Generalizations of the Kerr-Newman solution

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- Generalizations of the Kerr-Newman solution

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2 Brief description

One of the most important metrics in general relativity is the Kerr-Newman solution that describes the gravitational and electromagnetic fields of a rotating charged mass. For astrophysical purposes, however, it is necessary to take into account the effects due to the moment of inertia of the object. To attack this problem we have derived exact solutions of Einstein-Maxwell equations which posses an infinite set of gravitational and electromagnetic multipole moments. Several analysis were performed that investigate the physical effects generated by a rotating deformed mass distribution in which the angular momentum and the quadrupole determine the dominant multipole moments.

Tidal indicators are commonly associated with the electric and magnetic parts of the Riemann tensor (and its covariant derivatives) with respect to a given family of observers in a given spacetime. Recently, observer-dependent tidal effects have been extensively investigated with respect to a variety of special observers in the equatorial plane of the Kerr spacetime. This analysis was extended by considering a more general background solution to include the case of matter which is also endowed with an arbitrary mass quadrupole moment. Relations with curvature invariants and BelRobinson tensor, i.e. observer-dependent super-energy density and super-Poynting vector, are also investigated.

In an additional study, the neutrino oscillations in the field of a rotating deformed mass were investigated. The phase shift is evaluated in the case of weak field limit, slow rotation and small deformation. To this aim the HartleThorne metric is used, which is an approximate solution of the vacuum Einstein equations accurate to second order in the rotation parameter and to first order in the mass quadrupole moment. Implications on atmospheric, solar and astrophysical neutrinos are discussed.

We also study some exact and approximate solutions of Einstein's equations that can be used to describe the gravitational field of astrophysical compact objects in the limiting case of slow rotation and slight deformation. We show that none of the standard models obtained by using Fock's method can be used as an interior source for the approximate exterior Kerr solution. We then use Fock's method to derive a generalized interior solution, and also an exterior solution that turns out to be equivalent to the exterior Hartle-Thorne approximate solution that, in turn, is equivalent to an approximate limiting case of the exact Quevedo-Mashhoon solution. As a result we obtain an analytic approximate solution that describes the interior and exterior gravitational field of a slowly rotating and slightly deformed astrophysical object.

As for exact solutions of Einstein's equations, we show that the Zipoy-Voorhees line element represents the simplest metric ($q-$metric) which describes the exterior field of a static mass distribution with quadrupole q. We analyze the main properties of the $q-$metric and propose to use it as the starting point to search for exterior and interior stationary exact solutions.

In an additional study, we analyze a relativistic model describing a thin disk surrounded by a halo in presence of an electromagnetic field. The model is obtained by solving the Einstein-Maxwell equations on a particular conformastatic spacetime background and by using the distributional approach for the energy-momentum tensor. A class of solutions is obtained in which the gravitational and electromagnetic potentials are completely determined by a harmonic function only. A particular solution is given that is asymptotically flat and singularity-free, and satisfies all the energy conditions.

3 Introduction

It is hard to overemphasize the importance of the Kerr geometry not only for general relativity itself, but also for the very fundamentals of physics. It assumes this position as being the most physically relevant rotating generalization of the static Schwarzschild geometry. Its charged counterpart, the Kerr-Newman solution, representing the exterior gravitational and electromagnetic fields of a charged rotating object, is an exact solution of the Einstein-Maxwell equations.

Its line element in Boyer–Lindquist coordinates can be written as

$$\begin{aligned} ds^2 &= \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 \\ &\quad - \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[(r^2 + a^2)d\varphi - adt]^2 \\ &\quad - \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 , \end{aligned} \tag{3.0.1}$$

where M is the total mass of the object, $a = J/M$ is the specific angular momentum, and Q is the electric charge. In this particular coordinate system, the metric functions do not depend on the coordinates t and ϕ, indicating the existence of two Killing vector fields $\xi^I = \partial_t$ and $\xi^{II} = \partial_\varphi$ which represent the properties of stationarity and axial symmetry, respectively.

An important characteristic of this solution is that the source of gravity is surrounded by two horizons situated at a distance

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{3.0.2}$$

from the origin of coordinates. Inside the interior horizon, r_-, a ring singularity is present which, however, cannot be observed by any observer situated outside the exterior horizon. If the condition $M^2 < a^2 + Q^2$ is satisfied, no horizons are present and the Kerr–Newman spacetime represents the exterior field of a naked singularity.

Despite of its fundamental importance in general relativity, and its theoretical and mathematical interest, this solution has not been especially useful for describing astrophysical phenomena, first of all, because observed astrophysical objects do not possess an appreciable net electric charge. Furthermore, the limiting Kerr metric takes into account the mass and the rotation, but does not consider the moment of inertia of the object. For astrophysical applications it is, therefore, necessary to use more general solutions with higher multipole moments which are due not only to the rotation of the body but also to its shape. This means that even in the limiting case of a static spacetime, a solution is needed that takes into account possible deviations from spherically symmetry.

4 The general static vacuum solution

In general relativity, stationary axisymmetric solutions of Einstein's equations [1] play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of know exact solutions drastically increased after Ernst [2] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies lead finally to the development of solution generating techniques [1] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [4] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

4.1 Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindric coordinates (t, ρ, z, φ). Stationarity implies that t can be chosen as the time coordinate and the metric does not depend on time, i.e. $\partial g_{\mu\nu}/\partial t = 0$. Consequently, the corresponding timelike Killing vector has the components δ^{μ}_{t}.

A second Killing vector field is associated to the axial symmetry with respect to the axis $\rho = 0$. Then, choosing φ as the azimuthal angle, the metric satisfies the conditions $\partial g_{\mu\nu}/\partial\varphi = 0$, and the components of the corresponding spacelike Killing vector are δ^{μ}_{φ}.

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form $g_{\mu\nu} = g_{\mu\nu}(\rho, z)$, it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [4, 5, 6]

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] , \qquad (4.1.1)$$

where f, ω and γ are functions of ρ and z, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$\rho\partial_\rho\Omega = f^2\partial_z\omega , \qquad \rho\partial_z\Omega = -f^2\partial_\rho\omega , \qquad (4.1.2)$$

the vacuum field equations $R_{\mu\nu} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \partial_z^2 f + \frac{1}{f}[(\partial_\rho\Omega)^2 + (\partial_z\Omega)^2 - (\partial_\rho f)^2 - (\partial_z f)^2] = 0 , \qquad (4.1.3)$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Omega) + \partial_z^2\Omega - \frac{2}{f}\left(\partial_\rho f\,\partial_\rho\Omega + \partial_z f\,\partial_z\Omega\right) = 0 , \qquad (4.1.4)$$

$$\partial_\rho\gamma = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right] , \qquad (4.1.5)$$

$$\partial_z\gamma = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right) . \qquad (4.1.6)$$

It is clear that the field equations for γ can be integrated by quadratures, once f and Ω are known. For this reason, the equations (4.1.3) and (4.1.4) for f and Ω are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only. It is interesting to mention that this set of equations can be geometrically interpreted in the context of nonlinear sigma models [7].

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation $\varphi \to -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (4.1.1) with $\omega = 0$, and the field equations can be written as

$$\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi + \partial_z^2\psi = 0\,, \quad f = \exp(2\psi)\,, \tag{4.1.7}$$

$$\partial_\rho\gamma = \rho\left[(\partial_\rho\psi)^2 - (\partial_z\psi)^2\right]\,, \quad \partial_z\gamma = 2\rho\partial_\rho\psi\,\partial_z\psi\,. \tag{4.1.8}$$

We see that the main field equation (4.1.7) corresponds to the linear Laplace equation for the metric function ψ.

4.2 Static solution

The general solution of Laplace's equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as [4, 1]

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2+z^2)^{\frac{n+1}{2}}} P_n(\cos\theta)\,, \qquad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}\,, \tag{4.2.1}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The expression for the metric function γ can be calculated by quadratures by using the set of first order differential equations (4.1.8). Then

$$\gamma = -\sum_{n,m=0}^{\infty} \frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2+z^2)^{\frac{n+m+2}{2}}} \left(P_n P_m - P_{n+1}P_{m+1}\right)\,. \tag{4.2.2}$$

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one of the most interesting special solutions which is Schwarzschild's spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants a_n in such a way that the infinite sum (4.2.1) converges to the Schwarzschild solution in cylindric coordinates. But, or course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzchild's metric.

In fact, it turns out that to investigate the properties of solutions with multipole moments it is more convenient to use prolate spheroidal coordinates (t, x, y, φ) in which the line element can be written as

$$ds^2 = f dt^2 - \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right]$$

where

$$x = \frac{r_+ + r_-}{2\sigma}\ , \quad (x^2 \geq 1), \quad y = \frac{r_+ - r_-}{2\sigma}\ , \quad (y^2 \leq 1) \tag{4.2.3}$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2\ , \quad \sigma = const\ , \tag{4.2.4}$$

and the metric functions are f, ω, and γ depend on x and y, only. In this coordinate system, the general static solution which is also asymptotically flat can be expressed as

$$f = \exp(2\psi)\ , \quad \psi = \sum_{n=0}^{\infty} (-1)^{n+1} q_n P_n(y) Q_n(x)\ , \quad q_n = const$$

where $P_n(y)$ are the Legendre polynomials, and $Q_n(x)$ are the Legendre functions of second kind. In particular,

$$P_0 = 1, \quad P_1 = y, \quad P_2 = \frac{1}{2}(3y^2 - 1)\ , ...$$

$$Q_0 = \frac{1}{2} \ln \frac{x+1}{x-1}\ , \quad Q_1 = \frac{1}{2} x \ln \frac{x+1}{x-1} - 1\ ,$$

$$Q_2 = \frac{1}{2}(3x^2 - 1) \ln \frac{x+1}{x-1} - \frac{3}{2} x\ , ...$$

The corresponding function γ can be calculated by quadratures and its general expression has been explicitly derived in [8]. The most important special cases contained in this general solution are the Schwarzschild metric

$$\psi = -q_0 P_0(y) Q_0(x)\ , \quad \gamma = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2}\ ,$$

and the Erez-Rosen metric [9]

$$\psi = -q_0 P_0(y) Q_0(x) - q_2 P_2(y) Q_2(x)\ , \quad \gamma = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2} + ...$$

In the last case, the constant parameter q_2 turns out to determine the quadrupole moment. In general, the constants q_n represent an infinite set of parameters that determines an infinite set of mass multipole moments.

5 Stationary generalization

The solution generating techniques [12] can be applied, in particular, to any static seed solution in order to obtain the corresponding stationary generalization. One of the most powerful techniques is the inverse method (ISM) developed by Belinski and Zakharov [13]. We used a particular case of the ISM, which is known as the Hoenselaers–Kinnersley-Xanthopoulos (HKX) transformation to derive the stationary generalization of the general static solution in prolate spheroidal coordinates.

5.1 Ernst representation

In the general stationary case ($\omega \neq 0$) with line element

$$ds^2 = f(dt - \omega d\varphi)^2$$
$$- \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right]$$

it is useful to introduce the the Ernst potentials

$$E = f + i\Omega\,, \quad \xi = \frac{1-E}{1+E}\,,$$

where the function Ω is now determined by the equations

$$\sigma(x^2-1)\Omega_x = f^2\omega_y\,, \quad \sigma(1-y^2)\Omega_y = -f^2\omega_x\,.$$

Then, the main field equations can be represented in a compact and symmetric form:

$$(\xi\xi^* - 1)\left\{[(x^2-1)\xi_x]_x + [(1-y^2)\xi_y]_y\right\} = 2\xi^*[(x^2-1)\xi_x^2 + (1-y^2)\xi_y^2]\,.$$

This equation is invariant with respect to the transformation $x \leftrightarrow y$. Then, since the particular solution

$$\xi = \frac{1}{x} \to \Omega = 0 \to \omega = 0 \to \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}$$

represents the Schwarzschild spacetime, the choice $\xi^{-1} = y$ is also an exact solution. Furthermore, if we take the linear combination $\xi^{-1} = c_1 x + c_2 y$ and introduce it into the field equation, we obtain the new solution

$$\xi^{-1} = \frac{\sigma}{M}x + i\frac{a}{M}y \, , \; \sigma = \sqrt{M^2 - a^2} \, ,$$

which corresponds to the Kerr metric in prolate spheroidal coordinates.

In the case of the Einstein-Maxwell theory, the main field equations can be expressed as

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\xi \, ,$$

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\mathcal{F} = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\mathcal{F}$$

where ∇ represents the gradient operator in prolate spheroidal coordinates. Moreover, the gravitational potential ξ and the electromagnetic $\mathcal{F}$ Ernst potential are defined as

$$\xi = \frac{1 - f - i\Omega}{1 + f + i\Omega} \, , \quad \mathcal{F} = 2\frac{\Phi}{1 + f + i\Omega} \, .$$

The potential Φ can be shown to be determined uniquely by the electromagnetic potentials A_t and A_φ One can show that if ξ_0 is a vacuum solution, then the new potential

$$\xi = \xi_0\sqrt{1 - e^2}$$

represents a solution of the Einstein-Maxwell equations with effective electric charge e. This transformation is known in the literature as the Harrison transformation [10]. Accordingly, the Kerr–Newman solution in this representation acquires the simple form

$$\xi = \frac{\sqrt{1 - e^2}}{\frac{\sigma}{M}x + i\frac{a}{M}y} \, , \qquad e = \frac{Q}{M} \, , \qquad \sigma = \sqrt{M^2 - a^2 - Q^2} \, .$$

In this way, it is very easy to generalize any vacuum solution to include the case of electric charge. More general transformations of this type can be used in order to generate solutions with any desired set of gravitational and electromagnetic multipole moments [11].

5.2 Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds (M, γ) and (N, G) of dimension m and n, respectively. Let M be coordinatized by x^a, and N by X^μ, so that the metrics on M and N can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $G = G(X)$. A harmonic map is a smooth map $X : M \to N$, or in coordinates $X : x \longmapsto X$ so that X becomes a function of x, and the X's satisfy the motion equations following from the action [14]

$$S = \int d^m x \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \tag{5.2.1}$$

which sometimes is called the "energy" of the harmonic map X. The straightforward variation of S with respect to X^μ leads to the motion equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\, \gamma^{ab}\, \partial_a X^\nu\, \partial_b X^\lambda = 0\,, \tag{5.2.2}$$

where $\Gamma^\mu_{\ \nu\lambda}$ are the Christoffel symbols associated to the metric $G_{\mu\nu}$ of the target space N. If $G_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $G_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(\pm 1, ..., \pm 1)$, the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (5.2.1) is usually referred to as the Polyakov action [16].

Consider now the case in which the base space M is a stationary axisymmetric spacetime. Then, γ^{ab}, $a, b = 0, ..., 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (4.1.1), i.e.

$$\gamma_{ab} = \begin{pmatrix} f & 0 & 0 & -f\omega \\ 0 & -f^{-1}e^{2k} & 0 & 0 \\ 0 & 0 & -f^{-1}e^{2k} & 0 \\ -f\omega & 0 & 0 & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}. \tag{5.2.3}$$

Let the target space N be 2-dimensional with metric $G_{\mu\nu} = (1/2)f^{-2}\delta_{\mu\nu}$, $\mu, \nu = 1, 2$, and let the coordinates on N be $X^\mu = (f, \Omega)$. Then, it is straightforward to show that the action (5.2.1) becomes

$$S = \int \mathcal{L}\, dt d\varphi d\rho dz\,, \qquad \mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right]\,, \tag{5.2.4}$$

and the corresponding motion equations (5.2.2) are identical to the main field equations (4.1.3) and (4.1.4).

Notice that the field equations can also be obtained from (5.2.4) by a direct variation with respect to f and Ω. This interesting result was obtained originally by Ernst [2], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \to 2)-$nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2,R)/SO(2)$ [15], and the Lagrangian (5.2.4) can be written explicitly [17] in terms of the generators of the Lie group $SL(2,R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [14].

The form of the Lagrangian (5.2.4) with two gravitational field variables, f and Ω, depending on two coordinates, ρ and z, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (5.2.4). Indeed, if we consider γ^{ab} as a 2-dimensional metric that depends on the parameters ρ and z, the diagonal form of the Lagrangian (5.2.4) implies that $\sqrt{|\gamma|}\gamma^{ab} = \delta^{ab}$. Clearly, this choice is not compatible with the factor ρ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian (5.2.4) cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to "absorb" the unpleasant factor ρ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2,R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields [1]. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

5.3 Representation as a generalized harmonic map

Consider two (pseudo-)Riemannian manifolds (M,γ) and (N,G) of dimension m and n, respectively. Let x^a and X^μ be coordinates on M and N, respectively. This coordinatization implies that in general the metrics γ and G become functions of the corresponding coordinates. Let us assume that not only γ but also G can explicitly depend on the coordinates x^a, i.e. let $\gamma = \gamma(x)$ and $G = G(X,x)$. This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map $X : M \to N$ will be called an $(m \to n)-$generalized harmonic map if it satisfies the Euler-Lagrange equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\,\gamma^{ab}\,\partial_a X^\nu \partial_b X^\lambda + G^{\mu\lambda}\gamma^{ab}\,\partial_a X^\nu\,\partial_b G_{\lambda\nu} = 0\,, \tag{5.3.1}$$

which follow from the variation of the generalized action

$$S = \int d^m x \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a\, X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.2}$$

with respect to the fields X^μ. Here the Christoffel symbols, determined by the metric $G_{\mu\nu}$, are calculated in the standard manner, without considering the explicit dependence on x. Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term $G_{\mu\nu}(X,x)$ in the Lagrangian density implies that we are taking into account the "interaction" between the base space M and the target space N. This interaction leads to an extra term in the motion equations, as can be seen in (5.3.1). It turns out that this interaction is the result of the effective presence of the gravitational field.

Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (5.3.1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^\mu_{\ \nu\lambda}\ \partial_b X^\lambda + G^{\mu\lambda}\ \partial_b G_{\lambda\nu})\partial_a X^\nu = 0\,, \tag{5.3.3}$$

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma^\mu_{\ \nu\lambda} = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\lambda}\partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma^\mu_{\ \nu\lambda} \neq 0$ and (5.3.3) cannot be satisfied. In the general case of a curved target metric, conditions (5.3.3) represent a system of m first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

As we mentioned before, the generalized action (5.3.2) includes an interaction between the base space N and the target space M, reflected on the fact that $G_{\mu\nu}$ depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To see this explicitly we calculate the covariant derivative of the generalized Lagrangian density

$$\mathcal{L} = \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a\, X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.4}$$

and replace in the result the corresponding motion equations (5.3.1). Then, the final result can be written as

$$\nabla_b \widetilde{T}_a{}^b = -\frac{\partial \mathcal{L}}{\partial x^a} \tag{5.3.5}$$

where $\widetilde{T}_a{}^b$ represents the canonical energy-momentum tensor

$$\widetilde{T}_a{}^b = \frac{\partial \mathcal{L}}{\partial(\partial_b X^\mu)}(\partial_a X^\mu) - \delta_a^b \mathcal{L} = 2\sqrt{\gamma} G_{\mu\nu}\left(\gamma^{bc}\partial_a X^\mu\,\partial_c X^\nu - \frac{1}{2}\delta_a^b \gamma^{cd}\partial_c X^\mu\,\partial_d X^\nu\right). \tag{5.3.6}$$

The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space $\gamma_{ab} = \eta_{ab}$, the explicit dependence of the metric of the target space $G_{\mu\nu}(X, x)$ on x generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.

$$T_{ab} = \frac{\delta \mathcal{L}}{\delta \gamma^{ab}} \,. \tag{5.3.7}$$

A straightforward computation shows that for the action under consideration here we have that $\widetilde{T}_{ab} = 2T_{ab}$ so that the generalized conservation law (5.3.5) can be written as

$$\nabla_b T_a{}^b + \frac{1}{2}\frac{\partial \mathcal{L}}{\partial x^a} = 0 \,. \tag{5.3.8}$$

For a given metric on the base space, this represents in general a system of m differential equations for the “fields” X^μ which must be satisfied “on-shell”.

If the base space is 2-dimensional, we can use a reparametrization of x to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, $T_a{}^a = 0$.

In Section 5.1 we described stationary, axially symmetric, gravitational fields as a $(4 \to 2)-$nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \to 2)-$generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space M, and $X^\mu = (f, \Omega)$ the coordinates on the target space N. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

$$\gamma_{ab} = \delta_{ab} \qquad \text{and} \qquad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \qquad (a,b = 1,2;\ \mu,\nu = 1,2). \tag{5.3.9}$$

A straightforward computation shows that the generalized Lagrangian (5.3.4) coincides with the Lagrangian (5.2.4) for stationary axisymetric fields, and that the equations of motion (5.3.1) generate the main field equations (4.1.3) and (4.1.4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$. Then

$$T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right], \tag{5.3.10}$$

$$T_{\rho z} = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right). \tag{5.3.11}$$

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (5.3.8) on-shell:

$$\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\rho} = 0\,, \tag{5.3.12}$$

$$\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0\,. \tag{5.3.13}$$

Incidentally, the last equation coincides with the integrability condition for the metric function k, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs.(4.1.5,4.1.6) and (5.3.10,5.3.11), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (5.3.13) becomes an identity. Although we have eliminated from the starting Lagrangian (5.2.4) the variable k by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [17] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about k at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)-$generalized harmonic map with metrics given as in (5.3.9). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space M is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on M. Then, M represents the world-sheet of a bosonic string in which τ is measures the time and z is the parameter along the string. The string is "embedded" in the target space N whose metric is conformally flat and explicitly depends on the time parameter τ. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates ρ and z are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a\to\infty} f = 1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \omega = c_1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \Omega = O\left(\frac{1}{x^a}\right) \tag{5.3.14}$$

where c_1 is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate φ. If we choose the domain of the spatial coordinates as $\rho \in [0, \infty)$ and $z \in (-\infty, +\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space N satisfy the boundary conditions

$$\dot{X}^\mu(\rho, -\infty) = 0 = \dot{X}^\mu(\rho, \infty) , \quad X'^\mu(\rho, -\infty) = 0 = X'^\mu(\rho, \infty) \tag{5.3.15}$$

where the dot stands for a derivative with respect to ρ and the prime represents derivation with respect to z. These relationships are known in string theory [16] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume ρ as a "time" parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D-$branes situated at plus and minus infinity in the $z-$direction.

5.4 Dimensional extension

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space N, and study the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an $(m \to D)-$generalized harmonic map. As before we denote by $\{x^a\}$ the coordinates on M. Let $\{X^\mu, X^\alpha\}$ with $\mu = 1, 2$ and $\alpha = 3, 4, ..., D$ be the coordinates on N. The metric structure on M is again $\gamma = \gamma(x)$, whereas the metric on N can in general depend on all coordinates of M and N, i.e. $G = G(X^\mu, X^\alpha, x^a)$. The general structure of the corresponding field equations is as given in (5.3.1). They can be divided into one set of equations for X^μ and one set of equations for X^α. According to the results of the last section, the class of gravitational fields under consideration can be represented as a $(2 \rightarrow 2)-$generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates X^μ of the target space. Then, the gravitational sector of the target space will be contained in the components $G_{\mu\nu}$ $(\mu, \nu = 1, 2)$ of the metric, whereas the components $G_{\alpha\beta}$ $(\alpha, \beta = 3, 4, ..., D)$ represent the sector of the dimensional extension.

Clearly, the set of differential equations for X^μ also contains the variables X^α and its derivatives $\partial_a X^\alpha$. For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing X^α and its derivatives in the equations for X^μ. It is easy to show that this can be achieved by imposing the conditions

$$G_{\mu\alpha} = 0\,, \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0\,, \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0\,. \tag{5.4.1}$$

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., $G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a)$, $\gamma = 3, 4, ..., D$. Furthermore, the variables X^α must satisfy the differential equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\alpha\right) + \Gamma^\alpha_{\ \beta\gamma}\,\gamma^{ab}\,\partial_a X^\beta \partial_b X^\gamma + G^{\alpha\beta}\gamma^{ab}\,\partial_a X^\gamma\,\partial_b G_{\beta\gamma} = 0\,. \tag{5.4.2}$$

This shows that any given $(2 \rightarrow 2)-$generalized map can be extended, without affecting the field equations, to a $(2 \rightarrow D)-$generalized harmonic map.

It is worth mentioning that the fact that the target space N becomes split in two separate parts implies that the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. $T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)$. The generalized conservation law as given in (5.3.8) is satisfied by the sum of both parts.

Consider the example of stationary axisymmetric fields given the metrics (5.3.9). Taking into account the conditions (5.4.1), after a dimensional extension the metric of the target space becomes

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\ 0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\ . & . & \cdots & \cdots & \cdots \\ 0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x) \end{pmatrix}. \tag{5.4.3}$$

Clearly, to avoid that this metric becomes degenerate we must demand that $\det(G_{\alpha\beta}) \neq 0$, a condition that can be satisfied in view of the arbitrariness of the components of the metric. With the extended metric, the Lagrangian density gets an additional term

$$\mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right] + \left(\partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta\right) G_{\alpha\beta}\,, \tag{5.4.4}$$

which nevertheless does not affect the field equations for the gravitational variables f and Ω. On the other hand, the new fields must be solutions of the extra field equations

$$\left(\partial_\rho^2 + \partial_z^2\right) X^\alpha + \Gamma^\alpha{}_{\beta\gamma}\left(\partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma\right) + G^{\alpha\gamma}\left(\partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma}\right) = 0\,. \tag{5.4.5}$$

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice $G_{\alpha\beta} = \eta_{\alpha\beta}$ with additional fields X^α given as arbitrary harmonic functions. This choice opens the possibility of introducing a "time" coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section 4.1, these fields are obtained from the general stationary fields in the limiting case $\Omega = 0$ (or equivalently, $\omega = 0$). If we consider the representation as an $SL(2,R)/SO(2)$ nonlinear sigma model or as a $(2 \to 2)-$generalized harmonic map, we see immediately that the limit $\Omega = 0$ is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case $\Omega = 0$. In the most simple case of an extension with $G_{\alpha\beta} = \delta_{\alpha\beta}$, the resulting $(2 \to 2)-$generalized map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4.6}$$

where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable f. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string "living" in a D-dimensional target space N. The string world-sheet is parametrized by the coordinates ρ and z. The gravitational sector of the target space depends explicitly on the metric functions f and Ω and on the parameter ρ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)-$dimensional Minkowski spacetime with time parameter τ. Then, the string world-sheet is a 2-dimensional flat hypersurface which is "frozen" along the time τ.

5.5 The general solution

If we take as seed metric the general static solution, the application of two HXK transformations generates a stationary solution with an infinite number of gravitoelectric and gravitomagnetic multipole moments. The HKX method is applied at the level of the Ernst potential from which the metric functions can be calculated by using the definition of the Ernst potential E and the field equations for γ. The resulting expressions in the general case are quite cumbersome. We quote here only the special case in which only an arbitrary quadrupole parameter is present. In this case, the result can be written as

$$\begin{aligned} f &= \frac{R}{L} e^{-2qP_2Q_2} , \\ \omega &= -2a - 2\sigma \frac{\mathcal{M}}{R} e^{2qP_2Q_2} , \\ e^{2\gamma} &= \frac{1}{4}\left(1+\frac{M}{\sigma}\right)^2 \frac{R}{x^2-y^2} e^{2\hat{\gamma}} , \end{aligned} \tag{5.5.1}$$

where

$$\begin{aligned} R &= a_+a_- + b_+b_- , \qquad L = a_+^2 + b_+^2 , \\ \mathcal{M} &= \alpha x(1-y^2)(e^{2q\delta_+} + e^{2q\delta_-})a_+ + y(x^2-1)(1-\alpha^2 e^{2q(\delta_+ + \delta_-)})b_+ , \\ \hat{\gamma} &= \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 + q^2(1-P_2)\bigg[(1+P_2)(Q_1^2 - Q_2^2) \\ &\quad + \frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\bigg] . \end{aligned} \tag{5.5.2}$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$\begin{aligned} a_\pm &= x(1-\alpha^2 e^{2q(\delta_+ + \delta_-)}) \pm (1+\alpha^2 e^{2q(\delta_+ + \delta_-)}) , \\ b_\pm &= \alpha y(e^{2q\delta_+} + e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+} - e^{2q\delta_-}) , \\ \delta_\pm &= \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2 \mp xy) + \frac{3}{4}[x(1-y^2) \mp y(x^2-1)]\ln\frac{x-1}{x+1} , \end{aligned}$$

the quantity α being a constant

$$\alpha = \frac{\sigma - M}{a} , \qquad \sigma = \sqrt{M^2 - a^2} . \tag{5.5.3}$$

The physical significance of the parameters entering this metric can be clarified by calculating the Geroch-Hansen [18, 19] multipole moments

$$M_{2k+1} = J_{2k} = 0\,, \quad k = 0, 1, 2, ... \tag{5.5.4}$$

$$M_0 = M\,, \quad M_2 = -Ma^2 + \frac{2}{15}qM^3\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.5}$$

$$J_1 = Ma\,, \quad J_3 = -Ma^3 + \frac{4}{15}qM^3a\left(1 - \frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.6}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the symmetry with respect to the equatorial plane. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q.

We analyzed the geometric and physical properties of the above solution. The special cases contained in the general solution suggest that it can be used to describe the exterior asymptotically flat gravitational field of rotating body with arbitrary quadrupole moment. This is confirmed by the analysis of the motion of particles on the equatorial plane. The quadrupole moment turns out to drastically change the geometric structure of spacetime as well as the motion of particles, especially near the gravitational source.

We investigated in detail the properties of the Quevedo-Mashhoon (QM) spacetime which is a generalization of Kerr spacetime, including an arbitrary quadrupole. Our results show [20] that a deviation from spherical symmetry, corresponding to a non-zero electric quadrupole, completely changes the structure of spacetime. A similar behavior has been found in the case of the Erez-Rosen spacetime. In fact, a naked singularity appears that affects the ergosphere and introduces regions where closed timelike curves are allowed. Whereas in the Kerr spacetime the ergosphere corresponds to the boundary of a simply-connected region of spacetime, in the present case the ergosphere is distorted by the presence of the quadrupole and can even become transformed into non simply-connected regions. All these changes occur near the naked singularity which is situated at $x = 1$, a value that corresponds to the radial distance $r = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates. In the limiting case $a/M > 1$, the multipole moments and the metric become complex, indicating that the physical description breaks down. Consequently, the extreme Kerr black hole represents the limit of applicability of the QM spacetime.

