Symmetries in General Relativity

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3. Brief description

3.1. Spacetime splitting techniques in General Relativity

Spacetime splitting techniques play a central role and have fundamental interest in general relativity in view of extracting from the unified notion of spacetime the separate classical notions of space and time, at the foundation of all of our experience and intuition. Studying all the existing different approaches scattered in the literature has allowed the creation a unique framework encompassing all of them [1] and a more clear geometrical interpretation of the underlying "measurement process" for tensors and tensorial equations. "Gravitoelectromagnetism" is a convenient name for this framework because it helps explain the close relation between gravity and electromagnetism represented by the Coriolis and centrifugal forces on one side and the Lorentz force on the other side.

3.1.1. "1+3" splitting of the spacetime

During the last century, the various relativistic schools: Zelmanov, Landau, Lifshitz and the Russian school, Lichnerowicz in France, the British school, the Italian school (Cattaneo and Ferrarese), scattered Europeans (Ehlers and Trautman, for example) and the Americans (Wheeler, Misner, etc.), developed a number of different independent approaches to spacetime splitting almost without reference to each other.

R. Ruffini [2], a former student of Cattaneo and a collaborator of Wheeler, looking for a better understanding of black holes and their electromagnetic properties, stimulated Jantzen, Carini and Bini to approach the problem and to make an effort to clarify the interrelationships between these various approaches as well as to shed some light on the then confusing works of Abramowicz and others on relativistic centrifugal and Coriolis forces. By putting

them all in a common framework and clarifying the related geometrical aspects, some order was brought to the field [1, 3, 4].

3.1.2. Measurement process in general relativity

The investigations on the underlying geometrical structure of any spacetime splitting approach show that it is not relevant to ask which of these various splitting formalisms is the "best" or "correct" one, but to instead ask what exactly each one of them "measures" and which is specially suited to a particular application.

For instance, in certain situations a given approach can be more suitable than another to provide intuition about or simplify the presentation of the invariant spacetime geometry, even if all of them may always be used. These ideas were then used to try to understand better the geometry of circular orbits in stationary spacetimes and their physical properties where the connection between general relativity and its Newtonian progenitor are most natural.

The list of problems addressed and results obtained together can be found in Appendix A.

3.2. Motion of particles and extended bodies in general relativity

The features of test particle motion along a given orbit strongly depend on the nature of the background spacetime as well as on the model adopted for the description of the intrinsic properties of the particle itself (e.g., its charge or spin). As a basic assumption, the dimensions of the test particle are supposed to be very small compared with the characteristic length of the background field in such a way that the background metric is not modified by the presence of the particle (i.e., the back reaction is neglected), and that the gravitational radiation emitted by the particle in its motion is negligible. The particle can in turn be thought as a small extended body described by its own energy-momentum tensor, whose motion in a given background may be studied by treating the body via a multipole expansion. Thus, a single-pole particle is a test particle without any internal structure; a pole-dipole particle instead is a test particle whose internal structure is expressed by its spin, and so on. The

equations of motion are then obtained by applying Einsteins field equations together with conservation of the energy-momentum tensor describing the body. For a single-pole particle this leads to a free particle moving along the geodesics associated with the given background geometry. The motion of a pole-dipole particle is instead described by the Mathisson-Papapetrou-Dixon equations which couple background curvature and the spin tensor of the field. The motion of particles with an additional quadrupolar structure has been developed mostly by Dixon; because of its complexity, there are very few applications in the literature. Finally, the discussion of the case in which the test particle also has charge in addition to spin or mass quadrupole moment is due to Dixon and Souriau and this situation has been very poorly studied as well.

A complete list of the original results obtained and a deeper introduction to the models can be found in Appendix B.

3.2.1. Test particles

Since the 1990s we have been investigating the geometrical as well as physical properties of circular orbits in black hole spacetimes, selecting a number of special orbits for various reasons. These were already reviewed in a previous ICRANet report on activities. A recent work has instead been to consider a given gravitational background a (weak) radiation field superposed on it and a test particle interacting with both fields. Interesting effects arise like the Poynting-Robertson effect which have been considered in the framework of the full general relativistic theory for the first time.

Poynting-Robertson effect can be briefly described as follows.

For a small body orbiting a star, the radiation pressure of the light emitted by the star in addition to the direct effect of the outward radial force exerts a drag force on the body's motion which causes it to fall into the star unless the body is so small that the radiation pressure pushes it away from the star. Called the Poynting-Robertson effect, it was first investigated by J.H. Poynting in 1903 using Newtonian gravity and then later calculated in linearized general relativity by H.P. Robertson in 1937. These calculations were revisited by Wyatt and Whipple in 1950 for applications to meteor orbits, making more explicit Robertson's calculations for slowly evolving elliptical orbits and slightly extending them.

The drag force is easily naively understood as an aberration effect: if the

body is in a circular orbit, for example, the radiation pressure is radially outward from the star, but in the rest frame of the body, the radiation appears to be coming from a direction slightly towards its own direction of motion, and hence a backwards component of force is exerted on the body which acts as a drag force. If the drag force dominates the outward radial force, the body falls into the star. For the case in which a body is momentarily at rest, a critical luminosity similar to the Eddington limit for a star exists at which the inward gravitational force balances the outward radiation force, a critical value separating radial infall from radial escape. Similarly for a body initially in a circular orbit, there are two kinds of solutions: those in which the body spirals inward or spirals outward, depending on the strength of the radiation pressure.

We have considered [156, 162, 159] this problem in the context of a test body in orbit in a spherically symmetric Schwarzschild spacetime without the restriction of slow motion, and then in the larger context of an axially symmetric Kerr spacetime while developing the equations for a more general stationary axially symmetric spacetime. The finite size of the radiating body is ignored.

We have also developed applications to cylindrically symmetric Weyl class spacetimes (exhibiting a typical naked singularity structure) as well as to the Vaidya radiating spacetime where the photon field is not a test field but the source of the spacetime itself.

3.2.2. Spinning test particles

During the last five years we have investigated the motion of spinning test particles along special orbits in various spacetimes of astrophysical interest: black hole spacetimes as well as more "exotic" background fields representing naked singularities or the superposition of two or more axially symmetric bodies kept apart in a stable configuration by gravitationally inert singular structures.

In particular, we have focused on the so called "clock effect," defined by the difference in the arrival times between two massive particles (as well as photons) orbiting around a gravitating source in opposite directions after one complete loop with respect to a given observer [5, 6, 7].

We have also analyzed the motion of massless spinning test particles, according to an extended version of the Mathisson-Papapetrou model in a gen-

eral vacuum algebraically special spacetime using the Newman-Penrose formalism in the special case in which the multipole reduction world line is aligned with a principal null direction of the spacetime. Recent applications concern instead the study of the Poynting-Robertson effect for spinning particles.

3.2.3. Particles with both dipolar and quadrupolar structure (Dixon's model)

We have studied the motion of particles with both dipolar and quadrupolar structure in several different gravitational backgrounds (including Schwarzschild, Kerr, weak and strong gravitational waves, etc.) following Dixon's model and within certain restrictions (constant frame components for the spin and the quadrupole tensor, center of mass moving along a circular orbit, etc.).

We have found a number of interesting situations in which deviations from geodesic motion due to the internal structure of the particle can give rise to measurable effects.

3.2.4. Exact solutions representing extended bodies with quadrupolar structure

We have investigated geometrical as well as physical properties of exact solutions of Einstein's field equations representing extended bodies with structure up to the quadrupole mass moment, generalizing so the familiar black hole spacetimes of Schwarzschild and Kerr.

Recent results involve the use of the equivalence principle to compare geodesic motion in these spacetimes with nongeodesic motion of structured particles in Schwarzschild and Kerr spacetimes, allowing an interesting analysis which strongly support Dixon's model.

3.3. Perturbations

A discussion of curvature perturbations of black holes needs many different approaches and mathematical tools. For example, the Newman-Penrose formalism in the tetradic and spinor version, the Cahen-Debever-Defrise self-

dual theory, the properties of the spin-weighted angular harmonics, with particular attention to the related differential geometry and the group theory, some tools of complex analysis, etc. Furthermore, even using any of the above mentioned approaches, this remains a difficult problem to handle. It is not by chance, for instance, that the gravitational and electromagnetic perturbations of the Kerr-Newman rotating and charged black hole still represent an open problem in general relativity.

During the last years, however, modern computers and software have reached an exceptional computational level and one may re-visit some of these still open problems, where technical difficulties stopped the research in the past. Details can be found in Appendix C.

3.3.1. Curvature and metric perturbations in algebraically special spacetimes

Most of the work done when studying perturbations in General Relativity concerns curvature perturbations from one side or metric perturbations from the other side. In the first case, one can easily deal with gauge invariant quantities but the problem of finding frame-independent objects arises. Furthermore, the reconstruction of the metric once the curvature perturbations are known is a very difficult task. In the second case, instead, in order to start working with an explicit metric since the beginning, the choice of a gauge condition is necessary. Gauge independent quantities should therefore be determined properly.

There exist very few examples of works considering both cases of curvature and metric perturbations on the same level so that we have been motivated to start working in this direction.

3.3.2. Curvature perturbations in type D spacetimes

In the Kerr spacetime Teukolsky [8] has given a single "master equation" to deal with curvature perturbations by a field of any spin ("spin-weight," properly speaking). Then the problem of extending the results of Teukolsky to other spacetimes is raised.

Actually, a very important result that we have obtained framed the Teukolsky equation in the form of a linearized de-Rham Laplacian equation for the perturbing field [9, 10]. In addition, in all the cases (type D spacetime:

Taub-NUT, type D-Kasner, etc) in which an equation similar to the Teukolsky equation can be written down, one can study the various couplings between the spin of the perturbing field and the background parameters, i.e., spin-rotation, spin-acceleration couplings, etc., which can also be relevant in different contexts and from other points of view. We have obtained important results considering explicit applications to the Taub-NUT, Kerr-Taub-NUT, C-metric, spinning C-metric, Kasner and de Sitter spacetimes. For example in the Taub-NUT spacetime we have shown that the perturbing field acquires an effective spin which is simply related to the gravitomagnetic monopole parameter ℓ of the background [11]; in the C-metric case (uniformly accelerated black hole spacetime) we have been able to introduce a gravitational analog of the Stark effect, etc.

3.3.3. Metric perturbations in a Reissner-Nordström spacetime

A few years ago we solved the multiyear problem of a two-body system consisting of a ReissnerNordström black hole and a charged massive particle at rest at the first perturbative order. The expressions for the metric and the electromagnetic field, including the effects of the electromagnetically induced gravitational perturbation and of the gravitationally induced electromagnetic perturbation, have been presented in closed analytic formulas; the details are indicated in Appendix C. Motivated by our works an exact solution has then been found by Belinski and Alekseev [149] of which our solution is a linearization with respect to certain parameters.

3.3.4. Curvature perturbations due to spinning bodies on a Kerr background

A new scheme for computing dynamical evolutions and gravitational radiations for intermediate-mass-ratio inspirals (IMRIs) based on an effective one-body (EOB) dynamics plus Teukolsky perturbation theory has been recently derived by Wen-Biao Han and collaborators [169]. This research line is very promising in view of many possible applications ranging from the Post-Newtonian physics of binary systems to numerical relativity.

3.3.5. Metric perturbations due to spinning bodies on a Schwarzschild background

The full reconstruction of the perturbed metric by a pinning particle moving on a Schwarzschild background is possibile following the original Zerilli and Ruffini approach, at least perturbatively at various Post-Newtonian orders. This research project is expected to add corrections due to spin to the relativistic two body problem within the effective one-body formalism introduced by Damour.

3.3.6. Gravitational Self-force and Effective-One-Body model: synergies

In recent years, it has been understood that a useful strategy for studying the strong-field aspects of the dynamics of compact binaries is to combine, in a synergetic manner, information gathered from several different approximation methods, namely: the post-Newtonian (PN) formalism, the post-Minkowskian one, the gravitational self-force (SF) formalism, full numerical relativity simulations, and, the effective one-body (EOB) formalism. In particular, the EOB formalism appears to define a useful framework which can combine, in an efficient and accurate manner, information coming from all the other approximation schemes, while also adding genuinely new information coming from EOB theory.

A main motivation to pursue this synergetic effort is certainly the current development of gravitational wave detectors gives, which makes it urgent to improve our theoretical understanding of the general relativistic dynamics of compact binary systems, i.e., systems comprising black holes and/or neutron stars. Recent work has shown that tidal interactions have a significant influence on the late dynamics of coalescing neutron star binaries.

3.4. Cosmology

3.4.1. Mixmaster universe and the spectral index

We have recently revisited the Mixmaster dynamics in a new light, revealing a series of transitions in the complex scale invariant scalar invariant of the Weyl curvature tensor best represented by the speciality index S, which

gives a 4-dimensional measure of the evolution of the spacetime independent of all the 3-dimensional gauge-dependent variables except the time used to parametrize it.

Its graph versus time with typical spikes in its real and imaginary parts corresponding to curvature wall collisions serves as a sort of electrocardiogram of the Mixmaster universe, with each such spike pair arising from a single circuit or "pulse" around the origin in the complex plane. These pulses in the speciality index seem to invariantly characterize some of the so called spike solutions in inhomogeneous cosmology and should play an important role in the current investigations of inhomogeneous Mixmaster dynamics.

3.4.2. Wave equations in de Sitter spacetime

Wave propagation on a de Sitter background spacetime can be considered for both the electromagnetic and the gravitational case under the preliminar choice of a gauge conditions. Usually, even in the recent literature, the discussion is limited to special gauge conditions only, like the harmonic one. Recently, some interest has been raised instead for the development of a systematic study in terms of the de Donder gauge since this is close to the Lorentz gauge of the electromagnetic case. Due to the particular symmetries of the de Sitter spacetime we expect to be able to reconstruct the perturbed metric by the wave propagation, at least in the PN scheme.

3.4.3. Fluids obeying non-ideal equations of state

Recently, we have proposed a new class of cosmological models consisting of a FRW universe with a fluid source obeying a non-ideal equation of state, with the suitable property to support a phase transition between low and high density regimes, both characterized by an ideal gas behavior, i.e., pressure and density change in linear proportion to each other. This kind of equation of state was first introduced by Shan and Chen [177] in the context of lattice kinetic theory. We have first investigated the possibility to explain the growth of the dark energy component of the present universe, as a natural consequence of the fluid evolution equations. We have then developed an inflationary model based on a dark energy field described by a Shan-Chen-like equation of state. The results are summarized in Appendix D.

4. Publications (2012 – 2017)

Refereed journals

1. Bini D., Geralico A., Jantzen R.T.,
Separable geodesic action slicing in stationary spacetimes
General Relativity and Gravitation, vol. 44, 603-621, 2012.
Abstract
A simple observation about the action for geodesics in a stationary spacetime with separable geodesic equations leads to a natural class of slicings of that spacetime whose orthogonal geodesic trajectories represent the world lines of freely falling fiducial observers. The time coordinate function can then be taken to be the observer proper time, leading to a unit lapse function, although the time coordinate lines still follow Killing trajectories to retain the explicitly stationary nature of the coordinate grid. This explains some of the properties of the original Painlevé-Gullstrand coordinates on the Schwarzschild spacetime and their generalization to the Kerr-Newman family of spacetimes, reproducible also locally for the Gödel spacetime. For the static spherically symmetric case the slicing can be chosen to be intrinsically flat with spherically symmetric geodesic observers, leaving all the gravitational field information in the shift vector field.

2. Bini D., Esposito G., Geralico A.,
de Sitter spacetime: effects of metric perturbations on geodesic motion
General Relativity and Gravitation, vol. 44, 467-490 2012.
Abstract
Gravitational perturbations of the de Sitter spacetime are investigated using the Regge–Wheeler formalism. The set of perturbation equations is reduced to a single second order differential equation of the Heun-type for both electric and magnetic multipoles. The solution so obtained is used to study the deviation from an initially radial geodesic due to the perturbation. The spectral properties of the perturbed metric

are also analyzed. Finally, gauge- and tetrad-invariant first-order massless perturbations of any spin are explored following the approach of Teukolsky. The existence of closed-form, i.e. Liouvillian, solutions to the radial part of the Teukolsky master equation is discussed.

3. Bini D., Geralico A.,
Scattering by an electromagnetic radiation field
Physical Review D, vol. 85, 044001, 2012.
Abstract
Motion of test particles in the gravitational field associated with an electromagnetic plane wave is investigated. The interaction with the radiation field is modeled by a force term *à la* Poynting-Robertson entering the equations of motion given by the 4-momentum density of radiation observed in the particle's rest frame with a multiplicative constant factor expressing the strength of the interaction itself. Explicit analytical solutions are obtained. Scattering of fields by the electromagnetic wave, i.e., scalar (spin 0), massless spin $\frac{1}{2}$ and electromagnetic (spin 1) fields, is studied too.

4. Bini D., Falanga M., Geralico A., Stella L.,
The signal from an emitting source moving in a Schwarzschild spacetime under the influence of a radiation field
Classical and Quantum Gravity, vol. 29, 065014, 2012.
Abstract
The motion of matter immersed in a radiation field is affected by a radiation drag, as a result of scattering or absorption and re-emission. The resulting friction-like drag, also known as the PoyntingRobertson effect, has been recently studied in the general relativistic background of the Schwarzschild and Kerr metric, under the assumption that all photons in the radiation field possess the same angular momentum. We calculate here the signal produced by an emitting point-like specific source moving in a Schwarzschild spacetime under the influence of such a radiation field. We derive the flux, redshift factor and solid angle of the hot spot as a function of (coordinate) time, as well as the time-integrated image of the hot spot as seen by an observer at infinity. The results are then compared with those for a spot moving on a circular geodesic in a Schwarzschild metric.

5. Bini D., Gregoris D., Succi S.,

Radiation pressure vs friction effects in the description of the Poynting-Robertson scattering process
Europhysics Letters, vol. 97, 40007, 2012.
Abstract
The motion of a massive test particle inside a thermal (test) photon gas is studied near a Schwarzschild black hole, leading to a novel description of the effect of radiation scattering on the particle trajectory, responsible for half of the Poynting-Robertson effect: the azimuthal radiation drag. Lacking the outward directed radiation pressure of the latter effect, gravitationally bound orbits always decay, leading to capture by the black hole or the central object generating the exterior Schwarzschild field in which this discussion takes place.

6. Bini D., Geralico A., Succi S.,
Particle scattering by a test fluid on a Schwarzschild spacetime: the equation of state matters
The European Physical Journal C, vol. 72, Issue 3, 1913, 2012.
Abstract
The motion of a massive test particle in a Schwarzschild spacetime surrounded by a perfect fluid with equation of state $p_0 = w\rho_0$ is investigated. Deviations from geodesic motion are analyzed as a function of the parameter w, ranging from $w = 1$ which corresponds to the case of massive free scalar fields, down into the so-called "phantom" energy, with $w < -1$. It is found that the interaction with the fluid leads to capture (escape) of the particle trajectory in the case $1 + w > 0$ (< 0), respectively. Based on this result, it is argued that inspection of the trajectories of test particles in the vicinity of a Schwarzschild black hole with matter around may offer a new means of gaining insights into the nature of cosmic matter.

7. Bini D., Geralico A.,
Observer-dependent tidal indicators in the Kerr spacetime
Class. Quantum Grav., vol. 29, 055005, 2012.
Abstract
The observer-dependent tidal effects associated with the electric and magnetic parts of the Riemann tensor with respect to an arbitrary family of observers are discussed in a general spacetime in terms of certain tidal indicators. The features of such indicators are then explored by

specializing our considerations to the family of stationary circularly rotating observers in the equatorial plane of the Kerr spacetime. There exist a number of observer families which are special for several reasons and for each of them such indicators are evaluated. The transformation laws of tidal indicators when passing from one observer to another are also discussed, clarifying the interplay among them. Our analysis shows that no equatorial plane circularly rotating observer in the Kerr spacetime can ever measure a vanishing tidal electric indicator, whereas the family of Carters observers measures zero tidal magnetic indicator.

8. Bini D., Damour T., Faye G.,
Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description
Physical Review D, **85**, 124034, 2012.
Abstract
The gravitational-wave signal from inspiralling neutron-starneutron-star (or black-holeneutron-star) binaries will be influenced by tidal coupling in the system. An important science goal in the gravitational-wave detection of these systems is to obtain information about the equation of state of neutron star matter via the measurement of the tidal polarizability parameters of neutron stars. To extract this piece of information will require accurate analytical descriptions both of the motion and the radiation of tidally interacting binaries. We improve the analytical description of the late inspiral dynamics by computing the next-to-next-to-leadingorder relativistic correction to the tidal interaction energy. Our calculation is based on an effective-action approach to tidal interactions and on its transcription within the effective-one-body formalism. We find that second-order relativistic effects (quadratic in the relativistic gravitational potential $u = G(m_1 + m_2)/(c^2 r)$) significantly increase the effective tidal polarizability of neutron stars by a distance-dependent amplification factor of the form $1 + a_1 u + a_2 u^2 +$ where, say, for an equal-mass binary, $a_1 = 5/4 = 1.25$ (as previously known) and $a_2 = 85/14 = \approx 6.07143$ (as determined here for the first time). We argue that higher-order relativistic effects will lead to further amplification, and we suggest a Pad-type way of resumming them. We recommend testing our results by comparing resolution-extrapolated numerical simulations of inspiralling binary neutron stars to their effective one-body description.

9. Bini D., Gregoris D., Rosquist K., Succi S.,
Particle motion in a photon gas: friction matters
Gen. Rel. Grav., vol. 44, 2669-2680, 2012.
Abstract
The motion of a particle in the Tolman metric generated by a photon gas source is discussed. Both the case of geodesic motion and motion with nonzero friction, due to photon scattering effects, are analyzed. In the Minkowski limit, the particle moves along a straight line segment with a decelerated motion, reaching the endpoint at zero speed. The curved case shows a qualitatively different behavior; the geodesic motion consists of periodic orbits, confined within a specific radial interval. Under the effect of frictional drag, this radial interval closes up in time and in all our numerical simulations the particle ends up in the singularity at the center.

10. Bini D., Chicone C., Mashhoon B.,
Spacetime Splitting, Admissible Coordinates and Causality
Physical Review D, vol. 85, 104020, 2012.
arXiv:1203.3454
Abstract
To confront relativity theory with observation, it is necessary to split spacetime into its temporal and spatial components. The (1+3) timelike threading approach involves restrictions on the gravitational potentials ($g_{\mu\nu}$), while the (3+1) spacelike slicing approach involves restrictions on ($g^{\mu\nu}$). These latter coordinate conditions protect chronology within any such coordinate patch. While the threading coordinate conditions can be naturally integrated into the structure of Lorentzian geometry and constitute the standard coordinate conditions in general relativity, this circumstance does not extend to the slicing coordinate conditions. We explore the influence of chronology violation on wave motion. In particular, we consider the propagation of radiation parallel to the rotation axis of stationary Gödel-type universes characterized by parameters $\eta > 0$ and $\lambda > 0$ such that for $\eta < 1$ ($\eta > 1$) chronology is protected (violated). We show that in the WKB approximation such waves can freely propagate only when chronology is protected.

