Quantum Gravity and Unification Theories

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3. Brief Description of Quantum Gravity

3.1. The cosmological sector of Loop Quantum Gravity

In section "The cosmological sector of Loop Quantum Gravity" [30] a prescription is given to define in Loop Quantum Gravity the electric field operator related to the scale factor of an homogeneous and isotropic cosmological space-time. This procedure allows to link the fundamental theory with its cosmological implementation. In view of the conjugate relation existing between holonomies and fluxes, the edge length and the area of surfaces in the fiducial metric satisfy a duality condition. As a consequence, the area operator has a discrete spectrum also in Loop Quantum Cosmology. This feature makes the super-Hamiltonian regularization an open issue of the whole formulation.

The people involved in this line of research are Francesco Cianfrani and Giovanni Montani.

3.2. Semiclassical isotropization during a deSitter phase

In section "Semiclassical isotropization during a deSitter phase" semiclassical states for the Wheeler-DeWitt equation of a Bianchi type I model in the presence of a scalar field are analyzed [31]. It is outlined how this scheme can effectively describe more general situations, where the curvature of the Bianchi type IX model and a proper potential term for the scalar field are present. The introduction of a cosmological constant term accounts for the quasi-isotropization mechanism which bridges the proposed framework with a late isotropic phase. This result makes the semi-classical Bianchi I model a plausible scenario for the Universe pre-inflationary phase.

The people involved in this line of research are Francesco Cianfrani, Giovanni Montani and Marco Muccino.

3.3. Canonical Quantum Gravity without the time gauge

In section "Canonical Quantum Gravity without the time gauge" the description of gravitational degrees of freedom in a 4-bein formulation has been performed without any restriction on the local Lorentz frame. This analysis allows to investigate whether the SU(2) gauge structure, which arises in the Holst formulation with the time gauge, is an artifact of the gauge fixing or a proper feature of the gravitational field. The emergence of this SU(2) symmetry is one of the key points of Loop Quantum Gravity (LQG) and makes the Hamiltonian formulation for gravity close to the one of other fundamental interactions.

In particular, the analysis starts with vacuum gravity [23], for which it is outlined that, once second-class constraints are solved, boost degrees of freedom become non-dynamical and the Gauss constraints of a SU(2) gauge theory are inferred. This way, it is provided the extension of Ashtekar-Barbero-Immirzi connections to a generic local Lorentz frame. Then, matter fields are introduced and the corresponding Hamiltonian structure is analyzed. The case of a non-minimally coupled scalar field and the possible connection with $f(R)$ theories of gravity is discussed in [24]. The Immirzi field is added in [25] and it is demonstrated that the kinematical sector for such a model coincides with the one in which a minimally coupled scalar field is present, while the dynamics manifests interesting peculiar features. Among them, it is worth noting the possibility to explain the relaxation of the Immirzi field to a non-vanishing vacuum expectation value. Finally, the case in which spinor fields are present has been investigated[29].

The people involved in this line of research are Francesco Cianfrani and Giovanni Montani.

3.4. The problem of time in quantum gravity

In section "The problem of time in quantum gravity", several ways for defining a proper time variable in quantum gravity are discussed.

The so called Kučhar-Brown mechanism for a perfect fluid in the Schutz velocity potential representation is analyzed [11],[18]. This model is especially interesting in cosmology, since the Schutz fluid is a much more realistic description of the cosmological bath with respect to dust, especially when the cosmological singularity is approached. The Hamiltonian analysis is performed and second-class constraints are avoided by using Dirac brackets. Finally, a proper Hamiltonian can be defined by solving the super-Hamiltonian constraints and treating the Schutz fluid as a matter clock. Therefore, such a model of the cosmological bath provides a solution to the problem of time in Quantum Cosmology.

The people involved in this line of research are Francesco Cianfrani, Simone Zonetti and Giovanni Montani.

Moreover, the emergence of an evolutionary paradigm in canonical quantum geometro-dynamics from Hamiltonian equations is analyzed for Bianchi type I and type II cosmological models. The predicted dynamics are going to be compared with the behavior of Gaussian wave-packets expectation values.

The people involved in this line of research are Francesco Cianfrani, Marco Muccino and Giovanni Montani.

Finally, the extended phase space representation for gravity is developed for inhomogenous cosmological models. This representation is the starting point for a path integral formulation, in which the reference frame is expected to play a highly non-trivial role.

The people involved in this line of research are Francesco Cianfrani, Marco Renzelli and Giovanni Montani in collaboration with Tatyana Shestakova.

3.5. Quantum suppression of weak-anisotropies

In section "Quantum suppression of weak-anisotropies" we explain some results [21] of a research line in which a wave function of the inhomogeneous Mixmaster Universe, which has a meaningful probabilistic interpretation in agreement with the Copenhagen school, is obtained. To achieve this results, we followed an approach suggested by Vilenkin allowing us to write a Schröedinger-like equation of motion for the pure quantum part of the wave function of the Universe. Our result is that this wave function of the Universe is spread over all values of anisotropy near the cosmological singularity but, when the radius of the Universe grows, it is asymptotically peaked around the isotropic configuration. Therefore, the FRW cosmological model is naturally the privileged state when the Universe expands sufficiently and a semi-classical isotropization mechanism for the Universe naturally arises.

The people involved in this line of research are Riccardo Belvedere, Marco Valerio Battisti and Giovanni Montani.

3.6. Quantum behavior of the Universe for the small oscillations

In section "Quantum behavior of the Universe for the small oscillations", we deal with the analysis of the wave function of the inhomogeneous Mixmaster

Universe, in the case in which one of the two anisotropy parameter (β_-) is small on respect both the other one parameter (β_+) and the volume of the Universe (α). Following a WKB approximation we are able to regard α and β_+ as semi-classical variables, and β_- as a purely quantum one. The advantage in using this approximation is that it lead to a probabilistic interpretation in the Copenhagen school sense. What we obtain is that the quantum part of the wave function of the Universe seems approach the Taub model $\beta_- \rightarrow 0$ once the region in which the volume is not too big is investigated. At the same time we are now investigating the region in which the volume of the Universe is far from the cosmological singularity.

The people involved in this line of research are Riccardo Belvedere, Marco Valerio Battisti and Giovanni Montani.

3.7. Regularization and Quantization of Einstein-Cartan theory

In the Einstein-Cartan theory of torsion-free gravity coupling to massless fermions, the four-fermion interaction is induced and its strength is a function of the gravitational and gauge couplings, as well as the Immirzi parameter. We study the dynamics of the four-fermion interaction to determine whether effective bilinear terms of massive fermion fields are generated. Calculating one-particle-irreducible two-point functions of fermion fields, we identify three different phases and two critical points for phase transitions characterized by the strength of four-fermion interaction: (1) chiral symmetric phase for massive fermions in strong coupling regime; (2) chiral symmetric broken phase for massive fermions in intermediate coupling regime; (3) chiral symmetric phase for massless fermions in weak coupling regime. We discuss the scaling-invariant region for an effective theory of massive fermions coupled to torsion-free gravity in the *low-energy limit*.

The person involved in this line of research is She-Sheng Xue.

3.8. Quantum Regge Calculus of Einstein-Cartan theory

We study the Quantum Regge Calculus of Einstein-Cartan theory to describe quantum dynamics of Euclidean space-time discretized as a 4-simplices complex. Tetrad field $e_\mu(x)$ and spin-connection field $\omega_\mu(x)$ are assigned to each 1-simplex. Applying the torsion-free Cartan structure equation to each 2-simplex, we discuss parallel transports and construct a diffeomorphism and *local* gauge-invariant Einstein-Cartan action. Invariant holonomies of tetrad and spin-connection fields along large loops are also given. Quantization is defined by a bounded partition function with the measure of $SO(4)$-group valued $\omega_\mu(x)$ fields and Dirac-matrix valued $e_\mu(x)$ fields over 4-simplices complex.

In the 2-dimensional case (2-simplices complex), we calculate: (i) system's entropy and free-energy, being proportional to its surface; (ii) the average of regularized Einstein-Cartan action, implying that the Planck length sets the scale for the minimal distance between two space-time points. calculations of partition function, entropy and averaged EC action in 2-dimensional case.

The person involved in this line of research is She-Sheng Xue.

4. Brief Description of Unification Theories

In chapter "Unification theories", the geometrical description of gauge interactions in a Kaluza-Klein framework is investigated.

Within the unification picture provided by the Kaluza Klein (KK) theory, the 5- Dimensional (5D) model is the simplest one and the starting point for the investigation of the breaking of multidimensional gravity into the usual gravity plus Yang-Mills fields. It is characterized by an abelian structure; indeed, it provides the coupling between gravity, a U(1) gauge field and an extra scalar field. If the scalar field it is assumed to be constant from the beginning, the 5D model reproduces exactly the Einstein-Maxwell theory in vacuum. The research line about this topic is focused on following points:

- Brown-Kuchař approach in 5D Kaluza-Klein model;
- test-particles dynamics [50], 51;
- coupling with matter [52];
- geodesic deviation [53];
- massive test particles motion in Kaluza-Klein gravity.

The people involved in this research line are Riccardo Benini, Valentino Lacquaniti, Giovanni Montani, Francesco Vietri, Daniela Pugliese and Simone Zonetti.

The extension of the Kaluza-Klein framework to non-Abelian gauge interactions requires the introduction of more than one extra-dimension. In these cases huge mass terms for fermions fields are predicted when searching for a geometric gauge connection. However, in our analysis there are indications that such mass terms can be avoided and a proper phenomenology for a SU(2) gauge theory is achieved when Riemannian connections are considered and a suitable form for the extra-dimensional spinor is chosen.

The people involved in this research line are Alaudio Chingotuane, Francesco Cianfrani and Giovanni Montani.

5. Selected Publications before 2005

5.1. Quantum Gravity

[1] G. Montani, Canonical Quantization of Gravity without "Frozen Formalism", *Nucl. Phys. B*, **634**, 370 (2002).

We write down a quantum gravity equation which generalizes the WheelerDeWitt one in view of including a time dependence in the wave functional. The obtained equation provides a consistent canonical quantization of the 3-geometries resulting from a "gauge-fixing" (3 + 1)-slicing of the spacetime. Our leading idea relies on a criticism to the possibility that, in a quantum spacetime, the notion of a (3+1)-slicing formalism (underlying the WheelerDeWitt approach) has yet a precise physical meaning. As solution to this problem we propose of adding to the gravity-matter action the so-called kinematical action (indeed in its reduced form, as implemented in the quantum regime), and then we impose the new quantum constraints. As consequence of this revised approach, the quantization procedure of the 3-geometries takes place in a fixed reference frame and the wave functional acquires a time evolution along a one-parameter family of spatial hypersurfaces filling the spacetime. We show how the states of the new quantum dynamics can be arranged into an Hilbert space, whose associated inner product induces a conserved probability notion for the 3-geometries. Finally, since the constraints we quantize violate the classical symmetries (i.e., the vanishing nature of the super-Hamiltonian), then a key result is to find a (non-physical) restriction on the initial wave functional phase, ensuring that general relativity outcomes when taking the appropriate classical limit. However, we propose a physical interpretation of the kinematical variables which, based on the analogy with the so-called Gaussian reference fluid, makes allowance even for such classical symmetry violation.

[2] G. Montani, Cosmological Issues for revised canonical quantum gravity, *Int. J. Mod. Phys. D*, **12**, 8, 1445 (2003)

In a recent work we presented a reformulation of the canonical quantum gravity, based on adding the so-called kinematical term to the gravity-matter action. This revised approach leads to a self-consistent canonical quantization of the 3-geometries, which referred to the external time as provided via the added term. Here, we show how the kinematical term can be interpreted in terms of a non- relativistic dust fluid which plaies the role of a "real clock" for the quantum gravity theory, and, in the WKB limit of a cosmological problem, makes account for a dark matter component which, at present time, could play a dynamical role.

[3] G. Aprea, G. Montani and R. Ruffini, Test particles behavior in the framework of a Lagrangian geometric theory with propagating torsion, *Int. J. Mod. Phys. D*, **12**, 10, 1875 (2003)

Working in the Lagrangian framework, we develop a geometric theory in vacuum with propagating torsion; the antisymmetric and trace parts of the torsion tensor, considered as derived from local potential fields, are taken and, using the minimal action principle, their field equations are calculated. Actually these will show themselves to be just equations for propagating waves giving torsion a behavior similar to that of metric which, as known, propagates through gravitational waves. Then we establish a principle of minimal substitution to derive test particles equation of motion, obtaining, as result, that they move along autoparallels. We then calculate the analogous of the geodesic deviation for these trajectories and analyze their behavior in the nonrelativistic limit, showing that the torsion trace potential ϕ has a phenomenology which is indistinguishable from that of the gravitational Newtonian field; in this way we also give a reason for why there have never been evidence for it.

[4] G. Imponente and G. Montani, Mixmaster Chaoticity as Semiclassical Limit of the Canonical Quantum Dynamics, *Int. J. Mod. Phys. D*, **12**(6), 977-984 (2003).

Within a cosmological framework, we provide a Hamiltonian analysis of the Mixmaster Universe dynamics on the base of a standard Arnowitt-Deser-Misner approach, showing how the chaotic behavior characterizing the evolution of the system near the cosmological singularity can be obtained as the semiclassical limit of the canonical quantization of the model in the same dynamical representation. The relation between this intrinsic chaotic behavior and the indeterministic quantum dynamics is inferred through the coincidence between the microcanonical probability distribution and the semiclassical quantum one.

[5] S. Mercuri and G. Montani, Revised Canonical Quantum Gravity via the Frame Fixing, *Int. J. Mod. Phys. D*, **13**, 165 (2004).

We present a new reformulation of the canonical quantum geometrodynamics, which allows one to overcome the fundamental problem of the frozen formalism and, therefore, to construct an appropriate Hilbert space associate to the solution of the restated dynamics. More precisely, to remove the ambiguity contained in the Wheeler-DeWitt approach, with respect to the possibility of a (3 + 1)-splitting when space-time is in a quantum regime, we fix the reference frame (i.e. the lapse function and the shift vector) by introducing the so-called kinematical action. As a consequence the new super-Hamiltonian constraint becomes a parabolic one and we arrive to a Schrödingerlike approach for the quantum dynamics. In the semiclassical limit our theory provides General Relativity in the presence of an additional energy-momentum density contribution coming from non-zero eigenvalues of the Hamiltonian constraints. The interpretation of these new contributions comes out in natural way that soon as it is recognized that the kinematical action can be recasted in such a way that it describes a pressureless, but, in general, non-geodesic perfect fluid.

[6] S. Mercuri and G. Montani, Dualism between physical frames and time in quantum gravity, *Mod. Phys. Lett. A*, **19**, 20, 1519 (2004).

In this work we present a discussion of the existing links between the procedures of endowing the quantum gravity with a real time and of including in the theory a physical reference frame. More precisely, as a first step, we develop the canonical quantum dynamics, starting from the Einstein equations in presence of a dust fluid and arrive at a Schrödinger evolution. Then, by fixing the lapse function in the path-integral of gravity, we get a Schrödinger quantum dynamics, of which eigenvalues problem provides the appearance of a dust fluid in the classical limit. The main issue of our analysis is to claim that a theory, in which the time displacement invariance, on a quantum level, is broken, is indistinguishable from a theory for which this symmetry holds, but a real reference fluid is include.

[7] G. Montani, Minisuperspace model for revised canonical quantum gravity, *Int. J. Mod. Phys. D*, **13**, 8, 1703 (2004)

We present a reformulation of the canonical quantization of gravity, as referred to the minisuperspace; the new approach is based on fixing a Gaussian (or synchronous) reference frame and then quantizing the system via the reconstruction of a suitable constraint; then the quantum dynamics is re-stated in a generic coordinates system and it becomes dependent on the lapse function. The analysis follows a parallelism with the case of the non-relativistic particle and leads to the minisuperspace implementation of the so-called kinematical action as proposed in Ref. 1 (here almost coinciding also with the approach presented in Ref. 2). The new constraint leads to a Schrödinger equation for the system, i.e. to nonvanishing eigenvalues for the super-Hamiltonian operator; the physical interpretation of this feature relies on the appearance of a "dust fluid" (non-positive definite) energy density, i.e. a kind of "materialization" of the reference frame. As an example of minisuperspace model, we consider a Bianchi type IX Universe, for which some dynamical implications of the revised canonical quantum gravity are discussed. We also show how, on the classical limit, the presence of the dust fluid can have relevant cosmological issues. Finally we upgrade our analysis by its extension to the generic cosmological solution, which is performed in the so-called long-wavelength approximation. In fact, near the Big-Bang, we can neglect the spatial gradients of the dynamical variables and proceed to implement, in each space point, the same minisuperspace paradigm valid for the Bianchi IX model.

[8] G.V. Vereshchagin, On stability of simplest nonsingular inflationary cosmological models within general relativity and gauge theories of gravity, *Int. J. Mod. Phys. D*, **13**, 695 (2004).