Since standard astrophysical objects satisfy the condition $a/M < 1$, we can conclude that the QM metric can be used to describe their exterior gravitational field. Two alternative situations are possible. If the characteristic radius of the body is greater than the critical distance $M + \sqrt{M^2 - a^2}$, i.e.

$x > 1$, the exterior solution must be matched with an interior solution in order to describe the entire spacetime. If, however, the characteristic radius of the body is smaller than the critical distance $M + \sqrt{M^2 - a^2}$, the QM metric describes the field of a naked singularity.

The presence of a quadrupole and higher multipole moments leads to interesting consequences in the motion of test particles. In several works to be presented below, the For instance, repulsive effects can take place in a region very closed to the naked singularity. In that region stable circular orbits can exist. The limiting case of static particle is also allowed. Due to the complexity of the above solution, the investigation of naked singularities can be performed only numerically. To illustrate the effects of repulsive gravity analytically, we used the simplest possible case which corresponds to the Reissner-Nordströn spacetime.

6 Tidal indicators in the spacetime of a rotating deformed mass

6.1 Introduction

Relativistic tidal problems have been extensively studied in the literature in a variety of contexts. Tidal effects are responsible for deformations or even disruption of astrophysical objects (like ordinary stars but also compact objects like neutron stars) placed in strong gravitational fields. For instance, they play a central role in the merging of compact object binaries, which can be accurately modeled only by numerical simulations in full general relativity, solving the coupled Einstein-hydrodynamics equations needed to evolve relativistic, self-gravitating fluids [22, 23, 24]. Such tidal disruption events are expected to happen very frequently in the Universe, leading then to a possible detection of the associated emission of gravitational waves in the near future by ground-based detectors [25]. To this end, different analytical and semi-analytical approaches have been developed to properly describe at least part of the coalescence process and to study the associated gravitational wave signals [26, 27, 28]. These approaches usually require either Post-Newtonian techniques or first order perturbation theory. In fact, in this limit the motion of each individual compact object in the binary system can be treated as the motion of an extended body in a given gravitational field due to its companion under the assumption that it causes only a small perturbation on the background [29]. Finally, one can also be interested in studying the tidal disruption limit of ordinary stars and compact objects in the field of a black hole (see, e.g., Ref. [30] and references therein). Tidal problems of this kind can be treated within the so-called tidal approximation, i.e., by assuming that the mass of the star is much smaller than the black hole mass and that the stellar radius is smaller than the orbital radius, so that backreaction effects on the background field can be neglected. Therefore, the star is usually described as a self-gravitating Newtonian fluid and its center of mass is assumed to move around the black hole along a timelike geodesic path. The tidal field due to the black hole is then computed from the Riemann tensor in terms of the geodesic deviation equation.

The role of the observer in relativistic tidal problems has never received enough attention in the literature. Nevertheless, it is crucial in interpreting the results. In fact, if tidal forces are due to curvature, the latter is experienced by observers through the electric and magnetic parts of the Riemann tensor, which is the only true 4-dimensional invariant quantity. In contrast, its electric and magnetic parts depend by definition on the choice of the observers who perform the measurement. In a recent paper [31] we have addressed such an issue by providing all necessary tools to relate the measurement of tidal effects by different families of observers. We have considered two "tidal indicators" defined as the trace of the square of the electric and magnetic parts of the Riemann tensor, respectively. They are both curvature and observer dependent and we have investigated their properties by considering a number of special observer families in the equatorial plane of the Kerr spacetime. As an interesting feature we have shown that the electric-type indicator cannot be made as vanishing with respect to any such observers, whereas the family of Carter's observers is the only one which measures zero tidal magnetic indicator. We have argued that the explanation for this effect relies on the absence of a quadrupole moment independent on the rotational parameter for the Kerr solution. To answer this question, as well as to extend the previous analysis to a more general context, we consider here a solution of the vacuum Einstein field equations due to Quevedo and Mashhoon [8, 32], which generalizes the Kerr spacetime to include the case of matter with arbitrary mass quadrupole moment and is specified by three parameters, the mass M, the angular momentum per unit mass a and the quadrupole parameter q. It is its genuine quadrupole moment content which makes this solution of particular interest here. We will thus investigate how the shape deformation of the rotating source affects the properties of tidal indicators with respect to special family of observers, including static observers, ZAMOs (i.e., zero angular momentum observers) and geodesic observers.

Hereafter latin indices run from 1 to 3 whereas greek indices run from 0 to 3 and geometrical units are assumed. The metric signature is chosen as $+2$.

6.2 The gravitational field of a rotating deformed mass

The exterior gravitational field of a rotating deformed mass can be described by the Quevedo-Mashhoon (hereafter QM) solution [8, 32]. This is a stationary axisymmetric solution of the vacuum Einstein's equations belonging to the class of Weyl-Lewis-Papapetrou [33, 34, 35] and is characterized, in general, by the presence of a naked singularity. Although the general solution is characterized by an infinite set of gravitoelectric and gravitomagnetic multipoles, we consider here the special solution discussed in Ref. [36] that involves only three parameters: the mass M, the angular momentum per unit mass a and the mass quadrupole parameter q of the source.

The corresponding line element in prolate spheroidal coordinates (t, x, y, ϕ)

with $x \geq 1$, $-1 \leq y \leq 1$ is given by [1]

$$\begin{aligned} ds^2 &= -f(dt - \omega d\phi)^2 \\ &+\frac{\sigma^2}{f}\left\{e^{2\gamma}\left(x^2-y^2\right)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\phi^2\right\}, \end{aligned} \tag{6.2.1}$$

where f, ω and γ are functions of x and y only and σ is a constant. They have the form

$$\begin{aligned} f &= \frac{R}{L}e^{-2qP_2Q_2}, \qquad \omega = -2a - 2\sigma\frac{\mathfrak{M}}{R}e^{2qP_2Q_2}, \\ e^{2\gamma} &= \frac{1}{4}\left(1+\frac{M}{\sigma}\right)^2\frac{R}{x^2-1}e^{2\hat{\gamma}}, \end{aligned} \tag{6.2.2}$$

where

$$\begin{aligned} R &= a_+a_- + b_+b_-, \qquad L = a_+^2 + b_+^2, \\ \mathfrak{M} &= \alpha x(1-y^2)(e^{2q\delta_+}+e^{2q\delta_-})a_+ + y(x^2-1)(1-\alpha^2 e^{2q(\delta_++\delta_-)})b_+, \\ \hat{\gamma} &= \frac{1}{2}(1+q)^2\ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 + q^2(1-P_2)\Big[(1+P_2)(Q_1^2-Q_2^2) \\ &+\frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\Big]. \end{aligned} \tag{6.2.3}$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$\begin{aligned} a_\pm &= x(1-\alpha^2 e^{2q(\delta_++\delta_-)}) \pm (1+\alpha^2 e^{2q(\delta_++\delta_-)}), \\ b_\pm &= \alpha y(e^{2q\delta_+}+e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+}-e^{2q\delta_-}), \\ \delta_\pm &= \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2\mp xy) + \frac{3}{4}[x(1-y^2)\mp y(x^2-1)]\ln\frac{x-1}{x+1}, \end{aligned} \tag{6.2.4}$$

the quantity α being a constant

$$\alpha = \frac{\sigma - M}{a}, \qquad \sigma = \sqrt{M^2 - a^2}. \tag{6.2.5}$$

We limit our analysis here to the case $\sigma > 0$, i.e. $M > a$. In the case $\sigma = 0$ the solution reduces to the extreme Kerr spacetime irrespective of the value of q [32].

The Geroch-Hansen [18, 19] moments are given by

$$M_{2k+1} = J_{2k} = 0, \quad k = 0,1,2,... \tag{6.2.6}$$

$$M_0 = M\,, \quad M_2 = -Ma^2 + \frac{2}{15}qM^3\left(1-\frac{a^2}{M^2}\right)^{3/2}\,, \ldots \tag{6.2.7}$$

$$J_1 = Ma\,, \quad J_3 = -Ma^3 + \frac{4}{15}qM^3a\left(1-\frac{a^2}{M^2}\right)^{3/2}\,, \ldots. \tag{6.2.8}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the reflection symmetry of the solution about the hyperplane $y = 0$, which we will refer to as "symmetry" (or equivalently "equatorial") plane hereafter. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q [36].

Some geometric and physical properties of the QM solution have been analyzed in Ref. [37]. It turns out that the whole geometric structure of the QM spacetime is drastically changed in comparison with Kerr spacetime, leading to a number of previously unexplored physical effects strongly modifying the features of particle motion, especially near the gravitational source. In fact, the QM solution is characterized by a naked singularity at $x = 1$, whose existence critically depends on the value of the quadrupole parameter q. In the case $q = 0$ (Kerr solution) $x = 1$ represents instead an event horizon.

6.2.1 Limiting cases

The QM solution reduces to the Kerr spacetime in the limiting case $q \to 0$, to the Erez-Rosen spacetime when $a \to 0$ and to the Schwarzschild solution when both parameters vanish. Furthermore, it has been shown in Ref. [37] that the general form of the QM solution is equivalent, up to a coordinate transformation, to the exterior vacuum Hartle-Thorne solution once linearized to first order in the quadrupole parameter and to second order in the rotation parameter. The limiting cases contained in the general solution thus suggest that it can be used to describe the exterior asymptotically flat gravitational field of a rotating body with arbitrary quadrupole moment.

Kerr solution

For vanishing quadrupole parameter we recover the Kerr solution, with functions [1]

$$\begin{aligned} f_K &= \frac{c^2x^2+d^2y^2-1}{(cx+1)^2+d^2y^2}\,, \qquad \omega_K = 2a\frac{(cx+1)(1-y^2)}{c^2x^2+d^2y^2-1}\,, \\ \gamma_K &= \frac{1}{2}\ln\left(\frac{c^2x^2+d^2y^2-1}{c^2(x^2-y^2)}\right)\,, \end{aligned} \tag{6.2.9}$$

where

$$c = \frac{\sigma}{M}\,, \quad d = \frac{a}{M}\,, \quad c^2 + d^2 = 1\,, \tag{6.2.10}$$

so that $\alpha = (\sigma - M)/a = (c-1)/d$. Transition of this form of Kerr metric to the more familiar one associated with Boyer-Lindquist coordinates is accomplished by the map

$$t = t\,, \qquad x = \frac{r - M}{\sigma}\,, \qquad y = \cos\theta\,, \qquad \phi = \phi\,, \tag{6.2.11}$$

so that $x = 1$ corresponds to the outer horizon $r = r_+ = M + \sigma$.

Erez-Rosen solution

Similarly, for vanishing rotation parameter we recover the Erez-Rosen solution [38, 39, 40]. It is a solution of the static Weyl class of solutions (i.e. $\omega \equiv 0$) with functions

$$f_{ER} = \frac{x-1}{x+1} e^{-2qP_2Q_2}\,, \qquad \gamma_{ER} = \hat{\gamma}\,, \tag{6.2.12}$$

which reduce to the Schwarzschild solution

$$f_S = \frac{x-1}{x+1}\,, \qquad \gamma_S = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2-y^2}\right) \tag{6.2.13}$$

when $q = 0$ too.

Hartle-Thorne solution

The Hartle-Thorne solution is associated with the exterior field of a slowly rotating slightly deformed object [97]. It is an approximate solution of the vacuum Einstein equations accurate to second order in the rotation parameter a/M and to first order in the quadrupole parameter q, generalizing the Lense-Thirring spacetime [41]. The corresponding metric functions are given by

$$\begin{aligned}
f_{HT} &\simeq \frac{x-1}{x+1}\left[1 - q\left(2P_2Q_2 - \ln\frac{x-1}{x+1}\right)\right] - \frac{x^2+x-2y^2}{(x+1)^3}\left(\frac{a}{M}\right)^2\,, \\
\omega_{HT} &\simeq 2M\frac{1-y^2}{x-1}\left(\frac{a}{M}\right)\,, \\
\gamma_{HT} &\simeq \frac{1}{2}\ln\left(\frac{x^2-1}{x^2-y^2}\right) + 2q(1-P_2)Q_1 - \frac{1}{2}\frac{1-y^2}{x^2-1}\left(\frac{a}{M}\right)^2\,,
\end{aligned} \tag{6.2.14}$$

where terms of the order of $q(a/M)$ have also been neglected.

6.3 Circular orbits on the symmetry plane

Let us introduce the ZAMO family of fiducial observers, with four velocity

$$n = N^{-1}(\partial_t - N^\phi \partial_\phi)\,, \tag{6.3.1}$$

where $N = (-g^{tt})^{-1/2}$ and $N^\phi = g_{t\phi}/g_{\phi\phi}$ are the lapse and shift functions respectively. A suitable orthonormal frame adapted to ZAMOs is given by

$$e_{\hat t} = n, \quad e_{\hat x} = \frac{1}{\sqrt{g_{xx}}}\partial_x, \quad e_{\hat y} = \frac{1}{\sqrt{g_{yy}}}\partial_y, \quad e_{\hat \phi} = \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi, \tag{6.3.2}$$

with dual

$$\omega^{\hat t} = N\mathrm{d}t\,, \quad \omega^{\hat x} = \sqrt{g_{xx}}\mathrm{d}x\,, \quad \omega^{\hat y} = \sqrt{g_{yy}}\mathrm{d}y\,, \quad \omega^{\hat \phi} = \sqrt{g_{\phi\phi}}(\mathrm{d}\phi + N^\phi \mathrm{d}t)\,. \tag{6.3.3}$$

The 4-velocity U of uniformly rotating circular orbits can be parametrized either by the (constant) angular velocity with respect to infinity ζ or, equivalently, by the (constant) linear velocity ν with respect to ZAMOs

$$U = \Gamma[\partial_t + \zeta\partial_\phi] = \gamma[e_{\hat t} + \nu e_{\hat \phi}], \qquad \gamma = (1-\nu^2)^{-1/2}\,, \tag{6.3.4}$$

where Γ is a normalization factor which assures that $U_\alpha U^\alpha = -1$ given by

$$\Gamma = \left[N^2 - g_{\phi\phi}(\zeta + N^\phi)^2\right]^{-1/2} = \frac{\gamma}{N} \tag{6.3.5}$$

and

$$\zeta = -N^\phi + \frac{N}{\sqrt{g_{\phi\phi}}}\nu. \tag{6.3.6}$$

We limit our analysis to the motion on the symmetry plane ($y = 0$) of the solution (6.2.1)–(6.2.5), where there exists a large variety of special circular orbits [42, 43, 44, 45].

The prolate spheroidal coordinates in which the metric (6.2.1) is written are adapted to the Killing symmetries of the spacetime itself and automatically select the family of static or "threading" observers, i.e. those at rest with respect to the coordinates, following the time coordinate lines. Threading observers have zero angular velocity, whereas their relative velocity with respect to ZAMOs is

$$\zeta_{(\mathrm{thd})} = 0\,, \qquad \nu_{(\mathrm{thd})} = \frac{f\omega}{\sigma\sqrt{x^2-1}}\,. \tag{6.3.7}$$

ZAMOs are instead characterized by

$$\zeta_{(\text{zamo})} = -\frac{f^2\omega}{\sigma^2(x^2-1) - f^2\omega^2}\,, \qquad \nu_{(\text{zamo})} = 0\,. \tag{6.3.8}$$

Co-rotating $(+)$ and counter-rotating $(-)$ timelike circular geodesics are characterized by the following linear velocities

$$\nu_{(\text{geo})\pm} \equiv \nu_\pm = \frac{fC \pm \left[f^2\omega^2 - \sigma^2(x^2-1)\right]\sqrt{D}}{\sqrt{x^2-1}\sigma\{f_x[f^2\omega^2+\sigma^2(x^2-1)] + 2f(f^2\omega\omega_x - \sigma^2 x)\}}\,, \tag{6.3.9}$$

where

$$\begin{aligned} C &= -2\sigma^2(x^2-1)\omega f_x - f\{\omega_x[f^2\omega^2+\sigma^2(x^2-1)] - 2\sigma^2 x\omega\}\,, \\ D &= f^4\omega_x^2 - \sigma^2 f_x[f_x(x^2-1) - 2xf]\,. \end{aligned} \tag{6.3.10}$$

All quantities in the previous expressions are meant to be evaluated at $y = 0$. The corresponding timelike conditions $|\nu_\pm| < 1$ together with the reality condition $D \geq 0$ identify the allowed regions for the "radial" coordinate where co/counter-rotating geodesics exist. We refer to Ref. [37] for a detailed discussion about the effect of the quadrupole moment on the causality condition. There exists a finite range of values of q wherein timelike circular geodesics are allowed: $q_1 < q < q_3$ for co-rotating and $q_2 < q < q_3$ for counter-rotating circular geodesics. The critical values q_1, q_2 and q_3 of the quadrupole parameter can be (numerically) determined from Eq. (6.3.9). For instance, for a fixed distance parameter $x = 4$ from the source and the choice of the rotation parameter $a/M = 0.5$ we find $q_1 \approx -105.59$, $q_2 \approx -36.29$ and $q_3 \approx 87.68$.

6.4 Tidal indicators

We investigate here tidal forces, commonly associated with the Riemann tensor and more specifically with its electric and magnetic parts with respect to a generic timelike congruence. Let us denote by u the corresponding unit tangent vector.

The electric and magnetic parts of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ with respect to a generic timelike congruence with unit tangent vector u are defined as [46]

$$E(u)_{\alpha\beta} = C_{\alpha\mu\beta\nu}u^\mu u^\nu\,, \qquad H(u)_{\alpha\beta} = -C^*_{\alpha\mu\beta\nu}u^\mu u^\nu\,. \tag{6.4.1}$$

These spatial fields are both symmetric and tracefree. The electric part is associated with tidal gravity, whereas the magnetic part describes differential dragging of inertial frames. Some tools for visualizing the spacetime curvature through the electric and magnetic parts of the Weyl tensor have been recently introduced in Ref. [47], where the nonlinear dynamics of curved spacetime in merging black hole binaries has been investigated by using numerical simulations. We recall that in a vacuum spacetime, which is just the case we are considering, the Weyl and Riemann tensors coincide.

The simplest way to built up scalar quantities through the electric and magnetic parts of the Riemann tensor which are representative of them and serve as "tidal indicators" in the study of tidal effects is to take the trace of their square. One can then consider the following electric and magnetic tidal indicators [31]

$$\mathcal{T}_E(u) = \mathrm{Tr}[E(u)^2]\,, \qquad \mathcal{T}_H(u) = \mathrm{Tr}[H(u)^2]\,. \tag{6.4.2}$$

They are related to the curvature tensor as well as to the particle/observer undergoing tidal deformations. One could also consider other more involved tidal invariants constructed from the covariant derivative of the curvature tensor. Such invariants have received some attention in the recent literature in order to investigate both geometrical and topological properties of certain classes of static as well as stationary spacetimes (see, e.g., Refs. [48, 49, 50]). However, in the context of tidal problems differential invariants are of interest only when using Fermi coordinate tidal potential, as discussed in Ref. [30]. We will not address this problem in the present paper.

Let $u = n$ be the unit tangent vector to the ZAMO family of observer given by Eq. (6.3.1) with adapted frame (6.3.2). The relevant nonvanishing frame components of the electric and magnetic parts of the Riemann tensor are given by $E(n)_{11}$, $E(n)_{33}$ and $H(n)_{12}$ with

$$E(n)_{11} + E(n)_{22} = -E(n)_{33}\,. \tag{6.4.3}$$

They are listed in Appendix A. The tidal indicators (6.4.2) then turn out to be given by

$$\begin{aligned}
\mathcal{T}_E(n) &= 2\{[E(n)_{11}]^2 + [E(n)_{22}]^2 + E(n)_{11}E(n)_{22}\}\,, \\
\mathcal{T}_H(n) &= 2[H(n)_{12}]^2\,.
\end{aligned} \tag{6.4.4}$$

Let now U be tangent to a uniformly rotating timelike circular orbit on the symmetry plane. We find [1]

$$\begin{aligned}
\mathcal{T}_E(U) &= \gamma^4 \Big\{ \mathcal{T}_E(n)(\nu^4+1) - 4H(n)_{12}(E(n)_{11} - E(n)_{22})\nu(\nu^2+1) \\
&\quad -2\nu^2([E(n)_{11}]^2 + [E(n)_{22}]^2 + 4E(n)_{11}E(n)_{22} - 4[H(n)_{12}]^2) \Big\}\,, \\
\mathcal{T}_H(U) &= \mathcal{T}_H(n)\gamma^4(\nu-\nu_*)^2(\nu-\bar{\nu}_*)^2\,,
\end{aligned} \tag{6.4.5}$$

[1] These relations were first derived in Ref. [31]. Note that the last line of Eq. (4.5) there was misprinted by an overall minus sign, corrected here. Such a misprint also affected the equivalent form (4.6), which was but never used.

where

$$\nu_* = W - \sqrt{W^2 - 1}\,, \qquad W = \frac{E(n)_{11} - E(n)_{22}}{2H(n)_{12}}\,, \tag{6.4.6}$$

and we have used the notation $\bar{\nu}_* = 1/\nu_*$. After expliciting the Lorentz factor $\gamma^4 = 1/(1-\nu^2)^2$ and rearranging terms we can derive from Eq. (6.4.5) the following relation

$$\mathcal{T}_E(U) = \mathcal{T}_E(n) - \mathcal{T}_H(n) + \mathcal{T}_H(U)\,. \tag{6.4.7}$$

This is actually an invariance relation also involving the curvature invariants. In fact, it is possible to show that

$$\mathcal{T}_E(U) - \mathcal{T}_H(U) = \frac{K}{8} = \mathcal{T}_E(n) - \mathcal{T}_H(n)\,, \tag{6.4.8}$$

where K is the Kretschmann invariant of the spacetime (evaluated on the equatorial plane $y = 0$, see Appendix A), i.e.,

$$K = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\big|_{y=0}\,. \tag{6.4.9}$$

Its behavior as a function of the distance parameter is shown in Fig. 6.1(a) for a fixed value of the rotation parameter and different values of the quadrupole parameter. As a consequence, since K does not depend on ν, all along the family of circular orbits parametrized by ν both tidal indicators have their extremal values simultaneously, i.e.,

$$\frac{d\mathcal{T}_E(U)}{d\nu} = \frac{d\mathcal{T}_H(U)}{d\nu}\,. \tag{6.4.10}$$

6.4.1 Super-energy density and super-Poynting vector

Bel [51] and Robinson [52] first introduced the Bel-Robinson super-energy-momentum tensor for the gravitational field in vacuum in terms of the Weyl curvature tensor in analogy with electromagnetism (see also Refs. [53, 54])

$$T_{\alpha\beta}{}^{\gamma\delta} = \frac{1}{2}(C_{\alpha\rho\beta\sigma}C^{\gamma\rho\delta\sigma} + {}^*C_{\alpha\rho\beta\sigma}{}^*C^{\gamma\rho\delta\sigma})\,. \tag{6.4.11}$$

The super-energy density and the super-Poynting vector associated with a generic observer u are given by

$$\begin{aligned}
\mathcal{E}^{(g)}(u) &= T_{\alpha\beta\gamma\delta}u^\alpha u^\beta u^\gamma u^\delta = \frac{1}{2}[\mathcal{T}_E(u) + \mathcal{T}_H(u)]\,,\\
P^{(g)}(u)_\alpha &= T_{\alpha\beta\gamma\delta}u^\beta u^\gamma u^\delta = [E(u)\times_u H(u)]_\alpha\,,
\end{aligned} \tag{6.4.12}$$

where

$$[E(u) \times_u H(u)]_\alpha = \eta(u)_{\alpha\beta\gamma} E(u)^\beta{}_\delta H(u)^{\delta\gamma} \tag{6.4.13}$$

and the spatial unit-volume 3-form has been introduced, i.e.,

$$\eta(u)_{\alpha\beta\gamma} = u^\mu \eta_{\mu\alpha\beta\gamma} \,. \tag{6.4.14}$$

Bel showed that for Petrov types I and D, an observer always exists for which the super-Poynting vector vanishes: this observer aligns the electric and magnetic parts of the Weyl tensor in the sense that they are both diagonalized and therefore commute. For black hole spacetimes, the Carter's observer family plays this role at each spacetime point. The same property is also shared by the observers ν_* in this more general context. In fact, the super-Poynting vector for a circularly rotating orbit U turns out to have a single nonvanishing frame component along $e_{\hat\phi}$ given by

$$P^{(g)}(U)_{\hat\phi} = \frac{\gamma^4}{2\nu_*} \mathcal{T}_H(n)(\nu - \nu_*)(\nu - \bar\nu_*)[(1+\nu_*^2)(1+\nu^2) - 4\nu\nu_*] \,. \tag{6.4.15}$$

Furthermore, the orbit associated with ν_* is also characterized by the following relation involving the super-energy density (which can be easily evaluated from Eq. (6.4.5))

$$\frac{d\mathcal{E}^{(g)}(U)}{d\nu} = -4\gamma^2 P^{(g)}(U)_{\hat\phi} \,, \tag{6.4.16}$$

or in terms of the rapidity parameter α such that $\nu = \tanh\alpha$

$$\frac{d\mathcal{E}^{(g)}(U)}{d\alpha} = -4 P^{(g)}(U)_{\hat\phi} \,, \tag{6.4.17}$$

so remembering the structure of Hamilton's equations for conjugate variables. The observers ν_* thus correspond to vanishing super-Poynting vector and minimal super-energy density.

6.4.2 Discussion

Eq. (6.4.5) implies that $\mathcal{T}_H(U)$ vanishes for $\nu = \nu_*$, regardless of the value of the quadrupole parameter q. The family of observers identified by this 4-velocity plays the role of Carter's family in such a generalized Kerr spacetime. In the Kerr case in Boyer-Lindquist coordinates Carter's observer velocity is given by

$$\nu_{(\text{car})} = \frac{a\sqrt{\Delta}}{r^2 + a^2} \,. \tag{6.4.18}$$

The main property of Carter's observers world lines is to be the unique time-like world lines belonging to the intersection of the Killing two-plane (t, ϕ) with the two-plane spanned by the two independent principal null directions of the Kerr spacetime. In contrast, the QM solution is Petrov type I with four independent principal null directions and hence the above property is lost.

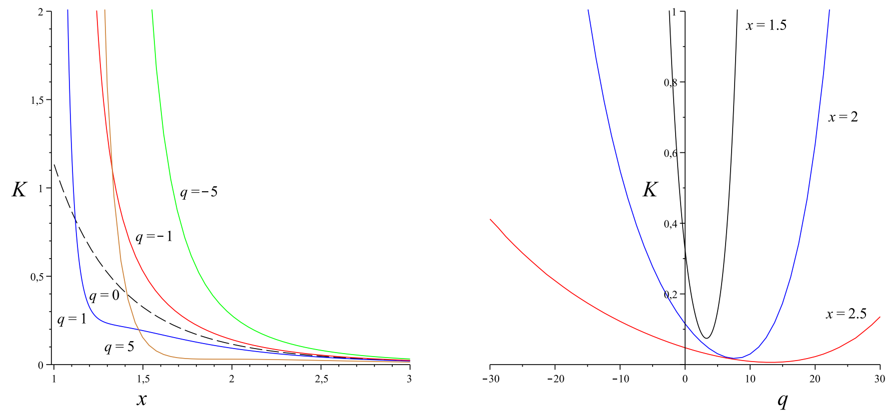
Fig. 6.2 shows how the behavior of the generalized Carter's observer velocity ν_* as a function of the distance parameter modifies due to the presence of the quadrupole. Carter's observers are defined outside the outer event horizon of the Kerr spacetime, corresponding to $x = 1$. For negative values of q, the observer horizon associated with such generalized Carter's observers is located at a certain value $x > 1$, which increases as q becomes increasingly negative. A similar situation occurs also for positive values of q, but additional allowed branches also appear in the inner region.

In contrast with the Kerr case where Carter's observers are the only ones which measure zero magnetic tidal indicator, this property is shared here also by other observer families for special values of q. This is evident from Figs. 6.3 and 6.4, where the behaviors of the tidal indicators as functions of ν are shown for a fixed value of the quadrupole and different values of the distance parameter in comparison with the Kerr case (Fig. 6.3) and for a fixed value of x and different values of q (Fig. 6.4). In fact, we see that $\mathcal{T}_H(U)$ vanishes many times for different values of ν corresponding either to different x with fixed q, or to different q with fixed x.

Correspondingly the electric type tidal indicator takes its minimum value, which can be further reduced by suitably choosing the quadrupole parameter, but cannot be made vanishing. In fact, the invariant relation (6.4.8) implies for instance that $\mathcal{T}_E(U)$ is minimum when $\mathcal{T}_H(U) = 0$ and K is mimimum as well. Fig. 6.1(b) shows the behavior of K as a function of q for different values of the distance parameter. K reaches an absolute minimum only for positive values of q, which increase for increasing x. The behaviors of the tidal indicators as measured by ZAMOs, static and geodesic observers are shown in Figs. 6.3–6.5, respectively, as functions of the distance parameter for different values of q. The behavior is Kerr-like for negative values of q. For positive values of q instead the situation significantly modifies with respect to the Kerr case, the magnetic tidal indicator vanishing many times and correspondingly the electric tidal indicator showing a damped oscillating behavior.

6.4.3 Limit of slow rotation and small deformation

Let us consider the limiting case of the Hartle-Thorne metric written in terms of the more familiar Boyer-Lindquist coordinates according to the transformation (6.2.11). Terms of the order a^3, q^2, aq and higher are then neglected in all formulas listed below. The relevant nonvanishing ZAMO frame com-



(a) (b)

Figure 6.1: The behavior of the Kretschmann invariant K evaluated on the symmetry plane is shown as a function of the distance parameter for the choice $a/M = 0.5$ and different values of $q = [-5, -1, 0, 1, 5]$ in panel (a). Panel (b) shows instead its behavior as a function of q for different values of $x = [1.5, 2, 2.5]$ Units on the vertical axis are chosen so that $M = 1$.