11. Bini D., Boshkayev K., Ruffini R., Siutsou I.,
Equatorial Circular Geodesics in the Hartle-Thorne Spacetime

Proceedings of the 12th Italian-Korean meeting July 4-8, 2011. Pescara (Italy). To appear on il Nuovo Cimento, 2012
Abstract
We investigate the influence of the quadrupole moment of a rotating source on the motion of a test particle in the strong field regime. For this purpose the Hartle-Thorne metric, that is an approximate solution of vacuum Einstein field equations that describes the exterior of any slowly rotating, stationary and axially symmetric body, is used. The metric is given with accuracy up to the second order terms in the body's angular momentum, and first order terms in its quadrupole moment. We give, with the same accuracy, analytic equations for equatorial circular geodesics in the Hartle-Thorne spacetime and integrate them numerically.

12. Bini D., Boshkayev K., Geralico A.,
Tidal indicators in the spacetime of a rotating deformed mass
Class. Quantum Grav., vol. 29, 145003, 2012.
Abstract
Tidal indicators are commonly associated with the electric and magnetic parts of the Riemann tensor (and its covariant derivatives) with respect to a given family of observers in a given spacetime. Recently, observer-dependent tidal effects have been extensively investigated with respect to a variety of special observers in the equatorial plane of the Kerr spacetime. This analysis is extended here by considering a more general background solution to include the case of matter which is also endowed with an arbitrary mass quadrupole moment. Relations with curvature invariants and BelRobinson tensor, i.e. observer-dependent super-energy density and super-Poynting vector, are also investigated.

13. Bini D., Geralico A., Haney M., Jantzen R.T.,
Scattering of particles by radiation fields: a comparative analysis
Phys. Rev. D., vol 86, 064016, 2012.
Abstract
The features of the scattering of massive neutral particles propagating in the field of a gravitational plane wave are compared with those characterizing their interaction with an electromagnetic radiation field. The motion is geodesic in the former case, whereas in the case of an electromagnetic pulse it is accelerated by the radiation field filling the as-

sociated spacetime region. The interaction with the radiation field is modeled by a force term entering the equations of motion proportional to the 4-momentum density of radiation observed in the particles rest frame. The corresponding classical scattering cross sections are evaluated too.

14. Bini D., Geralico A.,
Equilibrium Orbits of Particles Undergoing Poynting-Robertson Effect in Schwarzschild Spacetime
Proceedings of the 2^{nd} GalileoXuGuangqi Meeting
International Journal of Modern Physics: Conference Series, vol. 12, issue 01, p. 247, 2012.
Abstract
Equilibrium orbits of particles moving on the equatorial plane of a Schwarzschild spacetime are investigated when a test radiation field is superposed to the background gravitational field. The radiation flux is endowed with a fixed but arbitrary (non-zero) angular momentum. It is found that multiple equilibrium circular orbit exist provided that the photon angular momentum is sufficiently high. The stability of such orbits is also analyzed.

15. Bini D., Damour T.,
Gravitational radiation reaction along general orbits in the effective one-body formalism
Phys. Rev. D, vol. 86, 124012, 2012.
Abstract
We derive the gravitational radiation-reaction force modifying the Effective One Body (EOB) description of the conservative dynamics of binary systems. Our result is applicable to general orbits (elliptic or hyperbolic) and keeps terms of fractional second post-Newtonian order (but does not include tail effects). Our derivation of radiation-reaction is based on a new way of requiring energy and angular momentum balance. We give several applications of our results, notably the value of the (minimal) "Schott" contribution to the energy, the radial component of the radiation-reaction force, and the radiative contribution to the angle of scattering during hyperbolic encounters. We present also new results about the conservative relativistic dynamics of hyperbolic motions.

16. Bini D., Geralico A.,
Slicing black hole spacetimes
Proceedings of the 13th Marcel Grossmann Meeting, July 2-8, 2012, Stockholm (Sweden).
Abstract
Some properties of both the intrinsic and extrinsic geometry of special slicings in black hole spacetimes are discussed. A special role is played by slicings associated with the so called Painlevé-Gullstrand observers.

17. Bini D., Cherubini C., Filippi S.,
On the analog gravity formalism applied to white dwarfs
Proceedings of the 13th Marcel Grossmann Meeting, July 2-8, 2012, Stockholm (Sweden).
abstract
The "effective geometry"formalism is here adopted to analyze the perturbations of a non rotating white dwarf. After the derivation of a compact analytical parametric approximate white dwarf solution via the Padé formalism, we use it to construct the effective acoustic metric governing general fluid perturbations. The equations of the theory, numerically integrated in the case of irrotational, spherical, pulsating problem, show the analog space-time structure of the acoustic metric. In particular it appears that the stellar surface exhibits a curvature singularity associated to the vanishing of density.

18. Bini D.,
Observers, observables and measurements in general relativity
Proceedings of the meeting "Relativity and Gravitation 100 Years after Einstein in Prague" June 2529, 2012, Prague (Czech Republic).
Abstract
To perform any physical measurement it is necessary to identify in a non ambiguous way both the observer and the observable. A given observable can be then the target of different observers: a suitable algorithm to compare among their measurements should necessarily be developed, either formally or operationally. This is the task of what we call "theory of measurement," which we discuss here in the framework of general relativity.

19. Bini D., Crosta M.T., de Felice F., Geralico A., Vecchiato A.,

The Erez-Rosen metric and the role of the quadrupole on light propagation
Class. Quantum Grav., vol. 30 045009(22pp), 2013
Abstract
The gravitational field of a static body with quadrupole moment is described by an exact solution found by Erez and Rosen. Here we investigate the role of the quadrupole in the motion, deflection and lensing of a light ray in the above metric. The standard lensing observables like image positions and magnification have been explicitly obtained in the weak field and small quadrupole limit. In this limit the spacetime metric appears as the natural generalization to quadrupole corrections of the metric form adopted also in current astrometric models. Hence, the corresponding analytical solution of the inverse ray tracing problem as well as the consistency with other approaches are also discussed.

20. Bini D., Gregoris D., Rosquist K., Succi S.,
Effects of friction forces on the motion of objects in smoothly matched interior/exterior spacetimes
Class. Quantum Grav., vol. 30, 025009, 2013.
Abstract
We investigate the non-geodesic motion of a test particle inside a gas in equilibrium, as described by the interior Schwarzschild solution, in the presence of a Poynting-Robertson-like dissipative force. The accelerated motion in the interior spacetime is (regularly) connected to the geodesic one in the corresponding external Schwarzschild background, mimicking physical conditions often occurring in spacetime regions close to stellar or compact objects. A similar analysis is repeated for the case in which the interior Schwarzschild solution is given by a Pant-Sah perfect-fluid solution. This latter case, supporting a polytropic equation of state for the interior fluid source, appears to be more appropriate for modeling situations of either stellar or cosmological interest.

21. Bini D., Geralico A.,
On the occurrence of Closed Timelike Curves and the observer's point of view
Proceedings of the meeting "The Time-Machine Factory", October 14-19, 2012, Torino (Italy) [Invited speaker].
EPJ Web of Conferences **58**, 01002, (2013)
DOI: 10.1051/epjconf/20135801002
Published by EDP Sciences, 2013.

Abstract
The existence of Closed Timelike Curves (CTCs) in a generic spacetime is often associated with a non-physical choice of coordinates and can be cured by limiting the admissibility of such coordinates. Lichnerowicz conditions, for instance, represent a criterion for admissibility. The result, however, is a very restrictive limitation which may imply removal" of important regions (with respect to the peculiarity of phenomena which may happen there) of the spacetime manifold. We consider here the point of view of a family of observers (Fundamental Slicing Observers, FSO) having their world lines orthogonal to the surfaces of constant coordinate time. We say that the time coordinate has not a global character if the associated FSO change their causality condition in the domain of validity of the coordinates themselves. Furthermore, in those regions where FSO have no more timelike world lines, CTCs are present and one may think of special devices or investigation tools apt to operationally detect them. We will discuss in detail theoretical approaches involving (scalar) waves or photons.

22. Bini D., Geralico A.,
Dynamics of quadrupolar bodies in a Schwarzschild spacetime
Phys. Rev. D, vol. 87, 024028, 2013.
Abstract
The dynamics of extended bodies endowed with multipolar structure up to the mass quadrupole moment is investigated in the Schwarzschild background according to Dixons model, extending previous works. The whole set of evolution equations is numerically integrated under the simplifying assumptions of constant frame components of the quadrupole tensor and that the motion of the center of mass be confined on the equatorial plane, the spin vector being orthogonal to it. The equations of motion are also solved analytically in the limit of small values of the characteristic length scales associated with the spin and quadrupole with respect to the background curvature characteristic length. The results are qualitatively and quantitatively different from previous analyses involving only spin structures. In particular, the presence of the quadrupole turns out to be responsible for the onset of a nonzero spin angular momentum, even if initially absent.

23. Bini D., Geralico A., Gregoris D., Succi S.,

Friction forces in cosmological models
EPJC, vol. 73, 2334, 2013.
Abstract
We investigate the dynamics of test particles undergoing friction forces in a Friedmann-Robertson-Walker (FRW) spacetime. The interaction with the background fluid is modeled by introducing a Poynting-Robertson-like friction force in the equations of motion, leading to measurable (at least in principle) deviations of the particle trajectories from geodesic motion. The effect on the peculiar velocities of the particles is investigated for various equations of state of the background fluid and different standard cosmological models. The friction force is found to have major effects on particle motion in closed FRW universes, where it turns the time-asymptotic value (approaching the recollapse) of the peculiar particle velocity from ultra-relativistic (close to light speed) to a co-moving one, i.e., zero peculiar speed. On the other hand, for open or flat universes the effect of the friction is not so significant, because the time-asymptotic peculiar particle speed is largely non-relativistic also in the geodesic case.

24. Bini D., Fortini P., Geralico A., Haney M., Ortolan A.,
Light scattering by radiation fields: the optical medium analogy
EPL, vol 102, 20006 (2013)
Abstract
The optical medium analogy of a radiation field generated by either an exact gravitational plane wave or an exact electromagnetic wave in the framework of general relativity is developed. The equivalent medium of the associated background field is inhomogeneous and anisotropic in the former case, whereas it is inhomogeneous but isotropic in the latter. The features of light scattering are investigated by assuming the interaction region to be sandwiched between two flat spacetime regions, where light rays propagate along straight lines. Standard tools of ordinary wave optics are used to study the deflection of photon paths due to the interaction with the radiation fields, allowing for a comparison between the optical properties of the equivalent media associated with the different background fields.

25. Bini D., Esposito G., Kiefer C., Krämer M., Pessina F.,
On the Modification of the Cosmic Microwave Background Anisotropy Spec-

trum from Canonical Quantum Gravity
Phys. Rev. D, vol. 87, 104008 (2013)
Abstract
We evaluate the modifications to the CMB anisotropy spectrum that result from a semiclassical expansion of the Wheeler–DeWitt equation. Recently, such an investigation in the case of a real scalar field coupled to gravity, has led to the prediction that the power at large scales is suppressed. We make here a more general analysis and show that there is an ambiguity in the choice of solution to the equations describing the quantum gravitational effects. Whereas one of the two solutions describes a suppression of power, the other one describes an enhancement. We investigate possible criteria for an appropriate choice of solution. The absolute value of the correction term is in both cases of the same order and currently not observable. We also obtain detailed formulae for arbitrary values of a complex parameter occurring in the general solution of the nonlinear equations of the model. We finally discuss the modification of the spectral index connected with the power spectrum and comment on the possibility of a quantum-gravity induced unitarity violation.

26. Bini D., Damour T.,
Analytic determination of the two-body gravitational interaction potential at the 4th post-Newtonian approximation
Phys. Rev. D. Rapid Communications, vol. 87, 121501, (2013)
Abstract
We complete the analytical determination, at the 4th post-Newtonian approximation, of the main radial potential describing (within the effective one-body formalism) the gravitational interaction of two bodies. The (non logarithmic) coefficient $a_5(\nu)$ measuring this 4th post-Newtonian interaction potential is found to be linear in the symmetric mass ratio ν. Its ν-independent part $a_5(0)$ is obtained by an analytical gravitational self-force calculation that unambiguously resolves the formal infrared divergencies which currently impede its direct post-Newtonian calculation. Its ν-linear part $a_5(\nu) - a_5(0)$ is deduced from recent results of Jaranowski and Schaefer, and is found to be significantly negative.

27. Bini D., Geralico A., Gregoris D., Succi S.,

Dark energy from cosmological fluids with asymptotic-free non-ideal equation of state
Phys. Rev. D., vol. 88, 063007, (2013)
Abstract
We consider a Friedmann-Robertson-Walker universe with a fluid source obeying a non-ideal equation of state with "asymptotic freedom," namely ideal gas behavior (pressure changes directly proportional to density changes) both at low and high density regimes, following a fluid dynamical model due to Shan and Chen. It is shown that, starting from an ordinary energy density component, such fluids naturally evolve towards a universe with a substantial dark energy component at the present time, with no need of invoking any cosmological constant. Moreover, we introduce a quantitative indicator of darkness abundance, which provides a consistent picture of the actual matter-energy content of the universe.

28. Bini D., de Felice F., Geralico A.,
On the spacetime acting as an optical medium: the observer-dependent approach
IJGMMP, vol. 11 (3) 1450024 (15 pages) 2014
Abstract
A local observer-dependent approach to the optical medium analogy of a general gravitational background is developed. The optical properties of the equivalent media associated with several stationary axisymmetric spacetimes are investigated with respect to different families of observers playing a special role as well as selected null orbits. Explicit examples among the class of both exact and approximate solutions of Einstein's field equations describing the gravitational field of sources with a multipolar structure up to the quadrupole are analyzed. The analytic contribution of the individual polarities to the refraction index associated with the corresponding effective media is deduced as a function of the spacetime parameters.

29. Bini D., Geralico A., Haney M.,
Refraction index analysis of light propagation in a colliding gravitational wave spacetime
Gen. Relativ. Gravit. (2014) 46:1644
Abstract

The optical medium analogy of a given spacetime was developed decades ago and has since then been widely applied to different gravitational contexts. Here we consider applications to light propagation in a colliding gravitational wave spacetime, generalizing previous results concerning single gravitational pulses. In view of complexity of the nonlinear interaction of two gravitational waves in the framework of general relativity, typically leading to the formation of either horizons or singularities, the optical medium analogy proves helpful. It allows us to simply capture some interesting effects we discuss here in terms of wave propagation, interferometric fringe motions, time delays, etc.

30. Bini D., Geralico A.,
Deviation of quadrupolar bodies from geodesic motion in a Kerr spacetime
Phys. Rev. D. vol. 89 (4), 044013 (22 pages), 2014.
Abstract
The deviation from geodesic motion of the world line of an extended body endowed with multipolar structure up to the mass quadrupole moment is studied in the Kerr background according to the Mathisson-Papapetrou-Dixon model. The properties of the quadrupole tensor are clarified by identifying the relevant components which enter the equations of motion, leading to the definition of an effective quadrupole tensor sharing its own algebraic symmetries, but also obeying those implied by the Mathisson-Papapetrou-Dixon model itself. The equations of motion are then solved analytically in the limit of small values of the characteristic length scales associated with the spin and quadrupole variables in comparison with the one associated with the background curvature and under special assumptions on bodys structure and motion. The resulting quasi-circular orbit is parametrized in a Keplerian-like form, so that temporal, radial and azimuthal eccentricities as well as semi-major axis, period and periastron advance are explicitly computed and expressed in terms of gauge-invariant variables in the weak field and slow motion limit. A companion numerical study of the equations of motion is performed too.

31. Bini D., Haney M., Geralico A., Ortolan A.,
Deviation effects induced by strong electromagnetic waves
Phys. Rev. D 89, 044013 (2014)
Abstract

Some strong field effects on test particle motion associated with the propagation of a plane electromagnetic wave in the exact theory of general relativity are investigated. The chosen (not empty) spacetime metric is conformally flat and of Petrov type O, differently from the background of a plane gravitational wave, which is of type N. The world line deviation between both uncharged and charged particles on different spacetime trajectories due to the combined effect of gravitational and electromagnetic forces is studied. The interaction of charged particles with the background radiation field is also discussed through a general relativistic description of the inverse Compton effect.

32. Bini D., Damour T.,
High-order post-Newtonian contributions to the two-body gravitational interaction potential from analytical self-force calculations
Phys. Rev. D 89, 064063 (2014)
Abstract
We extend the analytical determination of the main radial potential describing (within the effective one-body formalism) the gravitational interaction of two bodies beyond the fourth post-Newtonian approximation recently obtained by us. This extension is done to linear order in the mass ratio by applying analytical gravitational self-force theory (for a particle in circular orbit around a Schwarzschild black hole) to Detweilers gauge-invariant redshift variable. By using the version of black hole perturbation theory developed by Mano, Suzuki and Takasugi, we have pushed the analytical determination of the (linear in mass ratio) radial potential to the sixth post-Newtonian order (passing through 5 and 5.5 post-Newtonian terms). In principle, our analytical method can be extended to arbitrarily high post-Newtonian orders.

33. Bini D., Geralico A.,
Extended bodies in a Kerr spacetime: exploring the role of a general quadrupole tensor
Class. Quantum Grav. 31 (2014) 075024.
Abstract
The equatorial motion of extended bodies in a Kerr spacetime is investigated in the framework of the MathissonPapapetrouDixon model, including the full set of effective components of the quadrupole tensor. The numerical integration of the associated equations shows the specific

role of the mass and current quadrupole moment components. While most of the literature on this topic is limited to spin-induced (purely electric) quadrupole tensor, the present analysis highlights the effect of a completely general quadrupole tensor on the dynamics. The contribution of the magnetic-type components is indeed related to a number of interesting features, e.g., enhanced inward/outward spiraling behavior of the orbit and spin-flip-like effects, which may have observational counterparts. Finally, the validity limit of the MathissonPapapetrouDixon model is also discussed through explicit examples.

34. Bini D., Geralico A., Gregoris D., Succi S.,
Scalar field inflation and Shan-Chen fluid models
Phys. Rev. D 90, 044021 (2014)
Abstract
A scalar field equivalent to a nonideal dark energy fluid obeying a Shan-Chen-like equation of state is used as the background source of a flat Friedmann-Robertson-Walker cosmological spacetime to describe the inflationary epoch of our Universe. Within the slow-roll approximation, a number of interesting features are presented, including the possibility to fulfill current observational constraints as well as a graceful exit mechanism from the inflationary epoch.

35. Bini D., Damour T.,
Analytic determination of the eight-and-a-half post-Newtonian self-force contributions to the two-body gravitational interaction potential
Phys. Rev. D 89, 104047 (2014)
Abstract
We *analytically* compute, to the eight-and-a-half post-Newtonian order, and to linear order in the mass ratio, the radial potential describing (within the effective one-body formalism) the gravitational interaction of two bodies, thereby extending previous analytic results. These results are obtained by applying analytical gravitational self-force theory (for a particle in circular orbit around a Schwarzschild black hole) to Detweiler's gauge-invariant redshift variable. We emphasize the increase in "transcendentality" of the numbers entering the post-Newtonian expansion coefficients as the order increases, in particular we note the appearance of $\zeta(3)$ (as well as the square of Euler's constant γ) starting at the seventh post-Newtonian order. We study the convergence of the

post-Newtonian expansion as the expansion parameter $u = GM/(c^2 r)$ leaves the weak-field domain $u \ll 1$ to enter the strong field domain $u = O(1)$.

36. Bini D., Esposito G.,
Perturbative evaluation of scalar two-point function in the Cosmic Microwave Background power spectrum
Phys. Rev. D, vol. 89, 084032, 2014.
Abstract
Recent work in the literature has found a suppression or, instead, an enhancement of the Cosmic Microwave Background power spectrum in quantum gravity, although the effect is too small to be observed, in both cases. The present paper studies in detail the equations recently proposed for a Born-Oppenheimer-type analysis of the problem. By using a perturbative approach to the analysis of the nonlinear ordinary differential equation obeyed by the two-point function for scalar fluctuations, we find various explicit forms of such a two-point function, with the associated power spectrum. In particular, a new family of power spectra is obtained and studied. The theoretical prediction of power enhancement at large scales is hence confirmed.

37. Bini D., Damour T.,
Two-body gravitational spin-orbit interaction at linear order in the mass ratio
Phys. Rev. D 90, 024039 (2014)
Abstract
We analytically compute, to linear order in the mass-ratio, the geodetic spin precession fre- quency of a small spinning body orbiting a large (non-spinning) body to the eight-and-a-half post- Newtonian order, thereby extending previous analytical knowledge which was limited to the third post-Newtonian level. These results are obtained applying analytical gravitational self-force theory to the first-derivative level generalization of Detweilers gauge-invariant redshift variable. We compare our analytic results with strong-field numerical data recently obtained by S. R. Dolan et al. [Phys. Rev. D 89, 064011 (2014)]. Our new, high-post-Newtonian-order results capture the strong-field features exhibited by the numerical data. We argue that the spin-precession will diverge as $-0.14/(1-3y)$ as the light-ring is approached. We transcribe our kinematical spin-precession results into a corresponding improved

analytic knowledge of one of the two (gauge-invariant) effective gyro-gravitomagnetic ratios characterizing spin-orbit couplings within the effective-one-body formalism. We provide simple, accurate analytic fits both for spin-precession and the effective gyro-gravitomagnetic ratio. The latter fit predicts that the linear-in-mass-ratio correction to the gyro-gravitomagnetic ratio changes sign before reaching the light-ring. This strong-field prediction might be important for improving the analytic modeling of coalescing spinning binaries.

38. Bini D., Mashhoon B.,
Peculiar velocities in dynamic spacetimes
Phys. Rev. D 90, 024030 (2014)
Abstract
We investigate the asymptotic behavior of peculiar velocities in certain physically significant time-dependent gravitational fields. Previous studies of the motion of free test particles have focused on the collapse scenario, according to which a double-jet pattern with Lorentz factor γ develops asymptotically along the direction of complete gravitational collapse. In the present work, we identify a second wave scenario, in which a single-jet pattern with Lorentz factor γ develops asymptotically along the direction of wave propagation. The possibility of a connection between the two scenarios for the formation of cosmic jets is critically examined.

39. Bini D., Geralico A., Passamonti A.,
Radiation drag in the field of a non-spherical source
Monthly Notices of the Royal Astronomical Society **446** (3): 65-74 (2014).
Abstract
The motion of a test particle in the gravitational field of a non-spherical source en- dowed with both mass and mass quadrupole moment is investigated when a test radiation field is also present. The background is described by the Erez-Rosen so- lution, which is a static spacetime belonging to the Weyl class of solutions to the vacuum Einsteins field equations, and reduces to the familiar Schwarzschild solution when the quadrupole parameter vanishes. The radiation flux has a fixed but arbitrary (non-zero) angular momentum. The interaction with the radiation field is assumed to be Thomson-like, i.e., the particles absorb and re-emit radiation, thus suffering for a friction-like drag force. Such

an additional force is responsible for the Poynting- Robertson effect, which is well established in the framework of Newtonian gravity and has been recently extended to the general theory of relativity. The balance between gravitational attraction, centrifugal force and radiation drag leads to the occurrence of equilibrium circular orbits which are attractors for the surrounding matter for ev- ery fixed value of the interaction strength. The presence of the quadrupolar structure of the source introduces a further degree of freedom: there exists a whole family of equilibrium orbits parametrized by the quadrupole parameter, generalizing previous works. This scenario is expected to play a role in the context of accretion matter around compact objects.