In this paper we provide approximate analytical analysis of stability of nonsingular inflationary chaotic-type cosmological models. Initial conditions for nonsingular solutions at the bounce correspond to dominance of potential part of the energy density of the scalar field over its kinetic part both within general relativity and gauge theories of gravity. Moreover, scalar field at the bounce exceeds the planckian value and on expansion stage these models correspond to chaotic inflation. Such solutions can be well approximated by explicitly solvable model with constant effective potential (cosmological term) and massless scalar field during the bounce and on stages of quasi-exponential contraction and expansion. Perturbative analysis shows that nonsingular inflationary solutions are exponentially unstable during contraction stage. This result is compared with numerical calculations.

[9] G.V. Vereshchagin, Qualitative Approach to Semi-Classical Loop Quantum Cosmology, *JCAP*, **0407**, 013 (2004).

Recently the mechanism was found which allows avoidance of the cosmological singularity within the semi-classical formulation of Loop Quantum Gravity. Numerical studies show that the presence of self-interaction potential of the scalar field allows generation of initial conditions for successful slow-roll inflation. In this paper qualitative analysis of dynamical system, corresponding to cosmological equations of Loop Quantum Gravity is performed. The conclusion on singularity avoidance in positively curved cosmological models is confirmed. Two cases are considered, the massless (with flat potential) and massive scalar field. Explanation of initial conditions generation for inflation in models with massive scalar field is given. The bounce is discussed in models with zero spatial curvature and negative potentials.

5.2. Quantum Field on Classical Background

[10] G. Montani, A scenario for the dimensional compactification in eleven-dimensional space-time, *Int. J. Mod. Phys. D*, **13**, 6, 1029 (2004).

We discuss the inhomogeneous multidimensional mixmaster model in view of the appearing, near the cosmological singularity, of a scenario for the dimensional compactification in correspondence to an 11-dimensional spacetime. Our analysis candidates such a collapsing picture toward the singularity to describe the actual expanding 3-dimensional Universe and an associated collapsed 7-dimensional space. To this end, a conformal factor is determined in front of the 4-dimensional metric to remove the 4-curvature divergences and the resulting Universe expands with a power-law inflation. Thus we provide an additional peculiarity of the eleven space-time dimensions in view of implementing a geometrical theory of unification.

6. Publications 2005-2010

6.1. Quantum Gravity

[1] E. Cerasti and G. Montani, Generating functional for the gravitational filed: implementation of an evolutionary quantum dynamics, *Int. J. Mod. Phys. D*, **14**, 10, 1739 (2005)

We provide a generating functional for the gravitational field that is associated with the relaxation of the primary constraints by extending to the quantum sector. This requirement of the theory relies on the assumption that a suitable time variable exists, when taking the T-products of the dynamical variables. More precisely, we start from the gravitational field equations written in the Hamiltonian formalism and expressed via Misner-like variables; hence we construct the equation to which the T-products of the dynamical variables obey and transform this paradigm in terms of the generating functional, as taken on the theory phase-space. We show how the relaxation of the primary constraints (which corresponds to the breakdown of the invariance of the quantum theory under the four-diffeomorphisms) is summarized by a free functional taken on the Lagrangian multipliers, accounting for such constraints in the classical theory. The issue of our analysis is equivalent to a Gupta-Bleuler approach on the quantum implementation of all the gravitational constraints; in fact, in the limit of small $\hbar$, the quantum dynamics is described by a Schrödinger equation as soon as the mean values of the momenta, associated to the lapse function and the shift vector, are not vanishing. Finally we show how, in the classical limit, the evolutionary quantum gravity reduces to General Relativity in the presence of an Eckart fluid, which corresponds to the classical counterpart of the physical clock, introduced in the quantum theory.

[2] M.V. Battisti and G. Montani, Evolutionary Quantum Dynamics of a Generic Universe, *Phys. Lett. B*, **637**, 203 (2006).

The implications of an evolutionary quantum gravity are addressed in view of formulating a new dark matter candidate. We consider a Schroedinger dynamics for the gravitational field associated to a generic cosmological model and then we solve the corresponding eigenvalue problem, inferring its phenomenological issue for the actual universe. The spectrum of the super-Hamiltonian is determined including a free inflaton field, the ultrarelativistic thermal bath and a perfect gas into the dynamics. We show that, when a Planckian cut-off is imposed in the theory and the classical limit of the ground state is taken, then a dark matter contribution cannot arise because its critical parameter Ω_{dm} is negligible today when the appropriate cosmological implementation of the model is provided. Thus, we show that, from a phenomenological point of view, an evolutionary quantum cosmology overlaps the Wheeler-DeWitt approach and therefore it can be inferred as appropriate to describe early stages of the universe without significant traces on the later evolution.

[3] P. Singh, K. Vandersloot and G.V. Vereshchagin, Nonsingular bouncing universes in loop quantum cosmology *Phys. Rev. D*, **74**, 043510 (2006).

Nonperturbative quantum geometric effects in loop quantum cosmology (LQC) predict a ρ^2 modification to the Friedmann equation at high energies. The quadratic term is negative definite and can lead to generic bounces when the matter energy density becomes equal to a critical value of the order of the Planck density. The nonsingular bounce is achieved for arbitrary matter without violation of positive energy conditions. By performing a qualitative analysis we explore the nature of the bounce for inflationary and cyclic model potentials. For the former we show that inflationary trajectories are attractors of the dynamics after the bounce implying that inflation can be harmoniously embedded in LQC. For the latter difficulties associated with singularities in cyclic models can be overcome. We show that nonsingular cyclic models can be constructed with a small variation in the original cyclic model potential by making it slightly positive in the regime where scalar field is negative.

[4] M.V. Battisti and G. Montani, The big-bang singularity in the framework of a generalized uncertainty principle, *Phys. Lett. B*, **656**, 96 (2006).

We analyze the quantum dynamics of the FriedmannRobertsonWalker Universe in the context of a Generalized Uncertainty Principle. Since the isotropic Universe dynamics resembles that of a one-dimensional particle, we quantize it with the commutation relations associated to an extended formulation of the Heisenberg algebra. The evolution of the system is described in terms of a massless scalar field taken as a relational time. We construct suitable wave packets and analyze their dynamics from a quasi-classical region to the initial singularity. The appearance of a non-singular dynamics comes out as far as the behavior of the probability density is investigated. Furthermore, reliable indications arise about the absence of a big-bounce, as predicted in recent issues of loop quantum cosmology.

[5] M.V. Battisti and G. Montani, Evolutionary Quantization of Cosmological Models, *Nuovo Cimento B*, **122**, 179-184 (2007).

We consider a Schrödinger quantum dynamics for the gravitational field associated to a FRW spacetime and then we solve the corresponding eigenvalue problem. We show that, from a phenomenological point of view, an Evolutionary Quantum Cosmology overlaps the Wheeler-DeWitt approach. We also show how a so peculiar solution can be inferred to describe the more interesting case of a generic cosmological model.

[6] F. Cianfrani and G. Montani, Boost invariance of the gravitational field dynamics: quantization without time gauge, *Class. Quant. Grav.*, **24**, 4161 (2007).

We perform a canonical quantization of gravity in a second-order formulation, taking as configuration variables those describing a 4-bein, not adapted to the spacetime splitting. We outline how, if we either fix the Lorentz frame before quantizing or perform no gauge fixing at all, the invariance under boost transformations is affected by the quantization.

[7] R. Benini and G. Montani, Inhomogeneous Quantum Mixmaster: from Classical toward Quantum Mechanics, *Class. Quant. Grav.*, **24**, 387 (2007).

Starting from the Hamiltonian formulation for the inhomogeneous Mixmaster dynamics, we approach its quantum features through the link of the quasiclassical limit. We fix the proper operator-ordering which ensures that the WKB continuity equation overlaps the Liouville theorem as restricted to the configuration space. We describe the full quantum dynamics of the model in some detail, providing a characterization of the (discrete) spectrum with analytic expressions for the limit of high occupation number. One of the main achievements of our analysis relies on the description of the ground state morphology, showing how it is characterized by a non-vanishing zero-point energy associated with the universe anisotropy degrees of freedom.

[8] N. Carlevaro, O.M. Lecian and G. Montani, Macroscopic and microscopic paradigms for the torsion field: from the test-particles motion to a Lorentz gauge theory, *Ann. Fond. Louis de Broglie*, **32**, 281 (2007).

Torsion represents the most natural extension of General Relativity and it attracted interest over the years in view of its link with fundamental properties of particle motion. The bulk of the approaches concerning the torsion dynamics focus their attention on their geometrical nature and they are naturally lead to formulate a non-propagating theory. Here we review two different paradigms to describe the role of the torsion field, as far as a propagating feature of the resulting dynamics is concerned. However, these two proposals deal with different pictures, i.e., a macroscopic approach, based on the construction of suitable potentials for the torsion field, and a microscopic approach, which relies on the identification of torsion with the gauge field associated with the local Lorentz symmetry. We analyze in some detail both points of view and their implications on the coupling between torsion and matter. In particular, in the macroscopic case, we analyze the test-particle motion to fix the physical trajectory, while, in the microscopic approach, a natural coupling between torsion and the spin momentum of matter fields arises

[9] F. Cianfrani and G. Montani, The role of the time gauge in the 2nd order formalism, *Int. J. Mod. Phys. A*, **23**, 8, 1214 (2008).

We perform a canonical quantization of gravity in a second-order formulation, taking as configuration variables those describing a 4-bein, not adapted to the space-time splitting. We outline how, neither if we fix the Lorentz frame before quantizing, nor if we perform no gauge fixing at all, is invariance under boost transformations affected by the quantization.

[10] M. Castellana and G. Montani, Physical state condition in Quantum General Relativity as a consequence of BRST symmetry, *Class. Quant. Grav.*, **25**, 105018 (2008).

Quantization of systems with constraints can be carried on with several methods. In the Dirac formulation the classical generators of gauge transformations are required to annihilate physical quantum states to ensure their gauge invariance. Carrying on BRST symmetry it is possible to get a condition on physical states which, differently from the Dirac method, requires them to be invariant under the BRST transformation. Employing this method for the action of general relativity expressed in terms of the spin connection and tetrad fields with path integral methods, we construct the generator of BRST transformation associated with the underlying local Lorentz symmetry of the theory and write a physical state condition consequence of BRST invariance. We observe that this condition differs form the one obtained within Ashtekar's canonical formulation, showing how we recover the latter only by a suitable choice of the gauge fixing functionals. We finally discuss how it should be possible to obtain all the requested physical state conditions associated with all the underlying gauge symmetries of the classical theory using our approach.

[11] G. Montani and S. Zonetti, Parametrizing fluids in canonical quantum gravity, *Int. J. Mod. Phys. A*, **23**, 8, 1240-1243 (2008).

The problem of time is an unsolved issue of canonical General Relativity. A possible solution is the Brown-Kuchar mechanism which couples matter to the gravitational field and recovers a physical, i.e. non vanishing, observable Hamiltonian functional by manipulating the set of constraints. Two cases are analyzed. A generalized scalar fluid model provides an evolutionary picture, but only in a singular case. The Schutz' model provides an interesting singularity free result: the entropy per baryon enters the definition of the physical Hamiltonian. Moreover in the co-moving frame one is able to identify the time variable tau with the logarithm of entropy.

[12] M.V.Battisti and G.Montani, Quantum dynamics of the Taub Universe in a generalized uncertainty principle framework, *Phys. Rev. D*, **77**, 023518 (2008).

The implications of a Generalized Uncertainty Principle on the Taub cosmological model are investigated. The model is studied in the ADM reduction of the dynamics and therefore a time variable is ruled out. Such a variable is quantized in a canonical way and the only physical degree of freedom of the system (related to the Universe anisotropy) is quantized by means of a modified Heisenberg algebra. The analysis is performed at both classical and quantum level. In particular, at quantum level, the motion of wave packets is investigated. The two main results obtained are as follows. i) The classical singularity is probabilistically suppressed. The Universe exhibits a stationary behavior and the probability amplitude is peaked in a determinate region. ii) The GUP wave packets provide the right behavior in the establishment of a quasi-isotropic configuration for the Universe.

[13] G. Montani and F. Cianfrani, General Relativity as Classical Limit of Evolutionary Quantum Gravity, *Class. Quant. Grav.*, **25**, 065007 (2008).

In this paper we analyze the dynamics of the gravitational field when the covariance is restricted to a synchronous gauge. In the spirit of the Noether theorem, we determine the conservation law associated to the Lagrangian invariance and we outline that a non-vanishing behavior of the Hamiltonian comes out. We then interpret such resulting non-zero "energy" of the gravitational field in terms of a dust fluid. This new matter contribution is co-moving to the slicing and it accounts for the "materialization" of a synchronous reference from the corresponding gauge condition. Further, we analyze the quantum dynamics of a generic inhomogeneous Universe as described by this evolutionary scheme, asymptotically to the singularity. We show how the phenomenology of such a model overlaps the corresponding Wheeler-DeWitt picture. Finally, we study the possibility of a Schrödinger dynamics of the gravitational field as a consequence of the correspondence inferred between the ensemble dynamics of stochastic systems and the WKB limit of their quantum evolution. We demonstrate that the time dependence of the ensemble distribution is associated with the first order correction in $\hbar$ to the WKB expansion of the energy spectrum.

[14] F. Cianfrani and G. Montani, Synchronous Quantum Gravity, *Int. J. Mod. Phys. A*, **23**, 8, 1105-1112 (2008).

The implications of restricting the covariance principle within a Gaussian gauge are developed both on a classical and a quantum level. Hence, we investigate the cosmological issues of the obtained Schrödinger Quantum Gravity with respect to the asymptotically early dynamics of a generic Universe. A dualism between time and the reference frame fixing is then inferred.

[15] M.V. Battisti, O.M. Lecian and G. Montani, Quantum cosmology with a minimal length, *Int. J. Mod. Phys. A*, **23**, 1257-1265 (2008).

Quantum cosmology in the presence of a fundamental minimal length is analyzed in the context of the flat isotropic and the Taub cosmological models. Such minimal scale comes out from a generalized uncertainty principle and the quantization is performed in the minisuperspace representation. Both the quantum Universes are singularity-free and (i) in the isotropic model no evidences for a Big-Bounce appear; (ii) in the Taub one a quasi-isotropic configuration for the Universe is predicted by the model.

[16] N. Carlevaro, O.M. Lecian and G. Montani, Lorentz Gauge Theory and Spinor Interaction, *Int. J. Mod. Phys. A*, **23**(8), 1282 (2008).

A gauge theory of the Lorentz group, based on the different behavior of spinors and vectors under local transformations, is formulated in a flat space-time and the role of the torsion field within the generalization to curved space-time is briefly discussed. The spinor interaction with the new gauge field is then analyzed assuming the time gauge and stationary solutions, in the non-relativistic limit, are treated to generalize the Pauli equation.

[17] M.V. Battisti, O.M. Lecian and G. Montani, Polymer Quantum Dynamics of the Taub Universe, *Phys. Rev. D*, **78**, 103514 (2008).

Within the framework of non-standard (Weyl) representations of the canonical commutation relations, we investigate the polymer quantization of the Taub cosmological model. The Taub model is analyzed within the Arnowitt-Deser-Misner reduction of its dynamics, by which a time variable arises. While the energy variable and its conjugate momentum are treated as ordinary Heisenberg operators, the anisotropy variable and its conjugate momentum are represented by the polymer technique. The model is analyzed at both classical and quantum level. As a result, classical trajectories flatten with respect to the potential wall, and the cosmological singularity is not probabilistically removed. In fact, the dynamics of the wave packets is characterized by an interference phenomenon, which, however, is not able to stop the evolution towards the classical singularity.

[18] F. Cianfrani, G. Montani and S. Zonetti,Definition of a time variable with Entropy of a perfect fluid in Canonical Quantum Gravity, *Class. Quant. Grav.*, **26**, 125002 (2009).

The Brown-Kuchař mechanism is applied in the case of General Relativity coupled with the Schutz' model for a perfect fluid. Using the canonical formalism and manipulating the set of modified constraints one is able to recover the definition of a time evolution operator, *i.e.* a physical Hamiltonian, expressed as a functional of gravitational variables and the entropy.

[19] M.V. Battisti and G. Montani, The Mixmaster Universe in a generalized uncertainty principle framework, *Phys. Lett. B*, **681**, 179 (2009).

The Bianchi IX cosmological model is analyzed in a generalized uncertainty principle framework. The Arnowitt-Deser-Misner reduction of the dynamics is performed and a time-coordinate, namely the volume of the Universe, naturally arises. Such a variable is treated in the ordinary way while the anisotropies (the physical degrees of freedom of the Universe) are described by a deformed Heisenberg algebra. The analysis of the model (passing through Bianchi I and II) is performed at classical level by studying the modifications induced on the symplectic geometry by the deformed algebra. We show that, the triangular allowed domain is asymptotically stationary with respect to the particle (Universe) and that its bounces against the walls are not interrupted by the deformed effects. Furthermore, no reflection law can be in general obtained since the Bianchi II model is no longer analytically integrable. This way, the deformed Mixmaster Universe can be still considered a chaotic system.

[20] M.V. Battisti, Cosmological bounce from a deformed Heisenberg algebra, *Phys. Rev. D*, **79**, 083506 (2009).