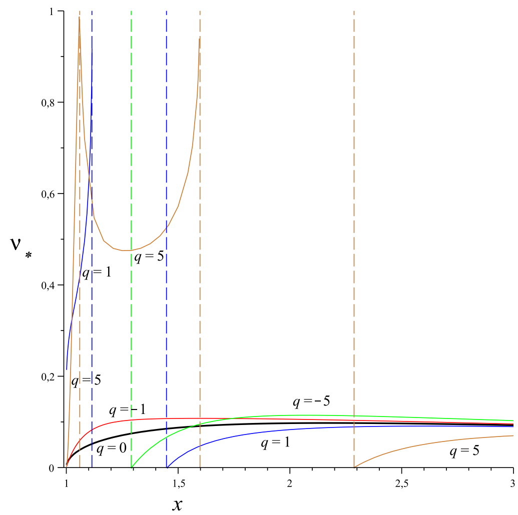


Figure 6.2: The behavior of the linear velocity ν_* as a function of the distance parameter is shown for the choice $a/M = 0.5$ and different values of the quadrupole $q = [-5, -1, 0, 1, 5]$. The thick black solid curve refers to the Kerr case ($q = 0$), i.e., to the Carter's 4-velocity. Dashed vertical lines correspond to observer horizons.

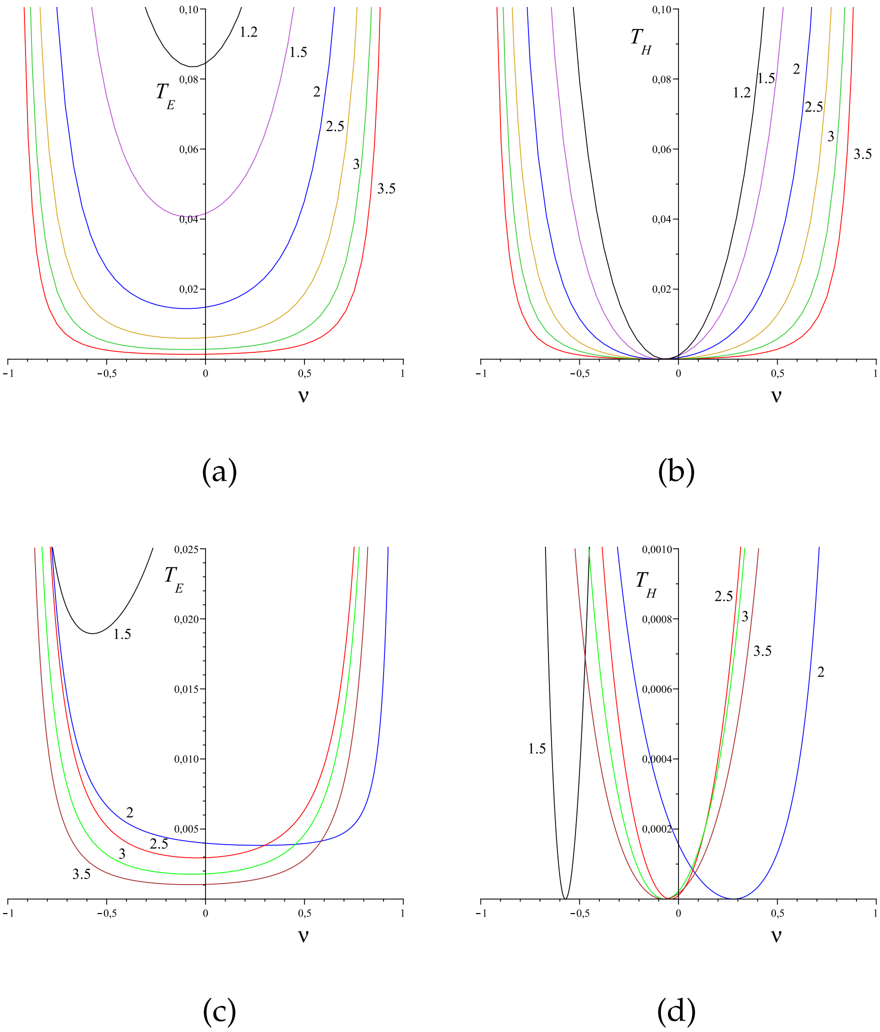
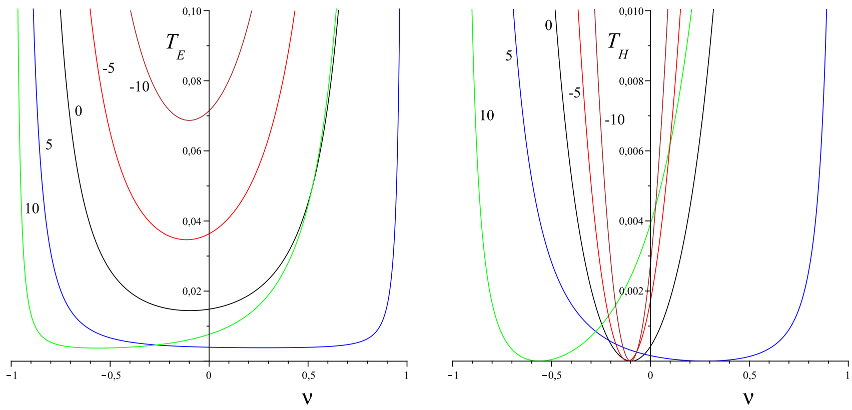
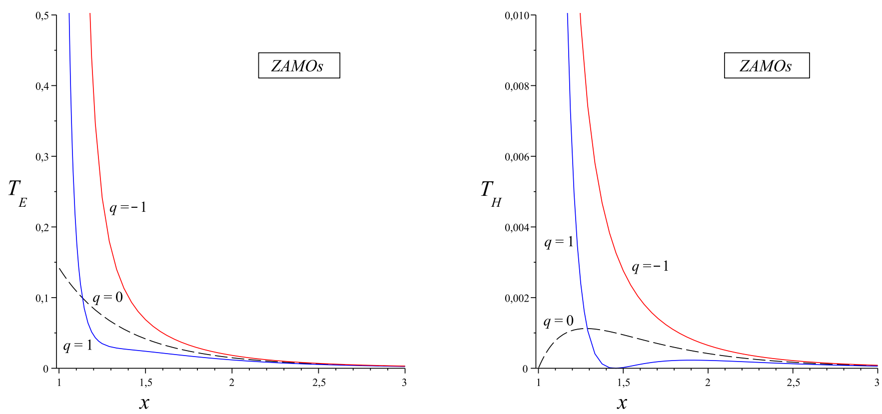


Figure 6.3: The behaviors of the tidal indicators $\mathcal{T}_E(U)$ and $\mathcal{T}_H(U)$ as functions of ν are shown in panels (a), (c) for $q = 0$ and (b), (d) for $q = 5$ respectively for the choice $a/M = 0.5$ and different values of the coordinate $x = [1.2, 1.5, 2, 2.5, 3, 3.5]$. For negative values of q the behaviors are very similar to the Kerr case. For q increasingly negative the values of $\mathcal{T}_E(U)$ at the minimum of each curve increase, whereas the curves corresponding to $\mathcal{T}_H(U)$ shrink to the vertical axis. Units on the vertical axis are chosen so that $M = 1$.



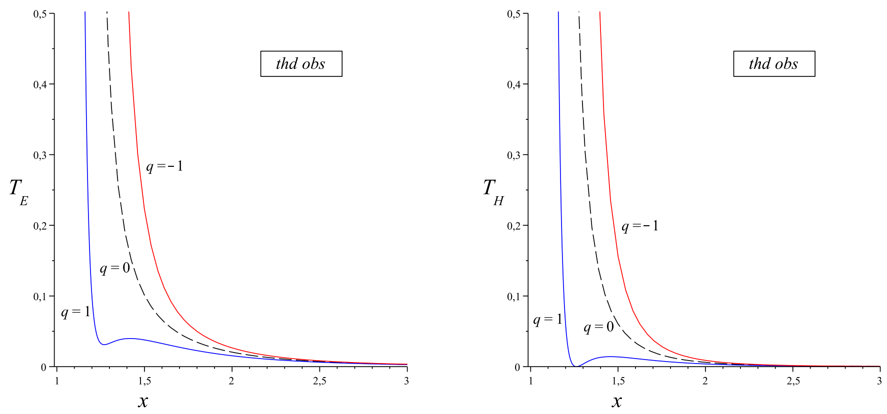
(a) (b)

Figure 6.4: The behaviors of the tidal indicators $\mathcal{T}_E(U)$ and $\mathcal{T}_H(U)$ as functions of ν are shown in panels (a) and (b) respectively for the choice $a/M = 0.5$, with a fixed value $x = 2$ of the distance parameter and different values of $q = [-10, -5, 0, 5, 10]$. Units on the vertical axis are chosen so that $M = 1$.



(a) (b)

Figure 6.5: The behaviors of the tidal indicators $\mathcal{T}_E(U)$ and $\mathcal{T}_H(U)$ as measured by ZAMOs are shown as functions of the distance parameter for the choice $a/M = 0.5$ and different values of $q = [-1, 0, 1]$. Dashed curves correspond to the Kerr case (i.e., $q = 0$). Units on the vertical axis are chosen so that $M = 1$.



(a) (b)

Figure 6.6: The behaviors of the tidal indicators $\mathcal{T}_E(U)$ and $\mathcal{T}_H(U)$ as measured by static observers are shown as functions of the distance parameter for the choice $a/M = 0.5$ and different values of $q = [-1, 0, 1]$. Dashed curves correspond to the Kerr case (i.e., $q = 0$). Units on the vertical axis are chosen so that $M = 1$.

ponents of the electric and magnetic parts of the Riemann tensor are given by

$$\begin{aligned} E(n)_{11} &= -\frac{2M}{r^3} - \frac{3a^2M}{r^5}N^2 + q\left[\frac{6(r^2-3Mr+4M^2)}{Mr^3}Q_1 - \frac{3r-10M}{r^3}Q_2 \right. \\ &\quad \left. + \frac{4M}{r^3}\ln N\right], \\ E(n)_{33} &= \frac{M}{r^3} + q\left[\frac{3(7M^3-9rM^2+5r^2M-r^3)}{Mr^3(r-2M)}Q_1 + \frac{3r^2-11Mr+7M^2}{r^3(r-2M)}Q_2 \right. \\ &\quad \left. - \frac{2M}{r^3}\ln N\right], \\ H(n)_{12} &= -\frac{3aM}{r^4}N, \end{aligned} \tag{6.4.19}$$

where $N = \sqrt{1-2M/r}$ and $Q_1 = Q_1(r/M-1)$ and $Q_2 = Q_2(r/M-1)$ are Legendre functions. The Kretschmann invariant on the symmetry plane is

$$K = \frac{48M^2}{r^6}\left\{1 - q\left[\frac{6(r^2-3Mr+4M^2)}{M^2}Q_1 - \frac{3r-10M}{M}Q_2 + 4\ln N\right]\right\}. \tag{6.4.20}$$

Generalized Carter's observers have 4-velocity $\nu_* = aN/r$, whereas static observers are characterized by $\nu_{(\mathrm{thd})} = -2aM/r^2N$ and geodesics observers

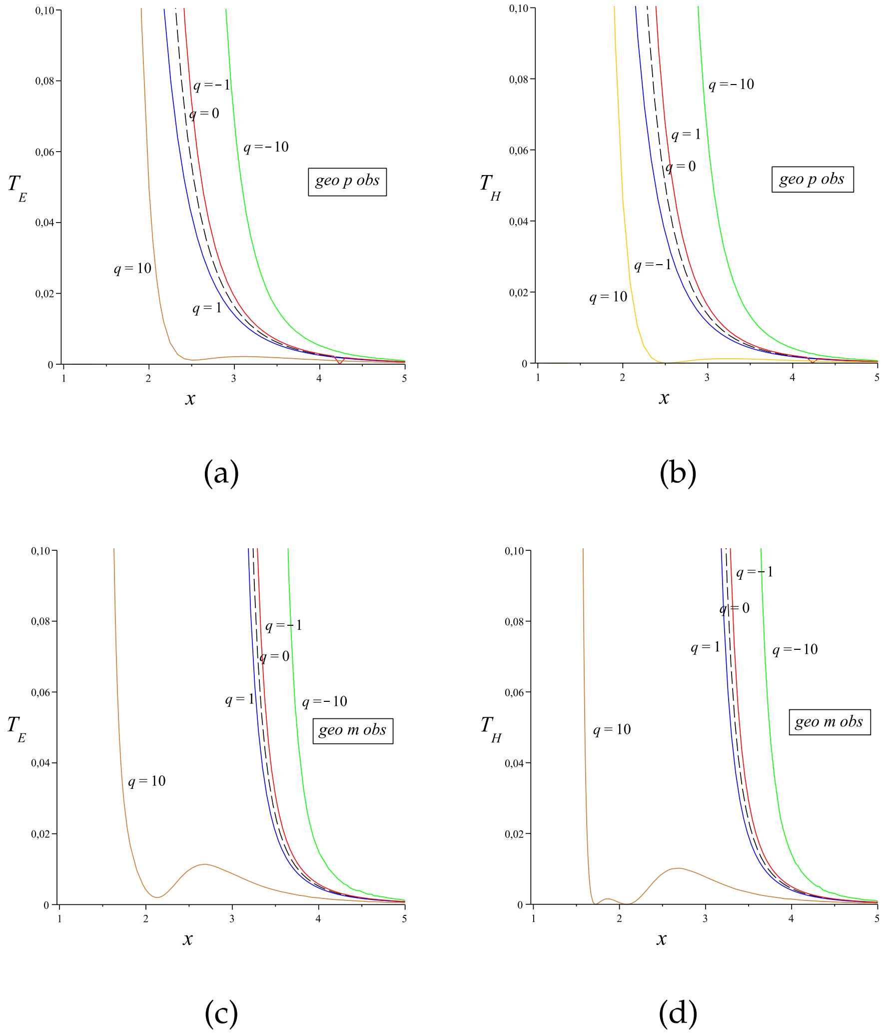


Figure 6.7: The behaviors of the tidal indicators $\mathcal{T}_E(U)$ and $\mathcal{T}_H(U)$ as measured by geodesic observers are shown as functions of the distance parameter for the choice $a/M = 0.5$ and different values of $q = [-10, -1, 0, 1, 10]$. Panels (a)–(b) and (c)–(d) correspond to the co-rotating and counter-rotating case respectively. Dashed curves correspond to the Kerr case (i.e., $q = 0$). Units on the vertical axis are chosen so that $M = 1$.

by

$$\begin{aligned}\nu_{(\text{geo})\pm} &= \pm\sqrt{\frac{M}{r}}\frac{1}{N}\left[1+\frac{a^2}{2N^2r^4}(r^2+2Mr-12M^2)\right]-\frac{3aM}{r^2N}\\ &\mp\frac{3}{2}q\frac{(r-M)^2}{N^3\sqrt{Mr^3}}\left[\left(\frac{r}{M}-1\right)Q_1-Q_2\right]\,. \end{aligned}\tag{6.4.21}$$

We list below the expression for $\mathcal{T}_H(U)$ corresponding to the different families of observers considered above:

$$\begin{aligned}\mathcal{T}_H(n) &= \frac{18a^2M^2}{r^8}N^2\,,\\ \mathcal{T}_H(u_{(\text{thd})}) &= \frac{18a^2M^2}{r^8}\frac{1}{N^2}\,,\\ \mathcal{T}_H(u_*) &= 0\,,\\ \mathcal{T}_H(u_{(\text{geo})\pm}) &= \frac{18M^3}{r^5}\frac{N^2}{(r-3M)^2}\mp 36aM^2\frac{N^2}{r^5}\sqrt{\frac{M}{r}}\frac{r-M}{(r-3M)^3}\\ &+\frac{18a^2M^2}{r^7}\frac{1}{(r-3M)^4}(r^3+Mr^2-13M^2r+15M^3)\\ &-q\left[\frac{72M^3}{r^5}\frac{N^2}{(r-3M)^2}\ln N\right.\\ &+\frac{18M}{r^6}\frac{1}{(r-3M)^3}(9r^4-60Mr^3+170M^2r^2-248M^3r+153M^4)Q_1\\ &\left.-\frac{18M^2}{r^6}\frac{1}{(r-3M)^3}(5r^3-29Mr^2+73M^2r-69M^3)Q_2\right]\,. \end{aligned}\tag{6.4.22}$$

Note that $\mathcal{T}_H(n)$ and $\mathcal{T}_H(u_{(\text{thd})})$ do not depend on q, in this limit, in contrast to the general case. This is a consequence of the series expansion in the parameters a/M and q, so that terms of the order of $q(a/M)$ have also been neglected.

Finally, in the weak field limit $M/r \ll 1$ the previous expressions have the following asymptotic forms (up to the order $(M/r)^{10}$)

$$\begin{aligned}\mathcal{T}_H(n) &\simeq \frac{18a^2M^2}{r^8}\left(1-\frac{2M}{r}\right)\,,\\ \mathcal{T}_H(u_{(\text{thd})}) &\simeq \frac{18a^2M^2}{r^8}\left(1+\frac{2M}{r}\right)\,,\\ \mathcal{T}_H(u_{(\text{geo})\pm}) &\simeq \frac{18M^3}{r^7}\left\{1-3q+\frac{M}{r}\left(4+\frac{a^2}{M^2}-9q\right)+\frac{M^2}{r^2}\left(15+13\frac{a^2}{M^2}-\frac{214}{5}q\right)\right.\\ &\left.\mp 2\frac{a}{M}\sqrt{\frac{M}{r}}\left(1+\frac{6M}{r}+29\frac{M^2}{r^2}\right)\right\}\,. \end{aligned}\tag{6.4.23}$$

6.5 Multipole moments, tidal Love numbers and Post-Newtonian theory

Although the merging of compact objects can be accurately modeled only by numerical simulations in full general relativity, there is a variety of analytical and semi-analytical approaches which allow to properly describe at least part of the coalescence process and to study the associated gravitational wave signals (see, e.g., Refs. [26, 27, 28]). The effect of the tidal interaction on the orbital motion and the gravitational wave signal is measured by a quantity known as the tidal Love number of each companion.

In Newtonian gravity, where it has been introduced [55], the Love number is a constant of proportionality between the external tidal field applied to the body and the resulting multipole moment of its mass distribution. In a relativistic context instead Flanagan and Hinderer [26, 27] estimated the tidal responses of a neutron star to the external tidal solicitation of its companion, showing that the Love number is potentially measurable in gravitational wave signals from the early regime of the inspiral through Earth-based detectors.

The relativistic theory of Love numbers has been developed by Binnington and Poisson [56] and Damour and Nagar [57]. They classified tidal Love numbers into two types: an electric-type Love number having a direct analogy with the Newtonian one, and a magnetic-type Love number with no analogue in Newtonian gravity, already introduced in Post-Newtonian theory by Damour, Soffel and Xu [58]. Damour and Lecian [59] also defined a third class of Love numbers, i.e., the "shape" Love numbers, measuring the distortion of the shape of the surface of a star by an external gravito-electric tidal field. The relativistic Love numbers are defined within the context of linear perturbation theory, in which an initially spherical body is perturbed slightly by an applied tidal field. Tidal fields are assumed to change slowly with time, so that only stationary perturbations are considered. Computing the Love numbers requires the construction of the metric also in the interior of the body and its matching with the external metric at the perturbed boundary of the matter distribution. Therefore, the internal problem depends on the choice of the stellar model, i.e., on the selected equation of state, whereas the external problem applies to a body of any kind.

Consider a massive body placed in a static, external tidal gravitational field, which is characterized by the electric part of the associated Riemann tensor. This tidal field will deform the body which will develop in response a gravitational mass quadrupole moment (and higher moments). The components of the tidal field, quadrupole moment and total mass of the body will enter as coefficients the power series expansion of the spacetime metric in the body local asymptotic rest frame [60, 27]. For an isolated body in a static situation these moments are uniquely defined. They are just the coordinate-independent moments defined by Geroch and Hansen [18, 19] for stationary, asymptotically flat spacetimes (see Eqs. (6.2.7)–(6.2.8)).

Tidal effects in relativistic binary system dynamics have been recently investigated in the framework of Post-Newtonian theory (see Ref. [61] and references therein). They have computed tidal indicators of both electric and magnetic types within the so called "effective one body approach" suitably modified to include tidal effects in the formalism, so improving the analytical description of the late inspiral dynamics with respect to previous works (see, e.g., Ref. [28]). In order to relate our results with the analysis done in Ref. [61], consider, for instance, the electric-type tidal indicator. In the case of geodesic orbits and in absence of rotation we find in the weak field limit the following approximate expression (up to terms of the order $(M/r)^8$)

$$\mathcal{T}_E(u_{(\mathrm{geo})\pm}) \simeq \frac{6M^2}{r^6}\left[1-2q+\frac{3M}{r}(1-q)\right] , \tag{6.5.1}$$

which can then be rewritten passing to harmonic coordinates $r = r_h + M$ and restoring the physical mass parameter [97], i.e., $\mathcal{M} = M(1-q)$, as

$$\mathcal{T}_E(u_{(\mathrm{geo})\pm}) \simeq \frac{6\mathcal{M}^2}{r_h^6}\left[1-\frac{3\mathcal{M}}{r_h}\right] . \tag{6.5.2}$$

The same quantity (termed J_a in Ref. [61], see Eq. (4.14) there) as above for the binary system at 1 PN order reads as

$$\mathcal{T}_E(u_{(\mathrm{geo})\pm}) \simeq \frac{6\mathcal{M}^2X_2^2}{r_h^6}\left[1+\frac{(X_1-3)\mathcal{M}}{r_h}\right] , \tag{6.5.3}$$

where evaluation is performed in the center of mass system using harmonic coordinates and the mass of the two bodies are encoded in the parameters $X_1 = m_1/\mathcal{M}$ and $X_2 = m_2/\mathcal{M}$, with $\mathcal{M} = m_1 + m_2$. In the limit $X_1 = 0$ ($X_2 = 1$) the two expressions coincide, as expected.

6.6 Concluding remarks

We have discussed the observer-dependent character of tidal effects associated with the electric and magnetic parts of the Riemann tensor with respect to an arbitrary family of observers in a generic spacetime. Our considerations have then been specialized to the Quevedo-Mashhoon solution describing the gravitational field of a rotating deformed mass and to the family of stationary circularly rotating observers on the equatorial plane. This family includes static, ZAMOs and geodesic observers and for each of them we have evaluated certain tidal indicators built up through the electric and magnetic parts of the Riemann tensor. The main difference from the Kerr case examined in a previous paper is due to the presence of a genuine quadrupolar structure of the background solution adopted here: the total quadrupole moment of the source is not depending on the rotation parameter only, but there is also a further contribution due to the shape deformation directly related to the mass through a new mass quadrupole parameter, q. The properties of tidal indicators strongly depend on this new parameter. We have found that there exists a family of circularly rotating orbits associated with $\nu = \nu_*$ along which the magnetic tidal indicator vanishes identically as in the Kerr case, playing the same role as Carter's observers there. For special values of q this property is also shared by other observer families, a novelty in comparison with the Kerr case. However, still no observer family can be found for which the electric tidal indicator vanishes, a fact that can be explained in terms of curvature invariants. The tidal electric indicator can be but extremized several times close to the source, showing also a damped oscillating behavior.

We have also investigated the relation between tidal indicators and Bel-Robinson tensor, i.e., observer-dependent super-energy density and super-Poynting vector. We have shown that the super-Poynting vector identically vanishes for $\nu = \nu_*$ leading to minimal gravitational super-energy as seen by such a generalized Carter's observer within the family of all circularly rotating observers at each spacetime point, a property already known to characterize Carter's observers in the case of black hole spacetimes.

Appendix: Relevant frame components of tidal tensors

We list below the relevant nonvanishing ZAMO frame components of the electric and magnetic parts of the Riemann tensor:

$$\begin{aligned}
E(n)_{11} &= \frac{e^{-2\gamma}}{4T\sigma^2x^2}\left\{\frac{X^2Sf_x^3}{2xf^2}+\frac{X}{x}f_x^2f\omega(X\omega_x+x\omega)-2X(Sf_{xx}+2f^3\omega\omega_{xx})\right.\\
&\quad -f_x\left[10Xf^2\omega\omega_x+\frac{2}{X}f^2\omega^2(X+2)+\frac{X}{2\sigma^2x}(4\sigma^4+Sf^2\omega_x^2)\right]\\
&\quad \left.-\frac{X}{x\sigma^2}f^3\omega\omega_x(f^2\omega_x^2-4\sigma^2)-\frac{f^3}{\sigma^2}\omega_x^2(S+f^2\omega^2)\right\}\,,\\
E(n)_{33} &= \frac{e^{-2\gamma}}{4\sigma^4fx^2}\left[\sigma^2f_x(2xf-Xf_x)+\omega_x^2f^4\right]\,,\\
H(n)_{12} &= \frac{e^{-2\gamma}\sqrt{X}}{8Tx^3\sigma^5f}\left\{2\omega\sigma^4[Xf_x^2(f_xX+xf)-6f^2f_x-2Xf^2(f_x+2xf_{xx})]-f^5\omega_x^3S\right.\\
&\quad +f\sigma^2[-2f^3\omega\omega_x^2(Xf_x+3xf)+4Sf^2(\omega_x-x\omega_{xx})\\
&\quad \left.+Sf_x\omega_x(Xf_x-10xf)]\right\}\,, \qquad (6.6.1)
\end{aligned}$$

where $X=x^2-1$, $S=\sigma^2X+f^2\omega^2$ and $T=S-2\sigma^2X$.

Furthermore, the Kretschmann invariant (6.4.9) of the QM spacetime evaluated on the equatorial plane $y=0$ is given by

$$\begin{aligned}
K &= e^{-4\gamma}\frac{4X^2}{\sigma^4x^4}\left\{f_{xx}^2-\frac{f^4\omega_{xx}^2}{X\sigma^2}+\frac{f^2\omega_x\omega_{xx}}{2xX\sigma^4}[\sigma^2f_x(Xf_x-10xf)-f^2(f^2\omega_x^2-4\sigma^2)]\right.\\
&\quad +f_{xx}\left[(X+3)\frac{f_x}{Xx}-\frac{f_x^2}{2f^2x}(Xf_x+xf)+\frac{f^2\omega_x^2}{2Xx\sigma^2}(Xf_x+3xf)\right]\\
&\quad +\frac{f_x^3}{16\sigma^2x^2f^2}\left[\frac{X\sigma^2}{f^2}f_x^2(Xf_x+2xf)-3f^2\omega_x^2(Xf_x-4xf)-8\sigma^2f_x\right.\\
&\quad \left.-\frac{8x\sigma^2f}{X}(2X+3)\right]+\frac{f^3f_x\omega_x^2}{2xX\sigma^2}\left(-\frac{7f^2\omega_x^2}{4\sigma^2}+13+\frac{6}{X}\right)\\
&\quad +f_x^2\left[\frac{3f^4\omega_x^4}{16\sigma^4x^2}-\frac{f^2\omega_x^2}{4X\sigma^2x^2}(29X+25)+\frac{X^2+3X+3}{X^2x^2}\right]\\
&\quad \left.-\frac{f^4\omega_x^2}{X\sigma^2x^2}\left[\frac{f^4\omega_x^4}{16\sigma^4}+1-\frac{f^2\omega_x^2}{4X\sigma^2}(5X+3)\right]\right\}\,. \qquad (6.6.2)
\end{aligned}$$

7 Neutrino oscillations in the field of a rotating deformed mass

7.1 Introduction

In the Standard Model with minimal particle content neutrinos are massless left-handed fermions. The question whether neutrinos have a non-vanishing rest mass influences research areas from particle physics up to cosmology, but it remains an open issue [62]. At present all hints for neutrino masses are connected with neutrino oscillation effects, namely the solar neutrino deficit, the atmospheric neutrino anomaly and the evidence from the LSND experiment [63]. Possible extensions of the Standard Model to generate neutrino masses are reviewed, e.g., in Ref. [64].

Mass neutrino mixing and oscillation in flat spacetime were proposed by Pontecorvo [65]. Later on Mikheyev, Smirnov and Wolfenstein [66] investigated the effect of transformation of one neutrino flavor into another in a medium with varying density. There have been many experimental studies exploring the evidence for oscillations of both atmospheric and solar neutrinos as well as imposing limits on their masses and mixing angle (see, e.g., Ref. [67] and references therein).

The possibility to detect CP violation effects in neutrino oscillations by future experiments has also been considered in recent years [68, 69, 70, 71]. Neutrino oscillation experiments are expected to provide stringent bounds on many quantum gravity models entailing violation of Lorentz invariance, so allowing to test quantum gravity theories [72, 73]. Planck scale-induced deviations from the standard oscillation length may be observable for ultra-high-energy neutrinos emitted by galactic and extragalactic sources by means of the next generation neutrino detectors such as IceCube and ANITA [74]. Furthermore, since neutrinos can propagate freely over large distances and can therefore pile up minimal length effects beyond detectable thresholds, there is the possibility to explore the presence of a quantum-gravity-induced minimal length using neutrino oscillation probabilities [75].

The effect of gravitation on the neutrino oscillations has been extensively investigated in the recent literature, starting from the pionering work of Stodolsky [76]. The correction to the phase difference of neutrino mass eigenstates due to the spherically symmetric gravitational field described by the Schwarzschild metric was calculated in various papers within the WKB approximation [77, 78, 79, 80, 81]. The results obtained in these papers differ from each other due to different methods used to perform the calculation. For instance, calculating the phase along the timelike geodesic line will produce a factor of 2 in the high energy limit, compared with the value along the null line [82]. A different method was proposed by Linet and Teyssandier [83], based on the world function developed by Synge [84] and defined as half the square of the spacetime distance between two generic points connected by a geodesic path. Unfortunately, the calculation of the world function is not a trivial task. In general, it is performed perturbatively unless the solution of the geodesic equations is explicitly known, as in the very special cases of Minkowski, Gödel, de Sitter spacetimes and the metric of a homogeneous gravitational field [85]. The effect of spacetime rotation on neutrino oscillations has been investigated in Ref. [86], where the Kerr solution was considered. A mechanism to generate pulsar kicks based on the spin flavor conversion of neutrinos propagating in a slowly rotating Kerr spacetime described by the Lense-Thirring metric has been recently proposed [87]. Furthermore, the neutrino geometrical optics in a gravitational field and in particular in a Lense-Thirring background has been investigated [88]. Finally, in Ref. [89] the generalization to the case of a Kerr-Newman spacetime has been discussed.