40. Bini D., Damour T.,
Gravitational self-force corrections to two-body tidal interactions and the effective one-body formalism
Phys. Rev. D 90, 124037 (2014)
Abstract
Tidal interactions have a significant influence on the late dynamics of compact binary systems, which constitute the prime targets of the upcoming network of gravitational-wave detectors. We refine the theoretical description of tidal interactions (hitherto known only to the second post- Newtonian level) by extending our recently developed analytic self-force formalism, for extreme mass-ratio binary systems, to the computation of several tidal invariants. Specifically, we compute, to linear order in the mass ratio and to the 7.5th post-Newtonian order, the following tidal invariants: the square and the cube of the gravitoelectric quadrupolar tidal tensor, the square of the gravito- magnetic quadrupolar tidal tensor, and the square of the gravitoelectric octupolar tidal tensor. Our high-accuracy analytic results are compared to recent numerical self-force tidal data by Dolan et al. [1], and, notably, provide an analytic understanding of the light ring asymptotic behavior found by them. We transcribe our kinematical tidal-invariant results in the more dynamically significant effective one-body description of the tidal interaction energy. By combining, in a synergetic man- ner, analytical and numerical results, we provide simple, accurate analytic representations of the global, strong-field behavior of the gravitoelectric quadrupolar tidal factor. A striking finding is that the linear-in-mass-ratio piece in the latter tidal factor changes sign in the strong-field domain, to be-

come negative (while its previously known second post-Newtonian approximant was always positive). We, however, argue that this will be more than compensated by a probable fast growth, in the strong-field domain, of the nonlinear-in-mass-ratio contributions in the tidal factor.

41. Bini D., Geralico A., Jantzen R.T., Semerak O.,
Particles under radiation thrust in Schwarzschild space-time: a flux perpendicular to the equatorial plane
Monthly Notices of the Royal Astronomical Society **446**, 1907 (2015).
Abstract
Motivated by the picture of a thin accretion disc around a black hole, radiating mainly in the direction perpendicular to its plane, we study the motion of test particles interacting with a test geodesic radiation flux propagating perpendicular to the equatorial plane in a Schwarzschild space-time. We assume that the interaction (kind of Poynting-Robertson effect) is modelled by an effective term corresponding to a Thomson-type radiation drag. After approximating the individual photon trajectories in quite an accurate way, we solve the continuity equation (up to linear order in M) in order to find a consistent radiation-flux density, prescribing a certain plausible equatorial profile. The combined effect of gravitation and radiation is illustrated on several figures; they confirm that the particles are generically strongly influenced by the flux, in particular, they are both collimated and accelerated in the direction perpendicular to the disc, but the acceleration received in this manner is not enough to explain highly relativistic outflows emanating from some black-holedisc sources. Main improvement needed is a more realistic description of the radiation-particle interaction, allowing for Compton-type frequency-dependent effect and particle heating/cooling.

42. Bini D. and Geralico A.,
Effect of an arbitrary spin orientation on the quadrupolar structure of an extended body in a Schwarzschild spacetime
Phys. Rev. D, 91, 104036 (2015)
Abstract
The influence of an arbitrary spin orientation on the quadrupolar structure of an extended body moving in a Schwarzschild spacetime is investigated. The body dynamics is described by the Mathisson-Papapetrou-Dixon model, without any restriction on the motion or simplifying as-

sumption on the associated spin vector and quadrupole tensor, generalizing previous works. The equations of motion are solved analytically in the limit of small values of the characteristic length scales associated with the spin and quadrupole variables with respect to the characteristic length of the background curvature. The solution provides all corrections to the circular geodesic on the equatorial plane taken as the reference trajectory due to both dipolar and quadrupolar structure of the body as well as the conditions which the nonvanishing components of the quadrupole tensor must fulfill in order that the problem be self-consistent.

43. Bini D. and Geralico A.,
Tidal invariants along the world line of an extended body in the Kerr spacetime
Phys. Rev. D **91**, no. 8, 084012 (2015).
Abstract
An extended body orbiting a compact object undergoes tidal deformations by the background gravitational field. Tidal invariants built up with the Riemann tensor and their derivatives evaluated along the worldline of the body are essential tools to investigate both geometrical and physical properties of the tidal interaction. For example, one can determine the tidal potential in the neighborhood of the body by constructing a body-fixed frame, which requires Fermi-type coordinates attached to the body itself, the latter being in turn related to the spacetime metric and curvature along the considered worldline. Similarly, in an effective field theory description of extended bodies, finite size effects are taken into account by adding to the point mass action certain nonminimal couplings which involve integrals of tidal invariants along the orbit of the body. In both cases such a computation of tidal tensors is required. Here we consider the case of a spinning body also endowed with a nonvanishing quadrupole moment in a Kerr spacetime. The structure of the body is modeled by a multipolar expansion around the center-of-mass line according to the Mathisson-Papapetrou-Dixon model truncated at the quadrupolar order. The quadrupole tensor is assumed to be quadratic in spin, accounting for rotational deformations. The behavior of tidal invariants of both electric and magnetic type is discussed in terms of gauge-invariant quantities when the body is moving along a circular orbit as well as in the case of an arbitrary (equatorial) motion. The analysis is completed by examining the associated

eigenvalues and eigenvectors of the tidal tensors. The limiting situation of the Schwarzschild solution is also explored both in the strong field regime and in the weak field limit.

44. Bini D. and Damour T.,
Detweiler's gauge-invariant redshift variable: analytic determination of the nine and nine-and-a-half post-Newtonian self-force contributions
Phys. Rev. D 91, 064050 (2015)
Abstract
Continuing our analytic computation of the first-order self-force contribution to Detweilers redshift variable we provide the exact expressions of the ninth and ninth-and-a-half post-Newtonian terms.

45. Bini D. , Mashhoon B..
Weitzenböck's Torsion, Fermi Coordinates and Adapted Frames
Phys. Rev. D **91**, no. 8, 084026 (2015)
Abstract
We study Weitzenböcks torsion and discuss its properties. Specifically, we calculate the measured components of Weitzenböcks torsion tensor for a frame field adapted to static observers in a Fermi normal coordinate system that we establish along the world line of an arbitrary accelerated observer in general relativity. A similar calculation is carried out in the standard Schwarzschild-like coordinates for static observers in the exterior Kerr spacetime; we then compare our results with the corresponding curvature components. Our work supports the contention that in the extended general relativistic framework involving both the Levi-Civita and Weitzenbock connections, curvature and torsion provide complementary representations of the gravitational field.

46. Bini D. , Bittencourt E., Geralico A. and Jantzen R.T.,
Slicing black hole spacetimes
International Journal of Geometric Methods in Modern Physics Vol. 12 (2015) 1550070 (32 pages).
Abstract
A general framework is developed to investigate the properties of useful choices of stationary spacelike slicings of stationary spacetimes whose congruences of timelike orthogonal trajectories are interpreted as the world lines of an associated family of observers, the kinematical properties of which in turn may be used to geometrically characterize the

original slicings. On the other hand properties of the slicings themselves can directly characterize their utility motivated instead by other considerations like the initial value and evolution problems in the 3-plus-1 approach to general relativity. An attempt is made to categorize the various slicing conditions or time gauges used in the literature for the most familiar stationary spacetimes: black holes and their flat spacetime limit.

47. Bini D. and Damour T.,
Analytic determination of high-order post-Newtonian self-force contributions to gravitational spin precession
Phys. Rev. D 91, 064064 (2015)
Abstract
Continuing our analytic computation of the first-order self-force contribution to the geodetic spin precession frequency of a small spinning body orbiting a large (non-spinning) body we provide the exact expressions of the tenth and tenth-and-a-half post-Newtonian terms. We also introduce a new approach to the analytic computation of self-force regularization parameters based on a WKB analysis of the radial and angular equations satisfied by the metric perturbations.

48. Bini D., Iorio L. and Giordano D.
Orbital effects due to gravitational induction
General Relativity and Gravitation, vol. 47, 130 (2015).
Abstract
We study the motion of test particles in the metric of a localized and slowly rotating astronomical source, within the framework of linear gravitoelectromagnetism, grounded on a Post-Minkowskian approximation of general relativity. Special attention is paid to gravitational inductive effects due to time-varying gravitomagnetic fields. We show that, within the limits of the approximation mentioned above, there are cumulative effects on the orbit of the particles either for planetary sources or for binary systems. They turn out to be negligible.

49. Bini D., Bittencourt E. and Geralico A.
Massless Dirac particles in the vacuum C-metric
Classical and Quantum Gravity, **32**, 215010 (2015).
Abstract

We study the behavior of massless Dirac particles in the vacuum C-metric spacetime, representing the nonlinear superposition of the Schwarzschild black hole solution and the Rindler flat spacetime associated with uniformly accelerated observers. Under certain conditions, the C-metric can be considered as a unique laboratory to test the coupling between intrinsic properties of particles and fields with the background acceleration in the full (exact) strong-field regime. The Dirac equation is separable by using, e.g., a spherical-like coordinate system, reducing the problem to one-dimensional radial and angular parts. Both radial and angular equations can be solved exactly in terms of general Heun functions. We also provide perturbative solutions to first-order in a suitably defined acceleration parameter, and compute the acceleration-induced corrections to the particle absorption rate as well as to the angle-averaged cross section of the associated scattering problem in the low-frequency limit. Furthermore, we show that the angular eigenvalue problem can be put in one-to-one correspondence with the analogous problem for a Kerr spacetime, by identifying a map between these acceleration harmonics and Kerr spheroidal harmonics. Finally, in this respect we discuss the nature of the coupling between intrinsic spin and spacetime acceleration in comparison with the well known Kerr spin-rotation coupling.

50. Bini D. , de Felice F.
Chronology protection in the Kerr metric
General Relativity and Gravitation, vol. 47, (2015)
Abstract
We show that causality violation in a Kerr naked singularity spacetime is constrained by the existence of (radial) potential barriers. We extend to the class of vortical nonequatorial null geodesics confined to θ= constant hyperboloids (*boreal* orbits) previous results concerning timelike ones [M. Calvani et. al., Gen. Rel. Grav. **9**, 155 (1978)], showing that within this class of orbits, the causality principle is rigorously satisfied.

51. Bini D., Faye G. and Geralico A.
Dynamics of extended bodies in a Kerr spacetime with spin-induced quadrupole tensor
Phys. Rev. D **92**, 104003 (2015)
Abstract

The features of equatorial motion of an extended body in Kerr spacetime are investigated in the framework of the Mathisson-Papapetrou-Dixon model. The body is assumed to stay at quasiequilibrium and respond instantly to external perturbations. Besides the mass, it is completely determined by its spin, the multipolar expansion being truncated at the quadrupole order, with a spin-induced quadrupole tensor. The study of the radial effective potential allows to analytically determine the ISCO shift due to spin and the associated frequency of the last circular orbit.

52. Bini D. , Damour T. and Geralico A.
Spin-dependent two-body interactions from gravitational self-force computations
Phys. Rev. D, **92**, 124058 (2015)
Abstract
We analytically compute, through the eight-and-a-half post-Newtonian order and the fourth-order in spin, the gravitational self-force correction to Detweilers gauge invariant redshift function for a small mass in circular orbit around a Kerr black hole. Using the first law of mechanics for black hole binaries with spin [L. Blanchet, A. Buonanno and A. Le Tiec, Phys. Rev. D 87, 024030 (2013)] we transcribe our results into a knowledge of various spin-dependent couplings, as encoded within the spinning effective-one-body model of T. Damour and A. Nagar [Phys. Rev. D 90, 044018 (2014)]. We also compare our analytical results to the (corrected) numerical self-force results of A. G. Shah, J. L. Friedman and T. S. Keidl [Phys. Rev. D 86, 084059 (2012)], from which we show how to directly extract physically relevant spin-dependent couplings.

53. Bini D., Esposito G. and Geralico A.
Late time evolution of cosmological models with non-ideal fluids
Phys. Rev. D, vol. 93, 023511 (2016)
Abstract
Classical as well as quantum features of the late-time evolution of cosmological models with fluids obeying a Shan-Chen-like equation of state are studied. The latter is of the type $p = w_{\rm eff}(\rho)\,\rho$, and has been used in previous works to describe, e.g., a possible scenario for the growth of the dark-energy content of the present Universe. At the classical level the fluid dynamics in a spatially flat Friedmann-Robertson-Walker

background implies the existence of two equilibrium solutions depending on the model parameters, associated with (asymptotic) finite pressure and energy density. We show that no future cosmological singularity is developed during the evolution for this specific model. The corresponding quantum effects in the late-time behavior of the system are also investigated within the framework of quantum geometrodynamics, i.e., by solving the (minisuperspace) Wheeler-DeWitt equation in the Born-Oppenheimer approximation, by constructing wave-packets and by analyzing their behavior.

54. Bini D., Damour T. and Geralico A.
Confirming and improving post-Newtonian and effective-one-body results from self-force computations along eccentric orbits around a Schwarzschild black hole
Phys. Rev. D, **93**, no. 6, 064023 (2016)
Abstract
We analytically compute, through the six-and-a-half post-Newtonian order, the second-order-ineccentricity piece of the Detweiler-Barack-Sago gauge-invariant redshift function for a small mass in eccentric orbit around a Schwarzschild black hole. Using the first law of mechanics for eccentric orbits [A. Le Tiec, Phys. Rev. D 92, 084021 (2015)] we transcribe our result into a correspondingly accurate knowledge of the second radial potential of the effective-one-body formalism [A. Buonanno and T. Damour, Phys. Rev. D 59, 084006 (1999)]. We compare our newly acquired analytical information to several different numerical self-force data and find good agreement, within estimated error bars. We also obtain, for the first time, independent analytical checks of the recently derived, comparable-mass fourth-post-Newtonian order dynamics [T. Damour, P. Jaranowski and G. Shaefer, Phys. Rev. D 89, 064058 (2014)].

55. Bini D., Damour T. and Geralico A.
New gravitational self-force analytical results for eccentric orbits around a Schwarzschild black hole Phys. Rev. D, **93**, no. 10, 104017 (2016)
Abstract
We raise the analytical knowledge of the eccentricity expansion of the Detweiler-Barack-Sago redshift invariant in a Schwarzschild spacetime up to the 9.5th post-Newtonian order (included) for the e^2 and e^4 contributions, and up to the 4th post-Newtonian order for the higher eccen-

tricity contributions through e^{20}. We convert this information into an analytical knowledge of the effective-one-body radial potentials $\bar{d}(u)$, $\rho(u)$ and $q(u)$ through the 9.5th post-Newtonian order. We find that our analytical results are compatible with current corresponding numerical self-force data.

56. Punsly B. and Bini D.
General Relativistic Considerations of the Field Shedding Model of Fast Radio Bursts
Mon. Not. Roy. Astron. Soc. **459**, L41 (2016)
Abstract
Popular models of fast radio bursts (FRBs) involve the gravitational collapse of neutron star progenitors to black holes. It has been proposed that the shedding of the strong neutron star magnetic field (B) during the collapse is the power source for the radio emission. Previously, these models have utilized the simplicity of the Schwarzschild metric which has the restriction that the magnetic flux is magnetic hair that must be shed before final collapse. But, neutron stars have angular momentum and charge and a fully relativistic Kerr Newman solution exists in which B has its source inside of the event horizon. In this letter, we consider the magnetic flux to be shed as a consequence of the electric discharge of a metastable collapsed state of a Kerr Newman black hole. It has also been argued that the shedding model will not operate due to pair creation. By considering the pulsar death line, we find that for a neutron star with $B = 10^{11}$ to $10^{13}\,G$ and a long rotation period, $> 1\,s$ this is not a concern. We also discuss the observational evidence supporting the plausibility of magnetic flux shedding models of FRBs that are spawned from rapidly rotating progenitors.

57. Bini D. and Geralico A.
Scattering by a Schwarzschild black hole of particles undergoing drag force effects
General Relativity and Gravitation, **48**, 101 (2016)
Abstract
The scattering of massive particles by a Schwarzschild black hole also undergoing a drag force is considered. The latter is modeled as a viscous force acting on the orbital plane, with components proportional to the associated particle 4-velocity components. The energy and angular

momentum losses as well as the dependence of the hyperbolic scattering angle on the strength of the drag are investigated in situations where strong field effects cause large deflections.

58. Bini D. and Mashhoon B.
Nonlocal Gravity: Conformally Flat Spacetimes
J. Geom. Methods Mod. Phys. 13, 1650081 (2016)
Abstract
The field equations of the recent nonlocal generalization of Einsteins theory of gravitation are presented in a form that is reminiscent of general relativity. The implications of the nonlocal field equations are studied in the case of conformally flat spacetimes. Even in this simple case, the field equations are intractable. Therefore, to gain insight into the nature of these equations, we investigate the structure of nonlocal gravity in two-dimensional spacetimes. While any smooth 2D spacetime is conformally flat and satisfies Einsteins field equations, ${}^0G_{\mu\nu} = 0$, only a subset containing either a Killing vector or a homothetic Killing vector can satisfy the field equations of nonlocal gravity.

59. Bini D., Damour T. and Geralico A.
High post-Newtonian order gravitational self-force analytical results for eccentric orbits around a Kerr black hole
Phys. Rev. D, **93**, no. 12, 124058 (2016)
Abstract
We present the first analytic computation of the Detweiler-Barack-Sago gauge-invariant redshift function for a small mass in eccentric equatorial orbit around a spinning black hole. Our results give the redshift contributions that mix eccentricity and spin effects, through second order in eccentricity, second order in spin parameter, and the eight-and-a-half post-Newtonian order.

60. Bini D. and Geralico A.
Schwarzschild black hole embedded in a dust field: scattering of particles and drag force effects
Class. Quantum. Grav., **33**, 125024 (2016)
Abstract
A 'temporal analogue' of the standard PoyntingRobertson effect is analyzed as induced by a dust of particles (instead of a gas of photons) surrounding a Schwarzschild black hole. Test particles inside this cloud

undergo acceleration effects due to the presence of a friction force, so that the fate of their evolution can be completely different from the corresponding geodesic motion. Typical situations are discussed of hyperbolic motion of particles scattered by the black hole in the presence of a dust filling the whole spacetime region outside the horizon as well as particles which free fall radially crossing a corona located at a certain distance from the horizon. The existence of equilibrium orbits may prevent particles from either falling into the hole or escaping to infinity.

61. Bini D. and Damour T.
Conservative second-order gravitational self-force on circular orbits and the effective one-body formalism
Phys. Rev. D, **93**, 104040 (2016)
Abstract
We consider Detweiler's redshift variable z for a nonspinning mass m_1 in circular motion (with orbital frequency Ω) around a nonspinning mass m_2. We show how the combination of effective-one-body (EOB) theory with the first law of binary dynamics allows one to derive a simple, exact expression for the functional dependence of z on the (gauge-invariant) EOB gravitational potential $u = (m_1 + m_2)/R$. We then use the recently obtained high-post-Newtonian(PN)-order knowledge of the main EOB radial potential $A(u;\nu)$ [where $\nu = m_1 m_2/(m_1 + m_2)^2$] to decompose the second-self-force-order contribution to the function $z(m_2\Omega, m_1/m_2)$ into a known part (which goes beyond the 4PN level in including the 5PN logarithmic term, and the 5.5PN contribution), and an unknown one [depending on the yet unknown, 5PN, 6PN, ..., contributions to the $O(\nu^2)$ contribution to the EOB radial potential $A(u;\nu)$]. We apply our results to the second-self-force-order contribution to the frequency shift of the last stable orbit. We indicate the expected singular behaviors, near the lightring, of the second-self-force-order contributions to both the redshift and the EOB A potential. Our results should help both in extracting information of direct dynamical significance from ongoing second-self-force-order computations, and in parametrizing their global strong-field behaviors. We also advocate computing second-self-force-order conservative quantities by iterating the time-symmetric Green-function in the background spacetime.

62. Bini D., Mashhoon B.

Relativistic Gravity Gradiometry: The Mashhoon–Theiss Effect
e-Print: arXiv:1607.05473 [gr-qc]
Abstract
In general relativity, relativistic gravity gradiometry involves the measurement of the relativistic tidal matrix, which is theoretically obtained from the projection of the Riemann curvature tensor onto the orthonormal tetrad frame of an observer. The observers 4-velocity vector defines its local temporal axis and its local spatial frame is defined by a set of three orthonormal nonrotating gyro directions. The general tidal matrix for the timelike geodesics of Kerr spacetime has been calculated by Marck [17]. We are interested in the measured components of the curvature tensor along the inclined circular geodesic orbit of a test mass about a slowly rotating astronomical object of mass M and angular momentum J. Therefore, we specialize Marcks results to such a circular orbit that is tilted with respect to the equatorial plane of the Kerr source. To linear order in J, we recover the MashhoonTheiss effect, which is due to a small denominator (resonance) phenomenon involving the frequency of geodetic precession. The MashhoonTheiss effect shows up as a special long-period gravitomagnetic part of the relativistic tidal matrix. The physical interpretation of this effect is briefly discussed.

63. Bini D., Geralico A. and Jantzen R.T.
Gyroscope precession along bound equatorial plane orbits around a Kerr black hole
Phys. Rev. D **94**, 064066 (2016)
Abstract
The precession of a test gyroscope along stable bound equatorial plane orbits around a Kerr black hole is analyzed and the precession angular velocity of the gyros parallel transported spin vector and the increment in precession angle after one orbital period is evaluated. The parallel transported Marck frame which enters this discussion is shown to have an elegant geometrical explanation in terms of the electric and magnetic parts of the Killing-Yano 2-form and a Wigner rotation effect.

64. Bini D., Damour T. and Geralico A.
High-order post-Newtonian contributions to gravitational self-force effects in black hole spacetimes
Proceedings of the international meeting "INdAM Workshop on Inno-

vative Algorithms and Analysis,"
May 17-20, 2016, Rome (It). Ed. by Springer.
Abstract
The explicit analytical computation of first-order metric perturbations in black hole spacetimes is described in the case of a perturbing mass moving on an equatorial circular orbit. The perturbation equations can be separated into an angular part and a radial part. The latter satisfies a single inhomogeneous radial Schrödinger-like equation with a Dirac-delta singular source term, whose solutions are built up through Green's function techniques. Various types of approximate analytical homogeneous solutions (and corresponding Green's functions) can be constructed: Post-Newtonian solutions (expanded in powers of $1/c$), Mano-Suzuki-Takasugi solutions (expanded in series of hypergeometric functions), Wentzel-Kramers-Brillouin (WKB) solutions (large l expansion). The perturbed black-hole metric constructed by suitably combining these different kind of solutions can then be used to compute, in analytical form, gauge-invariant quantities. These include several "potentials" entering the effective-one-body formalism (shortly reviewed here). The latter formalism is a new way of describing the gravitational interaction of two masses which has played a crucial role in the recent detection of gravitational waves.

65. Bini D., Carvalho G. and Geralico A.
Scalar field self-force effects on a particle orbiting a Reissner-Nordstrom black hole
Phys. Rev. D **94**, 124028 (2016)
e-print: arXiv:1610.02235 [gr-qc]
Abstract
Scalar field self-force effects on a scalar charge orbiting a Reissner-Nordstrom black hole are investigated. The scalar wave equation is solved analytically in a post-Newtonian framework, and the solution is used to compute the self-field as well as the components of the self-force at the particles location up to 7.5 post-Newtonian order. The energy fluxes radiated to infinity and down the hole are also evaluated. Comparison with previous numerical results in the Schwarzschild case shows a good agreement in both strong-field and weak-field regimes.