The implications of a deformed Heisenberg algebra on the Friedmann-Robertson-Walker cosmological models are investigated. We consider the Snyder non-commutative space in which the translation group is undeformed and the rotational invariance preserved. When this framework is implemented to one-dimensional systems (which is this case) the modifications are uniquely fixed up to a sign. A cosmological quantum bounce 'a la loop quantum cosmology is then obtained. We also get the Randall-Sundrum braneworld scenario and this way a Snyder-deformed quantum cosmology can be considered as a common phenomenological description for both theories.

[21] M.V. Battisti, R. Belvedere and G. Montani, Semi-classical suppression of weak anisotropies of a generic Universe, *Europhys. Lett.*, **86**, 69001 (2009).

A semiclassical mechanism which suppresses the weak anisotropies of an inhomogeneous cosmological model is developed. In particular, a wave function of this Universe having a meaningful probabilistic interpretation is obtained that is in agreement with the Copenhagen School. It describes the evolution of the anisotropies with respect to the isotropic scale factor which is regarded as a semiclassical variable playing an observer-like role. Near the cosmological singularity the solution spreads over all values of the anisotropies while, when the Universe expands sufficiently, the closed Friedmann-Robertson-Walker model appears to be the favorite state.

[22] N. Carlevaro, O.M. Lecian and G. Montani, Fermion dynamics by internal and space-time symmetries, *Mod. Phys. Lett. A*, **24**, 415 (2009).

This manuscript is devoted to introduce a gauge theory of the Lorentz Group based on the ambiguity emerging in dealing with isometric diffeomorphism-induced Lorentz transformations. The behaviors under local transformations of fermion fields and spin connections (assumed to be ordinary world vectors) are analyzed in flat space-time and the role of the torsion field, within the generalization to curved space-time, is briefly discussed. The fermion dynamics is then analyzed including the new gauge fields and assuming time-gauge. Stationary solutions of the problem are also studied in the non-relativistic limit, to study the spinor structure of an hydrogen-like atom.

[23] F. Cianfrani and G. Montani, Towards Loop Quantum Gravity without the time gauge, *Phys. Rev. Lett.*, **102**, 091301 (2009).

The Hamiltonian formulation of the Holst action is reviewed and it is provided a solution of second-class constraints corresponding to a generic local Lorentz frame. Within this scheme the form of rotation constraints can be reduced to a Gauss-like one by a proper generalization of Ashtekar- Barbero-Immirzi connections. This result emphasizes that the Loop Quantum Gravity quantization procedure can be applied when the time-gauge condition does not stand.

[24] F. Cianfrani and G. Montani, Matter in Loop Quantum Gravity without time gauge: a non-minimally coupled scalar field , *Phys. Rev. D*, **80**, 084045 (2009).

We analyze the phase space of gravity non-minimally coupled to a scalar field in a generic local Lorentz frame. We reduce the set of constraints to a first-class one by fixing a specific hypersurfaces in the phase space. The main issue of our analysis is to extend the features of the vacuum case to the presence of scalar matter by recovering the emergence of an SU(2) gauge structure and the non-dynamical role of boost variables. Within this scheme, the super-momentum and the super- Hamiltonian are those ones associated with a scalar field minimally coupled to the metric in the Einstein frame. Hence, the kinematical Hilbert space is defined as in canonical Loop Quantum Gravity with a scalar field, but the differences in the area spectrum are outlined to be the same as in the time-gauge approach.

[25] F. Cianfrani and G. Montani, The Immirzi parameter from an external scalar field, *Phys. Rev. D*, **80**, 084040 (2009).

We promote the Immirzi parameter to be a minimally coupled scalar field and we analyzed the Hamiltonian constraints in the framework of Loop Quantum Gravity without the time gauge. Proper SU(2) connections can be defined and a term containing derivatives of the field β enters into their definition. Furthermore, boost degrees of freedom are non-dynamical, while the super-momentum constraints coincide with the scalar field case. Hence, the kinematical Hilbert space can be defined as for gravity in presence of a minimally coupled scalar field. Then, we analyzed the dynamical implications of this scenario and we outline how a dynamical relaxation to a non-vanishing vacuum expectation value is predicted, so recovering the standard Loop Quantum Gravity formulation.

[26] N. Carlevaro, O.M. Lecian and G. Montani, Fermion Dynamics by Internal and Space-Time Symmetries, *Mod. Phys. Lett. A*, **24**, 415 (2009).

This manuscript is devoted to introduce a gauge theory of the Lorentz Group based on the ambiguity emerging in dealing with isometric diffeomorphism-induced Lorentz transformations. The behaviors under local transformations of fermion fields and spin connections (assumed to be ordinary world vectors) are analyzed in flat space-time and the role of the torsion field, within the generalization to curved space-time, is briefly discussed. The fermion dynamics is then analyzed including the new gauge fields and assuming time-gauge. Stationary solutions of the problem are also analyzed in the non-relativistic limit, to study the spinor structure of an hydrogen-like atom.

[27] M.V. Battisti and G. Montani, Bianchi IX in the GUP approach, *Phys. Lett. B*, **681**, 179 (2009).

We have analyzed the Bianchi IX cosmological model (the Mixmaster Universe) in a generalized uncertainty principle framework. The Arnowitt-Deser-Misner reduction of the dynamics is performed and a time-coordinate, namely the volume of the Universe, naturally arises. Such a variable is treated in the ordinary way while the anisotropies (the physical degrees of freedom) are described by a deformed Heisenberg algebra. The analysis of the model (passing through Bianchi I and II) is performed at classical level by studying the modifications induced on the symplectic geometry by the deformed algebra. We show that, the Universe can not isotropize because of the deformed Kasner dynamics, the triangular allowed domain is asymptotically stationary with respect to the particle (Universe) and its bounces against the walls are not interrupted by the deformed effects. Furthermore, no reflection law can be in general obtained since the Bianchi II model is no longer analytically integrable. This way, the deformed Mixmaster Universe can be still considered a chaotic system.

[28] S.S. Xue, Quantum Regge Calculus of EinsteinCartan theory, *Phys. Lett. B*, **682**, 300 (2009).

We study the Quantum Regge Calculus of EinsteinCartan theory to describe quantum dynamics of Euclidean spacetime discretized as a 4-simplices complex. Tetrad field $e_\mu(x)$ and spin-connection field $\omega_\mu(x)$ are assigned to each 1-simplex. Applying the torsion-free Cartan structure equation to each 2-simplex, we discuss parallel transports and construct a diffeomorphism and local gauge-invariant EinsteinCartan action. Invariant holonomies of tetrad and spin-connection fields along large loops are also given. Quantization is defined by a bounded partition function with the measure of SO(4)-group valued $\omega_\mu(x)$ fields and Dirac-matrix valued $e_\mu(x)$ fields over 4-simplices complex.

[29] F. Cianfrani and G. Montani, Gravity in presence of fermions as a SU(2) gauge theory, *Phys. Rev. D*, **81**, 044015 (2010).

The Hamiltonian formulation of the Holst action in presence of a massless fermion field with a non-minimal Lagrangian is performed without any restriction on the local Lorentz frame. It is outlined that the phase space structure does not resemble that one of a background independent Lorentz gauge theory, as some additional constraints are present. Proper phase space coordinates are introduced, such that SU(2) connections can be defined and the vanishing of conjugate momenta to boost variables is predicted. Finally, it is demonstrated that for a particular value of the non-minimal parameter the kinematics coincides with

that one of a background independent SU(2) gauge theory and the Immirzi parameter becomes the coupling constant of such an interaction between fermions and the gravitational field.

[30] F. Cianfrani and G. Montani, Shortcomings of the Big Bounce derivation in Loop Quantum Cosmology, *Phys. Rev. D*, **82**, 021501 (2010).

We give a prescription to define in Loop Quantum Gravity the electric field operator related to the scale factor of an homogeneous and isotropic cosmological space-time. This procedure allows to link the fundamental theory with its cosmological implementation. In view of the conjugate relation existing between holonomies and fluxes, the edge length and the area of surfaces in the fiducial metric satisfy a duality condition. As a consequence, the area operator has a discrete spectrum also in Loop Quantum Cosmology. This feature makes the super-Hamiltonian regularization an open issue of the whole formulation.

[31] F. Cianfrani, G. Montani and M. Muccino, Semi-Classical Isotropization of the Universe during a de Sitter phase, *Phys. Rev. D*, in press.

Semi-classical states for the Wheeler-DeWitt equation of a Bianchi type I model in the presence of a scalar field are analyzed. It is outlined how this scheme can effectively describe more general situations, where the curvature of the Bianchi type IX model and a proper potential term for the scalar field are present. The introduction of a cosmological constant term accounts for the quasi-isotropization mechanism which bridges the proposed framework with a late isotropic phase. This result makes the semi-classical Bianchi I model a plausible scenario for the Universe pre-inflationary phase.

6.2. Quantum Field on Classical Background

[32] V. Belinski , On the existence of black hole evaporation yet again, *Phys. Lett. A*, **354**, 249 (2006).

A new argument is presented confirming the point of view that a Schwarzshild black hole formed during a collapse process does not radiate.

[33] F. Cianfrani, G. Montani, Curvature-spin coupling from the semi-classical limit of the Dirac equation, *Int. J. Mod. Phys. A*, **23**, 8, 1274-1277 (2008).

The notion of a classical particle is inferred from Dirac quantum fields on a curved space-time, by an eikonal approximation and a localization hypothesis for amplitudes. This procedure allows to define a semi-classical version of the spin-tensor from internal quantum degrees of freedom, which has a Papapetrou-like coupling with the curvature.

[34] F. Cianfrani and G. Montani, Dirac equations in curved space-time versus Papapetrou spinning particles, *Europhys. Lett.*, in press.

We recover classical particles, starting from Dirac quantum fields on a curved space-time, by an eikonal approximation and a localization hypothesis for amplitudes. We conclude that the semi-classical dynamics of spinors is neither a geodesics one, nor resembling a Papapetrou-like spinning body. However, the spin-curvature coupling predicted by the Papapetrou theory is recovered in the weak-gravitational-field limit, but still an additional contribution to the dynamics arises

6.3. Unification Theories

[35] G. Montani, Geometrization of the Gauge Connection within a Kaluza-Klein Theory, *Int. J. Theor. Phys.*, **44**, 43-52 (2005).

Within the framework of a Kaluza-Klein theory, we provide the geometrization of a generic (Abelian and non-Abelian) gauge coupling, which comes out by choosing a suitable matter fields dependence on the extra-coordinates. We start by the extension of the Nother theorem to a multidimensional spacetime being the direct sum of a 4-dimensional Minkowski space and of a compact homogeneous manifold (whose isometries reflect the gauge symmetry); we show, how on such a "vacuum" configuration, the extra-dimensional components of the field momentum correspond to the gauge charges. Then we analyze the structure of a Dirac algebra as referred to a spacetime with the Kaluza-Klein restrictions and, by splitting the corresponding free-field Lagrangian, we show how the gauge coupling terms outcome.

[36] E. Alesci and G. Montani, Can gravitational waves be markers for an extra-dimension?, *Int. J. Mod. Phys. D*, **14**, 6, 923 (2005).

The main issue of the present paper is to fix specific features (which turn out being independent of extradimension size) of gravitational waves generated before a dimensional compactification process. Valuable is the possibility to detect our prediction from gravitational wave experiment without high energy laboratory investigation. In particular we show how gravitational waves can bring information on the number of Universe dimensions. Within the framework of Kaluza-Klein hypotheses, a different morphology arises between waves generated before than the compactification process settled down and ordinary 4-dimensional waves. In the former case the scalar and tensor degrees of freedom cannot be resolved. As a consequence if gravitational waves having the feature predicted here were detected (anomalous polarization ampli-

tudes), then they would be reliable markers for the existence of an extra dimension.

[37] F. Cianfrani, A. Marrocco and G. Montani, Gauge Theories as a Geometrical Issue of a Kaluza-Klein Framework, *Int. J. Mod. Phys. D*, **14**(7), 1095 (2006).

We present a geometrical unification theory in a Kaluza-Klein approach that achieve the geometrization of a generic gauge theory bosonic component. We show how it is possible to derive gauge charge conservation from the invariance of the model under extra-dimensional translations and to geometrize gauge connections for spinors, in order to make possible to introducing matter just through free spinorial fields. Then we present the applications to (i) a pentadimensional manifold so reproducing the original Kaluza-Klein theory with some extensions related to the rule of the scalar field contained in the metric and to the introduction of matter through spinors with a phase dependance from the fifth coordinate, (ii) a seven-dimensional manifold, in which we geometrize the electroweak model by introducing two spinors for every leptonic family and quark generation and a scalar field with two components with opposite hypercharge responsible for spontaneous symmetry breaking.

[38] F. Cianfrani and G. Montani, Non Abelian gauge symmetries induced by the unobservability of extra-dimensions in a Kaluza-Klein approach, *Mod. Phys. Lett. A*, **21**(3), 265 (2006).

In this work we deal with the extension of the Kaluza-Klein approach to a non-Abelian gauge theory; we show how we need to consider the link between the n-dimensional model and a four-dimensional observer physics, in order to reproduce field equations and gauge transformations in the four-dimensional picture. More precisely, in field equations any dependence on extra coordinates is canceled out by an integration, as consequence of the unobservability of extra dimensions. Thus, by virtue of this extra dimension unobservability, we are able to recast the multidimensional Einstein equations into the four-dimensional Einstein-Yang-Mills ones, as well as all the right gauge transformations of fields are induced. The same analysis is performed for the Dirac equation describing the dynamics of the matter fields and, again, the gauge coupling with Yang-Mills fields are inferred from the multidimensional free fields theory, together with the proper spinors transformations.

[39] O.M. Lecian and G. Montani, On the Kaluza-Klein geometrization of the Electro-Weak model within a gauge theory of the 5-dimensional Lorentz group, *Int. J. Mod. Phys. D*, **15**, 717 (2006).

The geometrization of the Electroweak Model is achieved in a five-dimensional RiemannCartan framework. Matter spinorial fields are extended to 5 dimensions by the choice of a proper dependence on the extracoordinate and of a normalization factor. weak hypercharge gauge fields are obtained from a KaluzaKlein scheme, while the tetradic projections of the extradimensional contortion fields are interpreted as weak isospin gauge fields. generators are derived by the identification of the weak isospin current to the extradimensional current term in the Lagrangian density of the local Lorentz group. The geometrized U(1) and SU(2) groups will provide the proper transformation laws for bosonic and spinorial fields. Spin connections will be found to be purely Riemannian.

[40] V. Lacquaniti and Giovanni Montani, On the ADM decomposition of the 5D Kaluza-Klein model, *Int.J. Mod. Phys. D*, **15**, 559 (2006).

Our purpose is to recast the KK model in terms of ADM variables. We examine and solve the problem of the consistency of this approach, with particular care about the role of the cylindricity hypothesis. We show in detail how the KK reduction commutes with the ADM slicing procedure and how this leads to a well-defined and unique ADM reformulation. This allows us to consider the Hamiltonian formulation of the model and moreover it can be viewed as the first step for the Ashtekar reformulation of the KK scheme. Moreover, we show how the time component of the gage vector arises naturally from the geometrical constraints of the dynamics; this is a positive check for the autoconsistency of the KK theory and for an Hamiltonian description of the dynamics which will take into account the compactification scenario; this result enforces the physical meaning of the KK model.

[41] F. Cianfrani and G. Montani, Geometrization of the electro-weak model bosonic component, *Int. J. Theor. Phys.*, **46**(3), 471 (2007).

In this work we develop a geometrical unification theory for gravity and the electro-weak model in a Kaluza-Klein approach; in particular, from the curvature dimensional reduction Einstein-Yang-Mills action is obtained. We consider two possible space-time manifolds: 1)$V^4 \otimes S^1 \otimes S^2$ where isospin doublets are identified with spinors; 2) $V^4 \otimes S^1 \otimes S^3$ in which both quarks and leptons doublets can be recast into the same spinor, such that the equal number of quark generations and leptonic families is explained. Finally a self-interacting complex scalar field is introduced to reproduce the spontaneous symmetry breaking mechanism; in this respect, at the end we get an Higgs fields whose two components have got opposite hypercharges.

[42] F. Cianfrani and G. Montani, The Electro-Weak model as low-energy sector of 8-dimensional General Relativity, *Nuovo Cimento B*, **122**, 213 (2007).

In a Kaluza-Klein background $V^4 \otimes S^3$, we provide a way to reproduce, by the dimensional reduction, a 4-spinor with a SU(2) gauge coupling. Since additional gauge violating terms cannot be avoided, we compute their order of magnitude by virtue of the application to the Electro-Weak model.

[43] F. Cianfrani, I. Milillo and G. Montani, Dixon-Souriau equations from a 5-dimensional spinning particle in a Kaluza-Klein framework, *Phys. Lett. A*, **366**, 7 (2007).

The dimensional reduction of Papapetrou equations is performed in a 5-dimensional KaluzaKlein background and DixonSouriau results for the motion of a charged spinning body are obtained. The splitting provides an electric dipole moment, and, for elementary particles, the induced parity and time-reversal violations are explained.

[44] F. Cianfrani and G. Montani, Spinning particles in General Relativity, *Nuovo Cimento B*, **122**, 173 (2007).

We analyze the behavior of a spinning particle in gravity, both from a quantum and a classical point of view. We infer that, since the interaction between the space-time curvature and a spinning test particle is expected, then the main features of such an interaction can get light on which degrees of freedom have physical meaning in a quantum gravity theory with fermions. Finally, the dimensional reduction of Papapetrou equations is performed in a 5-dimensional Kaluza-Klein background and Dixon-Souriau results for the motion of a charged spinning body are obtained.

