encounters of two spinning black holes, at the fourth post-Newtonian approximation level for orbital effects, and at the next-to-next-to-leading order for spin-dependent effects. We find it convenient to compute the gauge-invariant scattering angle (expressed as a function of energy, orbital angular momentum and spins) by using the Effective-One-Body formalism. The contribution to scattering associated with nonlocal, tail effects is computed by generalizing to the case of unbound motions the method of timelocalization of the action introduced in the case of (small-eccentricity) bound motions by Damour, Jaranowski and Schafer [Phys. Rev. D 91, 084024 (2015)]. Reference (12).

• It was presented an analytical gravitational self-force calculation of the spin-orbit precession along an eccentric orbit around a Schwarzschild black hole, following closely the recent prescription of Akcay, Dempsey, and Dolan. We then transcribe this quantity within the Effective One-Body (EOB) formalism, thereby determining several new, linear-in-mass-ratio, contributions in the post-Newtonian expansion of the spin-orbit couplings entering the EOB

Hamiltonian. Reference (13).

### 0.4.3 Quantum Fields

• It was calculated the Chern numbers of SU(2)-homogeneous Einstein-Maxwell gravitational instantons with boundary at infinity. By restating these numbers as Chern-Simons invariants on the boundary apparent conflicting results emerge. We resolved this issue examining the topological stability of the self-gravitating Abelian fields. No quantization carrying physical meaning is found when the background is a Taub-NUT space. However the magnetic charge of dyons on Taub-Bolt spaces is found to be of topological quantum nature. In this framework electric charge is quantized by a consistency condition. Reference (5).

• It was studied the quantum fermionic billiard defined by the dynamics of a quantized supersymmetric squashed three-sphere (Bianchi IX cosmological model within D = 4 simple supergravity). The quantization of the homogeneous gravitino field leads to a 64-dimensional fermionic Hilbert space. We focus on the 15 - and 20 - dimensional subspaces (with fermion numbers NF = 2 and NF =3) where there exist propagating solutions of the supersymmetry constraints that carry (in the small-wavelength limit) a chaotic spinorial dynamics generalizing the Belinskii-Khalatnikov-Lifshitz classical "oscillatory" dynamics. By exactly solving the supersymmetry constraints near each one of the three dominant potential walls underlying the latter chaotic billiard dynamics, we compute the three operators that describe the corresponding three potential-wall reflections of the spinorial state describing, in supergravity, the quantum evolution of the universe. It is remarkably found that the latter, purely dynamically-defined, reflection operators satisfy generalized Coxeter relations which define a type of spinorial extension of the Weyl group of the rank-3 hyperbolic Kac-Moody algebra AE3. Reference (14).

• It was computed a subset of three, velocity-independent four-loop (and fourth post-Newtonian) contributions to the harmonic-coordinates effective action of a gravitationally interacting system of two point-masses. We found that, after summing the three terms, the coeffcient of the total contribution is rational. This result, obtained by a classical field-theory calculation, corrects the recent effective-field-theory-based calculation by Fo et al. [arXiv:1612.00482]. Besides showing the usefulness of the saddle-point approach to the evaluation of the effective action, and of x-space computations, our result brings a

further confirmation of the current knowledge of the fourth post-Newtonian effective action. We also show how the use of the generalized Riesz formula [Phys. Rev. D 57, 7274 (1998)] allows one to analytically compute a certain four-loop scalar master integral (represented by a four-spoked wheel diagram) which was, so far, only numerically computed. Reference (15).

### 0.4.4 Fundamental Relativity

• It was proposed a classification of thermodynamic systems in terms of the homogeneity properties of their fundamental equations. Ordinary systems correspond to homogeneous functions and non-ordinary systems are given by generalized homogeneous functions. This affects the explicit form of the Gibbs-Duhem relation and E uler's identity. We show that these generalized relations can be implemented in the formalism of black hole geometrothermodynamics in order to completely fix the arbitrariness present in Legendre invariant metrics. Reference (7).

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