66. Bini D., Geralico A. and Jantzen R. T.,

Gyroscope precession along unbound equatorial plane orbits around a Kerr black hole
Phys. Rev. D **94**, 124002 (2016)
e-Print: arXiv:1610.06513 [gr-qc].
Abstract
The precession of a test gyroscope along unbound equatorial plane geodesic orbits around a Kerr black hole is analyzed with respect to a static reference frame whose axes point towards the "fixed stars." The accumulated precession angle after a complete scattering process is evaluated and compared with the corresponding change in the orbital angle. Limiting results for the non-rotating Schwarzschild black hole case are also discussed.

67. Bini D., Geralico A., Jantzen R.T.,
Gyroscope precession along general timelike geodesics in a Kerr black hole spacetime
Phys. Rev. D **95**, 124022 (2017)
e-print arXiv:1703.09525 [gr-qc].
Abstract
The precession angular velocity of a gyroscope moving along a general geodesic in the Kerr spacetime is analyzed using the geometric properties of the spacetime. Natural frames along the gyroscope world line are explicitly constructed by boosting frames adapted to fundamental observers. A novel geometrical description is given to Marck's construction of a parallel propagated orthonormal frame along a general geodesic, identifying and clarifying the special role played by the Carter family of observers in this general context, thus extending previous discussion for the equatorial plane case.

68. Bini D., Geralico A., Ortolan A.,
Deviation and precession effects in the field of a weak gravitational wave
Phys. Rev. D **95**, 104044 (2017)
Abstract
Deviation and precession effects of a bunch of spinning particles in the field of a weak gravitational plane wave are studied according to the Mathisson-Papapetrou-Dixon (MPD) model. Before the passage of the wave the particles are at rest with an associated spin vector aligned along a given direction with constant magnitude. The interaction with

the gravitational wave causes the particles to keep moving on the 2-plane orthogonal to the direction of propagation of the wave, with the transverse spin vector undergoing oscillations around the initial orientation. The transport equations for both the deviation vector and spin vector between two neighboring world lines of such a congruence are then solved by a suitable extension of the MPD model off the spinning particle's world line. In order to obtain measurable physical quantities a "laboratory" is set up by constructing a Fermi coordinate system attached to a reference world line. The exact transformation between TT coordinates and Fermi coordinates is derived too.

69. Bini D., Chicone C., Mashhoon B.,
Relativistic Tidal Acceleration of Astrophysical Jets
Phys. Rev. D **95**, 104029 (2017)
Abstract
Within the framework of general relativity, we investigate the tidal acceleration of astrophysical jets relative to the central collapsed configuration ("Kerr source"). To simplify matters, we neglect electromagnetic forces throughout; however, these must be included in a complete analysis. The rest frame of the Kerr source is locally defined via the set of hypothetical static observers in the spacetime exterior to the source. Relative to such a fiducial observer fixed on the rotation axis of the Kerr source, jet particles are tidally accelerated to almost the speed of light if their outflow speed is above a certain threshold, given roughly by one-half of the Newtonian escape velocity at the location of the reference observer; otherwise, the particles reach a certain height, reverse direction and fall back toward the gravitational source.

70. Bini D., Geralico A.
Hyperbolic-like elastic scattering of spinning particles by a Schwarzschild black hole
Gen. Rel. Gravit. **49**, 84 (2017)
Abstract
The scattering of spinning test particles by a Schwarzschild black hole is studied. The motion is described according to the MathissonPapapetrouDixon model for extended bodies in a given gravitational background field. The equatorial plane is taken as the orbital plane, the spin vector being orthogonal to it with constant magnitude. The equations of

motion are solved analytically in closed form to first-order in spin and the solution is used to compute corrections to the standard geodesic scattering angle as well as capture cross section by the black hole.

71. Kavanagh C., Bini D., Damour T., Hopper S., Ottewill A.C., Wardell B.
Spin-orbit precession along eccentric orbits for extreme mass ratio black hole binaries and its effective-one-body transcription
Phys. Rev. D **96**064012 (2017)
e-print arXiv:1706.00459 [gr-qc].
Abstract
In this work we present an analytical gravitational self-force calculation of the spin-orbit precession along an eccentric orbit around a Schwarzschild black hole, following closely the recent prescription of Akcay, Dempsey, and Dolan, giving results to six post-Newtonian orders expanded in small eccentricity through e2. We then transcribe this quantity within the effective-one-body (EOB) formalism, thereby determining several new, linear-in-mass-ratio contributions in the post-Newtonian expansion of the spin-orbit couplings entering the EOB Hamiltonian. Namely, we determine the second gyrogravitomagnetic ratio gS *(r ,pr,p?) up to order pr2/r4 included.

72. Bini D., Damour T.,
Gravitational scattering of two black holes at the fourth post-Newtonian approximation
Phys. Rev. D, **96**, 064021 (2017)
e-print arXiv:1706.06877v1 [gr-qc]
Abstract
We compute the (center-of-mass frame) scattering angle ? of hyperboliclike encounters of two spinning black holes, at the fourth post-Newtonian approximation level for orbital effects, and at the next-to-next-to-leading order for spin-dependent effects. We find it convenient to compute the gauge-invariant scattering angle (expressed as a function of energy, orbital angular momentum and spins) by using the effective-one-body formalism. The contribution to scattering associated with nonlocal, tail effects is computed by generalizing to the case of unbound motions the method of time localization of the action introduced in the case of (small-eccentricity) bound motions by Damour et al. [Phys. Rev. D 91, 084024 (2015), 10.1103/PhysRevD.91.084024].

73. Bini D., Geralico A., Jantzen R.T.,
Position determination and strong field parallax effects for photon emitters in the Schwarzschild spacetime
Gen. Rel. Gravit. 49, 84 (2017)
e-print arXiv:1707.00955
Abstract
The scattering of spinning test particles by a Schwarzschild black hole is studied. The motion is described according to the Mathisson-Papapetrou-Dixon model for extended bodies in a given gravitational background field. The equatorial plane is taken as the orbital plane, the spin vector being orthogonal to it with constant magnitude. The equations of motion are solved analytically in closed form to first-order in spin and the solution is used to compute corrections to the standard geodesic scattering angle as well as capture cross section by the black hole.

74. Bini D., Geralico A., Vines J.,
Hyperbolic scattering of spinning particles by a Kerr black hole
Phys. Rev. D, **96**, no. 8, 084044 (2017)
e-print arXiv:1707.09814
Abstract
The scattering of spinning test particles by a Schwarzschild black hole is studied. The motion is described according to the Mathisson-Papapetrou-Dixon model for extended bodies in a given gravitational background field. The equatorial plane is taken as the orbital plane, the spin vector being orthogonal to it with constant magnitude. The equations of motion are solved analytically in closed form to first-order in spin and the solution is used to compute corrections to the standard geodesic scattering angle as well as capture cross section by the black hole.

75. Bini D., Chicone C., Mashhoon B.,
Anisotropic gravitational collapse and cosmic Jets
Phys. Rev. D 96, 084034 (2017).
[arXiv:1708.01040 [gr-qc]].
Abstract
Consider a dynamic general relativistic spacetime in which the proper infinitesimal interval along one spatial coordinate direction decreases monotonically with time, while the corresponding intervals increase along other spatial directions. In a system undergoing such complete

anisotropic collapse/expansion, we look for the formation of a cosmic double-jet configuration: free test particles in the ambient medium, relative to the collapsing system, gain energy from the gravitational field and asymptotically line up parallel and antiparallel to the direction of collapse such that their Lorentz factors approach infinity. A strong burst of electromagnetic radiation is expected to accompany this event if some of the free test particles carry electric charge. Previous work in this direction involved mainly Ricci-flat spacetimes; hence, we concentrate here on inhomogeneous perfect fluid spacetimes. We briefly explore the possible connection between these theoretical cosmic jets and astrophysical jets. We also discuss other general relativistic scenarios for the formation of cosmic jets.

76. Bini D., Damour T.,
Gravitational spin-orbit coupling in binary systems, post-Minkowskian approximation and effective one-body theory
Phys. Rev. D, **96**, 104038 (2017)
e-print arXiv: 1709.00590 [gr-qc]
Abstract
A novel approach for extracting gauge-invariant information about spin-orbit coupling in gravitationally interacting binary systems is introduced. This approach is based on the "scattering holonomy", i.e. the integration (from the infinite past to the infinite future) of the differential spin evolution along the two worldlines of a binary system in hyperboliclike motion. We apply this approach to the computation, at the first post-Minkowskian approximation (i.e. first order in G and all orders in v/c), of the values of the two gyrogravitomagnetic ratios describing spin-orbit coupling in the Effective One-Body formalism. These gyrogravitomagnetic ratios are found to tend to zero in the ultrarelativistic limit.

Books and book chapters

1. (**Book)** Ferrarese G., Bini D.,
Introduction to relativistic continuum mechanics,
Lecture Notes in Physics **727**, Ed. Springer, 2007.

2. (**Book)** De Felice F., Bini D.,
Classical Measurements in Curved Space-Times

Series: Cambridge Monographs on Mathematical Physics, Cambridge, UK, 2010

Brief description

The theory of relativity describes the laws of physics in a given space-time. However, a physical theory must provide observational predictions expressed in terms of measurements, which are the outcome of practical experiments and observations. Ideal for readers with a mathematical background and a basic knowledge of relativity, this book will help readers understand the physics behind the mathematical formalism of the theory of relativity. It explores the informative power of the theory of relativity, and highlights its uses in space physics, astrophysics and cosmology. Readers are given the tools to pick out from the mathematical formalism those quantities that have physical meaning and which can therefore be the result of a measurement. The book considers the complications that arise through the interpretation of a measurement, which is dependent on the observer who performs it. Specific examples of this are given to highlight the awkwardness of the problem.
Provides a large sample of observers and reference frames in space-times that can be applied to space physics, astrophysics and cosmology. Tackles the problems encountered in interpreting measurements, giving specific examples. Features advice to help readers understand the logic of a given theory and its limitations.

Contents
1. Introduction; 2. The theory of relativity: a mathematical overview; 3. Space-time splitting; 4. Special frames; 5. The world function; 6. Local measurements; 7. Non-local measurements; 8. Observers in physical relevant space-times; 9. Measurements in physically relevant space-times; 10. Measurements of spinning bodies.

APPENDICES

A. Spacetime splitting techniques in general relativity

The concept of a "gravitational force" modeled after the electromagnetic Lorentz force was born in the Newtonian context of centrifugal and Coriolis "fictitious" forces introduced by a rigidly rotating coordinate system in a flat Euclidean space. Bringing this idea first into linearized general relativity and then into its fully nonlinear form, it has found a number of closely related but distinct generalizations. Regardless of the details, this analogy between gravitation and electromagnetism has proven useful in interpreting the results of spacetime geometry in terms we can relate to, and has been illustrated in many research articles and textbooks over the past half century.

ICRANet has itself devoted a workshop and its proceedings to aspects of this topic in 2003 [2]. In the lengthy introduction to these proceedings, R. Ruffini has discussed a number of related topics, like "the gravitational analogue of the Coulomb-like interactions, of Hertz-like wave solutions, of the Oersted-Ampére-like magnetic interaction, etc.," supporting the thesis that treating gravitation in analogy with electromagnetism may help to better understand the main features of certain gravitational phenomena, at least when the gravitational field may be considered appropriately described by its linearized approximation [12, 13, 14, 15, 16, 17, 18]. A particularly long bibliography surveying most of the relevant literature through 2001 had been published earlier in the Proceedings of one of the annual Spanish Relativity Meetings [19].

In the 1990s, working in fully nonlinear general relativity, all of the various notions of "noninertial forces" (centrifugal and Coriolis forces) were put into a single framework by means of a unifying formalism dubbed "gravitoelectromagnetism" [1, 3, 4] which is a convenient framework to deal with these and curvature forces and related questions of their effect on test bodies moving in the gravitational field. More precisely, such a language is based on the splitting of spacetime into "space plus time," accomplished locally by means of an observer congruence, namely a congruence of timelike worldlines with

(future-pointing) unit tangent vector field u which may be interpreted as the 4-velocity field of a family of test observers filling some region of spacetime. The orthogonal decomposition of each tangent space into a local time direction along u and the orthogonal local rest space (LRS) is used to decompose all spacetime tensors and tensor equations into a "space plus time" representation; the latter representation is somehow equivalent to a geometrical "measurement" process. This leads to a family of "spatial" spacetime tensor fields which represent each spacetime field and a family of spatial equations which represent each spacetime equation. Dealing with spacetime splitting techniques as well as 3-dimensional-like quantities clearly permits a better interface of our intuition and experience with the 4-dimensional geometry in certain gravitational problems. It can be particularly useful in spacetimes which have a geometrically defined timelike congruence, either explicitly given or defined implicitly as the congruence of orthogonal trajectories to a slicing or foliation of spacetime by a family of privileged spacelike hypersurfaces.

For example, splitting techniques are useful in the following spacetimes:

1. Stationary spacetimes, having a preferred congruence of Killing trajectories associated with the stationary symmetry, which is timelike on a certain region of spacetime (usually an open region, the boundary of which corresponding to the case in which the Killing vector becomes null so that in the exterior region Killing trajectories are spacelike).

2. Stationary axially symmetric spacetimes having in addition a preferred slicing whose orthogonal trajectories coincide with the worldlines of locally nonrotating test observers.

3. Cosmological spacetimes with a spatial homogeneity subgroup, which have a preferred spacelike slicing by the orbits of this subgroup.

From the various schools of relativity that blossomed during the second half of the last century a number of different approaches to spacetime splitting were developed without reference to each other. During the 1950s efforts were initiated to better understand general relativity and the mathematical tools needed to flush out its consequences. Lifshitz and the Russian school, Lichnerowicz in France, the British school, scattered Europeans (Ehlers and Trautman, for example) and the Americans best represented by Wheeler initiated this wave of relativity which blossomed in the 1960s. The textbook of Landau and Lifshitz and articles of Zelmanov [20, 21, 22, 23] presented the

"threading point of view" of the Russian school and of Moller [21] which influenced Cattaneo in Rome and his successor Ferrarese [24, 25, 26, 27, 28], while a variation of this approach not relying on a complementary family of hypersurfaces (the "congruence point of view) began from work initially codified by Ehlers [17] and then taken up by Ellis [29, 30] in analyzing cosmological issues.

However, issues of quantum gravity lead to the higher profile of the "slicing point of view" in the 1960s initiated earlier by Lichnerowicz and developed by Arnowitt, Deser and Misner and later promoted by the influential textbook "Gravitation" by Misner, Thorne and Wheeler [31, 32, 33, 34] represents a splitting technique which is complementary to the threading point of view and its congruence variation, and proved quite useful in illuminating properties of black hole spacetimes.

R. Ruffini, a former student of Cattaneo and a collaborator of Wheeler, in his quest to better understand electromagnetic properties of black holes, awakened the curiosity of Jantzen and Carini at the end of the 1980s, later joined by Bini, who together made an effort to clarify the interrelationships between these various approaches as well as shed some light on the then confusing work of Abramowicz and others on relativistic centrifugal and Coriolis forces. By putting them all in a common framework, and describing what each measured in geometrical terms, and how each related to the others, some order was brought to the field [1, 3, 4].

The ICRANet people working on this subject have applied the main ideas underlying spacetime splitting techniques to concrete problems arising when studying test particle motion in black hole spacetimes. Among the various results obtained it is worth mentioning the relativistic and geometrically correct definition of inertial forces in general relativity [35, 36, 37, 38, 39], the definition of special world line congruences, relevant for the description of the motion of test particles along circular orbits in the Kerr spacetime (geodesic meeting point observers, extremely accelerated observers, etc.), the specification of all the geometrical properties concerning observer-adapted frames to the above mentioned special world line congruences [40, 41], the characterization of certain relevant tensors in black hole spacetimes (Simon tensor, Killing-Yano tensor) in terms of gravitoelectromagnetism [42, 43], etc. This research line is still ongoing and productive.

Over a period of several decades Jantzen, Bini and a number of students at the University of Rome "La Sapienza" under the umbrella of the Rome ICRA group have been working on this problem under the supervision of Ruffini.

The collaborators involved have been already listed and the most relevant papers produced are indicated in the references below [44]–[83]. In the present year 2010 a book by F. de Felice and D. Bini, including a detailed discussion of this and related topics, has been published by Cambridge University Press [157].

Let us now describe some fundamental notions of gravitoelectromagnetism.

A.1. Observer-orthogonal splitting

Let ${}^{(4)}\mathrm{g}$ (signature $-+++$ and components ${}^{(4)}g_{\alpha\beta}$, $\alpha, \beta, \ldots = 0, 1, 2, 3$) be the spacetime metric, ${}^{(4)}\nabla$ its associated covariant derivative operator, and ${}^{(4)}\eta$ the unit volume 4-form which orients spacetime (${}^{(4)}\eta_{0123} = {}^{(4)}g^{1/2}$ in an oriented frame, where ${}^{(4)}g \equiv |\det({}^{(4)}g_{\alpha\beta})|$). Assume the spacetime is also time oriented and let u be a future-pointing unit timelike vector field ($u^\alpha u_\alpha = -1$) representing the 4-velocity field of a family of test observers filling the spacetime (or some open submanifold of it).

If S is an arbitrary tensor field, let $S^\flat$ and $S^\sharp$ denote its totally covariant and totally contravariant forms with respect to the metric index-shifting operations. It is also convenient to introduce the right contraction notation $[S \llcorner X]^\alpha = S^\alpha{}_\beta X^\beta$ for the contraction of a vector field and the covariant index of a $\binom{1}{1}$-tensor field (left contraction notation being analogous).

A.1.1. The measurement process

The observer-orthogonal decomposition of the tangent space, and in turn of the algebra of spacetime tensor fields, is accomplished by the temporal projection operator $T(u)$ along u and the spatial projection operator $P(u)$ onto LRS_u, which may be identified with mixed second rank tensors acting by contraction

$$\begin{aligned} \delta^\alpha{}_\beta &= T(u)^\alpha{}_\beta + P(u)^\alpha{}_\beta \,, \\ T(u)^\alpha{}_\beta &= -u^\alpha u_\beta \,, \\ P(u)^\alpha{}_\beta &= \delta^\alpha{}_\beta + u^\alpha u_\beta \,. \end{aligned} \tag{A.1.1}$$

These satisfy the usual orthogonal projection relations $P(u)^2 = P(u)$, $T(u)^2 = T(u)$, and $T(u) \llcorner P(u) = P(u) \llcorner T(u) = 0$. Let

$$[P(u)S]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^{\alpha}{}_{\gamma} \cdots P(u)^{\delta}{}_{\beta} \cdots S^{\gamma\ldots}{}_{\delta\ldots} \tag{A.1.2}$$

denote the spatial projection of a tensor S on all indices.

The *measurement of* S by the observer congruence is the family of spatial tensor fields which result from the spatial projection of all possible contractions of S by any number of factors of u. For example, if S is a $\binom{1}{1}$-tensor, then its measurement

$$S^{\alpha}{}_{\beta} \leftrightarrow (\underbrace{u^{\delta}u_{\gamma}S^{\gamma}{}_{\delta}}_{scalar}, \underbrace{P(u)^{\alpha}{}_{\gamma}u^{\delta}S^{\gamma}{}_{\delta}}_{vector}, \underbrace{P(u)^{\delta}{}_{\alpha}u_{\gamma}S^{\gamma}{}_{\delta}}_{vector}, \underbrace{P(u)^{\alpha}{}_{\gamma}P(u)^{\delta}{}_{\beta}S^{\gamma}{}_{\delta}}_{tensor}) \tag{A.1.3}$$

results in a scalar field, a spatial vector field, a spatial 1-form and a spatial $\binom{1}{1}$-tensor field. It is exactly this family of fields which occur in the orthogonal "decomposition of S" with respect to the observer congruence

$$\begin{aligned} S^{\alpha}{}_{\beta} &= [T(u)^{\alpha}{}_{\gamma} + P(u)^{\alpha}{}_{\gamma}][T(u)^{\delta}{}_{\beta} + P(u)^{\delta}{}_{\beta}]S^{\gamma}{}_{\delta} \\ &= [u^{\delta}u_{\gamma}S^{\gamma}{}_{\delta}]u^{\alpha}u_{\beta} + \cdots + [P(u)S]^{\alpha}{}_{\beta}\,. \end{aligned} \tag{A.1.4}$$

A.2. Examples

1. Measurement of the spacetime metric and volume 4-form
   - spatial metric $\quad [P(u)^{(4)}\mathrm{g}]_{\alpha\beta} = P(u)_{\alpha\beta}$
   - spatial unit volume 3-form $\quad \eta(u)_{\alpha\beta\gamma} = u^{\delta}{}^{(4)}\eta_{\delta\alpha\beta\gamma}$;
     In a compact notation: $\eta(u) = [P(u)\, u \lrcorner {}^{(4)}\eta]$
2. Measurement of the Lie, exterior and covariant derivative
   - spatial Lie derivative $\pounds(u)_X = P(u)\pounds_X$
   - the spatial exterior derivative $d(u) = P(u)d$
   - the spatial covariant derivative $\nabla(u) = P(u)^{(4)}\nabla$
   - the spatial Fermi-Walker derivative (or Fermi-Walker temporal derivative) $\nabla_{(\mathrm{fw})}(u) = P(u)^{(4)}\nabla_u$ (when acting on spatial fields)

- the Lie temporal derivative $\nabla_{(\mathrm{lie})}(u) = P(u)\pounds_u = \pounds(u)_u$

Note that spatial differential operators do not obey the usual product rules for nonspatial fields since undifferentiated factors of u are killed by the spatial projection.

3. Notation for 3-dimensional operations

   It is convenient to introduce 3-dimensional vector notation for the spatial inner product and spatial cross product of two spatial vector fields X and Y. The inner product is just

$$X \cdot_u Y = P(u)_{\alpha\beta} X^\alpha Y^\beta \tag{A.2.1}$$

   while the cross product is

$$[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma} X^\beta Y^\gamma \,. \tag{A.2.2}$$

   With the "vector derivative operator" $\nabla(u)^\alpha$ one can introduce spatial gradient, curl and divergence operators for functions f and spatial vector fields X by

$$\begin{aligned} \mathrm{grad}_u f &= \nabla(u) f = [d(u) f]^\sharp \,, \\ \mathrm{curl}_u X &= \nabla(u) \times_u X = [{}^{*(u)} d(u) X^\flat]^\sharp \,, \\ \mathrm{div}_u X &= \nabla(u) \cdot_u X = {}^{*(u)}[d(u)\, {}^{*(u)} X^\flat] \,, \end{aligned} \tag{A.2.3}$$

   where ${}^{*(u)}$ is the spatial duality operation for antisymmetric tensor fields associated with the spatial volume form $\eta(u)$ in the usual way. These definitions enable one to mimic all the usual formulas of 3-dimensional vector analysis. For example, the spatial exterior derivative formula for the curl has the index form

$$[\mathrm{curl}_u X]^\alpha = \eta(u)^{\alpha\beta\gamma (4)}\nabla_\beta X_\gamma \tag{A.2.4}$$

   which also defines a useful operator for nonspatial vector fields X.