[45] O.M. Lecian and G. Montani, Electro-weak Model within the framework of Lorentz gauge theory: Ashtekar variables?, *Nuovo Cimento B*, **122**, 207-212 (2007).

The Electroweak (EW) model is geometrized in the framework of a 5D gauge theory of the Lorentz group, after the implementation of the Kaluza-Klein (KK) paradigm. The possibility of introducing Ashtekar variables on a 5D KK manifold is considered on the ground of its geometrical structure.

[46] V. Lacquaniti and G. Montani, Hamiltonian Formulation of 5-dimensional Kaluza-Klein Theory, *Nuovo Cimento B*, **122**, 201-206 (2007).

We analyze the consistency of the ADM approach to KK model; we prove that KK reduction commute with ADM splitting. This leads to

a well defined Hamiltonian; we provide the outcome. The electromagnetic constraint is derived from a geometrical one and this result enforces the physical meaning of KK model. Moreover we study the role of the extra scalar field we have in our model; classical hints from geodesic motion and cosmological solutions suggest that the scalar field can be an alternative time variable in the relational point of view.

[47] F. Cianfrani and G. Montani, Low-energy sector of 8-dimensional General Relativity: Electro-Weak model and neutrino mass, *Int. J. Mod. Phys. D*, **17**(5), 785 (2008).

In this paper we demonstrate that in a Kaluza-Klein space-time $V^4 \otimes S^3$ the dimensional reduction of spinors provides a 4-field, whose associated SU(2) gauge connections are geometrized. However, additional and gauge-violating terms arise, but they are highly suppressed by a factor β, which fixes the amount of the spinor dependence on extra-coordinates. The application of this framework to the Electro-Weak model is performed, thus giving a lower bound for β from the request of the electric charge conservation. Moreover, we emphasize that also the Higgs sector can be reproduced, but neutrino masses are predicted and the fine-tuning on the Higgs parameters can be explained, too.

[48] F. Cianfrani and G. Montani, Elementary particle interaction from a Kaluza-Klein scheme, *Int. J. Mod. Phys. A*, **23**, 8, 1182-1189 (2008).

We discuss properties of particles and fields in a multi-dimensional space-time, where the geometrization of gauge interactions can be performed. As far as spinors are concerned, we outline how the gauge coupling can be recognized by a proper dependence on extra-coordinates and by the dimensional reduction procedure. Finally applications to the Electro-Weak model are presented.

[49] O.M. Lecian and G. Montani, Fundamental Symmetries of the extended Spacetime, *Int. J. Mod. Phys. A*, **23**, 1266-1269 (2008).

On the basis of Fourier duality and Stone-von Neumann theorem, we will examine polymer-quantization techniques and modified uncertainty relations as possible 1-extraD compactification schemes for a phenomenological truncation of the extraD tower.

[50] V. Lacquaniti and G. Montani, On matter coupling in 5D Kaluza-Klein framework, *Int. J. Mod. Phys. A* **23**, 1270-1273 (2008).

We analyze some unphysical features of the geodesic approach to matter coupling in a compactified Kaluza-Klein scenario, like the q/m puzzle and the huge massive modes. We propose a new approach, based on Papapetrou multipole expansion, that provides a new equation for

the motion of a test particle. We show how this equation provides right couplings and does not generate huge massive modes.

[51] V. Lacquaniti and Giovanni Montani, Dynamics of Matter in a Compactified Kaluza-Klein Model, *Int.J. Mod. Phys. D*, **18**, 929 (2009).

A longstanding problem in Kaluza-Klein models is the description of matter dynamics. Within the 5D model, the dimensional reduction of the geodesic motion for a 5D free test particle formally restores electrodynamics, but the reduced 4D particle shows a charge-mass ratio that is upper bounded, such that it cannot fit to any kind of elementary particle. At the same time, from the quantum dynamics viewpoint, there is the problem of the huge massive modes generation. We present a criticism against the 5D geodesic approach and face the hypothesis that in Kaluza-Klein space the geodesic motion does not deal with the real dynamics of test particle. We propose a new approach: starting from the conservation equation for the 5D matter tensor, within the Papapetrou multipole expansion, we prove that the 5D dynamical equation differs from the 5D geodesic one. Our new equation provides right coupling terms without bounding and in such a scheme the tower of massive modes is removed.

[52] V. Lacquaniti and G. Montani, *Geometry and Matter Reduction in a 5D Kaluza-Klein Framework*, *Mod. Phys. Lett. A*, **24**, No. 20, 1565 (2009).

In this paper we consider the Kaluza-Klein fields equations in presence of a generic 5D matter tensor which is governed by a conservation equation due to 5D Bianchi identities. Following a previous work, we provide a consistent approach to matter where the problem of huge massive modes is removed, without relaxing the compactification hypotheses; therefore we perform the dimensional reduction either for metric fields and for matter, thus identifying a pure 4D tensor term, a 4D vector term and a scalar one. Hence we are able to write down a consistent set of equations for the complete dynamics of matter and fields; with respect to the pure Einstein-Maxwell system we now have two additional scalar fields: the usual dilaton one plus a scalar source term. Some significant scenarios involving these terms are discussed and perspectives for cosmological applications are suggested.

[53] V. Lacquaniti, G. Montani and F. Vietri, *Dimensional Reduction of the 5D Kaluza-Klein Geodesic Deviation Equation*, to appear on *Gen. Rel. Grav.*.

In a work of Kerner et al. (2001) the problem of the geodesic deviation in 5D Kaluza-Klein background is faced. The 4D space-time projection of the resulting equation coincides with the usual geodesic deviation equation in the presence of the Lorenz force, provided that the fifth component of the deviation vector satisfies an extra constraint which

takes into account the q/m conservation along the path. The analysis was performed setting as a constant the scalar field which appears in Kaluza-Klein model. Here we focus on the extension of such a work to the model where the presence of the scalar field is considered. Our results coincide with those of Kerner et al. when the minimal case $\phi = 1$ is considered, while it shows some departures in the general case. The novelty due to the presence of ϕ is that the variation of the q/m between the two geodesic line is not conserved during the motion; an exact law for such a behavior has been derived.

7. Quantum Gravity

7.1. The cosmological sector of Loop Quantum Gravity

A cosmological space-time is assumed to be homogeneous and isotropic. The metric compatible with these assumptions and with Einstein equations is the Friedman-Robertson-Walker one, *i.e.*

$$ds^2 = -dt^2 + a(t)^2\left(\frac{1}{1+kr}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right), \tag{7.1.1}$$

where $k = 1, 0, -1$ for a closed, flat and open Universe, respectively. It is worth noting that the scale factor a is the only dynamical variable, which on spatial hypersurfaces behaves as a conformal factor in front of the fiducial line element

$$^0dl^2 = \frac{1}{1+kr}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{7.1.2}$$

Loop Quantum Cosmology (LQC) is based on fixing Ashtekar-Barbero-Immirzi connections and densitized 3-bein vectors as follows

$$A^a_i = c\,{}^0e^a_i, \qquad E^i_a = p\sqrt{{}^0h}\,{}^0e^i_a, \tag{7.1.3}$$

where ${}^0e^a_i$ and ${}^0e^i_a$ denote 3-bein vectors of the fiducial metric ${}^0h_{ij}$ and their inverses, respectively, while

$$|p| = a^2, \qquad c = \frac{1}{2}(k + \gamma\dot{a}). \tag{7.1.4}$$

The most general connections and momenta compatible with the FRW metric (7.1.1) are obtained from the expressions (7.1.3) by a generic SU(2) transformation. This means that although the metric has been partially fixed, nevertheless the local SU(2) gauge symmetry is not lost (this is not surprising, because such gauge transformations are related with rotations in the tangent space).

Let us now depict a possible description of a cosmological space-time in terms of LQG variables. Holonomies h^a_α are now being evaluated along straight edges α parallel to ${}^0e^i_a$, so finding

$$h^a_\alpha = e^{i\mu c\, {}^j\tau_a}, \tag{7.1.5}$$

μ being the edge length, $\mu = \int_\alpha {}^0e^a_i \frac{d\alpha^i}{dt} dt$, while ${}^j\tau_a$ denotes the SU(2) generator in the j-representation. In what follows we will label the holonomies by h^a_μ.

Similarly, fluxes $E_a(S)$ are restricted to those ones across surfaces S, $x^i = x^i(u,v)$, whose normal coincide with ${}^0e^a_i$ and their classical expression reads

$$E_a(S) = p\Delta, \qquad \Delta = \int_S {}^0e^i_a \epsilon_{ijk} \partial_u x^j \partial_v x^k du dv, \tag{7.1.6}$$

where Δ gives the flux of ${}^0e^i_a$ through S, so it measures the area of S itself in the fiducial metric. In the following, Δ will be used as a label for E_a.

If S and α intersects, the action of fluxes on holonomies on a quantum level according with LQG gives

$$\hat{E}_a(\Delta) h^b_\mu = 8\pi\gamma l_P^2 h^b_\mu {}^j\tau_a \delta^a_b \text{sign}\Delta\mu \tag{7.1.7}$$

where in the last relation repeated indexes are not summed.

Substituting the expression for $E_a(S)$ in terms of p, one finds

$$\hat{p}\Delta h^a_\mu = 8\pi\gamma l_P^2 h^a_\mu {}^j\tau_a \text{sign}\Delta\mu, \tag{7.1.8}$$

but from the classical Poisson brackets expression, the operator p can be represented as

$$\hat{p} = -i\frac{8\pi\gamma l_P^2}{3V_0}\frac{d}{dc}, \tag{7.1.9}$$

whose action on holonomies (7.1.5) gives

$$\hat{p} h^a_\mu = \frac{8\pi\gamma l_P^2 \mu}{3V_0} h^a_\mu {}^j\tau_a. \tag{7.1.10}$$

Therefore, relations (7.1.8) and (7.1.10) are consistent when

$$|\Delta\mu| = 3V_0. \tag{7.1.11}$$

This relation fixes a fundamental duality between the length of edges across which holonomies are evaluated and the area of surfaces across which fluxes are defined.

Within this scheme it is possible to establish a clear correspondence between the Hilbert space of functions of holonomies (7.1.5) and the one of quasi-periodic functions proper of LQC. This correspondence is realized via the trace on SU(2) indexes.

In fact, tracing both sides of Eq. (7.1.7) one gets

$$tr(E_a(S)h^a_\mu) = 2\hat{p}|\Delta|\Sigma_{n=0}^{j-\theta}\cos\left(\mu c(n+\theta)\right) =$$
$$= 8\pi\gamma l_P^2 tr(h^{a\,j}_\mu\tau_a) = -16\pi\gamma l_P^2\Sigma_{n=0}^{j-\theta} n\theta\sin\left(\mu c(n+\theta)\right), \quad (7.1.12)$$

where $\theta = 1/2, 0$ for j half-integer and integer, respectively.

It is worth noting that after the trace has been performed, linear combinations of quasi-periodic functions come out.

As soon as the action of $\hat{p}$ on such quasi-periodic functions is concerned, one immediately finds

$$\hat{p}e^{i\widetilde{\mu}c} = \frac{8\pi\gamma l_p^2}{3V_0}\widetilde{\mu}e^{i\widetilde{\mu}c}. \quad (7.1.13)$$

It is worth noting that in LQG two kind of information are present, the one related with the edge length μ and the one giving the SU(2) quantum number n. These two notions are condensed in the factor $\widetilde{\mu} = n\mu$, such that the SU(2) gauge structure is not manifest. However such an information is required to infer the area spectrum.

In fact, within this scheme, the regularized area operator can be represented by the square root of $\hat{p}^2\Delta^2$, thus its action on quasi periodic functions is

$$\hat{A}e^{i\mu nc} = \sqrt{\hat{p}^2\Delta^2}e^{i\mu nc} = 8\pi\gamma l_p^2\theta|n|e^{i\mu nc}. \quad (7.1.14)$$

Hence, the area operator has a discrete spectrum whatever value takes the parameter μ. Indeed, the spectrum do not coincide with the one of the fundamental theory, which is related with the Casimir of the SU(2) group.

Therefore, the procedure adopted in LQC to infer the parameter $\bar{\mu}$ required for the super-Hamiltonian regularization cannot be justified on the level of the area discrete spectrum. By other words, the existence of a low-bound for μ is not a consequence of fundamental properties of LQG and this shortcoming of the previous derivation leaves open the question about the proper implementation of the dynamical constraint.

As soon as the super-Hamiltonian is concerned, the corresponding operator is inferred from the following expression one deals with in LQG

$$H = -\frac{1}{32\pi^2\gamma^3 l_P^4}\sum_v H_v, \quad (7.1.15)$$
$$H_v = -\epsilon^{ijk}Tr[h(s_{ij})h(s_k)[V, h^{-1}(s_k)]], \quad (7.1.16)$$

where the sum is on all vertices v of the graph on which H acts, while s_{ij} denotes the square starting in v with edges along directions ij and s_k the edge along k. All holonomies in the expression (7.1.16) are in the fundamental representation. V is the volume operator in the full space.

The restriction to a FRW space-time implies to replace V and $h(s_i)$ with

$\hat{p}^{3/2}V_0$ and $h^a_{\bar{\mu}}$, $\bar{\mu}$ being the value at which the regularization should take place, respectively. From Eq. (7.1.10) one finds

$$[V, h^a_{\bar{\mu}}] = V_0[\hat{p}^{3/2}, h^a_{\bar{\mu}}] = 8\pi\gamma\bar{\mu}l_P^2\hat{p}^{1/21/2}\tau_a h^a_{\bar{\mu}}, \tag{7.1.17}$$

which reproduces the following expression when inserted into the super-Hamiltonian (7.1.16)

$$H = -\sum_v \frac{3\bar{\mu}}{8\pi l_P^2\gamma^2}\hat{p}^{1/2}\hat{\sin}^2\bar{\mu}c. \tag{7.1.18}$$

If we assume that each vertex gives the same contribution, then H can be written as

$$H = -\frac{3N_v\bar{\mu}^3}{8\pi l_P^2\gamma^2\bar{\mu}^2}\hat{p}^{1/2}\hat{\sin}^2\bar{\mu}c, \tag{7.1.19}$$

N_v being the total number of vertices of the fundamental graph underlying the continuous space-time manifold. It is worth noting that the two expression (7.1.19) coincides with the analogous one in LQC if

$$V_0 = N_v\bar{\mu}^3 \rightarrow \bar{\mu} = \left(\frac{V_0}{N_v}\right)^{1/3}. \tag{7.1.20}$$

Therefore, the assumption that the regularized super-Hamiltonian retains the same expression as in LQC links $\bar{\mu}$ with the total number of vertices.

7.2. Semiclassical isotropization during a deSitter phase

When describing the early Universe dynamics, it is convenient to distinguish between the variables α, the isotropic component, and β_{ab}, the anisotropies. In the ADM decomposition, the metric of a generic cosmological model can be written in the form

$$ds^2 = N^2(t)dt^2 - e^{2\alpha}(e^{2\beta})_{ab}\,\omega^a \otimes \omega^b\,, \tag{7.2.1}$$

where α, N and β_{ab} are space-time functions, while ω^a ($a = 1, 2, 3$) denote the 1-forms of the spatial metric. The matrix β_{ab} is taken diagonal and with a vanishing trace, and it has only two independent components, the so-called

Misner variables $\beta_\pm$, which are defined in terms of β_{ab} as follows

$$\begin{aligned}\beta_{11} &= \beta_+ + \sqrt{3}\beta_- \\ \beta_{22} &= \beta_+ - \sqrt{3}\beta_- \\ \beta_{33} &= -2\beta_+ .\end{aligned} \quad (7.2.2)$$

When all spatial gradients are neglected, the super-Hamiltonian constraint in the presence of a scalar field is given by

$$-p_\alpha^2 + p_+^2 + p_-^2 + p_\phi^2 + e^{4\alpha}U(\beta_+, \beta_-) + e^{6\alpha}V(\phi) = 0\,, \quad (7.2.3)$$

in which, p_α and $p_\pm$ are the conjugate momenta to α and $\beta_\pm$, respectively, while V denotes the interaction potential of the scalar field ϕ.

The presence of the potential $U(\beta_+, \beta_-)$ is due to the spatial curvature of the specific model and it is negligible when the Kasner-like regime holds (Bianchi I model), while in the Bianchi IX case it reads as

$$\begin{aligned} U(\beta_+\beta_-) = e^{-8\beta_+} + e^{4(\beta_+ + \sqrt{3}\beta_-)} + e^{4(\beta_+ - \sqrt{3}\beta_-)} - \\ -2\left[e^{4\beta_+} + e^{-2(\beta_+ + \sqrt{3}\beta_-)} + e^{-2(\beta_+ - \sqrt{3}\beta_-)}\right] \,. \end{aligned} \quad (7.2.4)$$

It has been show that a classical limit for the gravitational field dynamics can not take place before the Mixmaster ends and therefore the scenario we are addressing here requires that inflation emerges from a quantum (or a semiclassical) phase of the Universe. Thus, despite its apparent simplicity, the following Hamiltonian constraint

$$-p_\alpha^2 + p_+^2 + p_-^2 + p_\phi^2 + e^{6\alpha}\rho_\Lambda(x^\gamma) = 0 \quad (7.2.5)$$

properly describes a real phase of the early Universe evolution.