In the present paper we calculate the phase shift in the gravitational field produced by a massive, slowly rotating and quasi-spherical object, described by the Hartle-Thorne metric. This is an approximate solution of the vacuum Einstein equations accurate to second order in the rotation parameter a/M and to first order in the mass quadrupole moment q, generalizing the Lense-Thirring metric. We then discuss possible implications on atmospheric, solar and astrophysical neutrinos. The units $G = c = \hbar = 1$ are used throughout the paper.

7.2 Stationary axisymmetric spacetimes and neutrino oscillation

The line element corresponding to a general stationary axisymmetric solution of the vacuum Einstein equations can be written in the Weyl-Lewis-Papapetrou [33, 34, 35] form as

$$\begin{aligned} \mathrm{d}s^2 &= -f(\mathrm{d}t - \omega \mathrm{d}\phi)^2 \\ &+ \frac{\sigma^2}{f}\left\{e^{2\gamma}\left(x^2-y^2\right)\left(\frac{\mathrm{d}x^2}{x^2-1}+\frac{\mathrm{d}y^2}{1-y^2}\right)+(x^2-1)(1-y^2)\mathrm{d}\phi^2\right\} \end{aligned} \tag{7.2.1}$$

by using prolate spheroidal coordinates (t, x, y, ϕ) with $x \geq 1$, $-1 \leq y \leq 1$; the quantities f, ω and γ are functions of x and y only and σ is a constant.

The relation to Boyer-Lindquist coordinates (t, r, θ, ϕ) is given by

$$t = t\,, \qquad x = \frac{r - M}{\sigma}\,, \qquad y = \cos\theta\,, \qquad \phi = \phi\,. \tag{7.2.2}$$

7.2.1 Geodesics

The geodesic motion of test particles is governed by the following equations [37]

$$\begin{aligned}
\dot{t} &= \frac{E}{f} + \frac{\omega f}{\sigma^2 X^2 Y^2}(L - \omega E)\,, \qquad \dot{\phi} = \frac{f}{\sigma^2 X^2 Y^2}(L - \omega E)\,, \\
\ddot{y} &= -\frac{1}{2}\frac{Y^2}{X^2}\left[\frac{f_y}{f} - 2\gamma_y + \frac{2y}{X^2 + Y^2}\right]\dot{x}^2 + \left[\frac{f_x}{f} - 2\gamma_x - \frac{2x}{X^2 + Y^2}\right]\dot{x}\dot{y} \\
&\quad + \frac{1}{2}\left[\frac{f_y}{f} - 2\gamma_y - \frac{2y}{X^2 + Y^2}\frac{X^2}{Y^2}\right]\dot{y}^2 \\
&\quad - \frac{1}{2}\frac{e^{-2\gamma}}{f\sigma^4 X^2 Y^2 (X^2 + Y^2)}\Big\{Y^2[f^2(L - \omega E)^2 + E^2\sigma^2 X^2 Y^2] f_y \\
&\quad + 2(L - \omega E) f^3 [y(L - \omega E) - E Y^2 \omega_y]\Big\}\,, \\
\dot{x}^2 &= -\frac{X^2}{Y^2}\dot{y}^2 + \frac{e^{-2\gamma} X^2}{\sigma^2 (X^2 + Y^2)}\left[E^2 - \mu^2 f - \frac{f^2}{\sigma^2 X^2 Y^2}(L - \omega E)^2\right]\,,
\end{aligned} \tag{7.2.3}$$

where Killing symmetries and the normalization condition $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = -\mu^2$ have been used. Here E and L are the conserved energy (associated with the Killing vector ∂_t) and angular momentum (associated with the Killing vector ∂_ϕ) of the test particle respectively, μ is the particle mass and a dot denotes differentiation with respect to the affine parameter λ along the curve; furthermore, the notation

$$X = \sqrt{x^2 - 1}\,, \quad Y = \sqrt{1 - y^2} \tag{7.2.4}$$

has been introduced. For timelike geodesics, λ can be identified with the proper time by setting $\mu = 1$. Let U be the associated 4-velocity vector ($U \cdot U = -1$). Null geodesics are characterized instead by $\mu = 0$. Let K be the associated tangent vector ($K \cdot K = 0$).

Let us consider the motion on the symmetry plane $y = 0$. If $y = 0$ and $\dot{y} = 0$ initially, the third equation of Eqs. (7.2.3) ensures that the motion will be confined on the symmetry plane, since the derivatives of the metric functions with respect to y, i.e., f_y, ω_y and γ_y, all vanish at $y = 0$, so that

$\ddot{y} = 0$ too. Eqs. (7.2.3) thus reduce to

$$\begin{aligned} \dot{t} &= \frac{E}{f} + \frac{\omega f}{\sigma^2 X^2}(L - \omega E)\,, \qquad \dot{\phi} = \frac{f}{\sigma^2 X^2}(L - \omega E)\,, \\ \dot{x} &= \pm \frac{e^{-\gamma} X}{\sigma\sqrt{1+X^2}}\left[E^2 - \mu^2 f - \frac{f^2}{\sigma^2 X^2}(L - \omega E)^2\right]^{1/2}\,, \end{aligned} \tag{7.2.5}$$

where metric functions are meant to be evaluated at $y = 0$.

7.2.2 Neutrino oscillations

The phase associated with neutrinos of different mass eigenstate is given by [76]

$$\Phi_k = \int_A^B P_{\mu\,(k)} \mathrm{d}x^\mu\,, \tag{7.2.6}$$

if the neutrino with 4-momentum $P = m_k U$ is produced at a spacetime point A and detected at B.

The standard assumptions usually applied to evaluate the phase are the following (see, e.g., Ref. [90]): a massless trajectory is assumed, which means that the neutrino travels along a null geodesic path; the mass eigenstates are taken to be the energy eigenstates, with a common energy E; the ultrarelativistic approximation $m_k \ll E$ is performed throughout, so that all quantities are evaluated up to first order in the ratio m_k/E.

The integral is carried out over a null path, so that Eq. (7.2.6) can be also written as

$$\Phi_k = \int_{\lambda_A}^{\lambda_B} P_{\mu\,(k)} K^\mu \mathrm{d}\lambda\,, \tag{7.2.7}$$

where K is a null vector tangent to the photon path. The components of P and K are thus obtained from Eq. (7.2.5) by setting $\mu = m_k$ and $\mu = 0$ respectively. In the case of equatorial motion the argument of the integral (7.2.7) depends on the coordinate x only, so that the integration over the affine parameter λ can be switched over x by

$$\Phi_k = \int_{x_A}^{x_B} P_{\mu\,(k)} \frac{K^\mu}{K^x} \mathrm{d}x\,, \tag{7.2.8}$$

where $K^x = \mathrm{d}x/\mathrm{d}\lambda$. By applying the relativistic condition $m_k \ll E$ we find

$$\Phi_k \simeq \mp \frac{1}{2}\sigma^2 \frac{m_k^2}{E} \int_{x_A}^{x_B} \frac{x e^\gamma}{\sqrt{\sigma^2(x^2-1) - f^2(b-\omega)^2}} \mathrm{d}x\,, \tag{7.2.9}$$

to first order in the expansion parameter $m_k/E \ll 1$, where E is the energy for a massless neutrino and $b = L/E$ the impact parameter.

Therefore, the phase shift responsible for the oscillation is given by

$$\Phi_{kj} = \Phi_k - \Phi_j \simeq \mp \frac{1}{2}\sigma^2 \frac{\Delta m^2_{kj}}{E} \int_{x_A}^{x_B} \frac{x e^{\gamma}}{\sqrt{\sigma^2(x^2-1) - f^2(b-\omega)^2}} \mathrm{d}x \,, \qquad (7.2.10)$$

where

$$\Delta m^2_{kj} = m^2_k - m^2_j \,. \qquad (7.2.11)$$

The question as to whether neutrino oscillations should be thought of as taking place between states of the same energy or the same momentum is still open. The various controversies concerning quantum-mechanical derivations of the oscillation formula as well as the contradictions between the existing field-theoretical approaches proposed to settle them are reviewed in Ref. [91]. The advantage of the equal-energy prescription is that the time dependence completely drops from the phase difference. This is also justified by the fact that in none of the neutrino oscillation experiments the time was measured, only distances between creation and detection points, as discussed by Lipkin [92] and Stodolsky [93]. They showed that in the plane wave approximation neutrino oscillations are experimentally observable only as a result of interference between neutrino states with different masses and the same energy. All interference effects between neutrino states having different energies are destroyed by the interaction between the incident neutrino and the neutrino detector [94]. The absence of clocks in these experiments allows to consider the behaviour of neutrinos as a "stationary" one, only the distance between the source and detector being known. Therefore, the time interval is not an observable and only the oscillation wave length is measured.

When the separation between source and detector is large enough, the coherence between the different mass eigenstates is expected to be lost. However, for atmospheric and solar neutrinos, where the source is free to move in distances many orders of magnitudes larger, the decoherence distance will be even larger [95]. This is in agreement with the result quoted in Ref. [96] that the coherence is lost only at astronomical distances much larger than the size of the solar system and that this coherence loss is relevant only for supernova neutrinos.

7.3 Neutrino oscillations in the Hartle-Thorne metric

The exterior field of a slowly rotating slightly deformed object is described by the Hartle-Thorne metric [97], whose line element can be written in Lewis-

Papapetrou form (7.2.1) with metric functions

$$
\begin{aligned}
f &\simeq f_S\left[1-q\left(2P_2Q_2+\ln\frac{x-1}{x+1}\right)\right]-\frac{x^2+x-2y^2}{(x+1)^3}\left(\frac{a}{M}\right)^2 , \\
\omega &\simeq 2M\frac{1-y^2}{x-1}\left(\frac{a}{M}\right) , \\
\gamma &\simeq \gamma_S+2q(1-P_2)Q_1-\frac{1}{2}\frac{1-y^2}{x^2-1}\left(\frac{a}{M}\right)^2 , \qquad (7.3.1)
\end{aligned}
$$

and $\sigma = \sqrt{M^2-a^2}$. Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively, while the functions

$$
f_S=\frac{x-1}{x+1} , \qquad \gamma_S=\frac{1}{2}\ln\left(\frac{x^2-1}{x^2-y^2}\right) \qquad (7.3.2)
$$

correspond to the Schwarzschild solution ($q = 0 = a$). The connection between this form of the metric and the original one as derived by Hartle and Thorne was discussed in Ref. [37].

When expressed in terms of the standard Boyer-Lindquist coordinates (7.2.2) the phase shift (7.2.10) becomes

$$
\begin{aligned}
\Phi_{kj} &\simeq \mp\frac{\Delta m^2_{kj}}{2E}\left[r-\frac{M^2}{r}\left(q+\frac{b^2-a^2}{2M^2}\right)-\frac{M}{2r^2}\left[q(b^2+2M^2)-(a+b)^2\right]\right]_{r_A}^{r_B} \\
&= \mp\frac{\Delta m^2_{kj}}{2E}(r_B-r_A)\left[1+\frac{M^2}{r_Br_A}\left(q+\frac{b^2-a^2}{2M^2}\right)\right. \\
&\qquad \left.+\frac{M(r_B+r_A)}{2r_B^2r_A^2}\left[q(b^2+2M^2)-(a+b)^2\right]\right] , \qquad (7.3.3)
\end{aligned}
$$

where terms of the order of $q(a/M)$ as well as higher order terms in the weak field expansion $M/r \ll 1$ have been neglected. In the limiting case of vanishing quadrupole parameter ($q = 0$) Eq. (7.3.3) reproduces the results of Ref. [86] for the slowly rotating Kerr spacetime. It is also useful to replace the parameters a/M and q by the Hartle-Thorne angular momentum $\mathcal{J}$ and mass quadrupole moment $\mathcal{Q}$ according to

$$
\mathcal{M}=M(1-q) , \qquad \mathcal{J}=-Ma , \qquad \mathcal{Q}=Ma^2+\frac{4}{5}M^3q . \qquad (7.3.4)
$$

For instance, for $b = 0$ Eq. (7.3.3) gives

$$\begin{aligned}\Phi_{kj} &\simeq \mp\frac{\Delta m_{kj}^2}{2E}(r_B - r_A)\left[1 + \frac{\mathcal{M}^2}{r_B r_A}\left(\frac{5}{4}\frac{\mathcal{Q} - 7\mathcal{J}^2/\mathcal{M}}{\mathcal{M}^3}\right)\left(1 + \frac{\mathcal{M}(r_B + r_A)}{r_B r_A}\right)\right] \\ &\equiv \Phi_{kj}^{(\mathrm{mono})} + \Phi_{kj}^{(\mathrm{dip})} + \Phi_{kj}^{(\mathrm{quad})}\,, \end{aligned} \tag{7.3.5}$$

where

$$\begin{aligned}\Phi_{kj}^{(\mathrm{mono})} &= \mp\frac{\Delta m_{kj}^2}{2E}(r_B - r_A)\,, \\ \Phi_{kj}^{(\mathrm{dip})} &= \Phi_{kj}^{(\mathrm{mono})}\frac{\mathcal{M}^2}{r_B r_A}\left(1 + \frac{\mathcal{M}(r_B + r_A)}{r_B r_A}\right)\left(-\frac{35}{4}\frac{\mathcal{J}^2}{\mathcal{M}^4}\right)\,, \\ \Phi_{kj}^{(\mathrm{quad})} &= \Phi_{kj}^{(\mathrm{mono})}\frac{\mathcal{M}^2}{r_B r_A}\left(1 + \frac{\mathcal{M}(r_B + r_A)}{r_B r_A}\right)\left(\frac{5}{4}\frac{\mathcal{Q}}{\mathcal{M}^3}\right)\,. \end{aligned} \tag{7.3.6}$$

The monopole term is the dominant one, due to the large distance between source and detector. However, describing the background gravitational field simply by using the spherically symmetric Schwarzschild solution is not satisfactory in most situations. In fact, astrophysical sources are expected to be rotating as well endowed with shape deformations leading to effects which cannot be neglected in general. The modification to the phase shift induced by spacetime rotation has been already taken into account in Ref. [86]. We will estimate below the contribution due to the quadrupole moment of the source with respect to that due to spin for solar, atmospheric and astrophysical neutrinos by evaluating the ratio

$$\frac{\Phi_{kj}^{(\mathrm{quad})}}{\Phi_{kj}^{(\mathrm{dip})}} = -\frac{1}{7}\left(\frac{\mathcal{Q}}{\mathcal{M}^3}\right)\left(\frac{\mathcal{M}^4}{\mathcal{J}^2}\right)\,. \tag{7.3.7}$$

In the case of the Sun we have $\mathcal{M}_\odot \approx 1.5 \times 10^5$ cm, $R_\odot \approx 7 \times 10^{10}$ cm and $\mathcal{J}_\odot \approx 5 \times 10^9$ cm^2, so that

$$\frac{\mathcal{M}_\odot}{R_\odot} \approx 2 \times 10^{-6}\,, \qquad \frac{\mathcal{J}_\odot}{R_\odot^2} \approx 10^{-12}\,, \qquad \frac{\mathcal{Q}_\odot}{R_\odot^3} \approx -4 \times 10^{-13}\,, \tag{7.3.8}$$

where the mass quadrupole moment has been evaluated through the relation $\mathcal{Q}_\odot = -J_2\mathcal{M}_\odot R_\odot^2$, with $J_2 \approx 2 \times 10^{-7}$ (see, e.g., Ref. [98]). Therefore, $[\mathcal{J}_\odot^2/\mathcal{M}_\odot]/\mathcal{M}_\odot^3 \approx 5 \times 10^{-2}$ and $\mathcal{Q}_\odot/\mathcal{M}_\odot^3 \approx -4.4 \times 10^4$, so that the quadrupole contribution is 10^5 times greater than that due to spin.

In the case of the Earth the quadrupole term is even more dominant. In fact, since $\mathcal{M}_\oplus \approx 0.45$ cm, $R_\oplus \approx 6.4 \times 10^8$ cm and $\mathcal{J}_\oplus \approx 1.5 \times 10^2$ cm^2, we have

$$\frac{\mathcal{M}_\oplus}{R_\oplus} \approx 7 \times 10^{-10}\,, \qquad \frac{\mathcal{J}_\oplus}{R_\oplus^2} \approx 3.7 \times 10^{-16}\,, \qquad \frac{\mathcal{Q}_\oplus}{R_\oplus^3} \approx -7 \times 10^{-13}\,, \quad (7.3.9)$$

with $\mathcal{Q}_\oplus = -J_2 \mathcal{M}_\oplus R_\oplus^2$ and $J_2 \approx 10^{-3}$, implying that $[\mathcal{J}_\oplus^2/\mathcal{M}_\oplus]/\mathcal{M}_\oplus^3 \approx 5.5 \times 10^5$ and $\mathcal{Q}_\oplus/\mathcal{M}_\oplus^3 \approx -2 \times 10^{15}$ leading to $\Phi_{kj}^{(\mathrm{quad})}/\Phi_{kj}^{(\mathrm{dip})} \approx 5 \times 10^8$.

The contribution due to rotation is expected to be comparable with that due to the deformation in the case of rotating neutron stars. Laarakkers and Poisson [99] numerically compute the mass quadrupole moment $\mathcal{Q}_{\mathrm{NS}}$ for several equations of state. They found that for fixed gravitational mass $\mathcal{M}_{\mathrm{NS}}$, the quadrupole moment is given as a simple quadratic fit, i.e.,

$$\mathcal{Q}_{\mathrm{NS}} \simeq -k \frac{\mathcal{J}_{\mathrm{NS}}^2}{\mathcal{M}_{\mathrm{NS}}}\,, \qquad (7.3.10)$$

where $\mathcal{J}_{\mathrm{NS}}$ is the angular momentum of the star and k is a dimensionless parameter which depends on the equation of state. It varies between $k \sim 2$ for very soft equations of state and $k \sim 8$ for very stiff ones, for a typical mass of $\mathcal{M}_{\mathrm{NS}} = 1.4 M_\odot$. Therefore, in this case the spin term in Eq. (7.3.5) is of the same order as the quadrupole term.

Finally, it is worth to note that the monopole term in Eq. (7.3.5) is always the leading one. In fact, for solar neutrinos the factor $\mathcal{M}_\odot^2/r_B r_A$ turns out to be about 2.3×10^{-12}, assuming $r_A \sim r_\oplus$ and $r_B \sim R_\oplus + d \sim d$, being $d \approx 1.5 \times 10^{13}$ cm the Earth-Sun distance. The first correction to the monopole term is thus of the order of 10^{-7}. For atmospheric neutrinos $\mathcal{M}_\oplus^2/r_B r_A \sim \mathcal{M}_\oplus^2/R_\oplus^2 \approx 5 \times 10^{-19}$, implying that the first correction is of the order of 10^{-3}.

7.4 Concluding remarks

The issue of the interaction of neutrinos with gravitational fields is timely and has a lot of implications in astrophysics and cosmology. Neutrinos are usually generated by radioactive decays or nuclear reactions such as those occurring in the Sun, stars, accelerators or nuclear reactors and in particular when cosmic rays hit atoms. Neutrinos are also expected to carry off the largest amount of energy of an exploding star in a supernova. Neutrinos from core collapse supernovae can be emitted from a rapidly accreting disk surrounding a black hole and have been suggested to be responsible for the cooling process [100]. In general the coalescence of compact objects produces hot disks. The neutrino flux from these objects is so large that it would be easily detected by currently online neutrino detectors [101].

Neutrino detectors of increasing sensitivity require a better understanding of propagation and oscillation properties of neutrinos in the neighborhood of massive astrophysical objects. In particular, it has been suggested that the gravitational oscillation phase might have a significant effect in supernova explosions [102].

We have evaluated the correction to the phase difference of neutrino mass eigenstates due to the gravitational field produced by a massive, slowly rotating and quasi-spherical object, described by the Hartle-Thorne metric, generalizing previous results. This is an approximate solution of the vacuum Einstein equations accurate to second order in the rotation parameter and to first order in the mass quadrupole moment of the source, generalizing the Lense-Thirring metric. The large distances covered by neutrinos from the emitting source to the detector allow to apply the weak field limit in the calculations, so that the Hartle-Thorne metric is enough to account for the leading gravitational effects. We have shown that, apart from the monopole term which is the dominant one, the contribution due to the quadrupole moment is much greater than that due to spin in the case of solar neutrinos and even more for atmospheric neutrinos. The effect of rotation is instead expected to be comparable with that due to deformation in the case of rotating neutron stars.

Our result can also be relevant in the context of CPT and lepton number violation in the neutrino sector induced by gravity (see, e.g., Ref. [103] and references therein). In fact, it has been recently argued that the generation of a neutrino asymmetry can arise in accretion disks around rotating black holes or more generally when neutrinos are propagating in background spacetimes which are not spherically symmetric [104]. The proposed CPT violation mechanism leading to neutrino oscillations would be due to the different spin-gravity coupling of neutrinos with respect to anti-neutrinos. Of course, this is only one of all possible scenarios. Observable neutrino oscillations may result from a combination of effects involving neutrino masses and Lorentz violation [105]. Similar effects can also arise from violations of the equivalence principle [106]. Both current and future experiments on neutrino oscillations are expected to clarify the interplay of gravity and neutrino physics.

8 Gravitational field of compact objects in general relativity

8.1 Introduction

In astrophysics, the term compact object is used to refer to objects which are small for their mass. In a wider sense, the class of compact objects is often defined to contain collectively planet-like objects, white dwarfs, neutron stars, other exotic dense stars, and black holes. It is well known that Newtonian theory of gravitation provides an adequate description of the gravitational field of conventional astrophysical objects. However, the discovery of exotic compact objects such as quasars and pulsars together with the possibility of continued gravitational collapse to a black hole points to the importance of relativistic gravitation in astrophysics. Moreover, advances in space exploration and the development of modern measuring techniques have made it necessary to take relativistic effects into account even in the Solar system. Probably the simplest way to study the relativistic gravitational field of astrophysical compact objects is by expressing it in terms of their multipole moments, in close analogy with the Newtonian theory, taking into account the rotation and the internal structure of the source.

In this context, the first exterior solution with only a monopole moment was discovered by Schwarzschild [107], soon after the formulation of Einstein's theory of gravity. In 1917, Weyl [4] showed that the problem of finding static axisymmetric vacuum solutions can generically be reduced to a single linear differential equation whose general solution can be represented as an infinite series. The explicit form of this solution resembles the corresponding solution in Newtonian's gravity, indicating the possibility of describing the gravitational field by means of multipole moments. In 1918, Lense and Thirring [108] discovered an approximate exterior solution which, apart from the mass monopole, contains an additional parameter that can be interpreted as representing the angular momentum of the massive body. From this solution it became clear that, in Einstein's relativistic theory, rotation generates a gravitational field that leads to the dragging of inertial frames (Lense-Thirring effect). This is the so–called gravitomagnetic field which is of especial importance in the case of rapidly rotating compact objects. The case of a static axisymmetric solution with monopole and quadrupole moment was analyzed in 1959 by Erez and Rosen [109] by using spheroidal coordinates which are specially adapted to describe the gravitational field of non-spherically symmetric bodies. The exact exterior solution which considers arbitrary values for the angular momentum was found by Kerr [110] only in 1963. The problem of finding exact solutions changed dramatically after Ernst [2] discovered in 1968 a new representation of the field equations for stationary axisymmetric vacuum solutions. In fact, this new representation was the starting point to investigate the Lie symmetries of the field equations. Today, it is known that for this special case the field equations are completely integrable and solutions can be obtained by using the modern solution generating techniques [111]. A comprehensive review on solution generating techniques and stationary axisymmetric global solutions of Einstein and Einstein-Maxwell equations is given in [112]. There are several solutions with higher multipole moments [114, 115, 113, 116, 117] with very interesting physical properties. In this work, we will analyze a particular class of solutions, derived by Quevedo and Mashhoon [118] in 1991, which in the most general case contains infinite sets of gravitational and electromagnetic multipole moments. Hereafter this solution will be denoted as the QM solution.

As for the interior gravitational field of compact objects, the situation is more complicated. There exists in the literature a reasonable number of interior spherically symmetric solutions [119] that can be matched with the exterior Schwarzschild metric. Nevertheless, a major problem of classical general relativity consists in finding a physically reasonable interior solution for the exterior Kerr metric. Although it is possible to match numerically the Kerr solution with the interior field of an infinitely tiny rotating disk of dust [120], such a solution cannot be used in general to describe astrophysical compact objects. It is now widely believed that the Kerr solution is not appropriate to describe the exterior field of rapidly rotating compact objects. Indeed, the Kerr metric takes into account the total mass and the angular momentum of the body. However, the quadrupole moment is an additional characteristic of any realistic body which should be considered in order to correctly describe the gravitational field. As a consequence, the multipole moments of the field created by a rapidly rotating compact object are different from the multipole moments of the Kerr metric [121]. For this reason a solution with arbitrary sets of multipole moments, such as the QM solution, can be used to describe the exterior field of arbitrarily rotating mass distributions.

In the case of slowly rotating compact objects it is possible to find approximate interior solutions with physically meaningful energy-momentum tensors and state equations. Because of its physical importance, in this work we will review the Hartle-Thorne [122, 123] interior solution which are coupled to an approximate exterior metric. Hereafter this solution will be denoted as the HT solution. One of the most important characteristics of this family of solutions is that the corresponding equation of state has been constructed using realistic models for the internal structure of relativistic stars. Semi-analytical and numerical generalizations of the HT metrics with more sophisticated equations of state have been proposed by different authors [124]. A comprehensive review of these solutions is given in [121]. In all these cases, however, it is assumed that the multipole moments (quadrupole and octupole) are relatively small and that the rotation is slow.

To study the physical properties of solutions of Einstein's equations, Fock [125] proposed an alternative method in which the parameters entering the exterior metric are derived by using physical models for the internal structure of the body. In this manner, the significance of the exterior parameters become more plausible and the possibility appear of determining certain aspects of the interior structure of the object by using observations performed in the exterior region of the body. Fock's metric in its first-order approximation was recently generalized in 1985 by Abdildin [126, 127] to include the case of rotating objects.

In this work, we review the main exact and approximate metrics which can be used to study the interior and exterior gravitational field of compact objects and find the relationships between them. We will show that the exterior HT approximate solution is equivalent to a special case of the QM solution in the limit of a slowly rotating slightly deformed compact object (first order in the quadrupole and second order in the angular momentum). Moreover, we will show that a particular case of the extended Fock metric is equivalent to the approximate exterior HT solution. Furthermore, since those particular cases of the exterior HT metric that possess internal counterparts with plausible equations of state are also special cases of the exterior QM metric, we conclude that at least in those particular cases it should be possible to match the QM solution with an exact interior still unknown solution so that it describes globally the gravitational field of astrophysical compact bodies.

This paper is organized as follows. In Section 8.2 we review the HT solutions and briefly comment on their most important properties. In Section 8.3 we present Fock's extended metric, as first derived by Abdildin [126, 127] in harmonic coordinates, and introduce a set of new coordinates which makes it suitable for comparison with other exterior metrics. Moreover, we find explicitly the coordinate transformation that relates Fock's extended metric with the exterior HT solution.

In Section 8.4.2 we present a particular case of the QM metric which contains, in addition to the mass and angular momentum parameters, an additional parameter related to the mass quadrupole of the source. Here we show explicitly that a limiting case of the QM metric contains the HT metric. Finally, Section 8.5 contains discussions of our results and suggestions for further research.

8.2 The Hartle-Thorne metrics

To second order in the angular velocity, the structure of compact objects can be approximately described by the mass, angular momentum and quadrupole moment. An important consequence of this approximation is that the equilibrium equations reduce to a set of ordinary differential equations. Hartle and Thorne [122, 123] explored the gravitational field of rotating stars in this slow rotation approximation. This formalism can be applied to most compact objects including pulsars with millisecond rotational periods, but it shows "large" discrepancies in the case of rapidly rotating relativistic objects near the mass-shedding limit [121], i. e., when the angular velocity of the object reaches the angular velocity of a particle in a circular Keplerian orbit at the equator. In fact, recently in [128] and [129] it was shown that the second order rotation corrections of the HT metric are sufficient to describe the properties of stars with intermediate rotation rates. These results were generalized in [130] to include third order corrections. It turns out that third order corrections are irrelevant at the mass-shedding limit; however, they are important to study the moment of inertia of rapidly rotating neutron stars. Moreover, in [117] an analytical solution was derived that can be matched accurately with interior numerical solutions. On the other hand, an alternative numerical study [131] shows that in the case of uniformly rotating neutron stars the dimensionless specific angular momentum cannot exceed the value 0.7.

An additional property of this formalism is that it can be used to match an interior solution with an approximate exterior solution. In this connection, it is worth noticing that the problem of matching interior and exterior solutions implies many mathematical and physical issues [132, 133, 134, 135, 136, 137], including the performance of the metric functions and the coordinates at the matching surface as well as the physical behavior of the internal parameters like the density and pressure of the matter distribution. In the following subsections we will present the interior and the exterior metrics and introduce notations which will be used throughout the paper.

8.2.1 The interior solution

If a compact object is rotating slowly, the calculation of its equilibrium properties reduces drastically because it can be considered as a linear perturbation of an already-known non-rotating configuration. This is the main idea of Hartle's formalism [122]. To simplify the computation the following conditions are assumed to be satisfied.

1) Equation of state: The matter in equilibrium configuration is assumed to satisfy a one-parameter equation of state, $\mathcal{P} = \mathcal{P}(\mathcal{E})$, where $\mathcal{P}$ is the pressure and $\mathcal{E}$ is the mass-energy density.