4. Measurement of the covariant derivative of the observer four velocity

   Measurement of the covariant derivative $[{}^{(4)}\nabla u]^\alpha{}_\beta = u^\alpha{}_{;\beta}$ leads to two spatial fields, the acceleration vector field $a(u)$ and the kinematical mixed

tensor field $k(u)$

$$\begin{aligned} u^{\alpha}{}_{;\beta} &= -a(u)^{\alpha}u_{\beta} - k(u)^{\alpha}{}_{\beta}\,, \\ a(u) &= \nabla_{(\mathrm{fw})}(u)u\,, \\ k(u) &= -\nabla(u)u\,. \end{aligned} \tag{A.2.5}$$

The kinematical tensor field may be decomposed into its antisymmetric and symmetric parts:

$$k(u) = \omega(u) - \theta(u)\,, \tag{A.2.6}$$

with

$$\begin{aligned} [\omega(u)^{\flat}]_{\alpha\beta} &= P(u)^{\sigma}_{\alpha}P(u)^{\delta}_{\beta}u_{[\delta;\sigma]} \\ &= \tfrac{1}{2}[d(u)u^{\flat}]_{\alpha\beta}\,, \\ [\theta(u)^{\flat}]_{\alpha\beta} &= P(u)^{\sigma}_{\alpha}P(u)^{\delta}_{\beta}u_{(\delta;\sigma)} \\ &= \tfrac{1}{2}[\nabla_{(\mathrm{lie})}(u)P(u)^{\flat}]_{\alpha\beta} = \tfrac{1}{2}\pounds(u)u^{(4)}g_{\alpha\beta}\,, \end{aligned} \tag{A.2.7}$$

defining the mixed rotation or vorticity tensor field $\omega(u)$ (whose sign depends on convention) and the mixed expansion tensor field $\theta(u)$, the latter of which may itself be decomposed into its tracefree and pure trace parts

$$\theta(u) = \sigma(u) + \frac{1}{3}\Theta(u)P(u)\,, \tag{A.2.8}$$

where the mixed shear tensor field $\sigma(u)$ is tracefree ($\sigma(u)^{\alpha}{}_{\alpha} = 0$) and the expansion scalar is

$$\Theta(u) = u^{\alpha}{}_{;\alpha} = {}^{*(u)}[\nabla_{(\mathrm{lie})}(u)\eta(u)]\,. \tag{A.2.9}$$

Define also the rotation or vorticity vector field $\omega(u) = \frac{1}{2}\operatorname{curl}_u u$ as the spatial dual of the spatial rotation tensor field

$$\omega(u)^{\alpha} = \tfrac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} = \tfrac{1}{2}{}^{(4)}\eta^{\alpha\beta\gamma\delta}u_{\beta}u_{\gamma;\delta}\,. \tag{A.2.10}$$

5. Lie, Fermi-Walker and co-Fermi-Walker derivatives

The kinematical tensor describes the difference between the Lie and Fermi-Walker temporal derivative operators when acting on spatial ten-

sor fields. For example, for a spatial vector field X

$$\begin{aligned}\nabla_{(\mathrm{fw})}(u)X^\alpha &= \nabla_{(\mathrm{lie})}(u)X^\alpha - k(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha - \omega(u)^\alpha{}_\beta X^\beta + \theta(u)^\alpha{}_\beta X^\beta \,,\end{aligned} \tag{A.2.11}$$

where

$$\omega(u)^\alpha{}_\beta X^\beta = -\eta(u)^\alpha{}_{\beta\gamma}\omega(u)^\beta X^\gamma = -[\omega(u) \times_u X]^\alpha \,. \tag{A.2.12}$$

The kinematical quantities associated with u may be used to introduce two spacetime temporal derivatives, the Fermi-Walker derivative and the co-rotating Fermi-Walker derivative along u

$$\begin{aligned}{}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha &= {}^{(4)}\nabla_u X^\alpha + [a(u) \wedge u]^{\alpha\beta}X_\beta \,, \\ {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \,.\end{aligned} \tag{A.2.13}$$

These may be extended to arbitrary tensor fields in the usual way (so that they commute with contraction and tensor products) and they both commute with index shifting with respect to the metric and with duality operations on antisymmetric tensor fields since both ${}^{(4)}\mathrm{g}$ and ${}^{(4)}\eta$ have zero derivative with respect to both operators (as does u itself). For an arbitrary vector field X the following relations hold

$$\begin{aligned}\pounds_u X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + [\omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta]X^\beta \\ &= {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha + [-\theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta]X^\beta \,.\end{aligned} \tag{A.2.14}$$

A spatial co-rotating Fermi-Walker derivative $\nabla_{(\mathrm{cfw})}(u)$ ("co-rotating Fermi-Walker temporal derivative") may be defined in a way analogous to the ordinary one, such that the three temporal derivatives have the following relation when acting on a spatial vector field X

$$\begin{aligned}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= \nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha + \theta(u)^\alpha{}_\beta X^\beta \,,\end{aligned} \tag{A.2.15}$$

while $\nabla_{(\mathrm{cfw})}(u)[fu] = fa(u)$ determines its action on nonspatial fields.

It has been introduced an index notation to handle these three operators simultaneously

$$\{\nabla_{(\mathrm{tem})}(u)\}_{\mathrm{tem=fw,cfw,lie}} = \{\nabla_{(\mathrm{fw})}(u), \nabla_{(\mathrm{cfw})}(u), \nabla_{(\mathrm{lie})}(u)\} \,. \quad \text{(A.2.16)}$$

A.3. Comparing measurements by two observers in relative motion

Suppose U is another unit timelike vector field representing a different family of test observers. One can then consider relating the "observations" of each to the other. Their relative velocities are defined by

$$\begin{aligned} U &= \gamma(U,u)[u + v(U,u)] \,, \\ u &= \gamma(u,U)[U + v(u,U)] \,, \end{aligned} \quad \text{(A.3.1)}$$

where the relative velocity $v(U,u)$ of U with respect to u is spatial with respect to u and vice versa, both of which have the same magnitude $||v(U,u)|| = [v(U,u)_\alpha v(U,u)^\alpha]^{1/2}$, while the common gamma factor is related to that magnitude by

$$\gamma(U,u) = \gamma(u,U) = [1 - ||v(U,u)||^2]^{-1/2} = -U_\alpha u^\alpha \,. \quad \text{(A.3.2)}$$

Let $\hat{v}(U,u)$ be the unit vector giving the direction of the relative velocity $v(U,u)$. In addition to the natural parametrization of the worldlines of U by the proper time τ_U, one may introduce two new parametrizations: by a (Cattaneo) relative standard time $\tau_{(U,u)}$

$$d\tau_{(U,u)}/d\tau_U = \gamma(U,u) \,, \quad \text{(A.3.3)}$$

which corresponds to the sequence of proper times of the family of observers from the u congruence which cross paths with a given worldline of the U congruence, and by a relative standard lenght $\ell_{(U,u)}$

$$d\ell_{(U,u)}/d\tau_U = \gamma(U,u)||v(U,u)|| = ||v(U,u)||d\tau_{(U,u)}/d\tau_U \,, \quad \text{(A.3.4)}$$

which corresponds to the spatial arc lenght along U as observed by u.

Eqs. (A.3.1) describe a unique active "relative observer boost" $B(U,u)$ in

the "relative observer plane" spanned by u and U such that

$$B(U,u)u = U\,, \quad B(U,u)v(U,u) = -v(u,U) \tag{A.3.5}$$

and which acts as the identity on the common subspace of the local rest spaces $LRS_u \cap LRS_U$ orthogonal to the direction of motion.

A.3.1. Maps between the LRSs of different observers

The projection $P(U)$ restricts to an invertible map when combined with $P(u)$ as follows

$$P(U,u) = P(U) \circ P(u) : LRS_u \to LRS_U \tag{A.3.6}$$

with inverse $P(U,u)^{-1} : LRS_U \to LRS_u$ and vice versa, and these maps also act as the identity on the common subspace of the local rest spaces.

Similarly the boost $B(U,u)$ restricts to an invertible map

$$B_{(\mathrm{lrs})}(U,u) \equiv P(U) \circ B(U,u) \circ P(u) \tag{A.3.7}$$

between the local rest spaces which also acts as the identity on their common subspace. The boosts and projections between the local rest spaces differ only by a gamma factor along the direction of motion.

An expression for the inverse projection

If $Y \in LRS_u$, then the orthogonality condition $0 = u_\alpha Y^\alpha$ implies that Y has the form

$$Y = [v(u,U) \cdot_U P(U,u)Y]U + P(U,u)Y\,. \tag{A.3.8}$$

If $X = P(U,u)Y \in LRS_U$ is the field seen by U, then $Y = P(U,u)^{-1}X$ and

$$P(U,u)^{-1}X = [v(u,U) \cdot_U X]U + X = [P(U) + U \otimes v(u,U)^\flat] \llcorner X\,, \tag{A.3.9}$$

which gives a useful expression for the inverse projection.

This map appears in the transformation law for the electric and magnetic fields:

$$\begin{aligned} E(u) &= \gamma P(U,u)^{-1}[E(U) + v(u,U) \times_U B(U)]\,, \\ B(u) &= \gamma P(U,u)^{-1}[B(U) - v(u,U) \times_U E(U)]\,. \end{aligned} \tag{A.3.10}$$

A.4. Comparing measurements by three or more observers in relative motion

A typical situation is that of a fluid/particle whis is observed by two diferrent families of observers. In this case one deal with three timelike congruences (or two congruences and a single line): the rest frame of the fluid U and the two observer families u e u'.

All the previous formalism can be easily generalized. One has

$$\begin{aligned} U &= \gamma(U,u)[u+\nu(U,u)]\,, \\ U &= \gamma(U,u')[u'+\nu(U,u')]\,, \\ u' &= \gamma(u',u)[u+\nu(u',u)]\,, \\ u &= \gamma(u,u')[u'+\nu(u,u')]\,. \end{aligned} \tag{A.4.1}$$

and mixed projectors involving the various four-velocities can be introduced. They are summarized in the following table:

PROJECTORS	
$P(u,U,u)$	$P(u)+\gamma(U,u)^2\nu(U,u)\otimes\nu(U,u)$
$P(u,U,u)^{-1}$	$P(u)-\nu(U,u)\otimes\nu(U,u)$
$P(u,U,u')$	$P(u,u')+\gamma(U,u)\gamma(U,u')\nu(U,u)\otimes\nu(U,u')$
$P(u,U,u')^{-1}$	$P(u',u)+\gamma(u,u')[(\nu(u,u')-\nu(U,u'))\otimes\nu(U,u)$ $+\nu(U,u')\otimes\nu(u',u)]$
$P(U,u)^{-1}P(U,u')$	$P(u,u')+\gamma(u,u')\nu(U,u)\otimes\nu(u,u')$
$P(u',u)P(U,u)^{-1}P(U,u')$	$P(u')+\delta(U,u,u')\nu(U,u')\otimes\nu(u,u')$
$P(u',u)P(u',U,u)^{-1}$	$P(u')+\delta(U,u,u')\nu(U,u')\otimes[\nu(u,u')-\nu(U,u')]$

where

$$\delta(U,u,u')=\frac{\gamma(U,u')\gamma(u',u)}{\gamma(U,u)}, \qquad \delta(U,u,u')^{-1}=\delta(u,U,u')\,, \tag{A.4.2}$$

and

$$P(u,U,u')=P(u,U)P(U,u')$$

A.5. Derivatives

Suppose one uses the suggestive notation

$$ {}^{(4)}D(U)/d\tau_U = {}^{(4)}\nabla_U \tag{A.5.1} $$

for the "total covariant derivative" along U. Its spatial projection with respect to u and rescaling corresponding to the reparametrization of Eq. (A.3.4) is then given by the "Fermi-Walker total spatial covariant derivative," defined by

$$ \begin{aligned} D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)} &= \gamma^{-1} D_{(\mathrm{fw},U,u)}/d\tau_U = \gamma^{-1} P(u)\,{}^{(4)}D(U)/d\tau_U \\ &= \nabla_{(\mathrm{fw})}(u) + \nabla(u)_{\nu(U,u)} \,. \end{aligned} \tag{A.5.2} $$

Extend this to two other similar derivative operators (the co-rotating Fermi-Walker and the Lie total spatial covariant derivatives) by

$$ D_{(\mathrm{tem},U,u)}/d\tau_{(U,u)} = \nabla_{(\mathrm{tem})}(u) + \nabla(u)_{\nu(U,u)} \,, \qquad \text{tem=fw,cfw,lie} \,, \tag{A.5.3} $$

which are then related to each other in the same way as the corresponding temporal derivative operators

$$ \begin{aligned} D_{(\mathrm{cfw},U,u)}X^\alpha/d\tau_{(U,u)} &= D_{(\mathrm{fw},U,u)}X^\alpha/d\tau_{(U,u)} + \omega(u)^\alpha{}_\beta X^\beta \\ &= D_{(\mathrm{lie},U,u)}X^\alpha/d\tau_{(U,u)} + \theta(u)^\alpha{}_\beta X^\beta \end{aligned} \tag{A.5.4} $$

when acting on a spatial vector field X. All of these derivative operators reduce to the ordinary parameter derivative $D/d\tau_{(U,u)} \equiv d/d\tau_{(U,u)}$ when acting on a function and extend in an obvious way to all tensor fields.

Introduce the ordinary and co-rotating Fermi-Walker and the Lie "relative accelerations" of U with respect to u by

$$ a_{(\mathrm{tem})}(U,u) = D_{(\mathrm{tem})}(U,u)\nu(U,u)/d\tau_{(U,u)} \,, \qquad \text{tem=fw,cfw,lie} \,. \tag{A.5.5} $$

These are related to each other in the same way as the corresponding derivative operators in Eq. (A.2.15).

The total spatial covariant derivative operators restrict in a natural way to a

single timelike worldline with 4-velocity U, where the $D/d\tau$ notation is most appropriate; ${}^{(4)}D(U)/d\tau_U$ is often called the absolute or intrinsic derivative along the worldline of U (associated with an induced connection along such a worldline).

A.6. Applications

A.6.1. Test-particle motion

Let's consider the motion of a unit mass test-particle with four velocity U, accelerated by an external force $f(U)$: $a(U) = f(U)$. A generic observer u can measure the particle four velocity U, obtaining its relative energy $E(U,u) = \gamma(U,u)$ and momentum $p(U,u) = \gamma(U,u)v(U,u)$,

$$U = E(U,u)[u + p(U,u)] = \gamma(U,u)[u + v(U,u)]. \tag{A.6.1}$$

Splitting the acceleration equation gives the evolution (along U) of the relative energy and momentum of the particle

$$\begin{aligned} \frac{dE(U,u)}{d\tau_{(U,u)}} &= [F^{(G)}_{(\mathrm{tem},U,u)} + F(U,u)] \cdot v(U,u) \\ &\quad + \epsilon_{(\mathrm{tem})}\gamma(U,u)v(U,u) \cdot (\theta(u) \llcorner v(U,u)) \\ \frac{D_{(\mathrm{tem})}p(U,u)}{d\tau_{(U,u)}} &= F^{(G)}_{(\mathrm{tem},U,u)} + F(U,u)\,, \end{aligned} \tag{A.6.2}$$

where tem=fw,cfw,lie,lie$^\flat$ refers to the various possible (i.e. geometrically meaningful) transport of vectors along U, $\epsilon_{(\mathrm{tem})} = (0,0,-1,1)$ respectively and

$$\begin{aligned} d\tau_{(U,u)} &= \gamma(U,u)d\tau_U \\ F^{(G)}_{(\mathrm{tem},U,u)} &= \gamma(U,u)[g(u) + H_{(\mathrm{tem},u)} \llcorner v(U,u)] \\ F(U,u) &= \gamma(U,u)^{-1}P(u,U)f(U) \end{aligned}$$

with

$$\begin{aligned} H_{(\mathrm{fw},u)} &= \omega(u) - \theta(u) & H_{(\mathrm{cfw},u)} &= 2\omega(u) - \theta(u) \\ H_{(\mathrm{lie},u)} &= 2\omega(u) - 2\theta(u) & H_{(\mathrm{lie}^\flat,u)} &= 2\omega(u)\,. \end{aligned} \tag{A.6.3}$$

The gravitoelectric vector field $g(u) = -a(u) = -\nabla_u u$ and the gravitomagnetic vector field $H(u) = 2[{}^{*(u)}\omega(u)^\flat]^\sharp$ of the observer u (sign-reversed acceleration and twice the vorticity vector field) are defined by the exterior derivative of u

$$du^\flat = [u \wedge g(u) + {}^{*(u)}H(u)]^\flat \,. \tag{A.6.4}$$

and will be essential in showing the analogy between the gravitational force $F^{(G)}_{(\mathrm{tem},U,u)}$ and the Lorentz force. The expansion scalar $\Theta(u) = \mathrm{Tr}\,\theta(u)$ appears in an additional term in the covariant derivative of u as the trace of the (mixed) expansion tensor $\theta(u)$, of which the shear tensor $\sigma(u) = \theta(u) - \frac{1}{3}\Theta(u)P(u)$ is its tracefree part

$$\nabla u = -a(u) \otimes u^\flat + \theta(u) - \omega(u) \,. \tag{A.6.5}$$

The term $D_{(\mathrm{tem})}p(U,u)/d\tau_{(U,u)}$ contains itself the "spatial geometry" contribution which must be added to the gravitational and the external force to reconstruct the spacetime point of view. Actually, this term comes out naturally and is significant all along the line of the particle: the single terms $\nabla_{(\mathrm{fw},u)}$ and $\nabla(u)_{\nu(U,u)}$, in which it can be further decomposed, are not individually meaningful unless one defines some extension for the spatial momentum $p(U,u)$ off the line of the particle, which of course is unnecessary at all.

From this spatial geometry contribution a general relativistic version of inertial forces can be further extracted.

A.6.2. Maxwell's equations

Maxwell's equations can be expressed covariantly in many ways. For instance, in differential form language one has

$$dF = 0 \,, \qquad d^*F = -4\pi^* J^\flat \,, \tag{A.6.6}$$

where F is the Faraday electromagnetic 2-form and J is the current vector field, obeying the conservation law

$$\delta J^\flat = {}^*d^*J^\flat = 0 \,. \tag{A.6.7}$$

The splitting of the electromagnetic 2-form F by any observer family (with unit 4-velocity vector field u) gives the associated electric and magnetic vector fields $E(u)$ and $B(u)$ as measured by those observers through the Lorentz force law on a test charge, and the relative charge and current density $\rho(u)$ and $J(u)$. The "relative observer decomposition" of F and its dual 2-form ${}^*F$ is

$$\begin{aligned} F &= [u \wedge E(u) + {}^{*(u)}B(u)]^\flat \,, \\ {}^*F &= [-u \wedge B(u) + {}^{*(u)}E(u)]^\flat \,, \end{aligned}$$

while J has the representation

$$J = \rho(u)u + J(u) \,. \tag{A.6.8}$$

If U is the 4-velocity of any test particle with charge q and nonzero rest mass m, it has the orthogonal decomposition

$$U = \gamma(U,u)[u + v(U,u)] \,. \tag{A.6.9}$$

Its absolute derivative with respect to a proper time parametrization of its world line is its 4-acceleration $a(U) = DU/d\tau_U$. The Lorentz force law then takes the form

$$ma(U) = q\gamma(U,u)[E(u) + v(U,u) \times_u B(u)] \,. \tag{A.6.10}$$

The relative observer formulation of Maxwell's equations is well known. Projection of the differential form equations (A.6.6) along and orthogonal to u gives the spatial scalar (divergence) and spatial vector (curl) equations:

$$\begin{aligned} &\mathrm{div}_u B(u) + \vec{H}(u) \cdot_u E(u) = 0 \,, \\ &\mathrm{curl}_u E(u) - \vec{g}(u) \times_u E(u) + [\pounds(u)_u + \Theta(u)]B(u) = 0 \,, \\ &\mathrm{div}_u E(u) - \vec{H}(u) \cdot_u B(u) = 4\pi\rho(u) \,, \\ &\mathrm{curl}_u B(u) - \vec{g}(u) \times_u B(u) - [\pounds(u)_u + \Theta(u)]E(u) = 4\pi J(u) \,, \end{aligned} \tag{A.6.11}$$

This representation of Maxwell's equations differs from the Ellis representation only in the use of the spatially projected Lie derivative rather than the spatially projected covariant derivative along u (spatial Fermi-Walker deriva-

tive). These two derivative operators are related by the following identity for a spatial vector field X (orthogonal to u)

$$[\pounds(u)_u + \Theta(u)]X = [\nabla(u)_u + \{-\sigma(u) + \omega(u)\}\sqcup]X\,. \tag{A.6.12}$$

It is clear, at this point, that for any spacetime tensor equation the "1+3" associated version allows one to read it in a Newtonian form and to interpret it quasi-classically.

For instance one can consider motion of test fields in a given gravitational background (i.e. neglecting backreaction) as described by spacetime equations and look at their "1+3" counterpart. Over the last ten years, in a similar way in which we have discussed the splitting of Maxwell's equations in integral formulation, we have studied scalar field, spinorial field (Dirac fields), fluid motions, etc.

B. Motion of particles and extended bodies in General Relativity

The motion of an extended body in a given background may be studied by treating the body via a multipole expansion. The starting point of this method is the covariant conservation law

$$\nabla_\mu T^\mu{}_\nu = 0 \,, \tag{B.0.1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor describing the body. The body sweeps out a narrow tube in spacetime as it moves. Let L be a line inside this tube representing the motion of the body. Denote the coordinates of the points of this line by X^α, and define the displacement $\delta x^\alpha = X^\alpha - x^\alpha$, where x^α are the coordinates of the points of the body. Let us consider now the quantities

$$\int T^{\mu\nu} \mathrm{d}V \,, \qquad \int \delta x^\lambda T^{\mu\nu} \mathrm{d}V \,, \qquad \int \delta x^\lambda \delta x^\rho T^{\mu\nu} \mathrm{d}V \,, \quad \ldots \tag{B.0.2}$$

where the integrations are carried out on the 3-dimensional hypersurfaces of fixed time $t = X^0 = const$, the tensor $T^{\mu\nu}$ being different from zero only inside the world tube: these are the successive terms of the multipole expansion. A single-pole particle is defined as a particle that has nonvanishing at least some of the integrals in the first (monopole) term, assuming that all the integrals containing δx^μ vanish. A pole-dipole particle, instead, is defined as a particle for which all the integrals with more than one factor of δx^μ (dipole term) vanish. Higher order approximations may be defined in a similar way. Thus, a single-pole particle is a test particle without any internal structure. A pole-dipole particle, instead, is a test particle whose internal structure is

expressed by its spin, an antisymmetric second-rank tensor defined by

$$S^{\mu\nu} \equiv \int \left[\delta x^{\mu} T^{0\nu} - \delta x^{\nu} T^{0\mu}\right] \mathrm{d}V \ . \tag{B.0.3}$$

The equations of motion are, then, obtained by applying the Einstein's field equations together with conservation of the energy-momentum tensor (B.0.1) describing the body. For a single-pole particle this leads to a free particle moving along the geodesics associated with the given background field. For the motion of a pole-dipole particle, instead, the corresponding set of equations was derived by Papapetrou [85] by using the above procedure. Obviously, the model is worked out under the assumption that the dimensions of the test particle are very small compared with the characteristic length of the basic field (i.e., with backreaction neglected), and that the gravitational radiation emitted by the particle in its motion is negligible. As a final remark, note that this model can be extended to charged bodies by considering in addition the conservation law of the current density.