The canonical quantum dynamics is implemented by the requirement that the constraint (7.2.3) is translated into an operator annihilating the local state function $\psi_x(\alpha, \beta_\pm)$. The Universe wavefunctional is then obtained as the infinite product of state functions taken on independent horizons, say $\Psi = \Pi_x \psi_x$. To avoid many of the subtle questions concerning the inhomogeneous functional sector, we address our main goal, the possibility that the vacuum energy isotropizes the Universe on a quantum or a semiclassical level, by the analysis of the homogeneous Bianchi IX cosmological model. Apart from the heuristic character of the long-wavelength approximation and some technicalities concerning the supermomentum constraint, we are really confident that the simplified homogeneous analysis already contains all the physical ingredients to qualitatively describe the sub-horizon physics even in the generic inhomogeneous case.

Let us now investigate the quantum dynamics of the Bianchi IX model, starting from the Bianchi type I.

At first, momenta are replaced by derivatives in the conjugate coordinates. Then, the super-Hamiltonian constraint $H = 0$ leads to the Wheeler-DeWitt equation (WDW), *i.e.*

$$\left[e^{c\alpha}\partial_\alpha\left(e^{-2b\alpha}\partial_\alpha e^{c\alpha}\right) - e^{-3\alpha}\Delta + e^{3\alpha}V(\phi)\right]\Psi_I(\alpha,\beta^r) = 0 \tag{7.2.6}$$

in which the Laplacian in the variables $\beta^r = \beta_\pm, \phi$ is

$$\Delta = \partial_+^2 + \partial_-^2 + \partial_\phi^2 \,.$$

In Eq. (7.2.6) we wrote a symmetric super-Hamiltonian operator, introducing generic parameters b and $c = b - \frac{3}{2}$. Such a choice for the operator ordering is not the most general one, but it captures several cases. Moreover, we will emphasize that the value of the parameter b does not affect the proposed semi-classical picture.

Approaching the singularity the scalar field potential term can be neglected and the solution is given by

$$\Psi_I(\alpha,\beta_\pm,\phi) = e^{\frac{3}{2}\alpha}\int dk_+ dk_- dk_\phi \sqrt{\frac{2}{3\,K_k}}\left(a_k\, e^{\frac{3}{2}\imath K\alpha + \imath k_r\beta^r} + b_k\, e^{-\frac{3}{2}\imath K_k\alpha + \imath k_r\beta^r}\right), \tag{7.2.7}$$

in which $k_r = \{k_\pm, k_\phi\}$, while a_k and b_k denote weights of the Fourier expansion. The conjugate momentum to the variable α is

$$K = \frac{2}{3}\sqrt{\varepsilon^2 - b^2} \quad , \quad \varepsilon^2 = k_+^2 + k_-^2 + k_\phi^2 \,. \tag{7.2.8}$$

It is worth noting that the parameter b labeling the operator ordering enters merely the definition of K. This fact means that b fixes the interval in the ϵ-line where the solution has an oscillatory behavior. Because, we are interested in developing wave-packets, we assume wave-functions to be negligible for $\epsilon \lesssim b$. In this regime, expectation values have actually no b-dependence.

The Hilbert space is $\mathcal{L}^2(\beta^r, d\mu)$, where the scalar product is given by

$$<\psi_2|\psi_1> = \frac{\imath}{2}\int e^{-3\alpha}\left[\Psi_2^*(\partial_\alpha\Psi_1) - (\partial_\alpha\Psi_2^*)\,\Psi_1\right] d\beta_+ d\beta_- d\phi \tag{7.2.9}$$

which is positive-defined as far as the proper separation of frequencies occurs by fixing $a_k = 0$ (such that the case of an expanding Universe is selected out).

The localization of the solution (7.2.7) in the phase space at a given initial position β_0^r is achieved by the following Gaussian wave packets

$$b_k = \frac{1}{(2\pi\,\sigma^2)^{\frac{3}{4}}} \exp\left(-\frac{\delta k_+^2 + \delta k_-^2 + \delta k_\phi^2}{4\,\sigma^2} \right) e^{-ik_r \beta_0^r}\,, \tag{7.2.10}$$

where $\delta k_r = k_r - \bar{k}_r$, while the variances have the same values σ.

The evolution of wave-packets is investigated by virtue of a saddle point expansion around $\bar{k}_r$ and a proper semi-classical behavior is found. In particular, the evaluation of expectation values and variances gives

$$\langle \beta^r \rangle = \beta_0^r - \bar{v}_r\,\alpha + \mathcal{O}(\bar{\varepsilon}^{-\frac{3}{2}}), \qquad \Sigma_r \bar{v}_r^2 = 1, \tag{7.2.11}$$

$$\langle \Delta\beta^{r\,2} \rangle = \frac{1}{4\sigma^2} + \frac{\sigma^2\,\alpha^2}{\bar{\varepsilon}^2} + \mathcal{O}(\bar{\varepsilon}^{-3}). \tag{7.2.12}$$

Hence, for sufficient high values of $\bar{\varepsilon} = \varepsilon(\bar{k}^r)$ the behavior of expectation values is approximated by the Kasner-like dynamics. The restriction to high values of $\bar{\varepsilon}$ corresponds to the well-known result that the semi-classical picture is inferred only for high values of the initial momenta. In what follows, we will consider the corrections of the $\bar{\varepsilon}^{-\frac{3}{2}}$ order to be negligible.

In this scheme a measure of the spread of wave packets is given by the ratio of the square root of variances with $\langle \beta^r \rangle$. In particular, such a quantity at late times goes as

$$\frac{\sqrt{\langle \Delta\beta^{r\,2} \rangle}}{\langle \beta^r \rangle} \approx \frac{\sigma}{\bar{v}_r\,\bar{\varepsilon}} + \mathcal{O}(\bar{\varepsilon}^{-\frac{3}{2}})\,, \tag{7.2.13}$$

such that the wave-function spread tends to a constant value. This fact implies that proper initial conditions can be chosen for gaussian wave-packets such that they remain well-localized around expectation values and the semi-classical picture holds.

Therefore, Gaussian wave functions are proper semi-classical states for the Bianchi type I model in the presence of the scalar field.

The WDW equation for a Bianchi type IX model is modified by the presence of the 3-dimensional scalar curvature, which acts as a potential term and whose expectation value on semi-classical states gives

$$\lim_{\alpha\to-\infty} \langle e^{4\alpha} U \rangle \propto \lim_{\alpha\to-\infty} e^{\frac{8}{\sigma^2}} [e^{4\alpha(1+2\bar{v}_+)} + \\ + e^{4\alpha(1-\bar{v}_+-\sqrt{3}\bar{v}_-)} + e^{4\alpha(1-\bar{v}_++\sqrt{3}\bar{v}_-)}] + \mathcal{O}(\bar{\varepsilon}^{-\frac{3}{2}})\,. \tag{7.2.14}$$

This expression approaches 0 when

$$\begin{cases} 1+2\,\bar{v}_+ > 0 \\ 1-\bar{v}_+ - \sqrt{3}\,\bar{v}_- > 0 \\ 1-\bar{v}_+ + \sqrt{3}\,\bar{v}_- > 0 \end{cases} \Rightarrow \begin{cases} \bar{v}_+^2 < \frac{1}{4} \\ \bar{v}_-^2 < \frac{1}{12} \\ \frac{2}{3} < \bar{v}_\phi^2 < 1 \end{cases} . \tag{7.2.15}$$

The conditions above coincide with the relations found in a classical framework to remove the chaotic behavior. Hence, to restrict the domain of the parameters $\bar{v}_\pm$, $\bar{v}_\phi$ according with inequalities (7.2.15) guarantees that both the classical and the semi-classical dynamics of the Bianchi type IX model resembles that of a Bianchi type I space.

Therefore, the obtained results support the idea that the proposed scenario realizes a proper semi-classical description of the Early phases of the Universe.

A quasi-isotropization mechanism is required in order to suppress anisotropies, so reconciling the early Universe dynamics with its late evolution. Inflation can provide such a suppression on a classical level. Here we are going to realize the inflationary phase via the introduction of a scalar field (the inflaton), which acquires a non-vanishing vacuum expectation value modeled by a cosmological constant ρ_Λ.

The WDW equation associated with a Bianchi type I model in the presence of a scalar field and of a cosmological constant is given by (we fix $c = 0$, because as in the previous case a different operator ordering does not provide any significant modification to the quasi-isotropization mechanism)

$$e^{-3\alpha}\left[\partial_\alpha^2 - 3\partial_\alpha - \Delta + e^{6\alpha}\rho_\Lambda\right]\Psi(\alpha,\beta_\pm,\phi) = 0\,. \tag{7.2.16}$$

The solution of such an equation restricted to negative frequencies has the following form

$$\Psi(\alpha,\beta^r) = \int dk_+ dk_- dk_\phi b_k \frac{\Gamma(1+n_k)}{\sqrt{N_k}} J_{n_k}[z(\rho_\Lambda,\alpha)] e^{\frac{3}{2}\alpha - \imath k_r \beta^r}, \tag{7.2.17}$$

in which N_k is the normalization factor, K retains the form (7.2.8), while $\Gamma(1+n_k)$ and $J_{n_k}(z)$ denote the Gamma function and the Bessel function of the first kind, respectively, where $n_k = -\frac{\imath}{2}K$ and $z(\rho_\Lambda,\alpha) = \frac{\sqrt{\rho_\Lambda}}{3}e^{3\alpha}$.

Eq. (7.2.16) aims to describe the phase of the Universe when the transition from the anisotropic to the isotropic regime takes place. In order to characterize such a transition, the two relevant cases $z \ll 1$ and $z \gg 1$ are going to be discussed separately.

Let us consider the early phase, where $z = \frac{\sqrt{\rho_\Lambda}}{3}e^{3\alpha} \ll 1$ and the Bessel function can be expanded as follows

$$J_n(z) = \sum_{l=0}^{+\infty} \frac{\left(\frac{z}{2}\right)^{2l+n}}{\Gamma(1+l+n)\, l!} \longrightarrow \frac{\left(\frac{z}{2}\right)^n}{\Gamma(1+n)}\,, \tag{7.2.18}$$

such that the solution (7.2.17) can be approximated by the following asymptotic form

$$\Psi(\alpha', \beta^r) = e^{\frac{3}{2}\alpha'} \int dk_+ dk_- dk_\phi \sqrt{\frac{2}{3\,K}} b_k\, e^{-\imath\left(\frac{3}{2}K_k\alpha' + k_r\,\beta^r\right)}.$$

The expression above coincides to the solution of the Bianchi type I case in the presence of a scalar field (7.2.7) in terms of the re-defined isotropic variable α', which is given by

$$\alpha' = \alpha + \frac{1}{3}\ln\frac{\sqrt{\rho_\Lambda}}{6}. \tag{7.2.19}$$

The wave packets can be developed according with the procedure adopted in the previous cases and all quantities are now functions of α'.

The evaluation of expectation values for operators corresponding to phase-space coordinates can be carried on just like in the case of a Bianchi type I model. Hence, wave-packets remain well localized around the classical trajectory.

Therefore, for $\alpha \ll \frac{1}{3}\ln\frac{3}{\sqrt{\rho_\Lambda}}$, the presence of the cosmological constant term ρ_Λ does not modify significantly the semi-classical picture inferred for the early Universe dynamics, which can be described by the Bianchi type I model in the presence of a scalar field.

For $z \gg 1$, the Bessel functions can be approximated with the following expression

$$J_n(z) \approx \frac{1}{\sqrt{2\pi z}}\left[e^{\imath\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)} + e^{-\imath\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)}\right].$$

The solution of the WDW equation within this scheme turns out to be given by

$$\Psi(\alpha, \beta^r) = \int dk_+ dk_- dk_\phi \frac{b_k e^{-\imath(k_r\,\beta^r)}\, e^{-\imath\left(\frac{\sqrt{\rho_\Lambda}}{3}e^{3\alpha} - \frac{\pi}{4} - \frac{n_k}{2}\right)}}{\sqrt{\sqrt{\rho_\Lambda}\,\sinh(\imath n_k)}}. \tag{7.2.20}$$

The explicit computation of Gaussian wave-packets is performed by a saddle-point expansion around the expectation value. Finally, the behavior of the expectation values and of the variances at late time is given by

$$\langle\beta^r\rangle = \beta_0^r + \mathcal{O}(\bar{\varepsilon}^{-\frac{3}{2}}), \qquad \langle\Delta\beta^{r2}\rangle = \frac{1}{4\sigma^2} + \mathcal{O}(\bar{\varepsilon}^{-3}). \tag{7.2.21}$$

It is worth noting that in the adopted approximation scheme the evaluation of $\sqrt{\langle \Delta\beta^{r2} \rangle}/\langle \beta^r \rangle$ gives

$$\frac{\sqrt{\langle \Delta\beta^{r2} \rangle}}{\langle \beta^r \rangle} \approx \frac{1}{4\sigma^2 \beta_0^r} + \mathcal{O}(\bar{\varepsilon}^{-\frac{3}{2}}). \tag{7.2.22}$$

The quantity above does not depend on the time-like variable and it can be set smaller than any given quantity by a proper choice of initial conditions β_0^r and σ. This fact implies that the semi-classical picture is consistent with the adopted approximation scheme.

Furthermore, the expectation values of anisotropies freeze out to constant values, corresponding to the chosen initial conditions. Because constant β can always be avoided by a re-definition of the 1-form ω^a of the spatial metric, the final space does not contain anisotropies and this scenario offers a bridge between a Bianchi type I model and a late isotropic phase.

Hence, the quasi-isotropization mechanism works also in the semi-classical regime. Therefore, a cosmological constant term can determine the isotropization of the Universe, even if the spontaneous symmetry breaking process (by which a vacuum energy expectation value arises) takes place during the quantum phase of the Universe.

7.3. Canonical Quantum Gravity without the time gauge

The development of a proper Hilbert space representation for the diffeomorphism group is one of the most compelling issue in Quantum Gravity. The major achievements have been obtained by Loop Quantum Gravity (LQG), in which the action of spatial diffeomorphisms can be properly implemented on a quantum level and the invariant subspace can be defined. The situation with time re-parameterizations is different, since the associated generator can be represented on the kinematical Hilbert space, but the physical Hilbert space has not been achieved yet. For these reasons, LQG is the most promising approach to Quantum Gravity, but still not a definitive theory.

LQG is based on applying quantization techniques proper of lattice gauge theory, as soon as the emergence of a SU(2) gauge structure alá Yang-Mills at the Hamiltonian level has been recognized. This SU(2) symmetry has been inferred after a gauge fixing of the full local Lorentz group, via the so-called time-gauge condition. Our analysis is devoted to investigate whether this SU(2) gauge invariance can be find out without any restriction of the local Lorentz frame.

At first we consider the case of vacuum gravity [23], described by the Holst action, which reads as follows (in units $8\pi G = 1$)

$$S = \frac{1}{2}\int \sqrt{-g} e^{\mu}_{A} e^{\nu}_{B} R^{CD}_{\mu\nu}(\omega^{FG}_{\mu}) {}^{\gamma}p^{AB}{}_{CD} d^4x, \tag{7.3.1}$$

g being the determinant of the metric tensor $g_{\mu\nu}$ with 4-bein vectors e^A_μ and spinor connections ω^{AB}_μ, while the expressions for $R^{AB}_{\mu\nu}$ and ${}^{\gamma}p^{AB}{}_{CD}$ are

$$R^{AB}_{\mu\nu} = 2\partial_{[\mu}\omega^{AB}_{\nu]} - 2\omega^{A}{}_{C[\mu}\omega^{CB}_{\nu]}, \qquad {}^{\gamma}p^{AB}{}_{CD} = \delta^{AB}{}_{CD} - \frac{1}{2\gamma}\epsilon^{AB}{}_{CD}. \tag{7.3.2}$$

Here γ is the Immirzi parameter.

Let us take ω^{AB}_i as configuration variables. By a Legendre transformation, conjugate momenta ${}^{\gamma}\pi^i_{AB} = {}^{\gamma}p^{CD}{}_{AB}\pi^i_{CD}$ can be defined.