2) Axial and reflection symmetry: The configuration is symmetric with respect to an arbitrary axis which can be taken as the rotation axis. Further-

more, the rotating object should be invariant with respect to reflections about a plane perpendicular to the axis of rotation.
3) Uniform rotation: Only uniformly rotating configurations are considered since it is known that configurations that minimize the total mass-energy (e.g., all stable configurations) must rotate uniformly [138]. [1].
4) Slow rotation: This means that angular velocities Ω are small enough so that the fractional changes in pressure, energy density and gravitational field due to the rotation are all less than unity, i.e.

$$\Omega^2 \ll \left(\frac{c}{R'}\right)^2 \frac{GM'}{c^2R'} \tag{8.2.1}$$

where M' is the mass and R' is the radius of the non-rotating configuration. The above condition is equivalent to the physical requirement $\Omega \ll c/R'$.

When the equilibrium configuration described above is set into slow rotation, the geometry of space-time around it and its interior distribution of stress-energy are changed. With an appropriate choice of coordinates, the perturbed geometry is described by

$$ds^2 = e^{\nu}\left[1+2(h_0+h_2P_2)\right]dt^2 - \frac{\left[1+2(m_0+m_2P_2)/(R-2M')\right]}{1-2M'/R}dR^2 - \\ -R^2\left[1+2(v_2-h_2)P_2\right]\left[d\Theta^2+\sin^2\Theta(d\phi-\omega dt)^2\right]+O(\Omega^3) \tag{8.2.2}$$

Here M' is the mass of the non-rotating star, $P_2 = P_2(\cos\Theta)$ is the Legendre polynomial of second order, ω is the angular velocity of the local inertial frame, which is a function of R and is proportional to the star's angular velocity Ω, and, finally, h_0, h_2, m_0, m_2, v_2 are all functions of R that are proportional to Ω^2.

In the above coordinate system the fluid inside the star moves with a 4-velocity corresponding to a uniform and rigid rotation [141]. The contravariant components are

$$u^t = (g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi})^{-1/2}, \quad u^{\phi} = \Omega u^t, \quad u^R = u^{\Theta} = 0. \tag{8.2.3}$$

The quantity Ω that appears in the expression for u^t is so defined that $\bar{\omega} \equiv \Omega - \omega$ is the angular velocity of the fluid relative to the local inertial frame.

The energy density and the pressure of the fluid are affected by the rotation because it deforms the compact object. In the interior of the object at a given (R,Θ), in a reference frame that is momentarily moving with the fluid, the pressure is

$$\mathcal{P} \equiv P + (E+P)(p_0^* + p_2^* P_2) = P + \Delta P; \tag{8.2.4}$$

[1] Notice, however, that stability is also possible in the case differentially rotating configurations. See, for instance, [139, 140]

the density of mass-energy is

$$\mathcal{E} \equiv E + (E+P)(dE/dP)(p_0^* + p_2^* P_2) = E + \Delta E. \tag{8.2.5}$$

Here, p_0^* and p_2^* are dimensionless functions of R that are proportional to Ω^2, and describe the pressure perturbation, P is the pressure and E is the energy density of the non-rotating configuration. The stress-energy tensor for the fluid of the rotating object is

$$T_\mu^\nu = (\mathcal{E} + \mathcal{P}) u_\mu u^\nu - \mathcal{P}\delta_\mu^\nu. \tag{8.2.6}$$

The rotational perturbations of the objects's structure are described by the functions $\bar{\omega}, h_0, m_0, p_0^*, h_2, m_2, v_2, p_2^*$. These functions are calculated from Einstein's field equations (for details see [122, 123]).

8.2.2 The Exterior Solution

The HT metric describing the exterior field of a slowly rotating slightly deformed object is given by

$$\begin{aligned} ds^2 = {} & \left(1 - \frac{2\mathcal{M}}{R}\right)\left[1 + 2k_1 P_2(\cos\Theta) + 2\left(1 - \frac{2\mathcal{M}}{R}\right)^{-1} \frac{J^2}{R^4}(2\cos^2\Theta - 1)\right] dt^2 \\ & - \left(1 - \frac{2\mathcal{M}}{R}\right)^{-1}\left[1 - 2k_2 P_2(\cos\Theta) - 2\left(1 - \frac{2\mathcal{M}}{R}\right)^{-1}\frac{J^2}{R^4}\right] dR^2 \\ & - R^2[1 - 2k_3 P_2(\cos\Theta)](d\Theta^2 + \sin^2\Theta d\phi^2) + 4\frac{J}{R}\sin^2\Theta dt d\phi \end{aligned} \tag{8.2.7}$$

with

$$\begin{aligned} k_1 &= \frac{J^2}{\mathcal{M}R^3}\left(1 + \frac{\mathcal{M}}{R}\right) + \frac{5}{8}\frac{Q - J^2/\mathcal{M}}{\mathcal{M}^3} Q_2^2\left(\frac{R}{\mathcal{M}} - 1\right), \\ k_2 &= k_1 - \frac{6J^2}{R^4}, \\ k_3 &= k_1 + \frac{J^2}{R^4} + \frac{5}{4}\frac{Q - J^2/\mathcal{M}}{\mathcal{M}^2 R}\left(1 - \frac{2\mathcal{M}}{R}\right)^{-1/2} Q_2^1\left(\frac{R}{\mathcal{M}} - 1\right), \end{aligned}$$

where

$$Q_2^1(x) = (x^2-1)^{1/2}\left[\frac{3x}{2}\ln\frac{x+1}{x-1} - \frac{3x^2-2}{x^2-1}\right], \tag{8.2.8}$$

$$Q_2^2(x) = (x^2-1)\left[\frac{3}{2}\ln\frac{x+1}{x-1} - \frac{3x^3-5x}{(x^2-1)^2}\right], \tag{8.2.9}$$

are the associated Legendre functions of the second kind. The constants $\mathcal{M}$, J and Q are related to the total mass, angular momentum and mass quadrupole moment of the rotating object, respectively. This form of the metric corrects some misprints of the original paper by Hartle and Thorne [123] (see also [142] and [129]).

The total mass of a rotating configuration is defined as $\mathcal{M} = M' + \delta M$, where M' is the mass of non-rotating configuration and δM is the change in mass of the rotating from the non-rotating configuration with the same central density. It should be stressed that in the terms involving J^2 and Q the total mass $\mathcal{M}$ can be substituted by M' since δM is already a second order term in the angular velocity.

In general, the HT metric represents an approximate vacuum solution, accurate to second order in the angular momentum J and to first order in the quadrupole parameter Q. In the case of ordinary stars, such as the Sun, considering the gravitational constant G and the speed of light c, the metric (8.2.7) can be further simplified due to the smallness of the parameters:

$$\frac{G\mathcal{M}_{Sun}}{c^2\mathcal{R}_{Sun}} \approx 2\times 10^{-6}, \qquad \frac{GJ_{Sun}}{c^3\mathcal{R}_{Sun}^2} \approx 10^{-12}, \qquad \frac{GQ_{Sun}}{c^2\mathcal{R}_{Sun}^3} \approx 10^{-10}. \tag{8.2.10}$$

For this special case one can calculate the corresponding approximate metric from (8.2.7) in the limit $c \to \infty$. The computations are straightforward and lead to

$$\begin{aligned} ds^2 = &\left[1 - \frac{2G\mathcal{M}}{c^2R} + \frac{2GQ}{c^2R^3}P_2(\cos\Theta) + \frac{2G^2\mathcal{M}Q}{c^4R^4}P_2(\cos\Theta)\right]c^2dt^2 \\ &+\frac{4GJ}{c^2R}\sin^2\Theta dt d\phi - \left[1 + \frac{2G\mathcal{M}}{c^2R} - \frac{2GQ}{c^2R^3}P_2(\cos\Theta)\right]dR^2 \\ &- \left[1 - \frac{2GQ}{c^2R^3}P_2(\cos\Theta)\right]R^2(d\Theta^2 + \sin^2\Theta d\phi^2)\,. \end{aligned} \tag{8.2.11}$$

This metric describes the gravitational field for a wide range of compact objects, and only in the case of very dense ($G\mathcal{M} \sim c^2\mathcal{R}$) or very rapidly rotating ($GJ \sim c^3\mathcal{R}^2$) objects large discrepancies will appear.

8.3 The Fock's approach

Fock proposed in [125] a method to analyze Einstein's equations in the presence of matter and to derive approximate interior and exterior solutions. This approach takes into account the internal properties of the gravitational source, and reduces the problem of finding interior approximate solutions to the computation of some integrals that depend explicitly on the physical characteristics of the object. In this section, we present the main results of this approach, derive a particular interior approximate solution, and investigate the possibility of matching it with an exterior counterpart.

8.3.1 The interior solution

Fock's first-order approximation metric was recently re-derived and investigated by Abdildin [143] in a simple manner. Initially, this metric was written in its original form in a harmonic coordinate system [144, 145] as follows (a derivation of this metric is presented in the Appendix)

$$ds^2 = \left[c^2 - 2U + \frac{2U^2}{c^2} - \frac{2G}{c^2}\int \frac{\rho\left(\frac{3}{2}v^2 + \Pi - U\right) - p_{kk}}{|\vec{r}-\vec{r'}|}\left(dx'\right)^3\right] dt^2$$
$$- \left(1 + \frac{2U}{c^2}\right)\left({dx_1}^2 + {dx_2}^2 + {dx_3}^2\right) + \frac{8}{c^2}\left(U_1 dx_1 + U_2 dx_2 + U_3 dx_3\right) dt, \tag{8.3.1}$$

where U is the Newtonian gravitational potential, ρ is the mass density of the body, v is the speed of the particles inside the body, Π is the elastic energy per unit mass, p_{ik} is the stress tensor, $\vec{U}$ is the gravitational vector potential. Notice that the quantities ρ, v, Π and U that characterize the inner structure of the source depend only on the "inner" coordinates x'_i, which are defined inside the body only. To simplify the notations we omit the arguments that define this coordinate dependence.

The corresponding energy-momentum tensor is given as

$$T^{00} = \frac{\rho}{c^2}\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + \Pi - U\right)\right], \tag{8.3.2}$$
$$T^{0i} = \frac{\rho}{c^2}v^i, \quad T^{ij} = \frac{1}{c^2}\left(\rho v^i v^j - p^{ij}\right). \tag{8.3.3}$$

Newton's potential satisfies the equation $\nabla^2 U = -4\pi G\rho$. The solution of this equation that satisfies the asymptotically flatness condition at infinity can be written in the form of a volume integral:

$$U = G\int \frac{\rho}{|\vec{r}-\vec{r'}|} dx'_1 dx'_2 dx'_3 . \tag{8.3.4}$$

Furthermore, the vector potential must satisfy the equation $\nabla^2 U_i = -4\pi G\rho v_i$ whose general asymptotically flat solution can be represented as

$$U_i = G\int \frac{\rho v_i}{|\vec{r}-\vec{r'}|} dx'_1 dx'_2 dx'_3 . \tag{8.3.5}$$

Additional details about this metric can be found in [146] and [147].

It is worth noticing that Chandrasekhar, using the Fock method, obtained in [148] a solution similar to (8.3.1) that later on was used by Hartle and Sharp in [138]. However, it is not difficult to show that Chandrasekhar's solution is equivalent to (8.3.1). Indeed, the identification of the density

$$\rho = \rho_{Fock} = \rho_{Chandra}\left[1+\frac{1}{c^2}\left(3U+\frac{v^2}{2}\right)\right] \tag{8.3.6}$$

at the level of the energy-momentum tensor allows one to calculate the corresponding metric functions that show the equivalence of the metrics. Moreover, it has been shown in [148] that the solution for the non-rotating case can be matched with the well-known Schwarzschild solution, appropriately specialized to the case of spherical symmetry and hydrostatic equilibrium in the post Newtonian approximation.

8.3.2 The exterior Fock metric for slow rotation and spherically symmetric distribution of mass

In order to completely determine the metric, it is necessary to calculate the above integrals. Clearly, the result will depend on the internal structure of the body which is determined by the density ρ and velocity v_i distributions. Once these functions are given, the calculation of the integrals can be performed in accordance with the detailed formalism developed by Fock [125] and then extended and continued by Abdildin [126, 127] and Brumberg [149]. Consider, for instance, the case of a rotating sphere with total mass M. Then, the corresponding exterior metric in spherical-like (non harmonic) coordinates can be written as [126]

$$\begin{aligned} ds^2 = &\left[c^2 - \frac{2GM}{r} - \kappa\frac{GS_0^2}{c^2Mr^3}\left(1-3\cos^2\theta\right)\right]dt^2 - \left(1+\frac{2GM}{c^2r}\right)dr^2 \\ &-r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)+\frac{4GS_0}{c^2r}\sin^2\theta d\phi dt\ , \end{aligned} \tag{8.3.7}$$

where S_0 is the angular momentum of the body[2]. Here we added the constant κ and verified that in fact the above metric is an approximate solution for any arbitrary real value of κ. This simple observation allows us to interpret Fock's procedure as a method to find out how the internal structure of the object influences the values of the external parameters. For instance, the total mass in the above metric is M but it can decomposed as

$$M = m + \frac{\zeta}{c^2}, \tag{8.3.8}$$

[2]Notice the typos in the sign in front of S_0^2 in Eqs. (1.78) and (1.79) of [127]

where m is the rest mass of the body, and ζ is an arbitrary real constant which, as the constant κ, depends on the internal properties of the body. In particular, the cases of a liquid and a solid sphere have been analyzed in detail in [126, 127, 147] with the result

$$\zeta = \begin{cases} \frac{8}{3}T + \frac{2}{3}\varepsilon, & \text{for a liquid sphere,} \\ 4T + \frac{2}{3}\varepsilon, & \text{for a solid sphere,} \end{cases} \qquad \kappa = \begin{cases} \frac{4}{7}, & \text{for a liquid sphere,} \\ \frac{15}{28}, & \text{for a solid sphere.} \end{cases} \tag{8.3.9}$$

where T is the rotational kinetic energy of the body and ε is the energy of the mutual gravitational attraction of the particles inside the body. In Sec. 8.4.1, we will briefly explain how to obtain the above values.

8.3.3 The Kerr metric

To describe the gravitational field of the rotating sphere outside the source, it seems physically reasonable to assume that the exterior vacuum metric be asymptotically flat. In this case, the first obvious candidate is the Kerr solution in the corresponding limit. The Kerr metric [110] in Boyer-Lindquist coordinates [150, 149] can be written as

$$\begin{aligned} ds^2 = &\left(1 - \frac{2\mu\varrho}{\varrho^2 + a^2\cos^2\vartheta}\right) c^2dt^2 - \frac{\varrho^2 + a^2\cos^2\vartheta}{\varrho^2 - 2\mu\varrho + a^2} d\varrho^2 - \left(\varrho^2 + a^2\cos^2\vartheta\right) d\vartheta^2 \\ &- \left(\varrho^2 + a^2 + \frac{2\mu\varrho a^2\sin^2\vartheta}{\varrho^2 + a^2\cos^2\vartheta}\right)\sin^2\vartheta d\phi^2 - \frac{4\mu\varrho a\sin^2\vartheta}{\varrho^2 + a^2\cos^2\vartheta} cdtd\phi \,, \end{aligned} \tag{8.3.10}$$

where

$$\mu = \frac{GM}{c^2}, \; a = -\frac{S_0}{Mc}. \tag{8.3.11}$$

Expanding this metric to the order $\frac{1}{c^2}$, one obtains

$$\begin{aligned} ds^2 = &\left[c^2 - \frac{2GM}{\varrho} + \frac{2GMa^2}{\varrho^3}\cos^2\vartheta\right] dt^2 - \left(1 + \frac{2GM}{\varrho c^2} - \frac{a^2}{\varrho^2}\sin^2\vartheta\right) d\varrho^2 \\ &-\varrho^2\left(1 + \frac{a^2}{\varrho^2}\cos^2\vartheta\right) d\vartheta^2 - \varrho^2\left(1 + \frac{a^2}{\varrho^2}\right)\sin^2\vartheta d\phi^2 - \frac{4GMa}{\varrho c}\sin^2\vartheta d\phi dt \,. \end{aligned} \tag{8.3.12}$$

Furthermore, if we introduce new coordinates $\varrho = \varrho(r, \theta)$, $\vartheta = \vartheta(r, \theta)$ by means of the equations

$$\varrho = r - \frac{a^2\sin^2\theta}{2r}, \;\; \vartheta = \theta - \frac{a^2\sin\theta\cos\theta}{2r^2} \,, \tag{8.3.13}$$

then the approximate Kerr metric (8.3.12) can be reduced to the following form

$$ds^2 = \left[c^2 - \frac{2GM}{r} - \frac{GS_0^2}{c^2 M r^3}\left(1 - 3\cos^2\theta\right)\right] dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) dr^2$$
$$-r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{4GS_0}{c^2 r}\sin^2\theta d\phi dt\ , \quad (8.3.14)$$

which coincides with the metric (8.3.7) with $\kappa = 1$. Consequently, the extended Fock metric (8.3.7) can be interpreted as describing the exterior field of a rotating body to second order in the angular velocity. The advantage of using Fock's method to derive this approximate solution is that it allows the determination of the arbitrary constant κ. In fact, whereas $\kappa = \kappa_L = 4/7$ for a liquid sphere and $\kappa = \kappa_S = 15/28$ for a solid sphere, the value for the Kerr metric $\kappa = \kappa_K = 1$ does not seem to correspond to a concrete internal model. On the other hand, all the attempts to find a physically meaningful interior Kerr solution have been unsuccessful. Perhaps the relationship with Fock's formalism we have established here could shed some light into the structure of the interior counterpart of the Kerr metric.

Furthermore, the coordinate transformation [123]

$$\varrho = R - \frac{a^2}{2R}\left[\left(1 + \frac{2G\mathcal{M}}{c^2 R}\right)\left(1 - \frac{G\mathcal{M}}{c^2 R}\right) - \cos^2\Theta\left(1 - \frac{2G\mathcal{M}}{c^2 R}\right)\left(1 + \frac{3G\mathcal{M}}{c^2 R}\right)\right], \quad (8.3.15)$$

$$\vartheta = \Theta - \frac{a^2}{2R^2}\left(1 + \frac{2G\mathcal{M}}{c^2 R}\right)\cos\Theta\sin\Theta\ , \quad (8.3.16)$$

transforms the Kerr solution (8.3.10), expanded to second order in the angular momentum, (here one should set $G = c = 1$) into the HT solution (8.2.7) with $J = -\mathcal{M}a$, $M = \mathcal{M}$ and a particular quadrupole parameter $Q = J^2/\mathcal{M}$.

In this way, we have shown that the extended Fock metric coincides for $\kappa = 1$ with the approximate Kerr solution which, in turn, is equivalent to the exterior HT solution with a particular value of the quadrupole parameter. The fact that in the Kerr solution the quadrupole moment is completely specified by the angular momentum is an indication that it can be applied only to describe the gravitational field of a particular class of compact objects. A physically meaningful generalization of the Kerr solution should include a set or arbitrary multipole moments which are not completely determined by the angular momentum. In the next section, we present a particular exact solution characterized by an arbitrary quadrupole moment.

8.4 A solution with quadrupole moment

In this section, we will consider the case of deformed objects as, for example, a rotating ellipsoid. It is obvious that if the form of the body slightly deviates from spherical symmetry, it acquires multipole moments, in particular, a quadrupole moment; the moments of higher order are negligible, especially, for a slowly rotating ellipsoid. We will generalize Fock's metric so that the quadrupole moment appears explicitly from the integration of (8.3.1) and in the Newtonian potential. It should be mentioned that finding external and internal Newtonian potentials for a rotating ellipsoid is one of the classic problems of both physics and mathematical physics. Some examples for a homogeneous ellipsoid are considered in [151], but the most comprehensive details on this matter are given in [152] and more recently in [153]. As for the exterior counterpart, there are several exact solutions [114, 115, 113, 116, 117] with quadrupole moment and rotation parameter that could be used as possible candidates to be matched with the interior approximate solution. In this work, we limit ourselves to the study of a particular solution first proposed in [154] and then generalized in [155, 118].

8.4.1 The exterior Fock solution

Let us consider the first-order approximation metric (8.3.1). It is convenient to use the notation $x_1' = x$, $x_2' = y$, and $x_3' = z$. In general, the fact that the mass density $\rho = \rho(x, y, z)$ is a function of the coordinates does not allow us to find explicit expression for the internal Newtonian potential. It is possible only by numerical integration. However, for the case of uniform density there is in the literature a reasonable number of exact solutions for rotating ellipsoids. Since we consider slow rotation and the weak field approximation, we can use the expansion for the Newtonian potential [151], [156]

$$U(r, \theta) = G \int \frac{\rho}{|\vec{r} - \vec{r'}|} dxdydz = \frac{Gm}{r} + \frac{GD}{2r^3} P_2(\cos\theta) , \qquad (8.4.1)$$

where m is the rest mass of the ellipsoid, D is the Newtonian quadrupole moment, θ is the angle between $r' = \sqrt{x^2 + y^2 + z^2}$ and z — axis. The first term in the expression above is the potential of a sphere and the second one is responsible for the deviation from spherical symmetry. If one takes the z axis as a rotating axis then the quadrupole moment is defined by

$$D = \int \rho(2z^2 - x^2 - y^2) dxdydz. \qquad (8.4.2)$$

For the rotating ellipsoid with uniform density the quadrupole moment is

$$D = \frac{2m}{5}\left(r_p^2 - r_e^2\right) \,, \tag{8.4.3}$$

where r_p and r_e are the polar and equatorial radii of the ellipsoid, respectively. The mass of the ellipsoid is defined as the integral $m = \int \rho dxdydz$ that in the case of an ellipsoid with uniform density yields

$$m = \frac{4}{3}\pi\rho r_e^2 r_p \,. \tag{8.4.4}$$

Note that the integration is carried out in the ranges of $0 \leq x, y \leq r_e$ and $0 \leq z \leq r_p$. Using the same procedure one may write the integral in Fock's metric as follows

$$\int \frac{\rho\left(\frac{3}{2}v^2 + \Pi - U\right) - p_{kk}}{|\vec{r} - \vec{r}'|} dxdydz = \frac{\zeta}{r} + \frac{\mathcal{D}}{2r^3}P_2(\cos\theta), \tag{8.4.5}$$

where

$$\zeta = \int \left[\rho\left(\frac{3}{2}v^2 + \Pi - U\right) - p_{kk}\right] dxdydz \,, \tag{8.4.6}$$

$$\mathcal{D} = \int \left[\rho\left(\frac{3}{2}v^2 + \Pi - U\right) - p_{kk}\right](2z^2 - x^2 - y^2) dxdydz \,. \tag{8.4.7}$$

The quantity $\mathcal{D}/c^2$ is the relativistic correction to the Newtonian quadrupole moment D, i. e., the quadrupole moment due to rotation.

To evaluate the integrals we use the relation for a compressible elastic medium [125]

$$\rho\Pi - \rho U + p = \rho W \,, \tag{8.4.8}$$

where p is the isotropic pressure and W is the potential of the centrifugal forces determined by

$$W = \frac{1}{2}w^i{}_k w^{jk} x_i x_j, \quad i, j, k = 1, 2, 3 \tag{8.4.9}$$

or $v^2 = 2W$, which is a consequence of the equation $v_i = w^j{}_i x_j$ for the velocity within the body. In these equations w_{ik} is the three-dimensional antisymmetric tensor of the angular velocity of the body $\vec{\Omega}$. We may alternatively write its components as

$$w_{23} = \Omega_x = \Omega_1, \quad w_{31} = \Omega_y = \Omega_2, \quad w_{12} = \Omega_z = \Omega_3 = \Omega \,. \tag{8.4.10}$$

In our case $\vec{\Omega} = \{0, 0, \Omega\}$; hence the potential of the centrifugal forces is

$$W = \frac{(x^2 + y^2)}{2}\Omega^2. \tag{8.4.11}$$

If we assume that the stress tensor within the body reduces to an isotropic pressure p, i. e., $p_{ik} = -p\delta_{ik}$ (a liquid always satisfies this condition), and take Eq.(8.4.8) and $v^2 = 2W$ into account, Eqs. (8.4.6) and (8.4.7) reduce to the simple form

$$\zeta = 2\int [2\rho W + p]\, dxdydz\,, \tag{8.4.12}$$

$$\mathcal{D} = 2\int [2\rho W + p]\,(2z^2 - x^2 - y^2) dxdydz\,. \tag{8.4.13}$$

Furthermore, to evaluate these integrals we consider the following two cases that determine the inner structure of the body:

1) A liquid body with following the equation of internal motion [125]

$$\rho\left(\frac{\partial U}{\partial x_i} - w^i{}_k w^{jk} x_j\right) = \frac{\partial p}{\partial x_i}\,. \tag{8.4.14}$$

2) An absolute solid body with the following equation of internal motion [149]

$$\rho\frac{\partial U}{\partial x_i} = \frac{\partial p}{\partial x_i}\,. \tag{8.4.15}$$

These are the equations of hydrostatic equilibrium which are adopted by Fock [125] and Brumberg [149] to describe the internal structure of the object.

We limit ourselves to consider those cases in which the body rotates as a whole, in the manner of a rigid body. Then, for both liquid and solid bodies the rotational kinetic energy takes the form

$$T = \int \rho W dxdydz = \frac{I_{zz}\Omega^2}{2}\,, \tag{8.4.16}$$

where I_{zz} is the moment of inertia of the ellipsoid, which for a uniform density distribution, is equal to

$$I_{zz} = \int \rho(x^2 + y^2)\, dxdydz = \frac{2}{5}mr_e^2\,. \tag{8.4.17}$$

The pressure can be expressed as

$$\int p\, dxdydz = \begin{cases} \frac{1}{3}\left(\varepsilon - 2T\right), & \text{for a liquid body,} \\ \frac{1}{3}\varepsilon, & \text{for a solid body,} \end{cases} \tag{8.4.18}$$

where

$$\varepsilon = \frac{1}{2}\int \rho U \, dxdydz \,, \tag{8.4.19}$$

represents the negative of the energy of mutual attraction of the constituent particles of the body. For a uniform density it has form

$$\varepsilon = \frac{3Gm^2}{5\sqrt{r_e^2 - r_p^2}} \arccos\frac{r_p}{r_e}. \tag{8.4.20}$$

The second moments

$$T_{ik} = \int \rho W x'_i x'_k \, dxdydz \,, \tag{8.4.21}$$

can be computed by using the above expressions. Then, for the second moments of the pressure we obtain (see [143])

$$\int p x'_i x'_k \, dxdydz = \begin{cases} \frac{1}{2}\eta_{ik} - \frac{2}{5}T_{ik}, & \text{for a liquid body,} \\ -\frac{1}{2}T_{ik}, & \text{for a solid body,} \end{cases} \tag{8.4.22}$$

where (more details can be found in [143])

$$\eta_{ik} = -\frac{2}{5}\int \rho x'_i x'_k x'_j \frac{\partial U}{\partial x'_j} \, dxdydz \,. \tag{8.4.23}$$

After calculating all the integrals we have

$$\zeta = \begin{cases} \frac{8}{3}T + \frac{2}{3}\varepsilon, & \text{for a liquid body,} \\ 4T + \frac{2}{3}\varepsilon, & \text{for a solid body,} \end{cases} \tag{8.4.24}$$

$$\mathcal{D} = \begin{cases} \frac{28}{5}\frac{\kappa_L S_0^2}{I_{zz}^2}\left[\int \rho(x^2+y^2)(z^2-x^2)dxdydz\right] - \frac{4}{5}\int \rho(z^2-x^2)x'_j\frac{\partial U}{\partial x'_j}dxdydz, & \text{for a liquid body,} \\ \frac{28}{5}\frac{\kappa_S S_0^2}{I_{zz}^2}\left[\int \rho(x^2+y^2)(z^2-x^2)dxdydz\right], & \text{for a solid body,} \end{cases} \tag{8.4.25}$$

where S_0 is the angular momentum of the body, which is found from

$$\vec{S_0} = I_{zz}\vec{\Omega} \,, \tag{8.4.26}$$

and the numerical factors are

$$\kappa = \begin{cases} \kappa_L = \frac{4}{7}, & \text{for a liquid body,} \\ \kappa_S = \frac{15}{28}, & \text{for a solid body.} \end{cases} \tag{8.4.27}$$

Unlike the Newtonian scalar potential, the vector potential can be easily calculated from

$$\vec{U} = \frac{G}{2r^3}\left[\vec{S}_0 \times \vec{r}\right]. \tag{8.4.28}$$

Introducing the effective (total) mass as

$$M = m + \frac{\zeta}{c^2}, \tag{8.4.29}$$

for the Fock metric we obtain the following expression

$$ds^2 = \left[c^2 - 2\left(\frac{GM}{r} + \frac{GD}{2r^3}P_2(\cos\theta)\right) + \frac{2}{c^2}\left(\frac{GM}{r} + \frac{GD}{2r^3}P_2(\cos\theta)\right)^2 - \frac{G\mathcal{D}}{c^2r^3}P_2(\cos\theta)\right]dt^2 - \left[1 + \frac{2GM}{c^2r} + \frac{GD}{c^2r^3}P_2(\cos\theta)\right]\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] + \frac{4GS_0}{c^2r}\sin^2\theta d\phi dt\,, \tag{8.4.30}$$

in harmonic coordinates. In order to write it in Schwarzschild like (standard) spherical coordinates one should use the coordinate transformation

$$r \to R - \frac{GM}{c^2}, \quad \theta \to \Theta. \tag{8.4.31}$$

which transforms the metric (8.4.30) into

$$ds^2 = \left[c^2 - \frac{2GM}{R} - \left(D + \frac{\mathcal{D}}{c^2}\right)\frac{G}{R^3}P_2(\cos\Theta) - \frac{G^2DM}{c^2R^4}P_2(\cos\Theta)\right]dt^2 + \frac{4GS_0}{c^2R}\sin^2\Theta d\phi dt - \left[1 + \frac{2GM}{c^2R} + \frac{GD}{c^2R^3}P_2(\cos\Theta)\right]dR^2 - \left[1 + \frac{GD}{c^2R^3}P_2(\cos\Theta)\right]R^2(d\Theta^2 + \sin^2\Theta d\phi^2)\,, \tag{8.4.32}$$

where we have neglected quadratic terms in the quadrupole parameter D. In the limiting case with vanishing rotation $S_0 = 0$ and vanishing quadrupole moment $D = \mathcal{D} = 0$, this metric represents the approximate Schwarzschild solution.