B.1. The Mathisson-Papapetrou model

The equations of motion for a spinning (or pole-dipole) test particle in a given gravitational background were deduced by Mathisson and Papapetrou [85, 84] and read

$$\frac{DP^{\mu}}{\mathrm{d}\tau_U} = -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} U^{\nu} S^{\alpha\beta} \equiv F^{(\mathrm{spin})\mu} \ , \tag{B.1.1}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = P^{\mu} U^{\nu} - P^{\nu} U^{\mu} \ , \tag{B.1.2}$$

where P^{μ} is the total four-momentum of the particle, and $S^{\mu\nu}$ is a (antisymmetric) spin tensor; U is the timelike unit tangent vector of the "center of mass line" used to make the multipole reduction. Equations (B.1.1) and (B.1.2) define the evolution of P and S only along the world line of U, so a correct interpretation of U is that of being tangent to the *true* world-line of the spinning particle. The 4-momentum P and the spin tensor S are then defined as vector fields along the trajectory of U. By contracting both sides of Eq. (B.1.2)

with U_ν, one obtains the following expression for the total 4-momentum

$$P^\mu = -(U \cdot P)U^\mu - U_\nu \frac{DS^{\mu\nu}}{d\tau_U} \equiv mU^\mu + P_s^\mu \,, \tag{B.1.3}$$

where $m = -U \cdot P$ reduces to the ordinary mass in the case in which the particle is not spinning, and P_s is a 4-vector orthogonal to U.

The test character of the particle under consideration refers to its mass as well as to its spin, since both quantities should not be large enough to contribute to the background metric. In what follows, with the magnitude of the spin of the particle, with the mass and with a natural lengthscale associated with the gravitational background we will construct a dimensionless parameter as a smallness indicator, which we retain to the first order only so that the test character of the particle be fully satisfied. Moreover, in order to have a closed set of equations Eqs. (B.1.1) and (B.1.2) must be completed with supplementary conditions (SC), whose standard choices in the literature are the following

1. Corinaldesi-Papapetrou [86] conditions (CP): $S^{\mu\nu}(e_0)_\nu = 0$, where e_0 is the coordinate timelike direction given by the background;
2. Pirani [87] conditions (P): $S^{\mu\nu}U_\nu = 0$;
3. Tulczyjew [88] conditions (T): $S^{\mu\nu}P_\nu = 0$;

all of these are algebraic conditions.

Detailed studies concerning spinning test particles in General Relativity are due to Dixon [89, 90, 91, 92, 93], Taub [94], Mashhoon [95, 96] and Ehlers and Rudolph [97]. The Mathisson-Papapetrou model does not give *a priori* restrictions on the causal character of U and P and there is no agreement in the literature on how this point should be considered. For instance, Tod, de Felice and Calvani [98] consider P timelike, assuming that it represents the total energy momentum content of the particle, while they do not impose any causality condition on the world line U, which plays the role of a mere mathematical "tool" to perform the multipole reduction. Differently, according to Mashhoon [96], P can be considered analogously to the canonical momentum of the particle: hence, there should be not any meaning for its causality character, while the world line U has to be timelike (or eventually null) because it represents the center of mass line of the particle. This uncertainty in the model itself then reflects in the need for a supplementary condition, whose

choice among the three mentioned above is arbitrary, making the general relativistic description of a spinning test particle somehow unsatisfactory. When both U and P are timelike vectors as e_0, all of them can be taken as the 4-velocity field of a preferred observer family, and all the SC above state that for the corresponding observer the spin tensor is purely spatial. In a sense, only P and T supplementary conditions give "intrinsic" relations between the various unknown of the model and they should be somehow more physical conditions. In fact, the CP conditions are "coordinate dependent," being e_0 the coordinate timelike vector. It is worth to mention that grounded on physical reasons, Dixon has shown that the T conditions should be preferred with respect to the others.

B.2. The Dixon-Souriau model

The equations of motion for a charged spinning test particle in a given gravitational as well as electromagnetic background were deduced by Dixon-Souriau [99, 100, 101, 102]. They have the form

$$\frac{DP^\mu}{\mathrm{d}\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} + qF^\mu{}_\nu U^\nu - \frac{\lambda}{2}S^{\rho\sigma}\nabla^\mu F_{\rho\sigma} \equiv F^{(\mathrm{tot})\mu}\,, \quad \text{(B.2.1)}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = P^\mu U^\nu - P^\nu U^\mu + \lambda[S^{\mu\rho}F_\rho{}^\nu - S^{\nu\rho}F_\rho{}^\mu]\,, \quad \text{(B.2.2)}$$

where $F^{\mu\nu}$ is the electromagnetic field, P^μ is the total 4-momentum of the particle, and $S^{\mu\nu}$ is the spin tensor (antisymmetric); U is the timelike unit tangent vector of the "center of mass line" used to make the multipole reduction. As it has been shown by Souriau, the quantity λ is an arbitrary electromagnetic coupling scalar constant. We note that the special choice $\lambda = -q/m$ (see [46]) in flat spacetime corresponds to the Bargman-Michel-Telegdi [103] spin precession law.

B.3. Particles with quadrupolar structure

The equations of motion for an extended body in a given gravitational background were deduced by Dixon [89, 90, 91, 92, 93] in multipole approxima-

tion to any order. In the quadrupole approximation they read

$$\frac{DP^\mu}{\mathrm{d}\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}{}^{;\mu} \equiv F^{(\mathrm{spin})\mu} + F^{(\mathrm{quad})\mu} \quad \text{(B.3.1)}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = 2P^{[\mu}U^{\nu]} - \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]}{}_{\alpha\beta\gamma}\,, \quad \text{(B.3.2)}$$

where $P^\mu = mU_p^\mu$ (with $U_p \cdot U_p = -1$) is the total four-momentum of the particle, and $S^{\mu\nu}$ is a (antisymmetric) spin tensor; U is the timelike unit tangent vector of the "center of mass line" $\mathcal{C}_U$ used to make the multipole reduction, parametrized by the proper time τ_U. The tensor $J^{\alpha\beta\gamma\delta}$ is the quadrupole moment of the stress-energy tensor of the body, and has the same algebraic symmetries as the Riemann tensor. Using standard spacetime splitting techniques it can be reduced to the following form

$$J^{\alpha\beta\gamma\delta} = \Pi^{\alpha\beta\gamma\delta} - \bar{u}^{[\alpha}\pi^{\beta]\gamma\delta} - \bar{u}^{[\gamma}\pi^{\delta]\alpha\beta} - 3\bar{u}^{[\alpha}Q^{\beta][\gamma}\bar{u}^{\delta]}\,, \quad \text{(B.3.3)}$$

where $Q^{\alpha\beta} = Q^{(\alpha\beta)}$ represents the quadrupole moment of the mass distribution as measured by an observer with 4-velocity $\bar{u}$. Similarly $\pi^{\alpha\beta\gamma} = \pi^{\alpha[\beta\gamma]}$ (with the additional property $\pi^{[\alpha\beta\gamma]} = 0$) and $\Pi^{\alpha\beta\gamma\delta} = \Pi^{[\alpha\beta][\gamma\delta]}$ are essentially the body's momentum and stress quadrupoles. Moreover the various fields $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma}$ and $\Pi^{\alpha\beta\gamma\delta}$ are all spatial (i.e. give zero after any contraction by $\bar{u}$). The number of independent components of $J^{\alpha\beta\gamma\delta}$ is 20: 6 in $Q^{\alpha\beta}$, 6 in $\Pi^{\alpha\beta\gamma\delta}$ and 8 in $\pi^{\alpha\beta\gamma}$. When the observer $\bar{u} = U_p$, i.e. in the frame associated with the momentum of the particle, the tensors $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma}$ and $\Pi^{\alpha\beta\gamma\delta}$ have an intrinsic meaning.

There are no evolution equations for the quadrupole as well as higher multipoles as a consequence of the Dixon's construction, so their evolution is completely free, depending only on the considered body. Therefore the system of equations is not self-consistent, and one must assume that all unspecified quantities are known as intrinsic properties of the matter under consideration.

In order the model to be mathematically correct the following additional condition should be imposed to the spin tensor:

$$S^{\mu\nu}U_{p\nu} = 0. \quad \text{(B.3.4)}$$

Such supplementary conditions (or Tulczyjew-Dixon conditions [88, 89]) are

necessary to ensure the correct definition of the various multipolar terms.

Dixon's model for structured particles originated to complete and give a rigorous mathematical support to the previously introduced Mathisson-Papapetrou model [85, 84, 86, 87], i.e. a multipole approximation to any order which includes evolutional equations along the "center line" for all the various structural quantities. The models are then different and a comparison between the two is possible at the dipolar order but not once the involved order is the quadrupole.

Here we limit our considerations to Dixon's model under the further simplifying assumption[94, 97] that the only contribution to the complete quadrupole moment $J^{\alpha\beta\gamma\delta}$ stems from the mass quadrupole moment $Q^{\alpha\beta}$, so that $\pi^{\alpha\beta\gamma} = 0 = \Pi^{\alpha\beta\gamma\delta}$ and

$$J^{\alpha\beta\gamma\delta} = -3U_p^{[\alpha}Q^{\beta][\gamma}U_p^{\delta]}\,, \qquad Q^{\alpha\beta}U_{p\beta} = 0\,; \tag{B.3.5}$$

The assumption that the particle under consideration is a test particle means that its mass, its spin as well as its quadrupole moments must all be small enough not to contribute significantly to the background metric. Otherwise, backreaction must be taken into account.

B.4. Null multipole reduction world line: the massless case

The extension of the Mathisson-Papapetrou model to the case of a null multipole reduction world line l has been considered by Mashhoon [96]: the model equations have exactly the same form as (B.1.1) and (B.1.2), with U (timelike) replaced by l (null) for what concerns the multipole reduction world line and τ_U (proper time parametrization of the U line) replaced by λ (affine parameter along the l line):

$$\frac{DP^\alpha}{\mathrm{d}\lambda} = -\frac{1}{2}R^\alpha{}_{\beta\rho\sigma}l^\beta S^{\rho\sigma} \equiv F^{(\mathrm{spin})\alpha}\,, \tag{B.4.1}$$

$$\frac{DS^{\alpha\beta}}{\mathrm{d}\lambda} = [P\wedge l]^{\alpha\beta}\,. \tag{B.4.2}$$

Equations (B.4.1) and (B.4.2) should be then solved assuming some SC. Let us limit ourselves to the case of "intrinsic" SC, i.e. Pirani and Tulczyjew, with

Pirani's conditions now naturally generalized as $S^{\alpha\beta}l_\beta = 0$. Furthermore, we require $P \cdot l = 0$: in fact, we are interested to the massless limit of the Mathisson-Papapetrou equations, and as the mass of the particle is defined by $m = -P \cdot U$ the massless limit implies $-P \cdot l = 0$.

By denoting with $\{l = e_1, n = e_2, m = e_3, \bar{m} = e_4\}$ a complex null frame along the center line l, such that $l \cdot l = n \cdot n = m \cdot m = 0$, $l \cdot n = 1$, $l \cdot m = l \cdot m = 0$ and $m \cdot \bar{m} = -1$, it is possible to parametrize the path so that

$$\begin{aligned} \frac{Dl^\mu}{\mathrm{d}\lambda} &= \bar{b}m^\mu + b\bar{m}^\mu \,, \\ \frac{Dn^\mu}{\mathrm{d}\lambda} &= \bar{a}m^\mu + a\bar{m}^\mu \,, \\ \frac{Dm^\mu}{\mathrm{d}\lambda} &= al^\mu + bn^\mu + icm^\mu \,, \end{aligned} \tag{B.4.3}$$

where a, b, c are functions of λ and c is real. The metric signature is assumed now $+---$ in order to follow standard notation of Newman-Penrose formalism, and the bar over a quantity denotes complex conjugation. Equations (B.4.3) are the analogous of the FS relations for null lines so that, repeating exactly the above procedure, one gets the final set of equations. Since for a massless spinning test particle we have $m = -P \cdot l = 0$, the total 4-momentum P has the following decomposition:

$$P^\mu = -[Bl^\mu + Am^\mu + \bar{A}\bar{m}^\mu] \,. \tag{B.4.4}$$

Following Mashhoon [96], Tulczyjew's conditions $S^{\alpha\beta}P_\beta = 0$ are in general inconsistent in the presence of a gravitational background if in addition one has P lightlike: $P \cdot P = 0$. Thus, even if these inconsistencies concern only the case of null P, we are clearly forced to consider Pirani's SC as the only physically meaningful supplementary conditions. Using the P supplementary conditions (implying $b = 0$), Mashhoon has shown that l is necessarily geodesics: $Dl^\mu/\mathrm{d}\lambda = 0$ and

$$S^{\mu\nu} = f(\lambda)[l \wedge m]^{\mu\nu} + \bar{f}(\lambda)[l \wedge \bar{m}]^{\mu\nu} + ig(\lambda)[m \wedge \bar{m}]^{\mu\nu} \,, \tag{B.4.5}$$

with B real and

$$A = \frac{\mathrm{d}f}{\mathrm{d}\lambda} + icf - ig\bar{a} \,, \qquad P^\mu P_\mu = -2|A|^2 \,. \tag{B.4.6}$$

so that P is in general spacelike or eventually null. Furthermore, he has shown that the spin vector defined by

$$\mathcal{S}^{\mu} = \frac{1}{2}\eta^{\mu\nu\alpha\beta} l_{\nu} S_{\alpha\beta} \tag{B.4.7}$$

is constant along l and either parallel or antiparallel to l.

Finally, the generalized momentum of the particle should be determined by solving equations (B.4.1) and (B.4.2) supplemented by $S^{\alpha\beta} l_{\beta} = 0$. The other components of the spin tensor not summarized by the spin vector should be determined too. By assuming $a = 0$ (n parallel propagated along l) without any loss of the physical content of the solution, Mashhoon has obtained for f and B the following differential equations:

$$\begin{aligned} \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} + ic\right]^2 f &= f R_{1413} + \bar{f} R_{1414} + ig R_{1434} \,, \\ -\frac{\mathrm{d}B}{\mathrm{d}\lambda} &= f R_{1213} + \bar{f} R_{1214} + ig R_{1234} \,, \end{aligned} \tag{B.4.8}$$

which determine the total 4-momentum and the spin tensor along the path once they have been specified initially.

B.5. Applications

B.5.1. The special case of constant frame components of the spin tensor

Due to the mathematical complexity in treating the general case of non-constant frame components of the spin tensor, we have considered first the simplest case of massive spinning test particles moving uniformly along circular orbits with constant frame components of the spin tensor with respect to a naturally geometrically defined frame adapted to the stationary observers in the Schwarzschild spacetime [104] as well as in other spacetimes of astrophysical interest: Reissner-Nordström spacetime [105], Kerr spacetime [5], superposed static Weyl field [106], vacuum C metric [107]. A static spin vector is a very strong restriction on the solutions of the Mathisson-Papapetrou equations of motion. However, this assumption not only greatly simplifies the calculation, but seems to be not so restrictive, since, as previously demon-

strated at least in the Schwarzschild case, the spin tensor components still remain constant under the CP an T choices of supplementary conditions, starting from the more general non-constant case.

We have confined our attention to spatially circular equatorial orbits in Schwarzschild, Reissner-Nordström and Kerr spacetimes, and searched for observable effects which could eventually discriminate among the standard supplementary conditions. We have found that if the world line chosen for the multipole reduction and whose unit tangent we denote as U is a circular orbit, then also the generalized momentum P of the spinning test particle is tangent to a circular orbit even though P and U are not parallel 4-vectors. These orbits are shown to exist because the spin induced tidal forces provide the required acceleration no matter what supplementary condition we select. Of course, in the limit of a small spin the particle's orbit is close of being a circular geodesic and the (small) deviation of the angular velocities from the geodesic values can be of an arbitrary sign, corresponding to the possible spin-up and spin-down alignment to the z-axis. When two massive particles (as well as photons) orbit around a gravitating source in opposite directions, they make one loop with respect to a given static observer with different arrival times. This difference is termed clock effect (see [50, 108, 109, 110, 111] and references therein). Hereafter we shall refer the co/counter-rotation as with respect to a fixed sense of variation of the azimuthal angular coordinate. In the case of a static observer and of timelike spatially circular geodesics the coordinate time delay is given by

$$\Delta t_{(+,-)} = 2\pi \left(\frac{1}{\zeta_+} + \frac{1}{\zeta_-} \right) , \tag{B.5.1}$$

where $\zeta_\pm$ denote angular velocities of two opposite rotating geodesics. In the case of spinless neutral particles in geodesic motion on the equatorial plane of both Schwarzschild and Reissner-Nordström spacetimes one has $\zeta_+ = -\zeta_-$, and so the clock effect vanishes; in the Kerr case, instead, the clock effect reads $\Delta t_{(+,-)} = 4\pi a$, where a is the angular momentum per unit mass of the Kerr black hole. These results are well known in the literature. We have then extended the notion of clock effect to non geodesic circular trajectories considering co/counter-rotating spinning-up/spinning-down particles. In this case we have found that the time delay is nonzero for oppositely orbiting both spin-up or spin-down particles even in both Schwarzschild and Reissner-Nordström cases, and can be measured. In addition, we have found that

a nonzero gravitomagnetic clock effect appears in the Reissner-Nordström spacetime for spinless (oppositely) charged particles as well.

An analogous effect is found in the case of superposed Weyl fields corresponding to Chazy-Curzon particles and Schwarzschild black holes when the circular motion of spinning test particles is considered on particular symmetry hyperplanes, where the orbits are close to a geodesic for small values of the spin. In the case of the C metric, instead, we have found that the orbital frequency is in general spin-dependent, but there is no clock effect, in contrast to the limiting Schwarzschild case.

B.5.2. Spin precession in Schwarzschild and Kerr spacetimes

We have then studied the behaviour of spinning test particles moving along equatorial circular orbits in the Schwarzschild [6] as well as Kerr [7] spacetimes within the framework of the Mathisson-Papapetrou approach supplemented by standard conditions, in the general case in which the components of the spin tensor are not constant along the orbit. We have found that precession effects occur only if the Pirani's supplementary conditions are imposed, where one finds a Fermi-Walker transported spin vector along an accelerated center of mass world line. The remaining two supplementary conditions apparently force the test particle center of mass world line to deviate from a circular orbit because of the feedback of the precessing spin vector; in addition, under these choices of supplementary conditions the spin tensor components still remain constant. In reaching these conclusions, we only considered solutions for which both U and P are timelike vectors, in order to have a meaningful interpretation describing a spinning test particle with nonzero rest mass.

B.5.3. Massless spinning test particles in vacuum algebraically special spacetimes

As a final application, we have derived the equations of motion for massless spinning test particles in general vacuum algebraically special spacetimes, using the Newman-Penrose formalism, in the special case in which the multipole reduction world line is aligned with a principal null direction of the spacetime [112]. This situation gives very simple equations and their complete integration is straightforward. Explicit solutions corresponding to some

familiar Petrov type D and type N spacetimes (including Schwarzschild, Taub-NUT, Kerr, C metric, Kasner, single exact gravitational wave) are derived and discussed. Furthermore, we have investigated the motion along (null) circular orbits, providing explicit solutions in black hole spacetimes.

B.5.4. Quadrupole effects in black hole spacetimes

We have studied the motion of quadrupolar particles on a Schwarzschild as well as Kerr backgrounds [113, 114] following Dixon's model. In the simplified situation of constant frame components (with respect to a natural orthonormal frame) of both the spin and the quadrupole tensor of the particle we have found the kinematical conditions to be imposed to the particle's structure in order the orbit of the particle itself be circular and confined on the equatorial plane. Co-rotating and counter-rotating particles result to have a non-symmetric speed in spite of the spherical symmetry of the background, due to their internal structure. This fact has been anticipated when studying spinning particles only, i.e. with vanishing quadrupole moments. We show modifications due to the quadrupole which could be eventually observed in experiments. Such experiment, however, cannot concern standard clock effects, because in this case we have shown that there are no contributions arising from the quadrupolar structure of the body. In contrast, the effect of the quadrupole terms could be important when considering the period of revolution of an extended body around the central source: measuring the period will provide an estimate of the quantities determining the quadrupolar structure of the body, if its spin is known.

It would be of great interest to extend this analysis to systems with varying quadrupolar structure and emitting gravitational waves without perturbing significantly the background spacetime.

B.5.5. Quadrupole effects in gravitational wave spacetimes

We have studied how a small extended body at rest interacts with an incoming single plane gravitational wave. The body is spinning and also endowed with a quadrupolar structure, so that due to the latter property it can be thus considered as a good model for a gravitational wave antenna.

We have first discussed the motion of such an extended body by assuming that it can be described according to Dixon's model and that the gravitational

field of the wave is weak, i.e. the "reaction" (induced motion) of a "gravitational wave antenna" (the extended body) to the passage of the wave, and then the case of an exact plane gravitational wave. We have found that in general, even if initially absent, the body acquires a dipolar moment induced by the quadrupole tensor, a property never pointed out before in the literature.

Special situations may occur in which certain spin components change their magnitude leading to effects (e.g. spin-flip) which can be eventually observed. This interesting feature recalls the phenomenon of glitches observed in pulsars: a sudden increase in the rotation frequency, often accompanied by an increase in slow-down rate. The physical mechanism triggering glitches is not well understood yet, even if these are commonly thought to be caused by internal processes. If one models a pulsar by a Dixon's extended body, then the present analysis shows that a sort of glitch can be generated by the passage of a strong gravitational wave, due to the pulsar quadrupole structure. In fact, we have found that the profile of a polarization function can be suitably selected in order to fit observed glitches and in particular to describe the post-glitch behavior.

B.5.6. Quadrupolar particles and the equivalence principle

We have compared the two "reciprocal " situations of motion of an extended body endowed with structure up to the mass quadrupole moment in a Shwarzschild background spacetime (as described by Dixon's model) with that of a test particle in geodesic motion in the background of an exact solution of Einstein's field equations describing a source with quadrupolar structure (for a more detailed study of this kind of solutions generalizing Schwarzschild, Kerr and Kerr-Newman spacetimes see also the section "Generalizations of the Kerr-Newman solution," included in the present report). Under certain conditions the two situations give perfect corresponding results a fact which has been interpreted as an argument in favour of the validity of Dixon's model.

B.5.7. Poynting-Robertson-like effects

Test particle motion in realistic gravitational fields is of obvious astrophysical importance and at the same time it provides reliable evidence of the properties of those gravitational fields. However, in many actual astrophysical sys-

tems the particles are not moving freely but are influenced by ambient matter, electromagnetic fields and radiation. In typical situations, these "physical" effects are probably even more important than fine details of the spacetime geometry alone. The most remarkable conditions, from the point of view of general relativity as well as astrophysics, appear near very compact objects where both the pure gravitational and other "physical" effects typically become extraordinarily strong.

In a series of papers, recently, we have focused on the motion of test particles in a spherically symmetric gravitational field, under the action of a Thomson-type interaction with radiation emitted or accreted by a compact center. This kind of problem was first investigated by Poynting using Newtonian gravity and then in the framework of linearized general relativity by Robertson (see [165] and the references therein). It involves competition between gravity and radiation drag, which may lead to interesting types of motion which do not occur in strictly vacuum circumstances. In particular, there arises the question of whether equilibrium behavior like circular orbit motion or even "staying at rest" are possible in some cases. Theoretical aspects of the Poynting-Robertson effect as well as its astrophysical relevance in specific situations have been studied by many authors since the original pioneering work. We first considered this effect on test particles orbiting in the equatorial plane of a Schwarzschild or Kerr black hole, assuming that the source of radiation is located symmetrically not far from the horizon (in the case of outgoing flux). Successively, we have generalized these results by including in our discussion other relevant spacetimes, e.g. Vaidya, or considering spinning particles undergoing Poynting-Robertson-like effect.