The full Hamiltonian turns out to be

$$\mathcal{H} = \int \left[\frac{1}{eg^{tt}}H - \frac{g^{ti}}{g^{tt}}H_i - \omega^{AB}_t {}^{\gamma}p^{CD}{}_{AB}G_{CD} + \lambda_{ij}C^{ij} + \eta_{ij}D^{ij}\right] d^3x, \tag{7.3.3}$$

where $1/eg^{tt}$, g^{ti}/g^{tt}, ${}^{\gamma}p^{CD}{}_{AB}\omega^{AB}_t$, λ_{ij}, are η_{ij} behave as Lagrangian multipliers, while constraints are given by

$$\begin{cases} H = \pi^i_{CF}\pi^{jF}{}_{D}\,{}^{\gamma}p^{CD}{}_{AB}R^{AB}_{ij} = 0 \\ H_i = {}^{\gamma}p_{AB}{}^{CD}\pi^j_{CD}R^{AB}_{ij} = 0 \\ G_{AB} = D_i\pi^i_{AB} = \partial_i\pi^i_{AB} - 2\omega_{i[A}{}^{C}\pi^i_{|C|B]} = 0 \\ C^{ij} = \epsilon^{ABCD}\pi^{(i}_{AB}\pi^{j)}_{CD} = 0 \\ D^{ij} = \epsilon^{ABCD}\pi^k_{AF}\pi^{(iF}{}_{B}D_k\pi^{j)}_{CD} = 0 \end{cases}. \tag{7.3.4}$$

H and H_i denote the super-Hamiltonian and the super-momentum, respectively, and their vanishing accounts for the invariance under time re-parameterizations and spatial diffeomorphisms, respectively. $G_{AB} = 0$ are the Gauss constraints of the Lorentz symmetry and the whole Hamiltonian formulation looks close to a Yang-Mills gauge theory for the local Lorentz group. But the presence of $C^{ij} = 0$ and $D^{ij} = 0$ makes the constraint algebra second-class and before performing the analysis of constraints the reduction to a first class set must be provided.

This reduction is performed by fixing $\omega^{ab}{}_i$ and π^i_{ab} such that the conditions

$C^{ij} = 0$ and $D^{ij} = 0$ hold identically. In particular we set

$$\omega_{a\ i}^{\ b} = {}^{\pi}\omega_{a\ i}^{\ b} + \chi_a \omega^{0b} + \chi^b(\omega_{a\ i}^{\ 0} - {}^{\pi}D_i\chi_a), \qquad \pi^i_{ab} = \chi_{[a}\pi^i_{b]} \tag{7.3.5}$$

π^i_b being π^i_{0b}, while ${}^{\pi}\omega_{a\ i}^{\ b} = \frac{1}{\pi^{1/2}}\pi^{b3}_l\nabla_i(\pi^{1/2}\pi^l_a)$ with π the determinant of π^a_i and $T^{-1}_{ab} = \eta_{ab} + \chi_a\chi_b$.

The function χ_a are three arbitrary space-time functions which gives the components e^t_a of the frame. Hence, χ_a are promoted to configuration variables, such that no gauge fixing of the local Lorentz frame occurs, while second-class constraints are solved.

Therefore, we defined the new phase coordinates as χ_a and conjugate momenta π^a, while other variables are fixed such that the associated momenta are densitized 3-bein of the spatial metric $\widetilde{\pi}^i_a$, which read as follows

$$\widetilde{\pi}^i_a = S^b_a\pi^i_b, \qquad S^a_b = \sqrt{1+\chi^2}\delta^a_b + \frac{1-\sqrt{1+\chi^2}}{\chi^2}\chi_a\chi_b, \tag{7.3.6}$$

From the analysis of the induced symplectic form, remaining configuration variables are

$$\begin{aligned} {}^{(\gamma)}\widetilde{A}^a_i = S^{-1a}_{b}\bigg(&\gamma(1+\chi^2)T^{bc}\left(\omega_{0ci} + {}^{\pi}D_i\chi_c\right) - \\ &-\frac{1}{2}\epsilon^b_{\ cd}\,{}^{\pi}\omega^{cf}_{\ \ i}T^{-1d}_{\ \ f} + \frac{2+\chi^2-2\sqrt{1+\chi^2}}{2\chi^2}\epsilon^{bcd}\partial_i\chi_c\chi_d\bigg). \end{aligned} \tag{7.3.7}$$

In the adopted set of coordinates the conditions $G_{AB} = 0$ are equivalent to

$$G_a = \partial_i{}^{(\gamma)}\widetilde{\pi}^i_a + \epsilon_{abc}{}^{(\gamma)}\widetilde{A}^b_i{}^{(\gamma)}\widetilde{\pi}^i_c = 0. \qquad \pi^a = 0. \tag{7.3.8}$$

It is worth noting that

- the Gauss constraints of a SU(2) gauge structure arises also when the time-gauge condition is relaxed, such that $({}^{(\gamma)}\widetilde{A}^a_i$ are *generalized Ashtekar-Barbero-Immirzi connections*;

- χ_a are non-dynamical variables.

Summarizing the previous analysis, the action of GR with the Holst modification can be written in a generic local Lorenz frame as follows

$$S = \int d^4x\left[{}^{(\gamma)}\widetilde{\pi}^i_a\partial_t{}^{(\gamma)}\widetilde{A}^a_i + \pi^a\partial_t\chi_a - \frac{1}{\sqrt{g g^{tt}}}H + \frac{g^{ti}}{g^{tt}}H_i + \eta^a G_a + \lambda_a\pi^a\right].$$

The set of kinematical Hamiltonian constraints reproduces a background-independent SU(2) gauge theory.

As soon as a quantum description is addressed, any dependence from χ_a variables can be avoided (as for the lapse function and the shift vector). Therefore, *the LQG standard quantization in terms of holonomies and fluxes of the SU(2) group works, even though no gauge fixing of the local Lorentz frame has been performed.* As a consequence, the discrete geometrical operator spectra which are inferred on a quantum level are invariant under local Lorentz transformations and no modification of the local Lorentz symmetry is expected to be induced by the existence of a minimal length.

The same achievements are obtained in presence of fundamental matter fields:

- in [24] a non-minimally coupled scalar field ϕ is added. This case is interesting since it mimics some feature of $f(R)$ model for gravity, as soon as a scalar-tensor representation is addressed.

  The full action action reads

$$S = \int \sqrt{-g}\left[F(\phi)e^{\mu}_{A}e^{\nu}_{B}R^{CD}_{\mu\nu}{}^{\gamma}p^{AB}{}_{CD} + \frac{1}{2}g^{\mu\nu}K(\phi)\partial_{\mu}\phi\partial_{\nu}\phi - V(\phi)\right]d^4x, \tag{7.3.9}$$

  V being the potential, while a non-standard kinetic term is considered, by taking an arbitrary function $K(\phi)$.

  The function $F(\phi) - 1$ gives the amount of the non-minimal coupling between the geometry and the scalar field.

  The Hamiltonian analysis is performed along the lines of the vacuum case and the conditions (7.3.8) still arise. Therefore, the LQG quantization can be applied also in this case and this feature opens interesting perspectives for the quantum analysis of $f(R)$-models.

  It is worth noting that in view of the non-minimal coupling the fundamental quantity on a quantum level is the re-scaled metric ${}^{\phi}h_{ij} = F(\phi)h_{ij}$, such that the field ϕ enters the spectra of geometrical operators.

- in [25] $\beta = 1/\gamma$ is promoted to be an external scalar field, in order to remove the Immirzi ambiguity. The associated Lagrangian density is developed via a canonical kinetic term and a potential $V(\beta)$, such that the full action reads

$$S = \int \sqrt{-g}\left[e^{\mu}_{A}e^{\nu}_{B}\left(R^{AB}_{\mu\nu} - \frac{\beta}{2}R^{CD}_{\mu\nu}\epsilon^{AB}{}_{CD}\right) + \frac{1}{2}g^{\mu\nu}\partial_{\mu}\beta\partial_{\nu}\beta - V(\beta)\right]d^4x. \tag{7.3.10}$$

To carry on the procedure adopted in the previous cases, second-class constraints solutions have to be generalized as follows

$$\omega_{a\ i}^{\ b} = {}^{\pi}\omega_{a\ i}^{\ b} + \chi_a\omega^{0b} + \chi^b(\omega_{a\ i}^{\ 0} - {}^{\pi}D_i\chi_a) + {}^{1}\omega_{a\ i}^{\ b} \qquad \pi^i_{ab} = 2\chi_{[a}\pi^i_{b]} \tag{7.3.11}$$

where the modification reads

$${}^{1}\omega^{ab}_{i} = T_c^{[a}\left(-\frac{2(1+\chi^2)^2}{\chi^4+2\chi^2+2}\eta^{b]d} + \frac{2+\chi^2}{\chi^4+2\chi^2+2}\chi^{b]}\chi^d\right)\pi^c_i\pi^j_d\frac{\beta\partial_j\beta}{\beta^2+1}. \tag{7.3.12}$$

The new set of canonically conjugate variables is given by $\{{}^{(\beta)}\widetilde{A}^a_i, \chi_b, {}^{(\beta)}\widetilde{\pi}^j_c, \pi^d, \beta, {}^{(\beta)}\widetilde{\pi}\}$, where configuration variables associated with the geometry take the expressions

$$\begin{aligned}{}^{(\beta)}\widetilde{A}^a_i = S_b^{-1a}\Bigg(\frac{1+\chi^2}{\beta}T^{bc}\left(\omega_{0ci} + {}^{\pi}D_i\chi_c - \frac{1}{1+\chi^2}{}^{1}\omega_{c\ i}^{\ d}\chi_d\right) - \\ -\frac{1}{2}\epsilon^b_{\ cd}({}^{\pi}\omega^{cf}_{\ \ i}T^{-1d}_{f} + {}^{1}\omega^{cd}_{\ \ i}) + \frac{2+\chi^2-2\sqrt{1+\chi^2}}{2\chi^2}\epsilon^{abc}\partial_i\chi_b\chi_c\Bigg),\end{aligned} \tag{7.3.13}$$

while conjugate momentum to β changes as

$${}^{(\beta)}\widetilde{\pi} = {}^{\beta}\pi - \frac{1}{\beta}{}^{(\beta)}\widetilde{\pi}^i_a\left({}^{(\beta)}\widetilde{A}^a_i + \frac{1}{2}\epsilon^a_{\ bc}{}^{(\beta)}\widetilde{\omega}^{bc}_i\right). \tag{7.3.14}$$

Also in this scheme geometrical variables ${}^{(\beta)}\widetilde{\pi}^i_a$ describe the fictitious rescaled spatial metric ${}^{\beta}h_{ij} = \beta h_{ij}$ and the true one h_{ij} is a derived quantity.

Finally, the kinematical sector is given by the constraints

$$G_a = \partial_i{}^{(\beta)}\widetilde{\pi}^i_a + \epsilon_{ab}^{\ \ c}{}^{(\beta)}\widetilde{A}^b_i{}^{(\beta)}\widetilde{\pi}^i_c = 0, \qquad \pi^a = 0, \qquad H_i = {}^{(\beta)}\widetilde{\pi}^j_a{}^{\beta}\widetilde{F}^a_{ij} + {}^{(\beta)}\widetilde{\pi}\partial_i\beta = 0. \tag{7.3.15}$$

The emergence of the SU(2) Gauss constraints makes the whole LQG quantization procedure well-grounded, while χ_a do not play any dynamical role. As for the super-momentum, it coincides with the one of gravity in presence of a scalar field non-minimally coupled to gravity, so the kinematical Hilbert space is the same as for a scalar field.

The super-Hamiltonian is given by

$$\begin{aligned}H = \frac{{}^{(\beta)}\widetilde{\pi}^i_a{}^{(\beta)}\widetilde{\pi}^j_b}{2}\epsilon_{ab}^{\ \ c}{}^{(\beta)}\widetilde{F}^c_{ij} - \frac{(\beta^2+1)}{\beta^2}{}^{(\beta)}\widetilde{\pi}^i_a{}^{(\beta)}\widetilde{\pi}^j_b\left(\partial_{[i}{}^{(\beta)}\omega^{ab}_{\ \ j]} - {}^{(\beta)}\omega^{ac}_{\ \ [i}{}^{(\beta)}\omega_{c\ j]}^{\ b}\right) + \\ +\frac{1}{2}{}^{\beta}\pi^2 - \beta^{\beta}h^{\beta}h^{ij\,(\beta)}\widetilde{\nabla}_i\partial_j\left(\frac{1}{\beta}\right) + \frac{{}^{\beta}h^{\beta}h^{ij}}{2}\partial_i\beta\partial_j\beta + {}^{\beta}h\frac{V}{\beta^3},\end{aligned} \tag{7.3.16}$$

and it differs significantly with respect to the case a scalar field is present. Hence the Immirzi field has peculiar dynamical features, whose analysis will allow to identify it as a distinctive component of the cosmological bath, once the implementation of this scenario in the framework of Loop Quantum Cosmology is addressed.

Among these peculiarities, it is worth noting that if a quartic potential is assumed for β, *i.e.* $V(\beta) = \mu^2\beta^2 + \frac{1}{4}\lambda\beta^4$, a non-vanishing minimum is predicted for the effective potential

$$V_{eff}(\beta) = \frac{V}{\beta^3} = \frac{\mu^2}{\beta} + \frac{\lambda}{4}\beta \rightarrow \beta^2_{min} = 4\frac{\mu^2}{\lambda}. \tag{7.3.17}$$

Hence neglecting spatial gradients and the interaction with the geometry, *a dynamical relaxation to a non-vanishing expectation value is predicted for the Immirzi field*. This relaxation is able to explain the parametric role of γ in standard LQG. The contributions that oscillations around this minimum give to the dynamics of the gravitation field are actually under investigation.

- in [29] it has been added a spinor field with a nonminimal lagrangian density, whose action reads

$$S_\psi = \frac{i}{2}\int \sqrt{-g}\left[(\bar{\psi}\gamma^\mu(1+i\alpha\gamma_5)D_\mu\psi - D_\mu\bar{\psi}(1+i\alpha\gamma_5)\gamma^\mu\psi)\right] d^4x, \tag{7.3.18}$$

α being the nonminimal parameter.

The Hamiltonian analysis outlines that the constraints $C^{ij} = D^{ij} = 0$ are modified, such that new solutions are given by the following expressions

$$\pi^i_{ab} = 2\chi_{[a}\pi^i_{b]}, \qquad \omega_a{}^b{}_i = {}^\pi\omega_a{}^c{}_i T_c^{-1b} + \chi_a\omega^{0b}{}_i + \chi^b(\omega_a{}^0{}_i - \partial_i\chi_a) + {}^\psi\omega_a{}^b{}_i, \tag{7.3.19}$$

where

$${}^\psi\omega_a{}^b{}_i = \frac{1}{4}\frac{\gamma(\gamma-\alpha)}{(1+\gamma^2)\sqrt{1+\chi^2}}\epsilon^{ab}{}_c\pi^c_i(J^0+\chi_d J^d) - \frac{1}{2}\frac{\gamma(1+\alpha\gamma)}{\gamma^2+1}\pi^c_i T_c^{-1[a}\eta^{b]d}(J_d - \chi_d J^0), \tag{7.3.20}$$

with $J^A = \sqrt{h}\bar{\psi}\gamma_5\gamma^A\psi$.

By redefining connections as follows

$$\widetilde{A}_i^a = S_b^{-1a}\left((1+\chi^2)T^{bc}\left(\omega_{0ci} + {}^{\pi}D_i\chi_c\right) - \frac{1}{2\gamma}\epsilon^b{}_{cd}({}^{\pi}\omega^{cf}{}_i T^{-1d}_f + \right.$$
$$\left. + {}^{\psi}\omega^{cd}{}_i) + \frac{2+\chi^2-2\sqrt{1+\chi^2}}{2\gamma\chi^2}\epsilon^{bcd}\partial_i\chi_c\chi_d\right)\tag{7.3.21}$$

and boosting spinors, *i.e.* $\psi = e^{i\chi^a\Sigma_{0a}}\psi^*$, the SU(2) Gauss constraints with a source term arise, while χ_a turn out to be nondynamical,

$$\partial_i\widetilde{\pi}^i_a + \gamma\epsilon_{ab}{}^c\widetilde{A}^b_i\widetilde{\pi}^i_c = -\frac{\gamma}{2}J^*_a, \qquad \pi^a = 0, \tag{7.3.22}$$

where $J^*_A = \sqrt{h}\bar{\psi}^*\gamma_A\gamma_5\psi^*$ is the axial component of the fermion current.

The super-momentum and the super-Hamiltonian for the spinor part get simplified for $\alpha = \gamma$,

$$H^{\psi}_i = \frac{i}{2}\sqrt{h}(\bar{\psi}^*\gamma^0(1+i\gamma\gamma_5){}^{(A)}D_i\psi^* - {}^{(A)}D_i\bar{\psi}^*(1+i\gamma\gamma_5)\gamma^0\psi^*),$$
$$H^{\psi} = \frac{i}{2}\widetilde{\pi}^i_a\left(\bar{\psi}^*\gamma^a(1+i\gamma\gamma_5){}^{(A)}D_i\psi^* - {}^{(A)}D_i\bar{\psi}^*(1+i\gamma\gamma_5)\gamma^a\psi^*\right) - \frac{1+\gamma^2}{16\sqrt{h}}{}^AJ_c{}^AJ^c.$$

where ${}^{(A)}D_i\psi = \partial_i\psi - \frac{i}{2}\gamma\widetilde{A}^a_i T_a\psi$ and the gauge generator is $T_a = \epsilon_a{}^{bc}\Sigma_{bc}$.

Within this scheme, the Immirzi parameter resembles the coupling constant for a Yang-Mills SU(2) interaction. Moreover, it is worth noting the presence of the 4-fermion terms, which makes the theory nonrenormalizable in the perturbative approach.

7.4. The problem of time in quantum gravity

The definition of a proper time variable is among the most compelling issues in Quantum Gravity. This problem originates from the 4-diffeomorphism invariance of General Relativity, which implies that the total Hamiltonian is a linear combination of constraints and it vanishes on physical states. Therefore, no evolution at all is predicted as soon as the constraints are implemented alà Dirac on the wave-function.