An examination of the metric (8.4.30) shows that the rough approximation with $r_e \approx r_p \approx r_{sphere}$ and $S_0 \neq 0$ leads to the approximate Fock metric considered in Sec. 8.3 with the total mass M, for a slowly rotating spherically symmetric body with

$$D = 0, \quad \mathcal{D} = -\frac{2\kappa S_0^2}{M}\,. \tag{8.4.33}$$

It should be noted that an analogous result was obtained by Laarakkers and Poisson [157]. They numerically computed the scalar quadrupole moment $\mathcal{Q}$ of rotating neutron stars for several equations of state (EoS). They found that for fixed gravitational mass M, the quadrupole moment is given as a simple quadratic fit

$$\mathcal{Q} = -K\frac{J^2}{Mc^2} \tag{8.4.34}$$

where J is the angular momentum of the star and K is a dimensionless quantity that depends on the EoS. Note that the scalar quadrupole moment $\mathcal{Q}$ of Laarakkers and Poisson is related to the one of Hartle and Thorne as follows $\mathcal{Q} = -Q$. The above quadratic fit reproduces $\mathcal{Q}$ with remarkable accuracy. The quantity K varies between $K \approx 2$ for very soft EoS's and $K \approx 7.4$ for very stiff EoS's, for $M = 1.4M_{Sun}$ as in neutron stars. This is considerably different from a Kerr black hole, for which $K = 1$ (see [121, 158]). Recently, the results of [157] were modified taking into account the correct definition of multipole moments [134]. Therefore the value of the K parameter in the numerical fit (8.4.34) is slightly different from that given in [157]. In our case, we have similar, but not the same results, since the Fock solution is not valid in the limit of strong gravitational fields (like in neutron stars) and fast rotation. The values for the constant κ are obtained from qualitative analyses in the limit of weak field and slow rotation. In order to find exact values for κ one should specify the EoS's and perform numerical integrations. This task, however, is out of the scope of the present work.

8.4.2 The exterior Quevedo-Mashhoon solution

In this section, we study the general metric describing the gravitational field of a rotating deformed mass found by Quevedo and Mashhoon [154, 155, 159, 118], which is a stationary axisymmetric solution of the vacuum Einstein's equations belonging to the class of Weyl-Lewis-Papapetrou [160, 161, 162]. For the sake of simplicity we consider here a particular solution involving only four parameters: the mass parameter M, the angular momentum parameter a, the quadrupole parameter q, and the additional Zipoy-Voorhees [163, 164] constant δ. For brevity, in this section we use geometric units with $G = c = 1$. The corresponding line element in spheroidal coordinates (t, r, θ, ϕ) with $r \geq \sigma + M_0$, $0 \leq \theta \leq \frac{\pi}{2}$ is given by [165]

$$\begin{aligned} ds^2 = \quad & f(dt - \omega d\phi)^2 \\ & -\frac{e^{2\gamma}}{f}\left(d\theta^2 + \frac{dr^2}{r^2 - 2M_0 r + a^2}\right)\left[(M_0 - r)^2 - (M_0^2 - a^2)\cos^2\theta\right] \\ & +\frac{1}{f}\left(r^2 - 2M_0 r + a^2\right)\sin^2\theta d\phi^2, \end{aligned} \tag{8.4.35}$$

where f, ω and γ are functions of r and θ only, and σ is a constant. They have the form $[x = (r - M_0)/\sigma,\ y = \cos\theta]$

$$f = \frac{\tilde{R}}{L} e^{-2q\delta P_2 Q_2}, \tag{8.4.36}$$

$$\omega = -2a - 2\sigma \frac{\mathfrak{M}}{\tilde{R}} e^{2q\delta P_2 Q_2}, \tag{8.4.37}$$

$$e^{2\gamma} = \frac{1}{4}\left(1 + \frac{M}{\sigma}\right)^2 \frac{\tilde{R}}{(x^2-1)^{\delta}} e^{2\delta^2 \hat{\gamma}}, \tag{8.4.38}$$

where

$$\tilde{R} = a_+ a_- + b_+ b_-, \qquad L = a_+^2 + b_+^2, \tag{8.4.39}$$

$$\mathfrak{M} = (x+1)^{\delta-1}\left[x(1-y^2)(\lambda+\eta)a_+ + y(x^2-1)(1-\lambda\eta)b_+\right], \tag{8.4.40}$$

$$\begin{aligned}\hat{\gamma} = \quad & \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 \\ & + q^2(1-P_2)[(1+P_2)(Q_1^2 - Q_2^2) \\ & + \frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')].\end{aligned} \tag{8.4.41}$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind, respectively. Furthermore

$$a_\pm = (x \pm 1)^{\delta-1}[x(1-\lambda\eta) \pm (1+\lambda\eta)], \tag{8.4.42}$$

$$b_\pm = (x \pm 1)^{\delta-1}[y(\lambda+\eta) \mp (\lambda-\eta)], \tag{8.4.43}$$

with

$$\begin{aligned}\lambda &= \alpha(x^2-1)^{1-\delta}(x+y)^{2\delta-2}e^{2q\delta\delta_+}, \\ \eta &= \alpha(x^2-1)^{1-\delta}(x-y)^{2\delta-2}e^{2q\delta\delta_-}, \\ \delta_\pm &= \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2 \mp xy) + \frac{3}{4}[x(1-y^2) \mp y(x^2-1)]\ln\frac{x-1}{x+1}.\end{aligned} \tag{8.4.44}$$

Morover, α and σ are constants defined as

$$\alpha = \frac{\sigma - M}{a}, \qquad \sigma = \sqrt{M^2 - a^2}. \tag{8.4.45}$$

The physical meaning of the parameters entering this metric can be investigated in an invariant manner by calculating the Geroch-Hansen [18, 19] moments:

$$M_{2k+1} = J_{2k} = 0, \quad k = 0,1,2,\ldots, \tag{8.4.46}$$

$$M_0 = M + \sigma(\delta - 1), \tag{8.4.47}$$

$$M_2 = -Ma^2 + \frac{2}{15}q\sigma^3 - \frac{1}{15}\sigma(\delta-1) \times \left[45M^2 + 15M\sigma(\delta-1) - (30 + 2q + 10\delta - 5\delta^2)\sigma^2\right], \tag{8.4.48}$$

$$J_1 = Ma + 2a\sigma(\delta - 1), \tag{8.4.49}$$

$$J_3 = -Ma^3 + \frac{4}{15}aq\sigma^3 - \frac{1}{15}a\sigma(\delta-1) \times \left[60M^2 + 45M\sigma(\delta-1) - 2\sigma^2(15 + 2q + 10\delta - 5\delta^2)\right]. \tag{8.4.50}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the reflection symmetry with respect to the equatorial plane $\theta = \pi/2$. Note that in the limiting case $\delta = 1$, $M_0 = M$ is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q. In general, we see that the Zipoy-Voorhees parameter is related to the quadrupole moment of the source. In fact, even in the limiting static case with $a = 0$ and $q = 0$, the only non-vanishing parameters are $M = \sigma$ and δ so that all gravitomagnetic multipoles vanish and one obtains $M_0 = M\delta$ and $M_2 = -\frac{1}{3}M^3\delta(\delta^2 - 1)$ — the quadrupole moment that indicates a deviation from spherical symmetry. Some geometrical properties of (8.4.35) versus particle motion and tidal indicators in this spacetime were explored in [142] and [166], respectively.

Consider the limiting cases of the QM solution. For vanishing quadrupole parameter, $q = 0$, $\delta = 1$, and vanishing angular momentum $a = 0$, $\alpha = 0$, and $\sigma = M$, one recovers the Schwarzschild solution with the following metric functions:

$$f = 1 - \frac{2M}{r}, \quad \omega = 0, \quad \gamma = \frac{1}{2}\ln\frac{r(r-2M)}{(M-r)^2 - M^2\cos^2\theta}. \tag{8.4.51}$$

For vanishing quadrupole parameter and $\delta = 1$, one recovers the Kerr solution (8.3.10) with $\vartheta \to \theta$ and $\varrho \to r$ and functions

$$f = 1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}, \quad \omega = \frac{2aMr\sin^2\theta}{r^2 - 2Mr + a^2\cos^2\theta}, \tag{8.4.52}$$

$$\gamma = \frac{1}{2}\ln\frac{r(r-2M) + a^2\cos^2\theta}{(M-r)^2 - (M^2 - a^2)\cos^2\theta}. \tag{8.4.53}$$

The above limiting cases show that this solution describes the exact exterior field a rotating deformed object. To compute the case of a slowly rotating and slightly deformed body we choose the Zipoy-Voorhees parameter as $\delta = 1 + sq$, where s is a real constant. Then, expanding the metric (8.4.35) to first order in the quadrupole parameter q and to second order in the rotation parameter a, we obtain

$$\begin{aligned} f &= 1 - \frac{2M}{r} + \frac{2a^2 M \cos^2\theta}{r^3} + q(1+s)\left(1 - \frac{2M}{r}\right)\ln\left(1 - \frac{2M}{r}\right) \\ &\quad + 3q\left(\frac{r}{2M} - 1\right)\left[\left(1 - \frac{M}{r}\right)\left(3\cos^2\theta - 1\right)\right. \\ &\quad \left. + \left\{\left(\frac{r}{2M} - 1\right)(3\cos^2\theta - 1) - \frac{M}{r}\sin^2\theta\right\}\ln\left(1 - \frac{2M}{r}\right)\right], \end{aligned} \tag{8.4.54}$$

$$\omega = \frac{2aMr\sin^2\theta}{r - 2M}, \tag{8.4.55}$$

$$\begin{aligned} \gamma &= \frac{1}{2}\ln\frac{r(r-2M)}{(r-M)^2 - M^2\cos^2\theta} + \frac{a^2}{2}\left[\frac{M^2\cos^2\theta\sin^2\theta}{r(r-2M)((r-M)^2 - M^2\cos^2\theta)}\right] \\ &+ q(1+s)\ln\frac{r(r-2M)}{(r-M)^2 - M^2\cos^2\theta} - 3q\left[1 + \frac{1}{2}\left(\frac{r}{M} - 1\right)\ln\left(1 - \frac{2M}{r}\right)\right]\sin^2\theta. \end{aligned} \tag{8.4.56}$$

The further simplification $s = -1$, and the coordinate transformation [123, 142, 167]

$$\begin{aligned} r &= R + \mathcal{M}q + \frac{3}{2}\mathcal{M}q\sin^2\Theta\left[\frac{R}{\mathcal{M}} - 1 + \frac{R^2}{2\mathcal{M}^2}\left(1 - \frac{2\mathcal{M}}{R}\right)\ln\left(1 - \frac{2\mathcal{M}}{R}\right)\right] \\ &\quad - \frac{a^2}{2R}\left[\left(1 + \frac{2\mathcal{M}}{R}\right)\left(1 - \frac{\mathcal{M}}{R}\right) - \cos^2\Theta\left(1 - \frac{2\mathcal{M}}{R}\right)\left(1 + \frac{3\mathcal{M}}{R}\right)\right] \end{aligned} \tag{8.4.57}$$

$$\theta = \Theta - \sin\Theta\cos\Theta\left\{\frac{3}{2}q\left[2 + \left(\frac{R}{\mathcal{M}} - 1\right)\ln\left(1 - \frac{2\mathcal{M}}{R}\right)\right] + \frac{a^2}{2R}\left(1 + \frac{2\mathcal{M}}{R}\right)\right\} \tag{8.4.58}$$

transforms the approximate QM solution (8.4.54)–(8.4.56) into

$$ds^2 = \left[1 - \frac{2GM(1-q)}{c^2R} + \frac{2G}{c^2R^3}\left(\frac{J^2}{M} - \frac{4}{5}qM^3\right)\left(1 + \frac{GM(1-q)}{c^2R}\right)P_2(\cos\Theta)\right]c^2dt^2$$
$$-\frac{4GMa}{c^2R}\sin^2\Theta dtd\phi - \left[1 + \frac{2GM(1-q)}{c^2R} - \frac{2G}{c^2R^3}\left(\frac{J^2}{M} - \frac{4}{5}qM^3\right)P_2(\cos\Theta)\right]dR^2$$
$$-\left[1 - \frac{2G}{c^2R^3}\left(\frac{J^2}{M} - \frac{4}{5}qM^3\right)P_2(\cos\Theta)\right]R^2(d\Theta^2 + \sin^2\Theta d\phi^2)\,. \tag{8.4.59}$$

Here we introduced again all the necessary constants G and c in order to compare our results with previous metrics. Finally, if we redefine the parameters M, a, and q as

$$\mathcal{M} = M(1-q),\;\; J = -Ma,\;\; Q = \frac{J^2}{M} - \frac{4}{5}M^3q\,, \tag{8.4.60}$$

the approximate metric (8.4.59) coincides with the exterior HT metric (8.2.11) and, consequently, can be matched with the interior HT metric discussed in 8.2.

The above metric is equivalent to the exterior extended Fock metric discussed in the previous subsection. To see this one has to consider the exterior solution (8.4.32) which is written in the same coordinates as the exterior solutions (8.4.59) and (8.2.11). It is convenient to show first the equivalence with the exterior HT solution (8.2.11) that yields the conditions

$$\mathcal{M} = M, \qquad J = S_0, \qquad Q = -\frac{1}{2}\left(D + \frac{\mathcal{D}}{c^2}\right). \tag{8.4.61}$$

The equivalence with the approximate QM solution (8.4.59) follows then from the comparison of Eqs.(8.4.60) and (8.4.61). We obtain

$$q = \frac{5}{8}\frac{c^4}{G^2}\frac{1}{M^3}\left[D + \frac{1}{c^2}\left(\mathcal{D} + \frac{2J^2}{M}\right)\right]\,, \tag{8.4.62}$$

and for vanishing D

$$q = \frac{5}{4}\frac{c^2}{G^2}\frac{J^2}{M^4}(1-\kappa)\,. \tag{8.4.63}$$

This result is in accordance with the limiting case of the Kerr metric for which we obtained that $\kappa = 1$ and hence $q = 0$.

Thus we come to the conclusion that in the limit of a slowly rotating and slightly deformed body the QM approximate solution is equivalent to the exterior Fock solution.

8.5 Conclusions

In this work, we studied the gravitational field of slowly rotating, slightly deformed astrophysical compact objects. We presented the main exact and approximate solutions of Einstein's equations that can be used to describe the interior and the exterior gravitational field. In particular, we presented the method proposed by Hartle and Thorne to find interior and exterior approximate solutions, and the method proposed by Fock to derive approximate interior and exterior solutions. We derived an extension of the approximate exterior Fock metric that takes into account up to the first order the contribution of a quadrupole parameter that describes the deviation of the body from spherical symmetry. A particular parameter that enters the extended Fock metric turns out to have very specific values in the case of a liquid sphere and a solid sphere. In the case of the approximate Kerr metric, this parameter does not seem to correspond to any known interior model analyzed in the framework of Fock's formalism.

We found that a particular QM solution, which in general possesses an infinite set of gravitational and electromagnetic multipole moments, contains the exact Kerr metric and the approximate HT metric as special cases. Moreover, since the HT solution is endowed with its interior counterpart, we conclude that the approximate QM solution (to the second order in the angular momentum and to the first order in the quadrupole parameter) can be matched with the interior HT solution, indicating that it can be used to correctly describe the gravitational field of astrophysical compact objects. Moreover, we showed that the explicit form of the exterior Fock metric is equivalent to the approximate exterior QM solution.

To avoid the technical problems that are usually found in the process of matching solutions [137], we use the same set of coordinates inside and outside the body. In the cases presented here, this can be done in a relative easy way only because all the coordinate transformations are not calculated exactly, but with the same approximation as the metric functions. This approach allows us to reduce the matching problem to the comparison of the metrics on the matching surfaces in such a way that only algebraic conditions appear. Using this method, we could show that the approximate Kerr metric cannot be matched with an interior Fock solution. However, if we take into account an additional quadrupole parameter, the matching of the extended Fock metric can be carried out by using as exterior counterpart a particular approximate QM solution that contains the Kerr metric as special case. We conclude that the quadrupole parameter offers an additional degree of freedom that allows the matching. A first step in this direction was recently taken forward in [168]. It would be interesting to see if this is also true in the case of exact solutions. This could shed some light into the problem of finding a realistic gravitational source for the Kerr metric, a long-standing problem of classical general relativity.

Appendix: Abdildin's metric

In this Appendix, we present a review of the derivation of a generalization of Fock's metric, based upon the approach formulated by Abdildin in [127]. The original approximate metric derived by Fock in [125] can be written as

$$ds^2 = \left(c^2 - 2U\right) dt^2 - \left(1 + \frac{2U}{c^2}\right) \left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2\right) + \frac{8}{c^2} \left(U_1 dx_1 + U_2 dx_2 + U_3 dx_3\right) dt, \tag{8.5.1}$$

where U is the Newtonian gravitational potential that satisfies the equation $\nabla^2 U = -4\pi G\rho$, where ρ represents the matter density of the gravitational source. Moreover, the gravitational vector potential $\vec{U}$ satisfies the equation $\nabla^2 U_i = -4\pi G\rho v_i$, where v_i are the components of the 3-velocity of the particles inside the source. The coordinates x^μ are harmonic functions satisfying the D'Alambert equation $\Box x^\mu = 0$.

As noticed by Abdildin, the metric (8.5.1) presents certain difficulties. First, the components g_{0i} and g_{ij} contain a relativistic contribution that is absent in the component g_{00}. Second, if we use the metric (8.5.1) to investigate the motion of test particles in a Kepler potential, we obtain an expression for the perihelion shift that differs from the correct one by a factor of 1/2. Finally, in the case of a static field or for Gaussian-like coordinate systems $Udt^2 \sim dx_1^2 + dx_2^2 + dx_3^2$, i.e., the relativistic correction of g_{00} must be of the same order as that of g_{ij}.

From the above observations it follows that it is necessary to consider a more appropriate expression for the component

$$g^{00} = \frac{1}{c^2} + \frac{2U}{c^4} + \frac{\Phi}{c^6}\,, \tag{8.5.2}$$

where Φ is an unknown function which must satisfy the corresponding approximate Einstein equation in harmonic coordinates

$$R^{00} = \frac{1}{2}\nabla^2 g^{00} - \frac{2U}{c^6}\nabla^2 U - \frac{2}{c^6}\sum_i \left(\frac{\partial U}{\partial x_i}\right)^2 = -\frac{8\pi G}{c^2}\left(T^{00} - \frac{1}{2}g^{00}T\right)\,. \tag{8.5.3}$$

As for the components of a energy-momentum tensor, in the case of an elastic source one can use the expressions

$$T^{00} = \frac{\rho}{c^2}\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + \Pi - U\right)\right]\,, \quad T^{0i} = \frac{\rho}{c^2}v^i\,, \quad T^{ij} = \frac{1}{c^2}\left(\rho v^i v^j - p^{ij}\right)\,, \tag{8.5.4}$$

where Π is the elastic energy. It is then straightforward from Eqs.(8.5.3) and

(8.5.4) to conclude that

$$\Phi = 2U^2 + 2G \int \frac{\rho\left(\frac{3}{2}v^2 + \Pi - U\right)}{|\vec{r} - \vec{r}'|}(dx')^3 - 2G \int \frac{p_{kk}}{|\vec{r} - \vec{r}'|}(dx')^3 \tag{8.5.5}$$

where $p_{kk} = p_{11} + p_{22} + p_{33}$. Consequently, the generalized approximate metric is

$$ds^2 = \left[c^2 - 2U + \frac{2U^2}{c^2} - \frac{2G}{c^2} \int \frac{\rho\left(\frac{3}{2}v^2 + \Pi - U\right) - p_{kk}}{|\vec{r} - \vec{r'}|} \left(dx'\right)^3\right] dt^2$$
$$- \left(1 + \frac{2U}{c^2}\right) \left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2\right) + \frac{8}{c^2} \left(U_1 dx_1 + U_2 dx_2 + U_3 dx_3\right) dt \,. \tag{8.5.6}$$

This form of the metric overcome all the difficulties mentioned above for the original Fock metric (8.5.1), and is used everywhere in the present work to obtain the correct approximations.

9 Quadrupolar gravitational fields described by the $q-$metric

9.1 Introduction

The Zipoy–Voorhees metric [163, 164] was discovered more than forty years ago as a particular exact solution of Einstein's vacuum field equations that belongs to the Weyl class [1] of vacuum solutions. In this work, we will refer to the Zipoy-Voorhees solution as to the $q-$metric for a reason that will be explained below. Since its discovery, many works have been devoted to the investigation of its geometric and physical properties. In particular, it has been established that it describes an asymptotically flat spacetime, it possesses two commuting, hypersurface orthogonal Killing vector fields that imply that the spacetime is static and axially symmetric, it contains the Schwarzschild metric as a special case that turns out to be the only one with a true curvature singularity surrounded by an event horizon [169, 170, 171, 172, 173].

In a recent work [174], it was proposed to interpret the $q-$metric as describing the gravitational field of a distribution of mass whose non-spherically symmetric shape is represented by an independent quadrupole parameter. Moreover, the curvature singularities turn out to be localized inside a region situated very close to the origin of coordinates. Consequently, this metric can be used to describe the exterior gravitational field of deformed distributions of mass in which the quadrupole moment is the main parameter that describes the deformation. Tue question arises whether it is possible to find an interior metric that can be matched to the exterior one in such a way that the entire spacetime is described. To this end, it is usually assumed that the interior mass distribution can be described by means of a perfect fluid with two physical parameters, namely, energy density and pressure. The energy-momentum tensor of the perfect fluid is then used in the Einstein equations as the source of the gravitational field. It turns out that the system of the corresponding differential equations cannot be solved, because the number of equations is less than the number of unknown functions. This problem is usually solved by imposing equations of state that relate the pressure and density of the fluid. In this work, however, we will explore a different approach that was first proposed by Synge [84]. To apply this method, one first uses general physical considerations to postulate the form of the interior metric and then one evaluates the energy-momentum tensor of the source by using Einstein's equations. In this manner, any interior metric can be considered as an exact solution of the Einstein equations for some energy-momentum tensor. However, the main point of the procedure is to impose physical conditions on the resulting matter source so that it corresponds to a physical reasonable configuration. In general, one can impose the energy conditions, the matching conditions with the exterior metric, and conditions on the behavior of the metric functions near the center of the source and on the boundary with the exterior field.

9.2 The Zipoy–Voorhees transformation

Zipoy [163] and Voorhees [164] investigated static, axisymmetric vacuum solutions of Einstein's equations and found a simple transformation which allows to generate new solutions from a known solution. To illustrate the idea of the transformation, we use the general line element for static, axisymmetric vacuum gravitational fields in prolate spheroidal coordinates (t, x, y, φ):

$$ds^2 = e^{2\psi}dt^2 - \sigma^2 e^{-2\psi}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right] , \tag{9.2.1}$$

where the metric functions ψ and γ depend on the spatial coordinates x and y, only, and σ represents a non-zero real constant. The corresponding vacuum field equations can be written as

$$[(x^2-1)\psi_x]_x + [(1-y^2)\psi_y]_y = 0 , \quad \psi_x = \frac{\partial\psi}{\partial x} , \tag{9.2.2}$$

$$\gamma_x = \frac{1-y^2}{x^2-y^2}\left[x(x^2-1)\psi_x^2 - x(1-y^2)\psi_y^2 - 2y(x^2-1)\psi_x\psi_y\right] , \tag{9.2.3}$$

$$\gamma_y = \frac{1-y^2}{x^2-y^2}\left[y(x^2-1)\psi_x^2 - y(1-y^2)\psi_y^2 + 2x(1-y^2)\psi_x\psi_y\right] . \tag{9.2.4}$$

It can be seen that the function γ can be calculated by quadratures once ψ is known. If we demand that ψ be asymptotically flat, i.e.,

$$\lim_{x\to\infty} \psi(x,y) = 0 , \tag{9.2.5}$$

it can be shown [175] that using quadratures the asymptotically flat function γ can be calculated as

$$\gamma = (x^2-1)\int_{-1}^{y}(x^2-y^2)^{-1}\left[y(x^2-1)\psi_x^2 - y(1-y^2)\psi_y^2 + 2x(1-y^2)\psi_x\psi_y\right]dy . \tag{9.2.6}$$

Suppose that a solution ψ_0 and γ_0 of this system is known. It is then easy to see that $\psi = \delta\psi_0$ and $\gamma = \delta^2\gamma_0$ is also a solution for any constant δ. This is the Zipoy-Voorhees transformation that can be used to generate new solutions. The simplest example is

$$\psi = \frac{\delta}{2}\ln\frac{x-1}{x+1}\,, \quad \gamma = \frac{\delta^2}{2}\ln\frac{x^2-1}{x^2-y^2}\,, \tag{9.2.7}$$

which is generated from Schwarzschild solution ($\delta = 1$). This metric is known in the literature as the $\delta-$metric to emphasize the fact that it is obtained by applying a Zipoy-Voorhees transformation with constant δ.

A different representation can be obtained by using cylindrical coordinates that are defined as

$$\rho = \sigma\sqrt{(1-y^2)(x^2-1)}\,, \quad z = \sigma xy\,. \tag{9.2.8}$$

and in which the line element becomes

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right]\,. \tag{9.2.9}$$

Then, the vacuum field equations can be expressed as

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} = 0\,, \tag{9.2.10}$$

$$\gamma_\rho = \rho(\psi_\rho^2 - \psi_z^2)\,, \quad \gamma_z = 2\rho\psi_\rho\psi_z\,. \tag{9.2.11}$$

In this representation, the Zipoy-Voorhees metric can be expressed in the Weyl form [1]

$$\psi = \sum_{n=0}^{\infty}\frac{a_n}{(\rho^2+z^2)^{\frac{n+1}{2}}}P_n(\cos\theta)\,, \qquad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}\,, \tag{9.2.12}$$

$$\gamma = -\sum_{n,m=0}^{\infty}\frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2+z^2)^{\frac{n+m+2}{2}}}\left(P_nP_m - P_{n+1}P_{m+1}\right)\,. \tag{9.2.13}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The Zipoy-Voorhees metric can be obtained by choosing the constants a_n in such a way that the infinite sum (9.2.12) converges to (9.2.7) in cylindric coordinates. A simpler representation, however, is obtained in spherical coordinates which are defined by means of the relationships

$$\rho^2 = (r^2 - 2\sigma r)\sin^2\theta\,, \quad z = (r-\sigma)\cos\theta\,, \tag{9.2.14}$$

so that the metric becomes

$$ds^2 = \Delta^{\delta} dt^2 - \Delta^{1-\delta}\left[\Sigma^{1-\delta^2}\Delta^{\delta^2-1}\left(\frac{dr^2}{\Delta} + r^2 d\theta^2\right) + r^2\sin^2\theta d\varphi^2\right] , \quad (9.2.15)$$

$$\Delta = 1 - \frac{2\sigma}{r} , \quad \Sigma = 1 - \frac{2\sigma}{r} + \frac{\sigma^2}{r^2}\sin^2\theta . \quad (9.2.16)$$

An analysis of the Newtonian limit of this metric shows that it corresponds to a thin rod source of constant density δ, uniformly distributed along the $z-$axis from $z_1 = -\sigma$ to $z_2 = \sigma$. In the literature, usually a different constant γ is used instead of δ, and, therefore, the Zipoy-Voorhees metric in the representation (9.2.15) is known as the Gamma-metric.

9.3 The $q-$metric

If we start from the Schwarzschild solution and apply a Zipoy-Voorhees transformation with $\delta = 1 + q$, we obtain the metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{1+q} dt^2$$

$$- \left(1 - \frac{2m}{r}\right)^{-q}\left[\left(1 + \frac{m^2\sin^2\theta}{r^2 - 2mr}\right)^{-q(2+q)}\left(\frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2\right) + r^2\sin^2\theta d\varphi^2\right] . \quad (9.3.1)$$

In [174], it was shown that this is the simplest generalization of the Schwarzschild solution that contains the additional parameter q, which describes the deformation of the mass distribution. In fact, this can be shown explicitly by calculating the invariant Geroch multipoles [18]. The lowest mass multipole moments M_n, $n = 0, 1, \ldots$ are given by

$$M_0 = (1 + q)m , \quad M_2 = -\frac{m^3}{3} q(1 + q)(2 + q) , \quad (9.3.2)$$

whereas higher moments are proportional to mq and can be completely rewritten in terms of M_0 and M_2. Accordingly, the arbitrary parameters m and q determine the mass and quadrupole which are the only independent multipole moments of the solution. In the limiting case $q = 0$ only the monopole $M_0 = m$ survives, as in the Schwarzschild spacetime. In the limit $m = 0$, with $q \neq 0$, all moments vanish identically, implying that no mass distribution is present and the spacetime must be flat. The same is true in the limiting case $q \to -1$ which corresponds to the Minkowski metric. Notice that all odd multipole moments are zero because the solution possesses an additional reflection symmetry with respect to the equatorial plane $\theta = \pi/2$.

The deformation is described by the quadrupole moment M_2 which is positive for a prolate source and negative for an oblate source. This implies that the parameter q can be either positive or negative. Since the total mass M_0 of the source must be positive, we must assume that $q > -1$ for positive values of m, and $q < -1$ for negative values of m. We conclude that the above metric can be used to describe the exterior gravitational field of a static positive mass M_0 with a positive or negative quadrupole moment M_2. The behavior of the mass moments depends on the explicit value of q. We will refer to the metric (9.3.1) as to the $q-$metric to the emphasize its physical significance as the simplest solution with an independent quadrupole moment.