C. Metric and curvature perturbations in black hole spacetimes

C.1. Perturbations of charged and rotating black hole

The gravitational and electromagnetic perturbations of the Kerr-Newman metric represent still an open problem in General Relativity whose solution could have an enormous importance for the astrophysics of charged and rotating collapsed objects. A complete discussion about this problems needs a plenty of different mathematical tools: the Newman-Penrose formalism in the tetradic and spinor version, the Cahen-Debever-Defrise self dual theory, the properties of the spin-weighted angular harmonics, with particular attention to the related differential geometry and the group theory, some tools of complex analysis, etc, but in any case it is difficult to handle with the perturbative equations. Fortunately, during the last years, the modern computers and software have reached an optimal computational level which allows now to approach this problem from a completely new point of view.

The Kerr-Newman solution in Boyer-Lindquist coordinates is represented by the metric:

$$\begin{aligned} ds^2 &= \left(1-\frac{V}{\Sigma}\right)dt^2 + \frac{2a\sin^2\theta}{\Sigma}Vdtd\phi - \frac{\Sigma}{\Delta}dr^2 \\ &-\Sigma d\theta^2 - \left[r^2+a^2+\frac{a^2\sin^2\theta}{\Sigma}V\right]\sin^2\theta d\phi^2 \end{aligned} \tag{C.1.1}$$

where as usual:

$$\begin{aligned} V &\equiv 2Mr - Q^2 \\ \Delta &\equiv r^2 - 2Mr + a^2 + Q^2 \\ \Sigma &\equiv r^2 + a^2 \cos^2\theta \end{aligned} \tag{C.1.2}$$

and by the vector potential:

$$A^\flat = A_\mu \, dx^\mu = \frac{Qr}{\Sigma}(dt - a \sin^2\theta \, d\phi) \ . \tag{C.1.3}$$

To investigate the geometrical features of this metric it is convenient to introduce a symmetry-adapted tetrad. For any type D metric, and in particular for the Kerr-Newman solution, the best choice is a null tetrad with two "legs" aligned along the two repeated principal null directions of the Weyl tensor. The standard theory for analyzing different spin massless wave fields in a given background is represented by the spinorial tetradic formalism of Newman-Penrose (hereafter N-P)[115]. Here we follow the standard approach, pointing out that a more advanced reformulation of this formalism, called "GHP" [116] exists, allowing a more geometric comprehension of the theory. In the N-P formalism, this solution is represented by the following quantities [117] (in this section we use an A label over all quantities for a reason which will be clear later). The Kinnersley tetrad [118]:

$$\begin{aligned} (l^\mu)^A &= \frac{1}{\Delta}[r^2 + a^2, \Delta, 0, a] \\ (n^\mu)^A &= \frac{1}{2\Sigma}[r^2 + a^2, -\Delta, 0, a] \\ (m^\mu)^A &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\,[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}] \ , \end{aligned} \tag{C.1.4}$$

with the 4^{th} leg represented by the conjugate $(m^{*\mu})^A$, gives the metric tensor of Kerr-Newman spacetime the form:

$$\eta_{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{C.1.5}$$

The Weyl tensor is represented by:

$$\begin{gathered}\Psi_0^A = \Psi_1^A = \Psi_3^A = \Psi_4^A = 0 \\ \Psi_2^A = M\rho^3 + Q^2\rho^*\rho^3\end{gathered} \tag{C.1.6}$$

and the electromagnetic field is given by:

$$\phi_0^A = \phi_2^A = 0 \ , \ \phi_1^A = \frac{Q}{2(r - ia\cos\theta)^2} \ . \tag{C.1.7}$$

For the Ricci tensor and the curvature scalar we have:

$$\Lambda^A = 0 \ , \ \Phi_{nm}^A = 2\phi_m^A \phi_n^{*A} \quad (\mathrm{m,n} = 0,1,2) \tag{C.1.8}$$

so in Kerr-Newman, the only quantity different from zero is:

$$\Phi_{11}^A = \frac{Q^2}{2\Sigma^2} \ . \tag{C.1.9}$$

The spin coefficients, which are linear combination of the Ricci rotation coefficients, are given by:

$$\begin{aligned}&\kappa^A = \sigma^A = \lambda^A = \nu^A = \epsilon^A = 0 \ , \\ &\rho^A = \frac{-1}{(r - ia\cos\theta)} \quad , \quad \tau^A = \frac{-ia\rho^A\rho^{*A}\sin\theta}{\sqrt{2}} \ , \\ &\beta^A = \frac{-\rho^{*A}\cot\theta}{2\sqrt{2}} \quad , \quad \pi^A = \frac{ia(\rho^A)^2\sin\theta}{\sqrt{2}} \ , \\ &\mu^A = \frac{(\rho^A)^2\rho^{*A}\Delta}{2} \quad , \quad \gamma^A = \mu^A + \frac{\rho^A\rho^{*A}(r-M)}{2} \ , \\ &\alpha^A = \pi^A - \beta^{*A} \ .\end{aligned} \tag{C.1.10}$$

The directional derivatives are expressed by:

$$D = l^\mu\partial_\mu \ , \quad \Delta = n^\mu\partial_\mu \ , \quad \delta = m^\mu\partial_\mu \ , \quad \delta^* = m^{*\mu}\partial_\mu \ . \tag{C.1.11}$$

Unfortunately in the literature the same letter for (C.1.2) and for the directional derivative along **n** it is used. However the meaning of Δ will always

be clear from the context. The study of perturbations in the N-P formalism is achieved splitting all the relevant quantities in the form $l = l^A + l^B, \Psi_4 = \Psi_4^A + \Psi_4^B$, $\sigma = \sigma^A + \sigma^B, D = D^A + D^B$, etc., where the A terms are the background and the B's are small perturbations. The full set of perturbative equations is obtained inserting these quantities in the basic equations of the theory (Ricci and Bianchi identities, Maxwell, Dirac, Rarita-Schwinger equations etc.) and keeping only first order terms. After certain standard algebraic manipulations one usually obtains coupled linear PDE's involving curvature quantities. In the following, we will omit the A superscript for the background quantities. Comparing with the standard Regge-Wheeler-Zerilli [119, 120] approach which gives the equation for the metric, here one gets the equations for Weyl tensor components. This theory is known as *curvature perturbations*. In the case of Einstein-Maxwell perturbed metrics, one gets as in R-W-Z the well known phenomenon of the "gravitationally induced electromagnetic radiation and vice versa" [121], which couples gravitational and electromagnetic fields. In the first formulation, one gets a coupled system for $F^B_{\mu\nu}$ and $g^B_{\mu\nu}$ quantities. In the N-P approach one has the coupling between perturbed Weyl and Maxwell tensor components, although it's possible to recover the metric perturbations using the curvature one [122]. A discussion about the connections between these two approaches can be found in [123]. To make a long story short, taking in account the two Killing vectors of this spacetime, one can write the unknown functions in the form:

$$F(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} f(r, \theta) \ . \tag{C.1.12}$$

In the easier cases of Kerr, Reissner-Nordstrom and Schwarzchild, writing $f(r, \theta) = R(r)Y(\theta)$ one gets separability of the problem. For instance, the Reissner-Nordström case [124] is separable using the spin-weighted spherical harmonics:

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \left(\frac{m^2 + s^2 + 2ms\cos\theta}{\sin^2\theta}\right)\right]{}_sY^m{}_l(\theta) = -l(l+1){}_sY^m{}_l(\theta) \tag{C.1.13}$$

and their related laddering operators:

$$\left(\frac{d}{d\theta} - \frac{m}{\sin\theta} - s\frac{\cos\theta}{\sin\theta}\right){}_sY^m{}_l(\theta) = -\sqrt{(l-s)(l+s+1)}\,{}_{s+1}Y^m{}_l(\theta) \tag{C.1.14}$$

$$\left(\frac{d}{d\theta}+\frac{m}{\sin\theta}+s\frac{\cos\theta}{\sin\theta}\right){}_sY^m{}_l(\theta)=+\sqrt{(l+s)(l-s+1)}\,{}_{s-1}Y^m{}_l(\theta)\ . \quad \text{(C.1.15)}$$

The unknown functions can be cast in the form:

$$\begin{aligned}
\Psi_0^B &= e^{-i\omega t}e^{im\phi}\,{}_2Y^m{}_l(\theta)R_l^{(2)}(r)\\
\chi_1^B &= e^{-i\omega t}e^{im\phi}\,{}_1Y^m{}_l(\theta)R_l^{(1)}(r)\\
\chi_{-1}^B &= e^{-i\omega t}e^{im\phi}\,{}_{-1}Y^m{}_l(\theta)\frac{\Delta}{2r^2}R_l^{(-1)}(r)\\
\Psi_4^B &= e^{-i\omega t}e^{im\phi}\,{}_{-2}Y^m{}_l(\theta)\frac{\Delta^2}{4r^4}R_l^{(-2)}(r)
\end{aligned} \quad \text{(C.1.16)}$$

where $\Delta = r^2 - 2Mr + Q^2$, and after manipulations, one gets two sets of coupled ODE's. The first set is:

$$\begin{aligned}
&\left[-\omega^2\frac{r^4}{\Delta}+4i\omega r\left(-2+\frac{r(r-M)}{\Delta}+\frac{Q^2}{3Mr-4Q^2}\right)-\Delta\frac{d^2}{dr^2}\right.\\
&-\left\{6(r-M)-\frac{4Q^2\Delta}{r(3Mr-4Q^2)}\right\}\frac{d}{dr}-4-\frac{2Q^2}{r^2}\\
&\left.+\frac{4Q^2(r^2+2Mr-3Q^2)}{r^2(3Mr-4Q^2)}+\frac{3Mr-4Q^2}{3Mr-2Q^2}(l-1)(l+2)\right]R_l^{(2)} \quad \text{(C.1.17)}\\
&=\frac{2\sqrt{2}Q\sqrt{(l-1)(l+2)}r^3}{3Mr-2Q^2}\left(-i\omega\frac{r^2}{\Delta}+\frac{d}{dr}+\frac{4}{r}\right.\\
&\left.-\frac{4Q^2}{r(3Mr-4Q^2)}\right)R_l^{(1)}
\end{aligned}$$

$$\left[-\omega^2\frac{r^4}{\Delta}+2i\omega r\left(-2+\frac{r(r-M)}{\Delta}-\frac{Q^2}{3Mr-2Q^2}\right)-\Delta\frac{d^2}{dr^2}\right.$$
$$-\left\{\frac{6\Delta}{r}+4(r-M)-\frac{2Q^2\Delta}{r(3Mr-2Q^2)}\right\}\frac{d}{dr}-\frac{18r^2-24Mr+2Q^2}{r^2}$$
$$\left.+\frac{12Q^2\Delta}{r^2(3Mr-2Q^2)}+\frac{3Mr-2Q^2}{3Mr-4Q^2}(l-1)(l+2)\right]R_l^{(1)} \qquad \text{(C.1.18)}$$
$$=\frac{-\sqrt{2}Q^2\sqrt{(l-1)(l+2)}\Delta}{r^3(3Mr-4Q^2)}\left(i\omega\frac{r^2}{\Delta}+\frac{d}{dr}-\frac{2}{r}+\frac{4(r-M)}{\Delta}\right.$$
$$\left.-\frac{2Q^2}{r(3Mr-2Q^2)}\right)R_l^{(2)}\ .$$

The quantities $(R_l^{(-1)})^*$ e $(R_l^{(-2)})^*$ (from χ_{-1}^B e Ψ_4^B), satisfy the same equations of $R_l^{(1)}$ and $R_l^{(2)}$. At this point decoupling this system of ordinary differential equations is straightforward.

Similarly, the Kerr case is separable using but the so-called spin-weighted spheroidal harmonics [8, 125]:

$$(H_0+H_1)\Theta(\theta)=-E\Theta(\theta) \qquad \text{(C.1.19)}$$

where:

$$H_0=\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-\left(\frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta}\right)\right] \qquad \text{(C.1.20)}$$

$$H_1=a^2\omega^2\cos^2\theta-2a\omega s\cos\theta \qquad \text{(C.1.21)}$$

and E is the eigenvalue. We have factorized the spherical and the spheroidal parts to give the problem the form of a typical Quantum Mechanics exercise. In fact depending if the H_1 term is small or not, the way to approach the problem is very different. Unfortunately, in this case the laddering operators are not know [126] and this does not allow the same strategy used in the case of the Reissner-Nordström spacetime. In the case of the Kerr spacetime instead, this is not a problem because laddering operators are unnecessary to solve completely the problem. In the case of the Kerr-Newman spacetime this creates a "formal" problem. In fact the presence of the charge Q generates "ugly" terms which don't allow the separation of variables in all known coordinates.

A hypothetical separation of variables in these coordinates would have been stopped by the explicit absence of laddering operators. During the last 25 years there have been various attempts to solve this problem. One idea, proposed in Chandrasekhar's monography [126], is to decouple the PDE's before the separation of variables, obtaining 4th order or higher linear PDE's. This task could be accomplished only using a super-computer, because of the 4th order derivatives. Another formulation was developed using de Cahen-Debever-Defrise formalism, but a part some elegant conservative equations, the problem has not been solved [127, 128]. In conclusion the problem remains still open. A new approach has been developed [9, 10] for vacuum spacetimes which gives directly the full set of perturbative equations. The direct extension of this work to the case of Einstein-Maxwell or more complicate spacetimes can put in a new light this difficult problem.

After this short historical overview we can discuss the results obtained by ICRANet researchers in this field. In [129], due to Cherubini and Ruffini, gravitational and electromagnetic perturbations to the Kerr-Newman spacetime using Maple tensor package are shown; a detailed analysis for slightly charged, rotating and oblate black hole is presented too. Subsequent to this article there have been various studies regarding the Teukolsky Master Equations (TMEs) in General Relativity. To this aim, a new form is found for the Teukolsky Master Equation in Kerr and interpreted in terms of de Rham-Lichenrowicz laplacians. The exact form of these generalized wave equations in any vacuum spacetime is given for the Riemann and Maxwell tensors, and the equations are linearized at any order, obtaining a hierarchy. It is shown that the TME for any Petrov type D spacetime is nothing more than a component of this laplacian linearized and that the TME cannot be derived by variational principles [9, 10]. More in detail, the Teukolsky Master Equation in the Kerr case, can be cast in a more compact form (Bini-Cherubini-Jantzen-Ruffini form) by introducing a "connection vector" whose components are:

$$\begin{aligned}
\Gamma^t &= -\frac{1}{\Sigma}\left[\frac{M(r^2-a^2)}{\Delta} - (r + ia\cos\theta)\right] \\
\Gamma^r &= -\frac{1}{\Sigma}(r-M) \\
\Gamma^\theta &= 0 \\
\Gamma^\phi &= -\frac{1}{\Sigma}\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right] .
\end{aligned} \tag{C.1.22}$$

It's easy to prove that:

$$\nabla^\mu \Gamma_\mu = -\frac{1}{\Sigma} \qquad , \qquad \Gamma^\mu \Gamma_\mu = \frac{1}{\Sigma}\cot^2\theta + 4\psi_2^A \tag{C.1.23}$$

and consequently the Teukolsky Master Equation assumes the form:

$$[(\nabla^\mu + s\Gamma^\mu)(\nabla_\mu + s\Gamma_\mu) - 4s^2\psi_2^A]\psi^{(s)} = 4\pi T \tag{C.1.24}$$

where ψ_2^A is the only non vanishing NP component of the Weyl tensor in the Kerr background in the Kinnersley tetrad (C.1.5) (with $Q = 0$). Equation (C.1.24) gives a common structure for these massless fields in the Kerr background varying the "s" index. In fact, the first part in the lhs represents (formally) a D'Alembertian, corrected by taking into account the spin-weight, and the second part is a curvature (Weyl) term linked to the "s" index too. This particular form of the Teukolsky Master Equation forces us to extend this analysis in the next sections because it suggests a connection between the perturbation theory and a sort of generalized wave equations which differ from the standard ones by curvature terms. In fact generalized wave operators are know in the mathematical literature as De Rham-Lichnerowicz Laplacians and the curvature terms which make them different from the ordinary ones are given by the Weitzenböck formulas. Mostly known examples in electromagnetism are

- the wave equation for the vector potential A_μ:

$$\nabla_\alpha \nabla^\alpha A_\mu - R_\mu{}^\lambda A_\lambda = -4\pi J_\mu \qquad , \qquad \nabla^\alpha A_\alpha = 0 \tag{C.1.25}$$

- the wave equation for the Maxwell tensor :

$$\nabla^\mu \nabla_\mu F_{\nu\lambda} + R_{\rho\mu\nu\lambda}F^{\rho\mu} - R^\rho{}_\lambda F_{\nu\rho} + R^\rho{}_\nu F_{\lambda\rho} = -8\pi \nabla_{[\mu} J_{\nu]} \tag{C.1.26}$$

while for the gravitational case one has

- the wave equation for the metric perturbations:

$$\nabla_\alpha \nabla^\alpha \bar{h}_{\mu\nu} + 2R_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} - 2R_{\alpha(\mu}\bar{h}_{\nu)}{}^\alpha = 0,$$

$$\nabla_\alpha \bar{h}_\mu{}^\alpha = 0, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h_\alpha{}^\alpha \tag{C.1.27}$$

- the wave equation for the Riemann Tensor

$$R^{\alpha\beta}{}_{\gamma\delta;\epsilon}{}^{\epsilon} = 4R^{[\alpha}{}_{[\gamma;\delta]}{}^{\beta]} - 2R^{[\alpha}{}_{\epsilon}R^{\beta]\epsilon}{}_{\gamma\delta} - 2R^{\alpha\mu\beta\nu}R_{\mu\nu\gamma\delta} - 4R^{[\alpha}{}_{\mu\nu[\gamma}R^{\beta]\mu\nu}{}_{\delta]} \,. \tag{C.1.28}$$

These equations are "non minimal," in the sense that they cannot be recovered by a minimal substitution from their flat space counterparts. A similar situation holds in the standard Quantum Field Theory for the electromagnetic Dirac equation. In fact, applying for instance to the Dirac equation an "ad hoc" first order differential operator one gets the second order Dirac equation

$$\begin{aligned} &(i\partial\!\!\!/ - eA\!\!\!/ + m)(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = \\ &\left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right]\psi = 0, \end{aligned} \tag{C.1.29}$$

where the notation is obvious. It is easy to recognize in equation (C.1.29) a generalized Laplacian and a curvature (Maxwell) term applied to the spinor. Moreover this equation is "non minimal", in the sense that the curvature (Maxwell) term cannot be recovered by electromagnetic minimal substitution in the standard Klein-Gordon equation for the spinor components. The analogous second order Dirac equation in presence of a gravitational field also has a non minimal curvature term and reduces to the form:

$$(\nabla_\alpha\nabla^\alpha + m^2 + \frac{1}{4}R)\psi = 0 \quad . \tag{C.1.30}$$

The general TME formalism is applied to other exact solutions of the vacuum Einstein field equations of Petrov type D. A new analysis of the Kerr-Taub-NUT black hole is given, focussing on Mashhoon spin-coupling and superradiance [130, 59].

More in detail, in [130] Bini, Cherubini and Jantzen studied a single master equation describing spin $s = 0 - 2$ test field gauge and tetrad-invariant perturbations of the Taub-NUT spacetime. This solution of vacuum Einstein field equations describes a black hole with mass M and gravitomagnetic monopole moment ℓ. This equation can be separated into its radial and angular parts. The behaviour of the radial functions at infinity and near the

horizon is studied. The angular equation, solved in terms of hypergeometric functions, can be related both to spherical harmonics of suitable weight, resulting from the coupling of the spin-weight of the field and the gravitomagnetic monopole moment of the spacetime, and to the total angular momentum operator associated with the spacetime's rotational symmetry. The results are compared with the Teukolsky master equation for the Kerr spacetime.

In [59] instead Bini, Cherubini, Jantzen and Mashhoon have studied a single master equation describing spin $s \leq 2$ test fields that are gauge- and tetrad-invariant perturbations of the Kerr-Taub-NUT (Newman - Unti - Tamburino) spacetime representing a source with a mass M, gravitomagnetic monopole moment $-\ell$, and gravitomagnetic dipole moment (angular momentum) per unit mass a. This equation can be separated into its radial and angular parts. The behavior of the radial functions at infinity and near the horizon is studied and used to examine the influence of l on the phenomenon of superradiance, while the angular equation leads to spin-weighted spheroidal harmonic solutions generalizing those of the Kerr spacetime. Finally, the coupling between the spin of the perturbing field and the gravitomagnetic monopole moment is discussed.

In [69] instead Bini and Cherubini investigate the algebraically special frequencies of Taub-NUT black holes in detail in comparison with known results concerning the Schwarzschild case. The periodicity of the time coordinate, required for regularity of the solution, prevents algebraically special frequencies to be physically acceptable. In the more involved Kerr-Taub-NUT case, the relevant equations governing the problem are obtained. The formalism is applied to the C-metric, and physical speculations are presented concerning the spin-acceleration coupling.

In [70] Bini, Cherubini and Mashhoon study the vacuum C metric and its physical interpretation in terms of the exterior spacetime of a uniformly accelerating spherically-symmetric gravitational source. Wave phenomena on the linearized C metric background are investigated. It is shown that the scalar perturbations of the linearized C metric correspond to the gravitational Stark effect. This effect is studied in connection with the Pioneer anomaly.

In [71] instead Bini, Cherubini and Mashhoon analysed the massless field perturbations of the accelerating Minkowski and Schwarzschild spacetimes. The results are extended to the propagation of the Proca field in Rindler spacetime. They examine critically the possibility of existence of a general spinacceleration coupling in complete analogy with the well-known spinro-

tation coupling. They argue that such a direct coupling between spin and linear acceleration does not exist.

In [72] Cherubini, Bini, Bruni and Perjes consider vacuum Kasner spacetimes, focusing on those that can be parametrized as linear perturbations of the special Petrov type D case. In particular they analyze in detail the perturbations which map the Petrov type D Kasner spacetime into another Kasner spacetime of Petrov type I. For these 'quasi-D' Kasner models they first investigate the modification to some curvature invariants and the principal null directions, both related to the Petrov classification of the spacetime. This simple Kasner example allows one to clarify the fact that perturbed spacetimes do not retain in general the speciality character of the background. In fact, there are four distinct principal null directions, although they are not necessarily first-order perturbations of the background principal null directions. Then in the Kasner type D background they derive a Teukolsky master equation, a classical tool for studying black-hole perturbations of any spin. This further step allows one to control totally general cosmologies around such a background as well as to show, from a completely new point of view, the well-known absence of gravitational waves in Kasner spacetimes.

C.2. Perturbations of a Reissner-Nordström black hole by a massive point charge at rest and the "electric Meissner effect"

The problem of the effect of gravity on the electromagnetic field of a charged particle leading to the consideration of the Einstein-Maxwell equations has been one of the most extensively treated in the literature, resulting in exact solutions (see [131] and references therein) as well as in a variety of approximation methods [132]-[140].

The issue of the interaction of a massive charged particle of mass m and charge q with a Reissner-Nordström black hole with mass $\mathcal{M}$ and charge Q has been addressed by the ICRANet collaboration: Bini, Geralico and Ruffini [141, 142, 143]. We have solved this problem by the first order perturbation approach formulated by Zerilli [120] using the tensor harmonic expansion of the Einstein-Maxwell system of equations.