A possible solution consists is using some kind of matter as a physical clock. In this respect, Brown and Kučhar considered the case in which a dust-fluid is coupled with the gravitational field and they found that the super-Hamiltoian constraint can be re-written in the form of a meaningful

Schrödingher equation, *i.e.*

$$\pi(x) - h(q, P)(x) = 0 \tag{7.4.1}$$

where π is the momentum conjugate to one of the fluid variables, τ. This way one can write down the equation for the physical *evolutionary* quantum states:

$$-i\hbar\frac{d}{d\tau}\psi = \hat{h}\psi, \tag{7.4.2}$$

where the notion of time is recovered from the coupling of GR with the fluid. This is the so called Kučhar-Brown mechanism, which is applied here to a perfect fluid composed by baryons (Schutz fluid).

The equation of state describing such a fluid depends on two parameters and it can be wrtitten as

$$p = \left(\frac{\mu}{1+\Pi} - 1\right)\rho, \tag{7.4.3}$$

ρ being the density of total mass-energy, while Π is the specific internal energy. Hence the whole dynamical system is much more complicated than the standard dust fluid and, in a cosmological setting, we think it better approximates the behavior of the thermal bath.

The relativistic description of the fluid is addressed using six scalar fields, which enter the 4-velocity as follows

$$\boldsymbol{U}_\nu = \mu^{-1}(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}) = \mu^{-1}v_\nu. \tag{7.4.4}$$

The Lagrangian is given by

$$\mathcal{L}_F = \sqrt{-g}\rho_0(\sqrt{v^\mu v_\mu} - TS), \tag{7.4.5}$$

ρ_0 being the rest mass-energy distribution, while T is the temperature and the field S can be interpreted as the entropy per barion.

After defining conjugate momenta, the following second-class system of constraints is obtained

$$\chi_1 = p_\alpha = 0, \qquad \chi_2 = p_\beta - \alpha\pi = 0, \qquad \chi_3 = p_\theta = 0, \qquad \chi_4 = p_S - \theta\pi, \tag{7.4.6}$$

π being the conjugate momentum to ϕ. Finally, the Hamiltonian of the fluid reads

$$H_F = N\left(\sqrt{(\pi^2 - q\rho_0^2)V} + q\rho_0 TS\right) + N^a\pi v_a, \tag{7.4.7}$$

N and N^a being the lapse function and the shift vector, respectively.

In presence of gravity the total Hamiltonian contains the super-Hamiltonian

H^G and the super-momentum H^G_a of the gravitational field as follows

$$\mathcal{H} = \int d^3x(HN + H_a N^a) = \tag{7.4.8}$$

$$= \int d^3x \left[N\left(\sqrt{V(\pi^2 - q\rho_0^2)} + \sqrt{q}\rho_0 ST + H^G \right) + N^a(\pi v_a + H^G_a) \right] . \tag{7.4.9}$$

It can be shown, using Dirac brackets to account for the presence of the conditions (7.4.6), that *H and H_a preserve their role of generators of the diffeomorphisms and exhibit a closed algebra*.

The Brown-Kuchař mechanism can be applied by squaring the super-momentum and imposing it on the super-Hamiltonian. The resulting expression can be solved for π, so finding

$$\pi \pm \sqrt{\rho_0 q \frac{d}{d - \Xi^2}} = 0 = \pi - h, \tag{7.4.10}$$

where $\Xi = \sqrt{q}\rho_0 ST + H^G$ and $d = H^G_a H^G_b q^{ab}$. The smeared version of the h function is invariant under 3-diffeomorphisms and commutes with itself, so it has all the properties to be the physical Hamiltonian.

In the case of a co-moving frame, the time-variable so obtained coincides with the specific entropy of the fluid. In fact, in this case one has

$$Sp_S = \frac{\theta H^G}{T} = h, \tag{7.4.11}$$

which integrated over the spatial manifold gives the equation:

$$\{\bar{\mathcal{H}}_{phys}, \mathcal{O}_f(\tau)\} = \frac{d}{dlnS}\mathcal{O}_f(\tau). \tag{7.4.12}$$

So *one can identify the time parameter τ with the logarithm of the entropy per baryon*. This result fixes an intriguing correspondence between time in Quantum Gravity and the thermodynamical time. Therefore, the implementation of this model in a cosmological setting could give an insight on the interplay between matter and geometry in Quantum Cosmology.

7.5. Quantum suppression of weak-anisotropies

In this section we show how a semi-classical mechanism, which leads to an isotropic configuration for an inhomogeneous quasi-isotropic Universe, can be developed. In particular, we obtain a wave function of the Universe which has a clear probabilistic interpretation when the isotropic scale factor a of the Universe is regarded as a semi-classical variable, differently from the

anisotropy parameters that are regarded as purely quantum ones. The quantum part of this wave function describes the evolution of the anisotropies of the inhomogeneous Mixmaster Universe and its dynamics is traced with respect to a, which can be regarded as a semi-classical variable when the Universe expands sufficiently.
The scalar constraint $\mathcal{H} = 0$, in the Misner scheme, reads

$$\mathcal{H}(x^i) = \kappa \left[-\frac{p_a^2}{a} + \frac{1}{a^3}\left(p_+^2 + p_-^2\right) \right] + \frac{a}{4\kappa} V(\beta_\pm) + U(a) = 0. \tag{7.5.1}$$

where the potential term $V(\beta_\pm)$ accounts for the spatial curvature and the potential term $U(\alpha)$ is the isotropic one.
In agreement with the WKB approximation, assuming ab initio that the radius of the Universe is of different nature with respect to its shape changes, the wave functional of the Universe $\Psi = \Psi(a, \beta_\pm)$ reads

$$\Psi \xrightarrow{a\to 0} \prod_i \Psi_i(x^i), \quad \Psi_i = \psi_0 \chi = A(a) e^{iS(a)} \chi(a, \beta_\pm) \tag{7.5.2}$$

where the factorization is due to decoupling of the spatial point.
The Wheeler-DeWitt (WDW) equation for this model leads, considering (7.5.2), to three different equations. We obtain the Hamilton-Jacobi equation for S and the equation of motion for A, which respectively read

$$-\kappa A \left(S'\right)^2 + aUA + \mathcal{V}_q = 0, \qquad \frac{1}{A}\left(A^2 S'\right)' = 0. \tag{7.5.3}$$

Here $(\cdot)' = \partial_a$ and $\mathcal{V}_q = \kappa A''$ is the so-called quantum potential which is negligible far from the singularity even if the $\hbar \to 0$ limit is not taken into account. The action $S(a)$ defines a congruence of classical trajectories, while the second equation in (7.5.3) is the continuity equation for the amplitude $A(a)$.
The third equation we achieve, once simplified in the asymptotic region $a \gg \lambda/\sqrt{\Lambda}$, and once the quasi-isotropic regime, i.e. $|\beta_\pm| \ll 1$ is taken into account, describes the evolution of the quantum subsystem and is given by

$$i\partial_\tau \chi = \hat{H}_q \chi = \frac{1}{2}\left(-\Delta_\beta + \omega^2(\tau)(\beta_+^2 + \beta_-^2)\right)\chi, \tag{7.5.4}$$

where $\omega^2(\tau) = C/\tau^{4/3}$ is a time-dependent frequency, C being a constant, and where τ behaves as $\tau = (\kappa/12\sqrt{\Lambda})a^{-3} + \mathcal{O}(a^{-5})$ and it is chosen as time coordinate.
The dynamics of the Universe anisotropies subsystem can then be regarded as a time-dependent bi-dimensional harmonic oscillator with frequency $\omega(\tau)$.

The exact solution can be obtained by the use of the invariants method and

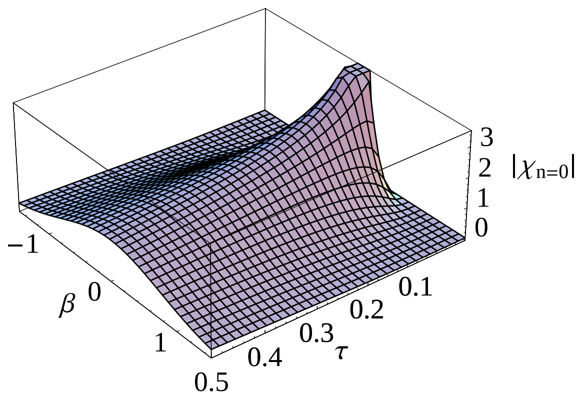

Figure 7.1.: The absolute value of the ground state of the wave function $\chi(\beta_\pm, \tau)$ far from the cosmological singularity. In the plot we take $C = 1$.

by means of some time-dependent transformations and is given by

$$\chi_{n_\pm}(\beta_\pm, \tau) = A\frac{e^{i\alpha_n(\tau)}}{\sqrt{\rho}} h_n(\xi_\pm) \exp\left[\frac{i}{2}\left(\dot{\rho}\rho^{-1} + i\rho^{-2}\right)\beta_\pm^2\right], \tag{7.5.5}$$

where $\rho = \rho(\tau)$ and $\alpha_n(\tau)$ are functions depending by the solving method, A is the normalization constant and h_n are the usual Hermite polynomial of order n.

The wave function of the Universe is spread over all values of anisotropy near the cosmological singularity but, when the radius of the Universe grows, it is asymptotically peaked around the isotropic configuration. In other words, the closed FRW model is naturally the privileged state when a sufficient large volume of the Universe is taken into account. This way, a semi-classical isotropization mechanism for the Universe is obtained.

7.6. Quantum behavior of the Universe for the small oscillations

In this section we study the behavior of the wave function of the Universe, when the small oscillations regime is approached. In particular, we obtain a wave function of the Universe which has a clear probabilistic interpretation when the volume α of the Universe and one of the anisotropy parameters β_+ are regarded as semi-classical variables, on respect the remaining anisotropy parameter that is regarded as purely quantum one. The quantum part of this wave function describes the evolution of the anisotropies of the inhomoge-

neous Mixmaster Universe and its dynamics is traced with respect to α.
In this regime, the potential term accounting for the curvature of the model, in the scalar constraint is approximated so that

$$V(\beta) \sim 1 + 16e^{4b_+}\beta_-^2 \quad ; \beta_+ \to +\infty \quad ; |\beta_-| \ll 1 \tag{7.6.1}$$

The scalar constraint $\mathcal{H} = 0$, in the Misner scheme, reads

$$\mathcal{H}(x^i) = e^{-3\alpha}\left(-p_\alpha^2 + p_+^2 + p_-^2\right) + e^\alpha\left[V(\beta) - 1\right] + \Lambda e^{3\alpha} = 0 \tag{7.6.2}$$

and with the last approximation on the $V(\beta)$ term, it transforms as

$$\mathcal{H}(x^i) = e^{-3\alpha}\left(-p_\alpha^2 + p_+^2 + p_-^2\right) + 16e^{\alpha+4\beta_+} + \Lambda e^{3\alpha} = 0 \tag{7.6.3}$$

Distinguishing between semi-classical and quantum variables, following the initial reasoning, the wave functional of the Universe reads:

$$\Psi = \Psi_0\chi = A(\alpha, \beta_+)e^{\frac{i}{\hbar}S(\alpha,\beta_+)}\chi(\alpha, \beta_+, \beta_-) \tag{7.6.4}$$

This wave function is WKB-like in α and β_+, the function χ depends on the quantum variable β_- and parametrically only on the scale factor and the other anisotropy parameter.

The Wheeler-DeWitt (WDW) equation for this model leads to three different equations. We obtain the Hamilton-Jacobi equation for S and the equation of motion for A, which respectively read

$$\frac{1}{\hbar^2}A\left[(\partial_+S)^2 - (\partial_\alpha S)^2\right] + \left(\partial_\alpha^2 A - \partial_+^2 A\right) + \Lambda e^{6\alpha}A = 0 \tag{7.6.5}$$

$$\frac{2}{\hbar}\left(\partial_\alpha A\partial_\alpha S - \partial_+ A\partial_+ S\right) + \frac{1}{\hbar}A\left(\partial_\alpha^2 S - \partial_+^2 S\right) = 0 \tag{7.6.6}$$

Lastly we achieve an equation describing the evolution of the quantum subsystem: We obtain:

$$\frac{2i}{\hbar}\left(\partial_\alpha S\partial_\alpha\chi - \partial_+ S\partial_+\chi\right) = -H_q\chi \tag{7.6.7}$$

where

$$-H_q = \partial_-^2 - 16e^{4(\alpha+\beta_+)}\beta_-^2 \tag{7.6.8}$$

Rescaling the lapse function to have a positive foliation of the space-time, introducing a new time coordinate τ related to α and connecting the semi-classical anisotropy parameter to the volume of the Universe, we obtain a Schröedinger-like equation for the quantum sub-system, ruled by the new time coordinate:

$$i\frac{\partial\chi}{\partial\tau} = -\partial_{-}^{2} + \Omega^{2}(\tau)\,\beta_{-}^{2} \tag{7.6.9}$$

where

$$\Omega(\tau) = B^{\frac{1}{2}}\tau^{\frac{1}{2}} \tag{7.6.10}$$

with $B = const$.

The exact solution can be obtained by the use of the invariants method and by means of some time-dependent transformations and is given by

$$\chi_{-}(\beta_{-},\tau) = e^{i\alpha_n(\tau)}\left(\frac{1}{\pi^{\frac{1}{2}}\hbar^{\frac{1}{2}}n_{-}!2_{-}^{n}\rho}\right)^{\frac{1}{2}}\left|Hn_{-}\left(\frac{\beta_{-}}{\hbar^{\frac{1}{2}}\rho}\right)\right|e^{\frac{i}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+\frac{i}{\rho^{2}}\right)\beta_{-}^{2}} \tag{7.6.11}$$

Analyzing the probability density related to this wave function we can see that it's peaked around small values of the anisotropy parameter β_{-}, or in other words, the Taub configuration is deeply favored from the probabilistic point of view once we are not too far from cosmological singularity.
The last step of this work that we are attempting to solve, is analyze the beahviour of the quantum sub-system once are investigated region far from the cosmological singularity.

7.7. Regularization and Quantization of Einstein-Cartan theory

The self-dual connection of a Yang-Mills gauge theory introduced in the Ashtekar formalism for General Relativity is crucial for the canonical quantization procedure, leading to the non-perturbative quantum theory of gravity, *Loop Quantum Gravity*. The complex Ashtekar's connection with reality condition and the real Barbero real connection are linked by a canonical transformation of the connection with the Immirzi parameter $\gamma \neq 0$, which has crucial effects on quantum gravity at the Planck energy scale, but does not affect the classical dynamics of torsion-free gravity. However, when fermion fields are present and coupled to gravity, yielding a non-vanishing torsion tensor, and the Einstein-Cartan theory for torsion-free gravity coupling to fermions should be modified. Indeed, the four-fermion interacting strength in the Einstein-Cartan theory is related to the Immirzi parameter, which can possibly lead to physical effects observable. Thus, it is worthwhile to study the dynamics of these quadralinear terms of fermion fields in terms of the four-fermion interacting strength to see whether effective bilinear terms of massive fermions are generated.

Einstein-Cartan theory.

To derive the Einstein-Cartan theory we use the following notations. $\mathcal{M}$ the 4-dimensional Euclidean space-time manifold, In the Ashtekar formalism for General Relativity and $g_{\mu\nu}$ space-time matrix with signature $(+,+,+,+)$. For the tetrad formalism, we fix a four-dimensional vector space V equipped with a fixed metric η^{ab} of signature $(+,+,+,+)$, which will serve as the 'internal space'. Orthonormal co-tetrats will be denoted by by e^a_μ; thus $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$. In this vector space, the conventions of Dirac γ-matrices are: communication $\{\gamma_a, \gamma_b\} = -2\eta_{ab}$; anti-hermitian $\gamma^\dagger_a = -\gamma_a$ and $\gamma^2_a = -1$ ($a = 0,1,2,3$). The hermitian γ_5-matrix $\gamma^\dagger_5 = \gamma_5$, $\gamma_5 = \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_0\gamma_1\gamma_2\gamma_3$ and $\gamma^2_5 = 1$. The hermitian spinor matrix $\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$, and $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{abcd}e^a_\mu e^b_\nu e^c_\rho e^d_\sigma$ is totally antisymmetric tensor.