The Kretschmann scalar

$$K = R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau} = \frac{16m^2(1+q)^2}{r^{4(2+2q+q^2)}}\frac{(r^2-2mr+m^2\sin^2\theta)^{2(2q+q^2)-1}}{(1-2m/r)^{2(q^2+q+1)}}L(r,\theta)\,, \tag{9.3.3}$$

$$\begin{aligned} L(r,\theta) = \quad & 3(r-2m-qm)^2(r^2-2mr+m^2\sin^2\theta) \\ & + m^2q(2+q)\sin^2\theta[m^2q(2+q)+3(r-m)(r-2m-qm)], \end{aligned} \tag{9.3.4}$$

can be used to explore the singularities of the spacetime. We can see that only the cases $q = -1$ and $m = 0$ are free of singularities. In fact, as noticed above, these cases correspond to a flat spacetime. A singularity exists at $r \to 0$ for any value of q and m. In fact, for negative values of m this is the only singular point of the spacetime which thus describes a naked singularity situated at the origin. In the range $2q(2+q) < 1$ with $m > 0$, there is singularity at those values of r that satisfy the condition $r^2 - 2mr + m^2\sin^2\theta = 0$, i.e, these singularities are all situated inside a sphere of radius $2m$. Finally, an additional singularity appears at the radius $r = 2m$ which, according to the metric (9.3.1), is also a horizon in the sense that the norm of the timelike Killing tensor vanishes at that radius. Outside the hypersurface $r = 2m$ no additional horizon exists, indicating that the singularities situated at $r = 2m$ and inside this sphere are naked. This result is in accordance with the black holes uniqueness theorems which establishes that the only compact object possessing an event horizon that covers the inner singularity is described by the Schwarschild solution.

The position of the outer most singularity situated at $r_s = 2m$ can be evaluated by using the expression for the invariant mass, i.e, $r_s = 2M_0/(1+q)$. In astrophysical compact objects, one expects that the quadrupole moment is small so that $q << 1$. Then the radius r_s of the singular sphere is of the order of magnitude of the Schwarzschild radius $2M_0$ of a compact object of mass M_0, which is usually located well inside the matter distribution. It follows that it should be possible to "eliminate" the naked singular sphere by finding the interior metric of an appropriate matter distribution that would fill completely the singular regions.

9.4 The interior metric

It is very difficult to find physically reasonable solutions in general relativity, because the underlying differential equations are highly nonlinear with very strong couplings between the metric functions. In [176], a numerical solution was derived for a particular choice of the interior static and axially symmetric line element

$$ds^2 = f dt^2 - \frac{e^{2k_0}}{f}\left(\frac{dr^2}{h} + d\theta^2\right) - \frac{\mu^2}{f} d\varphi^2 \,, \tag{9.4.1}$$

where

$$e^{2k_0} = (r^2 - 2mr + m^2 \cos^2\theta) e^{2k(r,\theta)} \,, \tag{9.4.2}$$

and $f = f(r,\theta)$, $h = h(r)$, and $\mu = \mu(r,\theta)$. To solve Einstein's equations with a perfect fluid source, the pressure and the energy must be functions of the coordinates r and θ. However, if we assume that $\rho =$ const, the complexity of the corresponding differential equations reduces drastically:

$$p_r = -\frac{1}{2}(p+\rho)\frac{f_r}{f} \,, \quad p_\theta = -\frac{1}{2}(p+\rho)\frac{f_\theta}{f} \,, \tag{9.4.3}$$

$$\mu_{rr} = -\frac{1}{2h}\left(2\mu_{\theta\theta} + h_r\mu_r - 32\pi p \frac{\mu e^{2\gamma_0}}{f}\right) \,, \tag{9.4.4}$$

$$f_{rr} = \frac{f_r^2}{f} - \left(\frac{h_r}{2h} + \frac{\mu_r}{\mu}\right) f_r + \frac{f_\theta^2}{hf} - \frac{\mu_\theta f_\theta}{\mu h} - \frac{f_{\theta\theta}}{h} + 8\pi\frac{(3p+\rho)e^{2\gamma_0}}{h} \,. \tag{9.4.5}$$

In addition, the function k is determined by a set of two partial differential equations which can be integrated by quadratures once f and μ are known. The integrability condition of these partial differential equations turns out to be satisfied identically by virtue of the remaining field equations. It is then possible to perform a numerical integration by imposing appropriate initial conditions. In particular, if we demand that the metric functions and the pressure are finite at the axis, it is possible to find a class of numerical solutions which can be matched with the exterior $q-$metric with a pressure that vanishes at the matching surface.

A different approach consists in postulating the interior line element and evaluating the energy-momentum tensor from the Einstein equations. This method was first proposed by Synge and has been applied very intensively to find approximate interior solutions [177, 178]. To find the interior metric we proceed as follows. Consider the case of a slightly deformed mass. This means that the parameter q can be considered as infinitesimal and this fact can be used to construct the interior metric functions. In fact, to the zeroth-order an interior line element can be obtained just by assuming that instead of the constant m, the function $\mu(r)$ appears in the metric. In the case of the

$q-$metric, the functions entering the metric can be separated as

$$\left(1-\frac{2m}{r}\right)\left(1-\frac{2m}{r}\right)^{c_1 q+c_2 q^2}, \tag{9.4.6}$$

where c_1 and c_2 are constants. Then, to the first order in q, we can approximate this combination of functions as

$$\left(1-\frac{2\mu}{r}\right)\left[1+c_1 q\alpha(r)\right]. \tag{9.4.7}$$

Following this procedure, an appropriate interior line element for the $q-$metric (9.3.1) can be expressed as

$$ds^2=\left(1-\frac{2\mu}{r}\right)(1+q\alpha)dt^2-(1+q\alpha+q\beta)\left(\frac{dr^2}{1-\frac{2\mu}{r}}+r^2d\theta^2\right)-r^2\sin^2\theta(1-q\alpha)d\varphi^2\,, \tag{9.4.8}$$

where $\mu=\mu(r)$, $\alpha=\alpha(r)$ and $\beta=\beta(r,\theta)$.

Let us now consider the boundary conditions at the matching surface by comparing the above interior metric (9.4.8) with the $q-$metric to first order in q, i.e.,

$$\begin{aligned} ds^2 = & \left(1-\frac{2m}{r}\right)\left[1+q\ln\left(1-\frac{2m}{r}\right)\right]dt^2-r^2\left[1-q\ln\left(1-\frac{2m}{r}\right)\right]d\varphi^2 \\ - & \left[1+q\ln\left(1-\frac{2m}{r}\right)-2q\ln\left(1-\frac{2m}{r}+\frac{m^2}{r^2}\sin^2\theta\right)\right]\left(\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right) \end{aligned} \tag{9.4.9}$$

A comparison of the metrics (9.4.8) and (9.4.9) shows that they coincide at the matching radius $r=r_m$, if the conditions

$$\mu(r_m)=m\,,\quad \alpha(r_m)=\ln\left(1-\frac{2m}{r_m}\right),\quad \beta(r_m,\theta)=-2\ln\left(1-\frac{2m}{r_m}+\frac{m^2}{r_m^2}\sin^2\theta\right), \tag{9.4.10}$$

are satisfied. Notice that we reach the desired matching by fixing only the radial coordinate as $r=r_m$, but it does not mean that the matching surface is a sphere. Indeed, the shape of matching surface is determined by the conditions $t=const$ and $r=r_m$ which, according to Eq.(9.4.9), determine a surface with explicit $\theta-$dependence.

Finally, we calculate the Einstein tensor G^μ_ν and find that the only non-diagonal component G^r_θ implies the equation

$$\beta_\theta=\frac{r\cos\theta}{\sin\theta}\frac{(r-2\mu)(2\alpha_r+\beta_r)}{r\mu_r-r+\mu} \tag{9.4.11}$$

which partially determines the function $\beta(r,\theta)$. Furthermore, the energy conditions $T_t^t \geq 0$ and $T_t^t - T_r^r \geq 0$ lead to

$$q\left[\beta_{\theta\theta} + r\beta_{rr}(r-2\mu) - \beta_r(r\mu_r + \mu - r) + 4\mu_r(\alpha+\beta) - 2r\alpha_r\right] \geq 4\mu_r \ , \quad (9.4.12)$$

$$q\left[\beta_{\theta\theta} + r\beta_{rr}(r-2\mu) + 4\beta\mu_r + 4\mu\alpha_r + \cot\theta\beta_\theta\right] \geq 0 \ , \quad (9.4.13)$$

respectively. A preliminary numerical analysis of these equations shows that it is possible to find solutions that satisfy the boundary conditions and the energy conditions simultaneously. In fact, the pressure and the energy density obtained in this way show a profile that is in accordance with the physical expectations. We conclude that by applying Synge's method it is possible to find physically reasonable interior solutions for the exterior $q-$metric. However, it will be necessary to further analyze the numerical solutions to find the ranges of boundary values of the main physical parameters that one can use to obtain physical configurations.

9.5 Final remarks

In this work, we discussed the Zipoy-Voorhees metric in different coordinate representations. We propose a different interpretation in terms of the quadrupole parameter q and, therefore, we designate it the $q-$metric. We found all the singularities of the underlying spacetime. It was shown that only the Minkowski spacetime is free of curvature singularities, and that only the Schwarzschild spacetime possesses an event horizon that separates the inner singularity from the exterior spacetime. For all the remaining cases with non-vanishing quadrupole moment, it was established that naked singularities are present inside a sphere with a radius which is of the same order or magnitude of the Schwarzschild radius for astrophysical compact objects.

We investigated the possibility of finding interior metrics that could be matched with the exterior $q-$metric. In particular, we postulated a specific line element for the interior metric and used Synge's method to derive the matter distribution. The matching conditions and the energy conditions were calculated explicitly in the case of a deformed source with a small quadrupole parameter. It was shown that the resulting system of differential equations is compatible and that particular solutions can be calculated by using numerical methods.

The resulting system of differential equations for the functions of the interior metric indicates that one can try to find analytical solutions, at least in the case of a slightly deformed mass distribution. To do this, it will be necessary to investigate in detail the mathematical properties of the differential equations. This is a task for future investigations.

Moreover, we expect to apply the same method in the case of rotating sources. The rotating $q-$metric was derived in [175], but no attempts have been made to investigate its physical properties and the possibility of matching it with a suitable interior metric.

10 Conformastatic disk-haloes in Einstein-Maxwell gravity

10.1 Introduction

A large number of galaxies and other astrophysical systems have extended mass distributions surrounded by a material halo. For practical reasons one can assume that many of these systems preserve axial symmetry and, therefore, they can be modeled in terms of relativistic thin disks with exterior halos, in particular when the gravity field is strong enough. Disks may also be used to model accretion disks, galaxies in thermodynamic equilibrium and the superposition of a black hole and a galaxy. Disk sources for stationary axially symmetric spacetimes with magnetic fields are also of astrophysical importance mainly in the study of neutron stars, white dwarfs and galaxy formation. To describe the gravitational and electromagnetic fields of such configurations we will use general relativity and Maxwell's theory. Consequently, we are interested in deriving and analyzing exact solutions of the Einstein-Maxwell equations. On the other hand, the study of axially symmetric solutions of the Einstein and Einstein-Maxwell field equations corresponding to shells and disk-like configurations of matter, apart from its astrophysical relevance, has a clear purely mathematical interest.

Exact solutions that have relativistic static thin disks as their sources were first studied by Bonnor and Sackfield [179] and Morgan and Morgan [180, 181]. Subsequently, several classes of exact solutions corresponding to static [182, 183, 184, 185, 186, 187, 188, 189, 190, 191] and stationary [192, 193, 194, 195, 196] thin disks have been obtained by different authors. The superposition of a static or stationary thin disk with a black hole has been considered in [197, 198, 199, 200, 201, 202, 203, 204, 205]. Thin disks around static black holes in a magnetic field have been studied in [206]. Relativistic disks embedded in an expanding Friedman-Lemaître-Roberton-Walker universe have been studied in [207], perfect fluid disks with halos in [208]. Furthermore, the stability of thin disks models has been investigated using a first order perturbation of the energy-momentum tensor in [209]. On the other hand, thin disks have been discussed as sources for Kerr-Newman fields [210, 211], magnetostatic axisymmetric fields [212, 213], and conformastatic and conformastationary metrics [214, 215, 222]. Also, models of electrovacuum static counterrotating dust disks were presented in [216], charged perfect fluid disks were studied in [217], and charged perfect fluid disks as sources of static and Taub-NUT-type spacetimes in [218, 219]. Also, monopole and dipole layers in curved spacetimes were analyzed in [220], and electromagnetic sources distributed on shells in a Schwarzschild background in [221].

Now, the thin disks with magnetic fields presented in [210, 211, 212] were obtained by means of the well-known "displace, cut and reflect" method that introduces a discontinuity in the first-order derivative of an otherwise smooth solution. The result is a solution with a singularity of the delta-function type in the entire $z = 0$ hypersurface, and so it can be interpreted as an infinite thin disk. On the other hand, solutions that can be interpreted as thin disks of finite extension can be obtained if an appropriate coordinate system is introduced. A coordinate system that adapts naturally to a finite source and presents the required discontinuous behavior is given by the oblate spheroidal coordinates. Some examples of finite thin disks obtained from vacuum solutions expressed in these coordinates can be found in references [179, 180, 182, 185], and from electrovacuum solutions in reference [222].

In a previous work [222], we presented an infinite family of conformastatic axially symmetric charged dust disks of finite extension with well-behaved surface energy and charge densities. These disks have a charge density that is equal, up to a sign, to their energy density, and so they are examples of the commonly named "electrically counterpoised dust" equilibrium configuration. The energy density of the disks is everywhere positive and well-behaved, vanishing at the edge. Furthermore, since the energy density of the disks is everywhere positive and the disks are made of dust, all the models are in a complete agreement with all the energy conditions, a fact of particular relevance in the study of relativistic thin disks models. In the present paper, we extend these studies to obtain a model corresponding to a system that is composed of a thin disk and an exterior halo. The main purpose of this work is, then, to extend the previous electric field to include an electromagnetic field, and the previous "isolated" thin disk to include a thin disk-halo system.

In this work, we present a relativistic model describing a thin disk surrounded by a halo in presence of an electromagnetic field. The model is obtained by solving the Einstein-Maxwell equations for a conformastatic spacetime, and by using the distributional approach under the assumption that the energy-momentum tensor can be expressed as the sum of two distributional contributions, one due to the electromagnetic part and the other one due to a "material" part. In this way, explicit expressions for the energy, pressure, electric current and electromagnetic field are obtained for the disk region and for the halo region. In order to obtain the solutions, an auxiliary function is introduced that determines the functional dependence of the metric and the electromagnetic potential. It is also assumed that the auxiliary function depends explicitly on an additional function which is taken as a solution of the Laplace equation. A simple thin disk-halo model is obtained from the Kuzmin solutions of the Laplace equation. The energy-momentum tensor of the system agrees with all the energy conditions.

The plan of our paper is as follows. First, in Section 10.2, the conformastatic line element is considered. The procedure to obtain electromagnetostatic, axially symmetric, relativistic thin disks surrounded by a material halo is also summarized in this section. Section 10.3 introduces a functional relationship dependence between the metric and electromagnetic potentials and an auxiliary function in order to obtain a family of solutions of the Einstein-Maxwell equations in terms of convenient solutions of the Laplace equations, modeling relativistic thin disk-halo systems. Next, the eigenvalue problem for disk with halos is studied and a particular model of a disk with halo is obtained from the Kuzmin solutions of the Laplace equation. In Section 10.3.3, the four Riemann invariants and the electromagnetic invariants are studied and the behavior of the Kuzmin disk with halo is analyzed. Finally, Section 10.4, is devoted to a discussion of the results.

10.2 The Einstein-Maxwell equations and the thin-disk-halo system

In order to formulate the Einstein-Maxwell equations for conformastatic axially symmetric spacetimes corresponding to an electromagnetized system constituted by a thin disk and a halo surrounding the exterior of the disk, we first introduce coordinates $x^a = (t, \varphi, r, z)$ in which the metric tensor and the electromagnetic potential only depend on r and z. We assume that these coordinates are quasicylindrical in the sense that the coordinate r vanishes on the axis of symmetry and, for fixed z, increases monotonically to infinity, while the coordinate z, for r fixed, increases monotonically in the interval $(-\infty, \infty)$. The azimuthal angle φ ranges in the interval $[0, 2\pi)$, as usual, [180, 181]. We assume that there exists an infinitesimally thin disk, located at the hypersurface $z = 0$, so that the components of the metric tensor g_{ab} and the components of the electromagnetic potential A_a are symmetrical functions of z and their first derivatives have a finite discontinuity at $z = 0$, accordingly,

$$g_{ab}(r,z) = g_{ab}(r,-z), \qquad A_a(r,z) = A_a(r,-z) \tag{10.2.1}$$

in such a way that, for $z \neq 0$,

$$g_{ab,z}(r,z) = -g_{ab,z}(r,-z), \qquad A_{a,z}(r,z) = -A_{a,z}(r,-z). \tag{10.2.2}$$

The metric tensor and the electromagnetic potential are continuous at $z = 0$,

$$[g_{ab}] = g_{ab}\big|_{z=0^+} - g_{ab}\big|_{z=0^-} = 0, \tag{10.2.3a}$$

$$[A_a] = A_a\big|_{z=0^+} - A_a\big|_{z=0^-} = 0, \tag{10.2.3b}$$

whereas the discontinuity in the derivatives of the metric tensor and the electromagnetic potential can be written, respectively, as

$$\gamma_{ab} = [g_{ab,z}], \tag{10.2.4a}$$

$$\zeta_a = [A_{a,z}], \tag{10.2.4b}$$

where the reflection symmetry with respect to $z = 0$ has been used. Then, by using the distributional approach [223, 224, 225] or the junction conditions on the extrinsic curvature of thin shell [226, 227, 228], we can write the metric and the electromagnetic potential as

$$g_{ab} = g^+_{ab}\theta(z) + g^-_{ab}\{1 - \theta(z)\}, \tag{10.2.5a}$$

$$A_a = A^+_a\theta(z) + A^-_a\{1 - \theta(z)\}, \tag{10.2.5b}$$

and thus the Ricci tensor reads

$$R_{ab} = R^+_{ab}\theta(z) + R^-_{ab}\{1 - \theta(z)\} + H_{ab}\delta(z), \tag{10.2.6}$$

where $\theta(z)$ and $\delta(z)$ are, respectively, the Heaveside and Dirac distributions with support on $z = 0$. Here $g^{\pm}_{ab}$ and $R^{\pm}_{ab}$ are the metric tensors and the Ricci tensors of the $z \geq 0$ and $z \leq 0$ regions, respectively, and

$$H_{ab} = \frac{1}{2}\{\gamma^z_a\delta^z_b + \gamma^z_b\delta^z_a - \gamma^c_c\delta^z_a\delta^z_b - g^{zz}\gamma_{ab}\}, \tag{10.2.7}$$

where all the quantities are evaluated at $z = 0^+$. In agreement with (10.2.6), the energy-momentum tensor and the electric current density can be expressed as

$$T_{ab} = T^+_{ab}\theta(z) + T^-_{ab}\{1 - \theta(z)\} + Q_{ab}\delta(z), \tag{10.2.8a}$$

$$J_a = J^+_a\theta(z) + J^-_a\{1 - \theta(z)\} + I_a\delta(z), \tag{10.2.8b}$$

where $T^{\pm}_{ab}$ and $J^{\pm}_{ab}$ are the energy-momentum tensors and electric current density of the $z \geq 0$ and $z \leq 0$ regions, respectively. Moreover, Q_{ab} and I_a represent the part of the energy-momentum tensor and the electric current density corresponding to the disk-like source.

To describe the physical properties of an electromagnetized system constituted by a thin disk surrounded by an exterior halo, $T^{\pm}_{ab}$ in (10.2.8a) can be written as

$$T^{\pm}_{ab} = E^{\pm}_{ab} + M^{\pm}_{ab}, \tag{10.2.9}$$

where $E^{\pm}_{ab}$ is the electomagnetic energy-momentum tensor

$$E_{ab} = \frac{1}{4\pi}\left\{F_{ac}F_{b}{}^{c} - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right\}, \tag{10.2.10}$$

with $F_{ab} = A_{b,a} - A_{a,b}$ and $M^{\pm}_{ab}$ is an unknown "material" energy-momentum tensor (MEMT) to be obtained. Accordingly, the Einstein-Maxwell equations, in geometrized units such that $c = 8\pi G = \mu_0 = \epsilon_0 = 1$, are equivalent to the system of equations

$$G^{\pm}_{ab} = R^{\pm}_{ab} - \frac{1}{2}g_{ab}R^{\pm} \;=\; E^{\pm}_{ab} + M^{\pm}_{ab}, \tag{10.2.11}$$

$$H_{ab} - \frac{1}{2}g_{ab}H \;=\; Q_{ab}, \tag{10.2.12}$$

$$\hat{F}^{ab}_{\pm\;\;,b} \;=\; \hat{J}^{a}_{\pm}, \tag{10.2.13}$$

$$[\hat{F}^{ab}]n_{_b} \;=\; \hat{I}^{a}, \tag{10.2.14}$$

where $H = g^{ab}H_{ab}$ and $\hat{a} = \sqrt{-g}a$, being g the determinant of the metric tensor. Here, "[ ]" in the expressions $[\hat{F}^{ab}]$ denotes the jump of $\hat{F}^{ab}$ across of the surface $z = 0$ and $n_{_b}$ denotes an unitary vector in the direction normal to it.

Now, in order to obtain explicit forms for the Einstein-Maxwell equations corresponding to the elctromagnetized disk-halo system, we take the metric tensor as given by the conformastatic line element [84, 1]

$$\mathrm{d}s^2 = -\mathrm{e}^{2\phi}\mathrm{d}t^2 + \mathrm{e}^{2\psi}[r^2\mathrm{d}\varphi^2 + \mathrm{d}r^2 + \mathrm{d}z^2], \tag{10.2.15}$$

where the metric functions ϕ and ψ depend only on r and z, and the electromagnetic potential as

$$A_{\alpha} = (A_0, A, 0, 0), \tag{10.2.16}$$

where it is also assumed that the electric potential A_0 and the magnetic potential A are independent of t.

Furthermore, from Eq.(10.2.13) we have for the current density in these regions,

$$\hat{J}^0_{\pm} = -re^{\psi-\phi}\{\nabla^2 A_0 - \nabla(\phi-\psi)\cdot\nabla A_0\}, \quad (10.2.17a)$$

$$\hat{J}^1_{\pm} = -r^{-1}e^{\phi-\psi}\{\nabla^2 A + \nabla(\phi-\psi)\cdot\nabla A - \frac{2}{r}A_{,r}\}, \quad (10.2.17b)$$

where, as we know, $\hat{J}^a_{\pm} = J^a_{\pm}\sqrt{-g}$. The "true" surface energy-momentum tensor of the disk, S_{ab}, can be obtained through the relation

$$S_{ab} = \int Q_{ab}\,\delta(z)\,ds_n = \sqrt{g_{zz}}\,Q_{ab}, \quad (10.2.18)$$

where $ds_n = \sqrt{g_{zz}}\,dz$ is the "physical measure" of length in the direction normal to the $z = 0$ plane. Accordingly, for the metric (10.2.15), the nonzero components of S_{ab} are are given by

$$S^0_{\ 0} = 4e^{-\psi}\psi_{,z}, \quad (10.2.19a)$$

$$S^1_{\ 1} = 2e^{-\psi}(\phi+\psi)_{,z}, \quad (10.2.19b)$$

$$S^2_{\ 2} = 2e^{-\psi}(\phi+\psi)_{,z}, \quad (10.2.19c)$$

where all the quantities are evaluated at $z = 0^+$. The "true" current density on the surface of the disk, $\mathcal{J}_a$, can be obtained through the relation

$$\mathcal{J}^a = \int I^a\delta(z)\sqrt{g_{zz}}dz, \quad (10.2.20)$$

or explicitly

$$\mathcal{J}^0 = e^{-(\psi+2\phi)}[A_{0,z}], \quad (10.2.21a)$$

$$\mathcal{J}^1 = -r^{-2}e^{-3\psi}[A_{,z}], \quad (10.2.21b)$$

where all the quantities are evaluated on the surface of the disk, and $[A_{a,z}]$ denotes the jump of the derivative of A_a across the surface $z = 0$. Now, in order to analyze the physical characteristics of the system it is convenient to express the energy momentum tensor T_{ab} and the electric current density in terms of an orthonormal tetrad. We will use the tetrad of the "locally static observers" (LSO) [215], i.e., observers at rest with respect to infinity, which is given by

$$e_{(b)}^{\ a} = \{V^a, W^a, X^a, Y^a\}, \quad (10.2.22)$$

where

$$\begin{aligned}
V^a &= = e_{(0)}^{\;a} = e^{-\phi}\delta_0^a, && (10.2.23a)\\
W^a &= = e_{(1)}^{\;a} = r^{-1}e^{-\psi}\delta_1^a, && (10.2.23b)\\
X^a &= = e_{(2)}^{\;a} = e^{-\psi}\delta_2^a, && (10.2.23c)\\
Y^a &= = e_{(3)}^{\;a} = e^{-\psi}\delta_3^a. && (10.2.23d)
\end{aligned}$$

In terms of this tetrad $M^{\pm}_{ab}$ and $\hat{J}^{\pm}_a$ can be expressed as

$$\begin{aligned}
M^{ab}_{\pm} &= M^{\pm}_{(0)(0)}V^aV^b + M^{\pm}_{(1)(1)}W^aW^b + M^{\pm}_{(2)(2)}X^aX^b \\
&\quad + M^{\pm}_{(3)(3)}Y^aY^b - M^{\pm}_{(0)(1)}\left\{V^aW^b + W^aV^b\right\} && (10.2.24a)\\
&\quad + M^{\pm}_{(2)(3)}\left\{X^aY^b + Y^aX^b\right\},\\
\hat{J}^a_{\pm} &= -\hat{J}^{\pm}_{(0)}V^a + \hat{J}^{\pm}_{(1)}W^a, && (10.2.24b)
\end{aligned}$$

In the same way, by using the LSO tetrad the surface energy-momentum tensor and the surface current density of the disk as well as the electric current on the disk can be written in the canonical form as

$$\begin{aligned}
S^{ab} &= S_{(0)(0)}V^aV^b + S_{(1)(1)}W^aW^b + S_{(2)(2)}X^aX^b, && (10.2.25a)\\
\mathcal{J}^a &= -\mathcal{J}_{(0)}V^a + \mathcal{J}_{(1)}W^a, && (10.2.25b)
\end{aligned}$$

with

$$\begin{aligned}
S_{(0)(0)} &= -4e^{-\psi}\psi_{,z}, && (10.2.26a)\\
S_{(1)(1)} &= 2e^{-\psi}(\phi+\psi)_{,z} = S_{(2)(2)}, && (10.2.26b)\\
\mathcal{J}_{(0)} &= -e^{-(\psi+\phi)}[A_{0,z}], && (10.2.26c)\\
\mathcal{J}_{(1)} &= -r^{-1}e^{-2\psi}[A,z], && (10.2.26d)
\end{aligned}$$

where we have used (10.2.19) and (10.2.20) and again all the quantities are evaluated at $z = 0^+$.