The results discussed in [141, 142] gave answer to a problem whose investigation started long ago by Hanni and Ruffini [137]. They obtained the so-

lution for a charged particle at rest in the field of a Schwarzschild black hole in the case of test field approximation, i.e. under the conditions $q/m \gg 1$, $m \approx 0$ and $q \ll \mathcal{M}$, $q \ll Q$, by using the vector harmonic expansion of the electromagnetic field in curved space. The conditions above imply the solution of the Maxwell equations only in a fixed Schwarzschild metric, since the perturbation to the background geometry given by the electromagnetic stress-energy tensor is second order in the particle's charge and the effect of the particle's mass is there neglected. As a result, no constraint on the position of the test particle follows from the Einstein equations and the Bianchi identities: the position of the particle is totally arbitrary.

This same test field approximation has been applied to the case of a Reissner-Nordström black hole by Leaute and Linet [140]. In analogy with the Schwarzschild case, they used the vector harmonic expansion of the electromagnetic field holding the background geometry fixed. However, this "test field approximation" is not valid in the present context. In fact, in addition to neglecting the effect of the particle mass on the background geometry, this treatment also neglects the electromagnetically induced gravitational perturbation terms linear in the charge of the particle which would contribute to modifying the metric as well.

The correct way to attack the problem is thus to solve the linearized Einstein-Maxwell equations following Zerilli's first order tensor harmonic analysis [120]. In fact the source terms of the Einstein equations comprise the energy-momentum tensor associated with the particle's mass, the electromagnetic energy-momentum tensor associated with the background field as well as additional interaction terms, to first order in m and q, proportional to the product of the square of the charge of the background geometry and the particle's mass ($\sim Q^2 m$) and to the product of the charges of both the particle and the black hole ($\sim qQ$). These terms give origin to the so called "electromagnetically induced gravitational perturbation" [144]. On the other hand, the source terms of the Maxwell equations contain the electromagnetic current associated with the particle's charge as well as interaction terms proportional to the product of the black hole's charge and the particle's mass ($\sim Qm$), giving origin to the "gravitationally induced electromagnetic perturbation" [145].

This has been explicitly done in [141, 142]. Let us briefly summarize the results and the properties of the solution derived there. In standard Schwarzschild-

like coordinates the Reissner-Nordström black hole metric is

$$\begin{aligned} ds^2 &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \\ f(r) &= 1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2} \,, \end{aligned} \tag{C.2.1}$$

with associated electromagnetic field

$$F = -\frac{Q}{r^2} dt \wedge dr \,. \tag{C.2.2}$$

The horizons are located at $r_\pm = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - Q^2} \equiv \mathcal{M} \pm \Gamma$; we consider the case $|Q| \leq \mathcal{M}$ and the region $r > r_+$ outside the outer horizon, with an extremely charged hole corresponding to $|Q| = \mathcal{M}$ (which implies $\Gamma = 0$) where the two horizons coalesce.

The only nonvanishing components of the stress-energy tensor and of the current density are given by

$$\begin{aligned} T^{\rm part}_{00} &= \frac{m}{2\pi b^2} f(b)^{3/2} \delta\left(r - b\right) \delta\left(\cos\theta - 1\right) \\ J^0_{\rm part} &= \frac{q}{2\pi b^2} \delta\left(r - b\right) \delta\left(\cos\theta - 1\right) \,, \end{aligned} \tag{C.2.3}$$

which enter the system of combined Einstein-Maxwell equations

$$\begin{aligned} \tilde{G}_{\mu\nu} &= 8\pi \left(T^{\rm part}_{\mu\nu} + \tilde{T}^{\rm em}_{\mu\nu} \right) \,, \\ \tilde{F}^{\mu\nu}{}_{;\nu} &= 4\pi J^\mu_{\rm part} \,, \quad {}^*\tilde{F}^{\alpha\beta}{}_{;\beta} = 0 \,. \end{aligned} \tag{C.2.4}$$

The quantities denoted by the tilde refer to the total electromagnetic and gravitational fields, to first order of the perturbation:

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu} \,, & \tilde{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} \,, \\ \tilde{T}^{\rm em}_{\mu\nu} &= \frac{1}{4\pi} \left[\tilde{g}^{\rho\sigma} \tilde{F}_{\rho\mu} \tilde{F}_{\sigma\nu} - \frac{1}{4} \tilde{g}_{\mu\nu} \tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma} \right] \,, \\ \tilde{G}_{\mu\nu} &= \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \,; \end{aligned} \tag{C.2.5}$$

note that the covariant derivative operation makes use of the perturbed metric $\tilde{g}_{\mu\nu}$ as well. The corresponding quantities without the tilde refer to the

background Reissner-Nordström geometry (C.2.1) and electromagnetic field (C.2.2). Following Zerilli's [120] procedure we expand the fields $h_{\mu\nu}$ and $f_{\mu\nu}$ as well as the source terms (C.2.3) in tensor harmonics, imposing then the Regge-Wheeler gauge [119] to simplify the description of the perturbation. The perturbation equations are then obtained from the system (C.2.4), keeping terms to first order in the mass m of the particle and its charge q which are assumed sufficiently small with respect to the black hole mass and charge. The axial symmetry of the problem about the z axis ($\theta = 0$) allows to put the azimuthal parameter equal to zero in the expansion, leading to a great simplification. Furthermore, it is sufficient to consider only electric-parity perturbations, since there are no magnetic sources [144, 145, 120].

The geometrical perturbations $h_{\mu\nu}$ for the electric multipoles in the Regge-Wheeler gauge are given by

$$||h_{\mu\nu}|| = \begin{bmatrix} e^{\nu} H_0 Y_{l0} & H_1 Y_{l0} & 0 & 0 \\ H_1 Y_{l0} & e^{-\nu} H_2 Y_{l0} & 0 & 0 \\ 0 & 0 & r^2 K Y_{l0} & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta K Y_{l0} \end{bmatrix} , \qquad \text{(C.2.6)}$$

where Y_{l0} are normalized spherical harmonics with azimuthal index equal to zero and $e^{\nu} = f(r)$ is Zerilli's notation. The electromagnetic field harmonics $f_{\mu\nu}$ for the electric multipoles are given by

$$||f_{\mu\nu}|| = \begin{bmatrix} 0 & \tilde{f}_{01} Y_{l0} & \tilde{f}_{02} \dfrac{\partial Y_{l0}}{\partial\theta} & 0 \\ \text{antisym} & 0 & \tilde{f}_{12} \dfrac{\partial Y_{l0}}{\partial\theta} & 0 \\ \text{antisym} & \text{antisym} & 0 & 0 \\ \text{antisym} & \text{antisym} & \text{antisym} & 0 \end{bmatrix} , \qquad \text{(C.2.7)}$$

where $\tilde{f}_{\mu\nu}$ denotes the θ-independent part of $f_{\mu\nu}$, and the symbol "antisym" indicates components obtainable by antisymmetry. The expansion of the source terms (C.2.3) gives the relations

$$\sum_l A_{00} Y_{l0} = 16\pi T_{00}^{\text{part}} , \qquad \sum_l v Y_{l0} = J^0_{\text{part}} , \qquad \text{(C.2.8)}$$

with

$$A_{00} = 8\sqrt{\pi}\frac{m\sqrt{2l+1}}{b^2} f(b)^{3/2}\delta\left(r-b\right), \; v = \frac{1}{2\sqrt{\pi}}\frac{q\sqrt{2l+1}}{b^2}\delta\left(r-b\right) . \quad \text{(C.2.9)}$$

The Einstein-Maxwell field equations (C.2.4) give rise to the following system of radial equations for values $l \geq 2$ of the multipoles (note that the cases $l = 0, 1$ must be treated separately)

$$\begin{aligned}
0 &= e^{2\nu}\left[2K'' - \frac{2}{r}W' + \left(\nu' + \frac{6}{r}\right)K' - 4\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)W\right] - \frac{2\lambda e^{\nu}}{r^2}(W+K) \\
&\quad -2\frac{Q^2 e^{\nu} W}{r^4} - 4\frac{Q e^{\nu}\tilde{f}_{01}}{r^2} + A_{00} , \\
0 &= \frac{2}{r}W' - \left(\nu' + \frac{2}{r}\right)K' - \frac{2\lambda e^{-\nu}}{r^2}(W-K) - 2\frac{Q^2 e^{-\nu} W}{r^4} + 4\frac{Q e^{-\nu}\tilde{f}_{01}}{r^2} , \\
0 &= K'' + \left(\nu' + \frac{2}{r}\right)K' - W'' - 2\left(\nu' + \frac{1}{r}\right)W' \\
&\quad + \left(\nu'' + \nu'^2 + \frac{2\nu'}{r}\right)(K-W) - 2\frac{Q^2 e^{-\nu} K}{r^4} + \frac{4Q e^{-\nu}}{r^2}\tilde{f}_{01} , \\
0 &= -W' + K' - \nu' W + 4\frac{Q e^{-\nu}\tilde{f}_{02}}{r^2} , \\
0 &= \tilde{f}_{01}{}' + \frac{2}{r}\tilde{f}_{01} - \frac{l\left(l+1\right)e^{-\nu}\tilde{f}_{02}}{r^2} - \frac{Q}{r^2}K' + 4\pi v , \\
0 &= \tilde{f}_{01} - \tilde{f}_{02}{}' ,
\end{aligned} \quad \text{(C.2.10)}$$

since $H_0 = H_2 \equiv W$, $H_1 \equiv 0$ and $\tilde{f}_{12} \equiv 0$, where $\lambda = \frac{1}{2}\left(l-1\right)\left(l+2\right)$ and a prime denotes differentiation with respect to r.

We have a system of 6 coupled ordinary differential equations for 4 unknown functions: K, W, $\tilde{f}_{01}$ and $\tilde{f}_{02}$. The compatibility of the system requires that these equations are not independent. Two equations can indeed be eliminated provided that the following stability condition holds

$$m = qQ\frac{b f(b)^{1/2}}{\mathcal{M}b - Q^2} , \quad \text{(C.2.11)}$$

involving the black hole and particle parameters as well as their separation distance b. If the black hole is extreme (i.e. $Q/\mathcal{M} = 1$), then the particle

must also have the same ratio $q/m = 1$, and equilibrium exists independent of the separation. In the general non-extreme case $Q/\mathcal{M} < 1$ there is instead only one position of the particle which corresponds to equilibrium, for given values of the charge-to-mass ratios of the bodies. In this case the particle charge-to-mass ratio must satisfy the condition $q/m > 1$. Note that quite surprisingly Eq. (C.2.11) coincides with the equilibrium condition for a charged test particle in the field of a Reissner-Nordström black hole which has been discussed by Bonnor [146] in the simplified approach of test field approximation, neglecting all the feedback terms.

We then succeed in the exact reconstruction of both the perturbed gravitational and electromagnetic fields by summing all multipoles [142]. The perturbed metric is given by

$$\begin{aligned} d\tilde{s}^2 &= -[1-\bar{\mathcal{H}}-k(r)]f(r)dt^2+[1+\bar{\mathcal{H}}+k(r)]f(r)^{-1}dr^2 \\ &\quad +(1+\bar{\mathcal{H}})r^2(d\theta^2+\sin^2\theta d\phi^2)\,, \\ k(r) &= \frac{\bar{\mathcal{H}}_0 Q^2}{r^2 f(r)}\,, \qquad \bar{\mathcal{H}}_0 = -2q\Gamma^2/[Q(\mathcal{M}b-Q^2)]\,, \end{aligned} \tag{C.2.12}$$

where

$$\begin{aligned} \bar{\mathcal{H}} &= 2\frac{m}{br}f(b)^{-1/2}\frac{(r-\mathcal{M})(b-\mathcal{M})-\Gamma^2\cos\theta}{\bar{\mathcal{D}}}\,, \\ \bar{\mathcal{D}} &= [(r-\mathcal{M})^2+(b-\mathcal{M})^2-2(r-\mathcal{M})(b-\mathcal{M})\cos\theta \\ &\quad -\Gamma^2\sin^2\theta]^{1/2}\,. \end{aligned} \tag{C.2.13}$$

It can be shown that this perturbed metric is spatially conformally flat; moreover, the solution remains valid as long as the condition $|\bar{\mathcal{H}}| \ll 1$ is satisfied. The total electromagnetic field to first order of the perturbation turns out to be

$$\tilde{F} = -\left[\frac{Q}{r^2}+E_r\right]dt\wedge dr - E_\theta dt\wedge d\theta\,, \tag{C.2.14}$$

with

$$
\begin{aligned}
E_r &= \frac{q}{r^3}\frac{\mathcal{M}r-Q^2}{\mathcal{M}b-Q^2}\frac{1}{\bar{\mathcal{D}}}\Bigg\{\Bigg[\mathcal{M}(b-\mathcal{M})+\Gamma^2\cos\theta \\
&\quad +\frac{Q^2[(r-\mathcal{M})(b-\mathcal{M})-\Gamma^2\cos\theta]}{\mathcal{M}r-Q^2}\Bigg] \\
&\quad -\frac{r[(r-\mathcal{M})(b-\mathcal{M})-\Gamma^2\cos\theta]}{\bar{\mathcal{D}}^2}[(r-\mathcal{M})-(b-\mathcal{M})\cos\theta]\Bigg\}\,, \\
E_\theta &= q\frac{\mathcal{M}r-Q^2}{\mathcal{M}b-Q^2}\frac{b^2f(b)f(r)}{\bar{\mathcal{D}}^3}\sin\theta\,. \qquad \text{(C.2.15)}
\end{aligned}
$$

Note that in the extreme case $Q/\mathcal{M} = q/m = 1$ this solution reduces to the linearized form of the well known exact solution by Majumdar and Papapetrou [147, 148] for two extreme Reissner-Nordström black holes. Furthermore, this solution satisfies Gauss' theorem

$$\Phi = \int_S {}^*\tilde{F}\wedge dS = 4\pi[Q+q\vartheta(r-b)]\,, \qquad \text{(C.2.16)}$$

where Φ is the flux of the electric field obtained by integrating the dual of the electromagnetic form (C.2.14) over a spherical 2-surface S centered at the origin where the black hole charge Q is placed and with variable radius (r greater or lesser than b), the function $\vartheta(x)$ denoting the step function.

Recently an important progress has been achieved by Belinski and Alekseev [149]. They have obtained an exact two-body solution of the Einstein-Maxwell equations in explicit analytic form for the system consisting of a Reissner-Nordström black hole and a naked singularity, by using the monodromy transform approach [150]. They have shown that an equilibrium without intervening struts or tensions is possible for such a system at selected values of the separating distance between the sources. Furthermore, their equilibrium condition exactly reduces to our equation (C.2.11) once linearized with respect to the mass and charge of the naked singularity. We have indeed been able to show explicitly the coincidence between the linearized form of their exact solution and our perturbative solution.

We have then analyzed in [143] the properties of the perturbed electric field with special attention to the construction of the lines of force of the electric field. The two cases have been considered of the sole particle, with the subtraction of the dominant contribution of the black hole, as well as of the total

field due to the black hole and the particle. As the black hole becomes extreme an effect similar to the ordinary Meissner effect for magnetic fields in the presence of superconductors arises: the electric field lines of the point charge are expelled outside the outer horizon. Note that this effective "electric Meissner effect" has no classical analogue, as far as we know, and is a pure general relativistic effect. Furthermore, it is suitable to a suggestive interpretation in terms of the nature of the Reissner-Nordström solution. In fact, as soon as the black hole is not extreme the point particle induces charge on the horizon, and accordingly the electric field lines terminate on it; when the black hole becomes extreme no further charge induction is possible (unless one turns the black hole into a naked singularity), and coherently the electric field lines no more cross the horizon. In a sense the black hole rejects to turn itself into a naked singularity and this might be thought of as an argument in favor of the Cosmic Censorship conjecture.

It is worth to mention that we are currently extending the above mentioned results to more general situations. For example, we are considering (with A. Geralico and G. Carvalho) the case in which the source is rotating uniformly and the backreaction on the metric invloves all electric and magnetic multipoles.

D. Cosmology

D.1. Mixmaster universe and the spectral index

The Bianchi type IX spatially homogeneous vacuum spacetime also known as the Mixmaster universe has served as a theoretical playground for many ideas in general relativity, one of which is the question of the nature of the chaotic behavior exhibited in some solutions of the vacuum Einstein equations and another is the question of whether or not one can interpret the spacetime as a closed gravitational wave. In particular, to describe the mathematical approach to an initial cosmological singularity, the exact Bianchi type IX dynamics leads to the BLK approximation involving the discrete BLK map which acts as the transition between phases of approximately Bianchi type I evolution. The parameters of this map are not so easily extracted from the numerical evolution of the metric variables. However, recently it has been realized that these parameters are directly related to transitions in the scale-free part of the Weyl tensor. In fact this leads to a whole new interpretation of what the BLK dynamics represents.

For a given foliation of any spacetime, one can always introduce the scale free part of the extrinsic curvature when its trace is nonzero by dividing by that trace. In the expansion-normalized approach to spatially homogeneous dynamics, this corresponds to the expansion-normalized gravitational velocity variables. This scale free extrinsic curvature tensor can be characterized by its eigenvalues, whose sum is 1 by definition: these define three functions of the time parametrizing the foliation which generalize the Kasner indices of Bianchi type I vacuum spacetimes. A phase of velocity-dominated evolution is loosely defined as an interval of time during which the spatial curvature terms in the spacetime curvature are negligible compared to the extrinsic curvature terms. Under these conditions the vacuum Einstein equations can be approximated by ordinary differential equations in the time. These lead to a simple scaling of the eigenvectors of the extrinsic curvature during which the generalized Kasner indices remain approximately constant and simulate the

Bianchi type I Kasner evolution.

The Weyl tensor can be also be repackaged as a second rank but complex spatial tensor with respect to the foliation and its scale free part is determined by a single complex scalar function of its eigenvalues, a number of particular representations for which are useful. In particular the so called speciality index is the natural choice for this variable which is independent of the permutations of the spatial axes used to order the eigenvalues, and so is a natural 4-dimensional tracker of the evolving gravitational field quotienting out all 3-dimensional gauge-dependent quantities. During a phase of velocity-dominated ("Kasner") evolution, the Weyl tensor is approximately determined by the extrinsic curvature alone, and hence the scalefree invariant part of the Weyl tensor is locked to the generalized Kasner indices exactly as in a Kasner spacetime. Of course during transitions between velocity-dominated evolution where the spatial curvature terms are important, the generalized Kasner indices and the Weyl tensor are uncoupled in their evolution, but the transition between one set of generalized Kasner indices and the next is locked to a transition in the scalefree Weyl tensor. This idealized mapping, approximated by the BKL map between Kasner triplets, can be reinterpreted as a continuous transition in the Weyl tensor whose scale invariant part can be followed through the transition directly. For spatially homogeneous vacuum spacetimes, the BLK transition is a consequence of a Bianchi type II phase of the dynamics which can be interpreted as a single bounce with a curvature wall in the Hamiltonian approach to the problem. One can in fact follow this transition in the Weyl tensor directly with an additional first order differential equation which is easily extracted from the Newman-Penrose equations expressed in a frame adapted both to the foliation and the Petrov type of the Weyl tensor.

This type of Weyl transition in the Mixmaster dynamics can be followed approximately using the Bianchi type II approximation to a curvature bounce, leading to a temporal spike in the real and imaginary parts of the speciality index which represents a circuit in the complex plane between the two real asymptotic Kasner points (a "pulse"). The graph of the speciality index versus time thus serves as a sort of electrocardiogram of the "heart" of the Mixmaster dynamics, stripping away all the gauge and frame dependent details of its evolution except for the choice of time parametrization, which is a recent nice result of our investigation.

D.2. Cosmological fluids obeying a non-ideal equation of state

Current improvements in cosmological measurements strongly favor the standard model of the universe being spatially flat, homogeneous and isotropic on large scales and dominated by dark energy consistently with the effect of a cosmological constant and cold dark matter. In the literature such a concordance model is referred to as Λ-Cold Dark Matter (ΛCDM) model: the universe is well described by a Friedmann-Robertson-Walker (FRW) metric, whose gravity source is a mixture of non-interacting perfect fluids including a cosmological constant. At early times the universe was radiation-dominated, but the present contribution of radiation is negligibly small. The dominant contribution to the mass-energy budget of the universe today is due to dark energy, obeying an equation of state $p_{\rm de} \simeq -\rho_{\rm de}$. The cosmological constant thus acts as an effective negative pressure, allowing the total energy density of the universe to remain constant even though the universe expands.

However, no theoretical model determining the nature of dark energy is available as yet, leaving its existence still unexplained. A big effort has been spent in recent years in formulating cosmological models with modified equation of state [170]. These models include, for instance, a decaying scalar field (quintessence) minimally coupled to gravity, similar to the one assumed by inflation [171], scalar field models with nonstandard kinetic terms ($k-$essence) [172], the Chaplygin gas [173], braneworld models and cosmological models from scalar-tensor theories of gravity (see, e.g., Refs. [174, 175] and references therein).

Following the same line of thinking, we have recently proposed in Ref. [176] a cosmological model with a fluid source obeying a non-ideal equation of state with "asymptotic freedom," first introduced by Shan and Chen (SC) in the context of lattice kinetic theory [177]. Such an equation of state supports a phase transition between low and high density regimes, both characterized by an ideal gas behavior, i.e., pressure and density change in linear proportion to each other. Similarly to the case of lattice kinetic theory, in which the stabilizing effect of hard-core repulsion is replaced by an asymptotic-free attraction, the repulsive effect of the cosmological constant is here replaced by a scalar field with asymptotic-free attraction. We have used these properties to model the growth of the dark matter-energy component of the universe, showing that a cosmological FRW fluid obeying a SC-like equa-

tion of state naturally evolves from an ordinary energy density component towards a present-day universe with a suitable dark energy component, with no need of invoking any cosmological constant. We have also provided some observational tests in support to our model. More precisely, we have drawn the Hubble diagram (distance modulus vs redshift) as well as the expansion history of the universe (Hubble parameter vs redshift), showing that they are consistent with current astronomical data.

We have then investigated in Ref. [178] the possibility that a SC-like equation of state may also be used to describe an early inflationary universe. Inflation is an epoch of accelerated expansion, which was originally assumed as a mechanism to solve several puzzles of the standard Big Bang scenario, e.g., the flatness and horizon problems [179, 180, 181, 182, 183, 184]. It also provides plausible scenarios for the origin of the large scale structure of the universe, as well as the formation of anisotropies in the cosmic microwave background radiation. Many different kinds of inflationary models have been developed so far, including recent attempts to construct consistent models of inflation based on superstring or supergravity models.

In the context of inflation, we have represented a SC fluid in a flat FRW universe filled by a scalar field in an external potential, whose energy density and pressure are identified (and fixed by) with the SC corresponding quantities. Therefore, the potential is completely determined and we have analyzed in detail its role in the slow-roll approximation of inflation. We have found that simple choices of the free parameters of the SC model are consistent with current Planck and WMAP data, i.e., the minimal viability request for any model. Furthermore, the equation of state undergoes a transition between $p/\rho < 0$ (exotic matter) during inflation to $p/\rho > 0$ (ordinary matter) for late times, thus providing a natural exit mechanism. As a result, a SC-like equation of state for the fluid source of the early universe may play a role also in the context of inflation.

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