10.3 Thin disk with an electromagnetized material halo

In the precedent section, we discussed a generalized formalism in which conformastatic axially symmetric solutions of the Einstein-Maxwell can be interpreted in terms of a thin disk placed at the surface $z = 0$ surrounded by a distribution of electrically charged matter located in the $z \geq 0$ and $z \leq 0$ regions, whose physical properties can be studied by analyzing the behavior of $S_{(a)(b)}$, $\mathcal{J}_{(a)}$, $M^{\pm}_{(a)(b)}$ and $J^{\pm}_{(a)}$. To this end, it is necessary to "choose" a convenient explicit form for the metric. It turns out that the assumption $\psi = -\phi$ leads to a considerable simplification of the problem. Indeed, the equations are

$$M^{\pm}_{(0)(0)} = \frac{1}{4} f^{-1}\{4f\nabla^2 f - 5\nabla f\cdot\nabla f - 2f\nabla A_0\cdot\nabla A_0 - 2r^{-2}f^3\nabla A\cdot\nabla A\}, \quad (10.3.1a)$$

$$M^{\pm}_{(0)(1)} = -r^{-1}f\nabla A_0\cdot\nabla A, \quad (10.3.1b)$$

$$M^{\pm}_{(1)(1)} = \frac{1}{4} f^{-1}\{\nabla f\cdot\nabla f - 2f\nabla A_0\cdot\nabla A_0 - 2r^{-2}f^3\nabla A\cdot\nabla A\}, \quad (10.3.1c)$$

$$M^{\pm}_{(2)(2)} = \frac{1}{4} f^{-1}\{-(f_{,r}^2 - f_{,z}^2) + 2f(A_{0,r}^2 - A_{0,z}^2) - 2r^{-2}f^3(A_{,r}^2 - A_{,z}^2)\}, \quad (10.3.1d)$$

$$M^{\pm}_{(3)(3)} = -M^{\pm}_{(2)(2)}, \quad (10.3.1e)$$

$$M^{\pm}_{(2)(3)} = -\frac{1}{2} f^{-1} f_{,r} f_{,z} + A_{0,r}A_{0,z} - r^{-2}f^2 A_{,r}A_{,z}, \quad (10.3.1f)$$

$$\hat{J}^{\pm}_{(0)} = rf^{1/2}\nabla\cdot(f^{-1}\nabla A_0), \quad (10.3.1g)$$

$$\hat{J}^{\pm}_{(1)} = -r^2 f^{-1/2}\nabla\cdot(r^{-2}f\nabla A). \quad (10.3.1h)$$

In the same way, for the non-zero components of the energy-momentum and the current density on the surface of the disk (10.2.26) we have, respectively,

$$S_{(0)(0)} = 4(f^{1/2})_{,z}, \quad (10.3.2)$$

and

$$\mathcal{J}_{(0)} = -[A_{0,z}], \quad (10.3.3a)$$

$$\mathcal{J}_{(1)} = -r^{-1}f[A_{,z}], \quad (10.3.3b)$$

where "[ ]" denotes the jump across of the disk, $f \equiv e^{2\phi}$ and all the quantities are evaluated on the surface of the disk. We will suppose that there is no electric current in the halo, i. e., we assume that $\hat{J}^{\pm}_{(\alpha)} \equiv 0$. Then, if $\hat{\mathbf{e}}_{\varphi}$ is a unit vector in the azimuthal direction and λ is any reasonable function independent of the azimuth, one has the identity

$$\nabla\cdot(r^{-1}\hat{\mathbf{e}}_{\varphi}\times\nabla\lambda) = 0. \quad (10.3.4)$$

Equation (10.3.1h) may be regarded as the integrability condition for the existence of the function λ defined by

$$r^{-2} f \nabla A = r^{-1} \hat{\mathbf{e}}_\varphi \times \nabla \lambda, \tag{10.3.5}$$

or, equivalently

$$-f^{-1} \nabla \lambda = r^{-1} \hat{\mathbf{e}}_\varphi \times \nabla A. \tag{10.3.6}$$

Hence, the identity (10.3.4) implies the equation

$$\nabla \cdot (f^{-1} \nabla \lambda) = 0 \tag{10.3.7}$$

for the new "potential" $\lambda(r, z)$. In order to obtain an explicit form of the metric and electromagnetic potential, we suppose that f, A_0 and A depend explicitly on λ. Then, we obtain from (10.3.1g)

$$(-f^{-1} f' A_0' + A_0'') \nabla \lambda \cdot \nabla \lambda + A_0' \nabla^2 \lambda = 0, \tag{10.3.8}$$

where $()'$ denotes derivatives respect to λ. Whereas, from (10.3.7), we have

$$-f^{-1} f' \nabla \lambda \cdot \nabla \lambda + \nabla^2 \lambda = 0. \tag{10.3.9}$$

Consequently, by inserting (10.3.9) into (10.3.8), we obtain $A_0'' = 0$, whose general solution is $A_0 = k_1 \lambda + k_2$, where k_1 and k_2 are constants. We now proceed to determine the function $\lambda(r, z)$. Let us assume in (10.3.9) the very useful simplification $f' f^{-1} = k$, where k is an arbitrary constant. Then, $f = k_3 e^{k\lambda}$, and

$$\nabla^2 \lambda = k \nabla \lambda \cdot \nabla \lambda, \tag{10.3.10}$$

where k_3 is a constant. Furthermore, if we now assume the existence of a function $U = k_4 e^{-k\lambda} + k_5$, with k_4 and k_5 arbitrary constants, then

$$\nabla^2 U = -k k_4 e^{-k\lambda} (\nabla^2 \lambda - k \nabla \lambda \cdot \nabla \lambda) = 0 \tag{10.3.11}$$

and, consequently, λ can be represented in terms of solutions of the Laplace equation:

$$e^{k\lambda} = \frac{k_4}{U - k_5}; \qquad \nabla^2 U = 0. \tag{10.3.12}$$

On the other hand, from Eq.(10.3.5) we obtain the following relationship between A and λ:

$$\nabla A = A_{,r} \hat{\mathbf{e}}_r + A_{,z} \hat{\mathbf{e}}_z = r f^{-1} \hat{\mathbf{e}}_\varphi \times (\lambda_{,r} \hat{\mathbf{e}}_r + \lambda_{,z} \hat{\mathbf{e}}_z), \tag{10.3.13}$$

that is, $A_{,r} = -rf^{-1}\lambda_{,z}$, and $A_{,z} = rf^{-1}\lambda_{,r}$, or, in terms of U, $A_{,r} = k_6 r U_{,z}$, and $A_{,z} = -k_6 r U_{,r}$, where $k_6 = 1/(kk_3k_4)$. Then, using this solution, for the nonzero components of $M^{\pm}_{(a)(b)}$ we have from (10.3.1):

$$M^{\pm}_{(0)(0)} = \frac{U_{,r}^2 + U_{,z}^2}{4(U-k_5)^2}\{3f - k_7\}, \tag{10.3.14a}$$

$$M^{\pm}_{(1)(1)} = \frac{U_{,r}^2 + U_{,z}^2}{4(U-k_5)^2}\{f - k_7\}, \tag{10.3.14b}$$

$$M^{\pm}_{(2)(2)} = -\frac{U_{,r}^2 - U_{,z}^2}{4(U-k_5)^2}\{f - k_7\}, \tag{10.3.14c}$$

$$M^{\pm}_{(2)(3)} = -\frac{U_{,r}U_{,z}}{2(U-k_5)^2}\{f - k_7\}, \tag{10.3.14d}$$

$$M^{\pm}_{(3)(3)} = -M^{\pm}_{(2)(2)}. \tag{10.3.14e}$$

Furthermore, for the non-zero components of the energy-momentum tensor and the current density on the surface of the disk we have from (10.3.2) and (10.3.3) respectively,

$$S_{(0)(0)} = 2(k_3k_4)^{1/2}\frac{U_{,z}}{(U-k_5)^{3/2}} \tag{10.3.15}$$

and

$$\mathcal{J}_{(0)} = \frac{k_1}{k}\left[\frac{U_{,z}}{U-k_5}\right], \tag{10.3.16a}$$

$$\mathcal{J}_{(1)} = \frac{[U_{,r}]}{k(U-k_5)}, \tag{10.3.16b}$$

with

$$f = \frac{k_3k_4}{U-k_5} \qquad \text{and} \qquad k_7 = \frac{2(k_1^2+1)}{k^2}.$$

As we can see, $S_{(0)(0)}$ is the only nonzero component of the surface energy-momentum tensor; we can then interpret it as the surface energy density of the disk $\epsilon(r)$ as seen from any the LSO. Likewise we interpret $\sigma(r) = \mathcal{J}_{(0)}$ and $\mathcal{I} = \mathcal{J}_{(1)}$ as the charge density and electric current density on the surface of the disk, respectively.

10.3.1 The eigenvalue problem for the energy-momentum tensor of the halo

When the tensor $M^{\pm}_{(a)(b)}$ in the LSO tetrad is diagonal, its interpretation is immediate. In our case, however, $M^{\pm}_{(2)(3)} \neq 0$ and, therefore, it is necessary to rewrite $M^{\pm}_{(a)(b)}$ in the canonical form. To this end, we must solve the eigenvalue problem for $M^{\pm}_{(a)(b)}$, i.e.,

$$M^{\pm}_{(a)(b)}\xi^{(b)}{}_{A} = \lambda_A \eta_{(a)(b)}\xi^{(b)}{}_{A}\,, \tag{10.3.17}$$

and express the physically relevant quantities in terms of the eigenvalues and eigenvectors. The solution of the eigenvalue problem in the LSO orthonormal tetrad leads to the eigenvalues

$$\lambda_0 = -M^{\pm}_{(0)(0)}, \tag{10.3.18a}$$
$$\lambda_1 = M^{\pm}_{(1)(1)}, \tag{10.3.18b}$$
$$\lambda_{\pm} = \pm\sqrt{D}, \tag{10.3.18c}$$

and the corresponding eigenvectors are given by

$$\xi^{(a)}{}_{0} = V^{(a)} = (1,0,0,0), \tag{10.3.19a}$$
$$\xi^{(a)}{}_{1} = X^{(a)} = (0,1,0,0), \tag{10.3.19b}$$
$$\xi^{(a)}{}_{+} = Y^{(a)} = N(0,0,1,-\omega), \tag{10.3.19c}$$
$$\xi^{(a)}{}_{-} = Z^{(a)} = N(0,0,\omega,1), \tag{10.3.19d}$$

where

$$D = (M^{\pm}_{(2)(2)})^2 + (M^{\pm}_{(2)(3)})^2,$$
$$\omega = \frac{M^{\pm}_{(2)(2)} - \sqrt{D}}{M^{\pm}_{(2)(3)}},$$
$$N = \frac{1}{\sqrt{1+\omega^2}}.$$

In terms of the tetrad $\xi^{(a)}{}_{A} = \{V^{(a)}, X^{(a)}, Y^{(a)}, Z^{(a)}\}$, the tensor $M^{\pm}_{(a)(b)}$ can be written in the canonical form

$$M^{\pm}_{(a)(b)} = \varepsilon V_{(a)}V_{(b)} + p_1 X_{(a)}X_{(b)} + p_2 Y_{(a)}Y_{(b)} + p_3 Z_{(a)}Z_{(b)}. \tag{10.3.20}$$

Consequently, we can interpret ε as the energy density of the halo and p_1, p_2 and p_3 as the pressure in the principal directions of the halo. So, we have that the energy density of the halo is given by

$$\varepsilon = M^{\pm}_{(0)(0)} \tag{10.3.21}$$

whereas for the principal pressure we have

$$p = p_1 = p_2 = -p_3 = M^{\pm}_{(1)(1)}, \tag{10.3.22}$$

and

$$\langle p \rangle = \frac{p_1 + p_2 + p_3}{3} = \frac{p}{3} \tag{10.3.23}$$

is the average value of the pressure.

10.3.2 The Kuzmin-like solution

One can find many different models for a relativistic thin disk surrounded by a material electromagnetized halo by choosing different kind of solutions U of the Laplace equation. Let us consider the particular case of Kuzmin's solution [229, 230]

$$U = -\frac{m}{\sqrt{r^2 + (|z| + a)^2}}, \qquad (a, m > 0). \tag{10.3.24}$$

in which, at points with $z < 0$, U is identical to the potential of a point mass m located at the point $(r, z) = (0, -a)$; and when $z > 0$, U coincides with the potential generated by a point mass at $(0, a)$. Hence $\nabla^2 U$ must vanish everywhere except on the plane $z = 0$. By applying Gausss theorem to a flat volume that contains a small portion of the plane $z = 0$, we conclude that U is generated by the surface density of a Newtonian mass

$$\rho(r, z = 0) = \frac{am}{2\pi (r^2 + a^2)^{3/2}}. \tag{10.3.25}$$

Then, in order to have an asymptotically flat spacetime at infinity, we must take $k_5 = -1$ in Eq.(10.3.12). Furthermore, without loss of generality we can choose the constants $k_3 = k_4 = 1$. Then, for the metric potential we have

$$e^{2\phi} = \frac{\sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2}}{\sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2} - \tilde{m}}, \tag{10.3.26}$$

where we introduced the dimensionless variables $\tilde{r} = r/a$, $\tilde{z} = z/a$, and $\tilde{m} = m/a$. With this metric potential it is straightforward to calculate the dimensionless radial and axial components of the electric field, $\tilde{E}_r = aE_r$ and $\tilde{E}_z = a^2 E_z$ and the dimensionless radial and axial components of the magnetic field, $\tilde{B}_r = B_r$ and $\tilde{B}_z = B_z$, giving

$$\tilde{E}_r = -\frac{k_1 \tilde{m}\tilde{r}}{k\left\{\tilde{r}^2 + (|\tilde{z}|+1)^2\right\}\left\{\sqrt{\tilde{r}^2 + (|\tilde{z}|+1)^2} - \tilde{m}\right\}}, \tag{10.3.27a}$$

$$\tilde{E}_z = -\frac{k_1 \tilde{m}|\tilde{z}|(\tilde{z}+1)}{k\tilde{z}\left\{\tilde{r}^2 + (|\tilde{z}|+1)^2\right\}\left\{\sqrt{\tilde{r}^2 + (|\tilde{z}|+1)^2} - \tilde{m}\right\}}, \tag{10.3.27b}$$

$$\tilde{B}_r = -\frac{\tilde{m}\tilde{r}^2}{k\left\{\tilde{r}^2 + (|\tilde{z}|+1)^2\right\}^{3/2}}, \tag{10.3.27c}$$

$$\tilde{B}_z = -\frac{\tilde{m}\tilde{r}|\tilde{z}|(\tilde{z}+1)}{k\tilde{z}\left\{\tilde{r}^2 + (|\tilde{z}|+1)^2\right\}^{3/2}}, \tag{10.3.27d}$$

were we have used the definitions $E_r = A_{0,r}, E_z = A_{0,z}, B_r = A_{,z}$ and $B_z = -A_{,r}$.

Substituting (10.3.24) into (10.3.15) and (10.3.16b) we obtain for the dimensionless energy and charge density on surface of the disk, $\tilde{\epsilon} = a\epsilon$, $\tilde{\sigma} = a\sigma$, respectively,

$$\tilde{\epsilon}(\tilde{r}) = \frac{2\tilde{m}}{(\tilde{r}^2+1)^{3/4}(\sqrt{\tilde{r}^2+1} - \tilde{m})^{3/2}}, \tag{10.3.28a}$$

$$\tilde{\sigma}(\tilde{r}) = \frac{2k_1\tilde{m}}{k(\tilde{r}^2+1)(\sqrt{\tilde{r}^2+1} - \tilde{m})}. \tag{10.3.28b}$$

Substituting (10.3.24) into (10.3.21) and (10.3.22) allow us to write the dimensionless energy of the halo $\tilde{\varepsilon}(\tilde{r}) = a^2\varepsilon(r)$ and the dimensionless pressure of the halo $\tilde{p}(\tilde{r}) = a^2 p(r)$ as

$$\tilde{\varepsilon}(\tilde{r}) = \tilde{T}\left\{\frac{3\sqrt{\tilde{r}^2 + (|\tilde{z}|+1)^2}}{\sqrt{\tilde{r}^2 + (|\tilde{z}|+1)^2} - \tilde{m}} - k_7\right\}, \tag{10.3.29a}$$

$$\tilde{p}(\tilde{r}) = \tilde{T}\left\{\frac{\sqrt{\tilde{r}^2 + (|\tilde{z}|+1)^2}}{\sqrt{\tilde{r}^2 + (|\tilde{z}|+1)^2} - \tilde{m}} - k_7\right\}, \tag{10.3.29b}$$

where

$$\tilde{T} = \frac{\tilde{m}}{4\left\{\tilde{r}^2 + (|\tilde{z}|+1)^2\right\}\left\{\sqrt{\tilde{r}^2 + (|\tilde{z}|+1)^2} - \tilde{m}\right\}^2}.$$

To fix the values of the constants that enter the solution, we consider the energy conditions at infinity, $\tilde{r}, \tilde{z} \longrightarrow \infty$, and at the center of the body, $\tilde{r}, \tilde{z} \longrightarrow$

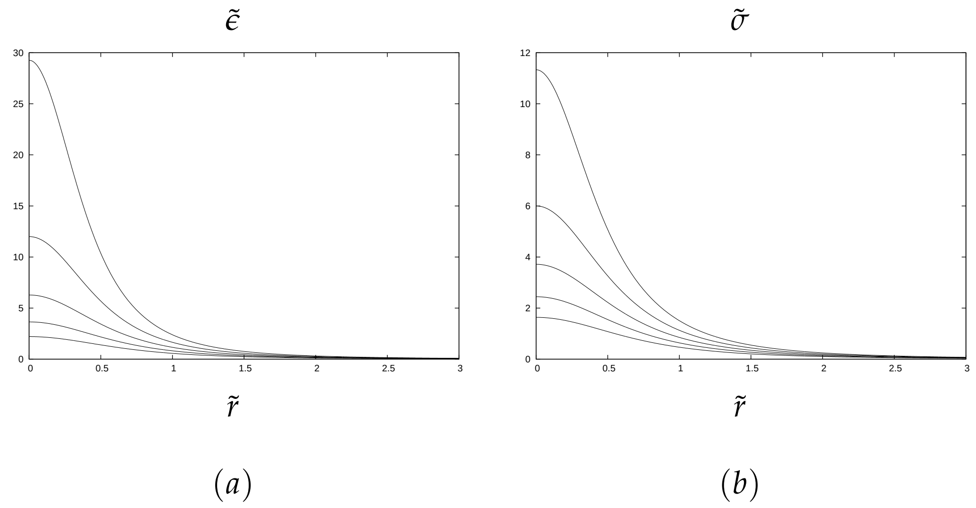


Figure 10.1: Dimensionless surface energy $\tilde{\epsilon}$ and charge $\tilde{\sigma}$ densities as a function of $\tilde{r}$. In each case, we plot $\tilde{\epsilon}(\tilde{r})$ and $\tilde{\sigma}(\tilde{r})$ for different values of the parameter $\tilde{m}$. First, we take $\tilde{m} = 0.45$ (the bottom curve in each plot) and then $0.55,\ 0.75$ and $\tilde{m} = 0.85$ (the top curve in each plot).

0. Then, it can be shown that all the energy conditions are satisfied at these extreme regions if

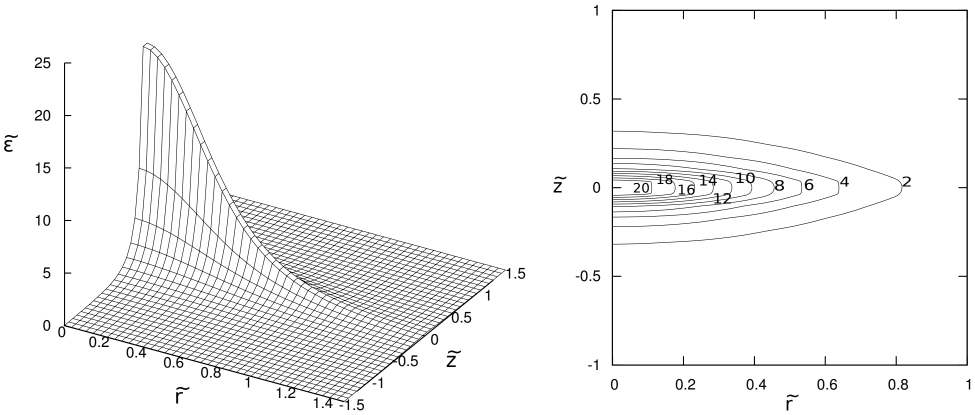
$$\tilde{m} < 1 \tag{10.3.30}$$

for the disk and, additionally,

$$-\sqrt{k_1^2+1} < k < \sqrt{k_1^2+1} \qquad (\text{that is} \quad k_7 < 2) \tag{10.3.31}$$

for the halo.

In Fig. 10.1(a), we show the dimensionless surface energy density on the disk $\tilde{\epsilon}$ as a function of $\tilde{r}$ and for for different values of the parameter $\tilde{m}$. First, we take $\tilde{m} = 0.45$ (the bottom curve in the plot) and then $0.55,\ 0.75$ and $\tilde{m} = 0.85$ (the top curve in the plot). It can be seen that the energy density is everywhere positive fulfilling the energy conditions. It can be observed that for all the values of $\tilde{m}$ the maximum of the energy density occurs at the center of the disk and that it vanishes sufficiently fast as r increases. It can also be observed that the energy density in the central region of the disk increases as the values of the parameter $\tilde{m}$ increases. We have also plotted in Fig. 10.1(b) the charge density $\tilde{\sigma}$ as a function of $\tilde{r}$. In each case, we plot $\tilde{\sigma}$ for different values of the parameter $\tilde{m}$. We observe that the electric charge density has a behavior similar to that of the energy. This is consistent with the fact that the mass and the charge are more densely concentrated in the center of the disk. We also computed these functions for other values of the parameters $\tilde{m}$ in the



(a) (b)

Figure 10.2: Surface plot an level curves of the energy density $\tilde{\varepsilon}$ on the exterior halo as a function of $\tilde{r}$ and $\tilde{z}$ with parameters $\tilde{m} = 0.75$ and $\tilde{k}_7 = 1$.

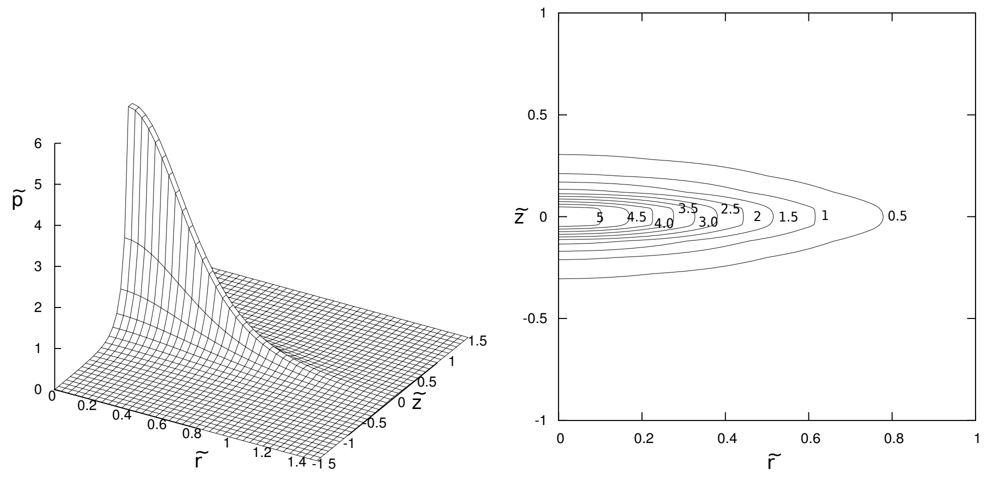
interval $(0, 1)$ and, in all cases, we found a similar behavior.

In Fig. 10.2(a) and in Fig. 10.2(b), we illustrate the behavior of the surfaces and level curves of the matter density in the halo around of the disk ($r \geq 0, z \geq 0$) for parameters $\tilde{m} = 075$ and $k_7 = 1$. We can see that the energy is everywhere positive, its maximum occurs around of the center of the disk, and it vanishes sufficiently fast as r increases.

Notice that in the limit $m \to 0$, the gravitational and electromagnetic fields vanish identically, and the metric becomes flat. This is an important limiting case because it indicates that m determines the mass of the disk and the halo, and that the electromagnetic field exists only in connection with the disk-halo configuration. Moreover, when $k_1 = 0$, we obtain $A_0 = k_2 =$ const., a "purely magnetic" solution. Then, k_1 determines the electric charge of the distribution whereas $k_6 = 1/k$ must be associated with the magnetic field.

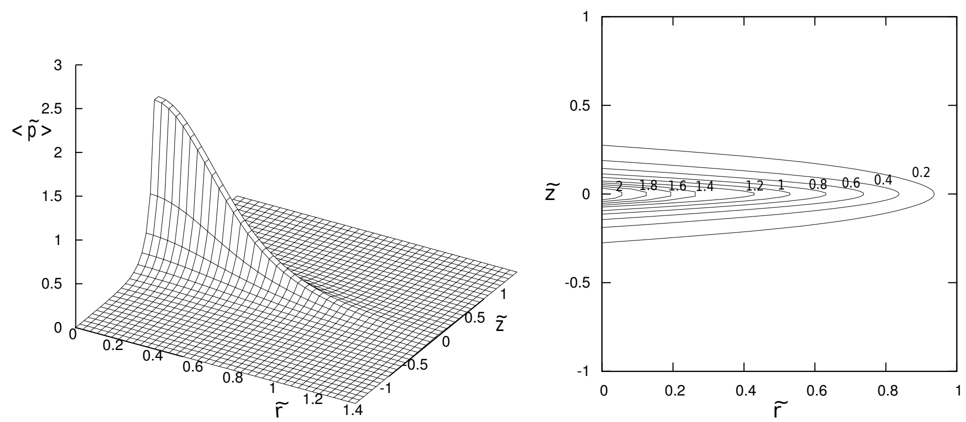
The energy condition $\tilde{m} < 1$ or, equivalently, $m < a$ imposes a maximum on the value of the mass parameter. Recall that in the Kuzmin solution a represents the distance along the axis between the equatorial plane and the point where the mass m is situated; accordingly, a can be interpreted as a parameter determining a proper length for the configuration. Then, the inequality $m/a < 1$ represents a condition on the "specific mass" of the system. This resembles the well-known Chandrasekhar limit for a spherically symmetric mass distribution, stating that the condition mass/radius $< 4/9$ must be satisfied in order to avoid gravitational collapse. The energy condition for the halo $k^2 < 1 + k_1^2$ also represents a relationship between the parameters that characterize the electric and magnetic properties of the system.

We conclude that all the relevant quantities show a physically reasonable



(a) (b)

Figure 10.3: Surface plot and level curves of the radial pressure $\tilde{p}$ on the exterior halo as a function of $\tilde{r}$ and $\tilde{z}$ with parameters $\tilde{m} = 0.75$ and $\tilde{k_7} = 1$.



(a) (b)

Figure 10.4: Surface plot an level curves of the average value of the radial pressure $< \tilde{p} >$ on the exterior halo as a function of $\tilde{r}$ and $\tilde{z}$ with parameters $\tilde{m} = 0.75$ and $\tilde{k_7} = 1$.

behavior within the allowed range of values of the parameters. This indicates that the solution presented here can be used to describe the gravitational field of a static thin disk surrounded by a material halo with a non-trivial electromagnetic field.

10.3.3 Singular behavior of the Kuzmin-like solution

In order to study the singularities that could be present in the space described by the Kuzmin-like solution derived in the last subsection, we compute the most important quadratic curvature scalars, namely, the Kretschmann $\mathcal{K}_I$, the Chern-Pontryagin $\mathcal{K}_{II}$ and the Euler invariants $\mathcal{K}_{III}$ defined as [231]

$$\mathcal{K}_I = R^{abcd}R_{abcd}, \tag{10.3.32a}$$

$$\mathcal{K}_{II} = [{}^*R]^{abcd}R_{abcd} = \frac{\epsilon^{ab}{}_{ij}R^{ijcd}R_{abcd}}{\sqrt{-g}}, \tag{10.3.32b}$$

$$\mathcal{K}_{III} = [{}^*R^*]^{abcd}R_{abcd} = \frac{\epsilon^{abij}\epsilon^{cdkl}R_{ijkl}R_{abcd}}{g}. \tag{10.3.32c}$$

As for the Maxwell field, we consider the electromagnetic invariants

$$\mathcal{F}_I = F_{ab}F^{ab} \tag{10.3.33a}$$

$$\mathcal{F}_{II} = F_{ab}F^{*ab}. \tag{10.3.33b}$$

Here $g = \det(g_{ab})$, ϵ^{abcd} is the Levi-Civita symbol and the asterisk denotes the dual operation. By using the solution (10.3.26) we can cast these invariants as

$$\mathcal{K}_I(\tilde{r},\tilde{z}) = \frac{\tilde{m}^2\left\{48\left[(|\tilde{z}|+1)^2+\tilde{r}^2\right] - 32\tilde{m}\sqrt{(|\tilde{z}|+1)^2+\tilde{r}^2} + 11\tilde{m}^2\right\}}{4\left[(|\tilde{z}|+1)^2+\tilde{r}^2\right]\left[\sqrt{(|\tilde{z}|+1)^2+\tilde{r}^2}-\tilde{m}\right]^6},$$

$$\mathcal{K}_{II}(\tilde{r},\tilde{z}) = 0,$$

$$\mathcal{K}_{III}(\tilde{r},\tilde{z}) = \frac{16\tilde{m}^2\left\{3[(|\tilde{z}|+1)^2+\tilde{r}^2] - 2\tilde{m}\sqrt{(|\tilde{z}|+1)^2+\tilde{r}^2}\right\}}{[(|\tilde{z}|+1)^2+\tilde{r}^2]\left[\sqrt{(|\tilde{z}|+1)^2+\tilde{r}^2}-\tilde{m}\right]^6},$$

whereas the electromagnetic invariants are

$$\mathcal{F}_I = \frac{2(1-k_1^2)\tilde{m}^2 a^2}{k^2\left[(|\tilde{z}|+1)^2+\tilde{r}^2\right]\left[\sqrt{(|\tilde{z}|+1)^2+\tilde{r}^2}-\tilde{m}\right]^2} \qquad (10.3.35a)$$

$$\mathcal{F}_{II} = \frac{-4k_1\tilde{m}^2 a^2}{k^2\left[(|\tilde{z}|+1)^2+\tilde{r}^2\right]\left[\sqrt{(|\tilde{z}|+1)^2+\tilde{r}^2}-\tilde{m}\right]^2}. \qquad (10.3.35b)$$

We see that there exists a singularity at the surface determined by the equation

$$(|\tilde{z}|+1)^2+\tilde{r}^2 = \tilde{m}^2\,, \qquad (10.3.36)$$

where all the non-trivial invariants diverge. The singularity equation allows real solutions only for $\tilde{m} > 1$. On the other hand, in the previous subsection we showed that the energy condition for the disk-halo configuration implies that $\tilde{m} < 1$. In fact, we can see from the expressions for the matter and charge density of the halo that they both diverge at the singular surface. Such divergencies are usually associated with the collapse of the gravitational configuration. This is an interesting result because it shows that the spacetime becomes singular as soon as the energy condition is violated. From a physical point this means that a disk-halo system can be constructed only if the condition $m/a < 1$ is satisfied. As noticed in the last subsection, this condition implies an upper bound on the value of the mass parameter m which is determined by the position of the mass along the symmetry axis. This, again, might be interpreted as a Chandrasekhar-like limit for the disk-halo system.

10.4 Concluding remarks

In this work, we derived a relativistic model describing a thin disk surrounded by a halo in presence of an electromagnetic field. The model was obtained by solving the Einstein-Maxwell equations on a particular confosmastatic spacetime in which only one independent metric function appears. For the energy-momentum tensor we used the distributional approach, and represented it as the sum of two distributional contributions, one due to the electromagnetic part and the other associated with a matter distribution. These assumptions allowed us to derive explicit expressions for the energy, pressure, electric current and electromagnetic field of the disk region and the halo as well. The main point of this approach is that it allows to write the gravitational and electromagnetic potentials in terms of a solution of Laplace's equation.

As a particular example, we used one of the simplest solutions of Laplace's equation, known as the Kuzmin solution, which contains two independent parameters, namely, the mass m and the parameter a that determines the proper length of the mass system. The resulting Kuzmin-like solution contains four independent parameters which determine the mass, proper length of the seed Kuzmin solution, electric charge and the magnetic field. The solution is asymptotically flat in general and turns out to be free of singularities if the ratio $m/a < 1$, an inequality that also guarantees the fulfillment of all the energy conditions. We interpret this condition as a Chandrasekhar-like limit for the disk-halo system.

Since all the relevant quantities show a physically reasonable behavior within the range $m/a < 1$, we conclude that the solution presented here can be used to describe the gravitational and electromagnetic fields of a think disk surrounded by a halo in the presence of an electromagnetic field.

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