In the Palatini framework, the basic gravitational variables constitute a pair of tetrat and spin-connection fields $(e^a_\mu, \omega^{ab}_\mu)$. They are 1-form fields on $\mathcal{M}$ the 4-dimensional Euclidean space-time manifold, taking values, respectively, in the vector space V and in the Lie algebra $so(\eta)$ of the group $SO(\eta)$ of the linear transformations of V preserving $\eta^{ab} = (+,+,+,+)$. The 2-form curvature associating with the spin-connection is

$$R^{ab} = d\omega^{ab} - \omega^{ae} \wedge \omega^b{}_e. \tag{7.7.1}$$

The Palatini action for gravitational field is given by,

$$S_P(e,\omega) = \frac{1}{4k}\int_{\mathcal{M}} d^4x \det(e)\epsilon_{abcd}e^a \wedge e^b \wedge R^{cd}, \tag{7.7.2}$$

where $k \equiv 8\pi G$. The relationship between spin-connection ω^{ab}_μ and the tetrat e^a_μ is determined by $\delta S_P(e,\omega)/\delta\omega = 0$, Cartan's structure equation,

$$de^a - \omega^{ab} \wedge e_b = 0, \tag{7.7.3}$$

which gives the torsion-free spin-connection: $\omega = \omega(e)$. Replacing ω in Eq. (7.7.2) by $\omega = \omega(e)$, the Palatini action $S_P[e, \omega(e)]$ reduces to the Einstein-Hilbert action and its variation with respect to the tetrad field e^a_μ leads to the Einstein field equation,

$$\epsilon_{abcd}e^b \wedge R^{cd}[\omega(e)] = 0. \tag{7.7.4}$$

Adding the Host modification with the Immirzi parameter γ, one has

$$S_H(e,\omega) = S_P(e,\omega) - \frac{1}{2k\gamma}\int_{\mathcal{M}} d^4x \det(e) e_a \wedge e_b \wedge R^{ab}. \tag{7.7.5}$$

Introducing massless Dirac fermions ψ coupled to the gravitational field described by $(e^a_\mu, \omega^{ab}_\mu)$, we adopt the fermion action of Ashtekar-Romano-Tate type,

$$S_F(e,\omega,\psi,\bar{\psi}) = \frac{1}{2}\int_{\mathcal{M}} d^4x \det(e)\left[\bar{\psi}e^\mu \mathcal{D}_\mu\psi + \text{h.c.}\right], \qquad (7.7.6)$$

where the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - \frac{i}{4}\beta\omega_\mu, \qquad (7.7.7)$$

β is the gauge coupling between fermion and spin-connection fields, Dirac-matrix valued tetrad and spin-connection fields are $e^\mu \equiv e^\mu_a\gamma^a$ and $\omega_\mu \equiv \omega^{ab}_\mu\sigma_{ab}$. The anti-hermitian Dirac matrix $\gamma^\dagger_a = -\gamma_a$, $\gamma^2_a = -1$ ($a = 0,1,2,3$), $\{\gamma_a,\gamma_b\} = -2\eta_{ab}$, and the hermitian spinor matrix $\sigma^{ab} = \frac{i}{2}[\gamma^a,\gamma^b]$. The actions (7.7.2,7.7.6) are invariant under the diffeomorphisms of the manifold $\mathcal{M}$, and can be separated into left- and right-handed parts, with respect to local $SU_L(2)-$ and $SU_R(2)-$ Lorentz symmetries.

This can be shown by writing Dirac fermion $\psi = \psi_L + \psi_R$, where Weyl fermions $\psi_{L,R} = P_{L,R}\psi$, $P_{L,R} = (1 \mp \gamma_5)/2$; and Dirac-matrix valued tetrad and spin-connection fields $e^\mu = P_L e^\mu + P_R e^\mu$ and $\omega_\mu = P_L\omega_\mu + P_R\omega_\mu$.

The Palatini action (7.7.2) and fermion action (7.7.6) give the Einstein-Cartan action,

$$S_{EC} = S_P(e,\omega) + S_F(e,\omega,\psi,\bar{\psi}). \qquad (7.7.8)$$

Analogously to Eq. (7.7.3), $\delta S_{EC}(e,\omega)/\delta\omega = 0$ gives Cartan's structure equation,

$$de^a - \omega^{ab}\wedge e_b - T^a = 0, \qquad (7.7.9)$$

where the non-vanishing torsion field $T^a = k\beta e_b \wedge e_c J^{ab,c}$, relating to the fermion spin-current

$$J^{ab,c} = \frac{i}{4}\bar{\psi}\{\sigma^{ab},\gamma^c\}\psi = \frac{1}{4}\epsilon^{abcd}\bar{\psi}\gamma_d\gamma^5\psi, \qquad (7.7.10)$$

and $\{\sigma^{ab},\gamma^c\} = i\epsilon^{abcd}\gamma^5\gamma_d$. The hermitian γ_5-matrix $\gamma_5 = \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_0\gamma_1\gamma_2\gamma_3$, $\gamma^\dagger_5 = \gamma_5$ and $\gamma^2_5 = 1$. The totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{abcd}e^a_\mu e^b_\nu e^c_\rho e^d_\sigma$. The solution to Eq. (7.7.9) is

$$\omega^{ab}_\mu = \omega^{ab}_\mu(e) + \tilde{\omega}^{ab}_\mu, \quad \tilde{\omega}^{ab}_\mu = k\beta e^c_\mu J^{ab}{}_c, \qquad (7.7.11)$$

where the connection $\omega^{ab}_\mu(e)$ obeys Eq. (7.7.3) for torsion-free case. The fermion spin-current (7.7.10) contributes only to the pseudo-trace axial vector of torsion tensor, which is one of irreducible parts of torsion tensor. Replacing the spin-connection ω in the Einstein-Cartan action (7.7.8) by (7.7.11),

$$S_P[e,\omega,\psi,\bar{\psi}] \rightarrow S_P[e,\omega(e)] - \frac{1}{16}k\beta^2\int_{\mathcal{M}} d^4x\,\det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi);\quad (7.7.12)$$

$$S_F[e,\omega,\psi,\bar{\psi}] \rightarrow S_F[e,\omega(e),\psi,\bar{\psi}] - \frac{2}{16}k\beta^2\int_{\mathcal{M}} d^4x\,\det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi)\quad (7.7.13)$$

one obtains the well-known Einstein-Cartan theory: the standard tetrad action of torsion-free gravity coupling to fermions,

$$\begin{aligned} S_{EC}[e,\omega(e),\psi,\bar{\psi}] &= S_P[e,\omega(e)] + S_F[e,\omega(e),\psi,\bar{\psi}] \\ &- \frac{3}{16}k\beta^2\int_{\mathcal{M}} d^4x\,\det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi).\end{aligned}\quad (7.7.14)$$

In the case of the Host action (7.7.5), the four-fermion interaction term is given by

$$-\frac{3}{16}\frac{\gamma^2}{\gamma^2+1}k\beta^2\int_{\mathcal{M}} d^4x\,\det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi). \quad (7.7.15)$$

As we can see from Eqs. (7.7.6) to (7.7.14), the bilinear term (7.7.6) of massless fermion fields coupled to the spin-connection (7.7.7) is bound to yield a non-vanishing torsion field T^a (7.7.9), which is local and static. As a result, the spin-connection ω is no longer torsion-free and acquires a torsion-related spin-connection $\tilde{\omega}^{ab}_\mu$ (7.7.11), in addition to the torsion-free spin-connection $\omega^{ab}_\mu(e)$. The torsion-related spin-connection $\tilde{\omega}^{ab}_\mu$ is related to the fermion spin-current (7.7.10). The quadratic term of the spin-connection field ω in Eq. (7.7.1) and the coupling between the spin-connection field and fermion spin-current in Eqs. (7.7.6,7.7.7) lead to the quadrilinear terms of fermion fields in Eqs. (7.7.12) and (7.7.13). Another way to see this is to treat the static torsion-related spin-connection $\tilde{\omega}^{ab}_\mu$ (7.7.11) as a static auxiliary field, which has its quadratic term and linear coupling to the spin-current of fermion fields. Performing the Gaussian integral of the static auxiliary field, we exactly obtain the quadrilinear term (7.7.14), in addition to the torsion-free action.

A postulation and fermion-mass generation. The gauge principle requires the action of gravitational and fermion fields be invariant under the diffeomorphisms of the manifold $\mathcal{M}$ and local Lorentz transformations. This leads to a pseudo-trace axial vector of static (non-dynamics) torsion field, which is related to the spin-current of fermion fields. The interaction between the pseudo-trace axial vector of torsion fields and the spin-current of fermion fields results in the four-fermion interaction (quadrilinear terms in massless fermion fields). As a consequence, the gauge-invariant action consists of the torsion-free action of gravitational and fermion fields and four-fermion interaction. We thus postulate that it is impossible to have any gauge-invariant theories made by the bilinear terms of massless fermion fields coupled to torsion-free gravitational field, and quadrilinear terms (or high-order terms) of massless fermion fields must be present.

The quadrilinear term (7.7.15) is a dimension-6 operator, and four-fermion couping is in terms of the Immirzi parameter γ, gravitational-coupling k and gauge-coupling β. If quantum gravity is taken into account, we expect non-local and high-dimensional operators ($d > 6$), which contain high-order derivatives. In this case, the torsion-related spin-connection $\tilde{\omega}^{ab}_{\mu}$ (7.7.11) is not completely static, rather has a mass of the order of the Planck mass, mediating in a few Planck length to form effective high-dimensional operators of massless fermion fields. The fundamental fields e, ω, ψ, and operators $\mathcal{O}(e, \omega, \psi, k, \beta)$ are functions of the couplings k, β, γ, depending on the energy-scale $\mathcal{E}$. First we should adopt appropriate vacuum expectational value (e.v.e) of operators $\langle \mathcal{O}(e, \omega, \psi, k, \beta)\rangle$ as order parameters, to describe different phases and phase transitions in the space of couplings k, β, and parameter γ. Second, we try to identify the scaling-invariant regime (ultraviolet fix points) for the *low-energy limit* ($\mathcal{E}/m_p \to 0$), where the variation of fundamental fields, couplings and operators as functions of the energy-scale $\mathcal{E}$ is govern by renormalization group equations. Third, in such scaling-invariant regime we try to determine the relevant and renormalizable operators that are effective dimension-4 operators, to obtain an effective low-energy theory for the present Universe.

In this Letter, we are interested in the one-particle-irreducible (1PI) two-point functions of fermion fields $\langle \psi(0)\bar{\psi}(x)\rangle$, since they contribute to effective operators for the energy-momentum tensor entering the right-hand side of the Einstein equation (7.7.4) for classical gravity. Our goal is limited to find non-trivial fermion-mass operators ($\langle \psi\bar{\psi}\rangle \not\equiv 0$) in terms of the four-fermion interacting strength. For convenience in calculations, using the Planck mass m_p, we rescale fermion fields $\psi \to \psi/m_p$ and rewrite four-fermion interaction (7.7.15) as

$$g \int_{\mathcal{M}} d^4x \det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi); \quad g = \frac{3}{16}\frac{\gamma^2}{\gamma^2+1}k\beta^2 m_p^4 \qquad (7.7.16)$$

where the four-fermion coupling g has dimension $[m_p^2]$. We assume the gauge-coupling β to be perturbatively small.

Weak four-fermion coupling. In the weak-coupling limit $g/m_p^2 \ll 1$, the dimension-3 fermion-mass operators $\langle \psi\bar{\psi}\rangle$ identically vanish ($\langle \psi\bar{\psi}\rangle \equiv 0$), the action (7.7.14) gives a weakly interacting, massless $SU_L(2) \otimes SU_R(2)$ fermion spectrum. We define this as the "weak-coupling symmetric phase". In the intermediate range of coupling g, there is a "broken phase" where spontaneous symmetry breaking occurs. Using large-N_f expansion technique [1] shows that the four-fermion interaction (7.7.14) undergoes Nambu-Jona-Lasinio (NJL) spontaneous chiral-symmetry breaking. In this symmetry broken phase, $SU_L(2) \otimes SU_R(2)$ chiral symmetry is violated by non-vanishing mass-operators

$$\frac{1}{2}\Sigma(p) = g \int d^4x \det(e) e^{-ipx} \langle \bar{\psi}(0) \cdot \psi(x) \rangle \neq 0, \tag{7.7.17}$$

where $\langle \cdots \rangle$ is the average with respect to the partition function Z of fermionic part of the action (7.7.14)

$$\langle \cdots \rangle = \frac{1}{Z} \int d\bar{\psi} d\psi \, (\cdots) \exp \{-S_{EC}[e, \omega(e), \psi, \bar{\psi}]\}. \tag{7.7.18}$$

The non-vanishing mass operator (7.7.17) obeys the NJL gap-equation,

$$\Sigma(p) = \tilde{g} \int \frac{d^4q}{(2\pi)^4} \frac{\Sigma(q)}{q^2 + (\Sigma(q)/m_p)^2}, \tag{7.7.19}$$

where momentum q and coupling $\tilde{g} = gN_f/m_p^2$ are dimensionless. The critical point $\tilde{g}_c = 8\pi^2$, which can be obtained by $\Sigma \to 0^+$, separates the "broken phase" ($\Sigma \neq 0, \tilde{g} > \tilde{g}_c$) from the "weak-coupling symmetric phase" ($\Sigma \equiv 0, \tilde{g} < \tilde{g}_c$). $\Sigma(p) \sim \mathcal{O}(m_p)$ for $\tilde{g} > \tilde{g}_c$. The inverse propagators of these fermions can then be written as,

$$S^{-1}(p) \;\; = \;\; i\gamma_\mu p^\mu + \Sigma(p). \tag{7.7.20}$$

The $SU_L(2) \otimes SU_R(2)$ chiral symmetry is realized to be $SU(2)$ with three Goldstone modes and a massive Higgs mode that are not presented here. Eq. (7.7.20) corresponds to the bilinear term of massive fermion fields in the effective action, which does not preserves chiral symmetries.

Strong four-fermion coupling. We turn to the strong-coupling region, where four-fermion coupling g in (7.7.14) is sufficiently larger than a certain critical value $g_{\rm crit}$, bound states of three fermions (three-fermion states) are formed

$$\Psi = \frac{m_p}{2}(\bar{\psi} \cdot \psi)\psi, \tag{7.7.21}$$

which can be understood as a bound state of one fermion and one composite boson $(\bar{\psi} \cdot \psi)$. These three-fermion states (7.7.21) carry the appropriate quantum numbers of the gauge group that accommodates ψ. The fermion-mass operator is $\bar{\psi}\Psi$ and thus massive fermion spectrum is consistent with the chiral symmetry $SU_L(2) \otimes SU_R(2)$.

[1] $g \ll 1, N_f \gg 1$ and gN_f fixed, N_f is the number of fermion flavors

For the purpose of understanding three-fermion states and their spectra, we henceforth focus on the strong-coupling region ($g/m_p^2 \gg 1$). We make a rescaling of fermion fields,

$$\psi(x) \to g^{1/4}\psi(x), \tag{7.7.22}$$

and rewrite the fermion action in terms of the new fermion fields

$$S_f(x) = \frac{1}{2g^{1/2}}\left[\bar{\psi}(x)\gamma_\mu\partial^\mu\psi(x) + \text{h.c.}\right] \tag{7.7.23}$$

$$S_i(x) = (\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi). \tag{7.7.24}$$

where the gauge coupling β is assumed to be weak. For the limit of strong coupling $g/m_p^2 \to \infty$, the kinetic terms $S_f(x)$ can be dropped and we calculate the partition function Z (7.7.18) in this strong-coupling limit. With $S_i(x)$ given in Eq. (7.7.24), the integral of $e^{-S_i(x)}$ is calculated by Grassmann anticommuting algebra,

$$Z = \Pi_x \int [d\bar{\psi}(x)d\psi(x)] \exp\left[-S_i(x)\right] = \Pi_x 2^4 \neq 0, \tag{7.7.25}$$

which shows a non-trivial strong-coupling limit. About this strong-coupling limit (7.7.25), we now can perform the strong-coupling expansion of $e^{-S_f(x)}$ in powers of $1/g$ to calculate Green-functions of fermion fields $\langle\psi(x_1)\psi(x_2)\cdot\cdot\cdot\psi(x_n)\rangle$. In order to do integral of Grassmann anticommuting algebra, we rewrite the kinetic term $S_f(x)$ (7.7.23) as a hopping term in the Planck spacing a^μ, $|a^\mu| = a = 1/m_p$,

$$S_f(x) = \frac{1}{2g^{1/2}a}\left[\bar{\psi}(x)\gamma_\mu\psi(x+a^\mu) - \bar{\psi}(x+a^\mu)\gamma_\mu\psi(x)\right]. \tag{7.7.26}$$

We consider the following two-point functions that form the propagator of the composite Dirac particle

$$S_{LL}(x) \equiv \langle\psi(0), \bar{\psi}(x)\rangle, \tag{7.7.27}$$

$$S_{ML}(x) \equiv (2a)\langle\psi(0), \bar{\Psi}(x)\rangle, \tag{7.7.28}$$

$$S_{MM}(x) \equiv (2a)^2\langle\Psi(0), \Psi(x)\rangle. \tag{7.7.29}$$

In the lowest non-trivial order $O(1/g)$, we obtain the following recursion relations

$$
\begin{aligned}
S_{LL}(x) &= \frac{1}{g}\left(\frac{1}{2a}\right)^3 \sum_\mu^\dagger S_{ML}(x+a^\mu)\gamma_\mu, && (7.7.30)\\
S_{ML}(x) &= \frac{\delta(x)}{2g} + \frac{1}{g}\left(\frac{1}{2a}\right) \sum_\mu^\dagger S_{LL}(x+a^\mu)\gamma_\mu, && (7.7.31)\\
S_{MM}(x) &= \frac{1}{g}\left(\frac{1}{2a}\right) \sum_\mu^\dagger \gamma_\mu\gamma_0 S^\dagger_{ML}(x+a^\mu)\gamma_0, && (7.7.32)
\end{aligned}
$$

where for an arbitrary function $f(x)$,

$$\sum_\mu^\dagger f(x) = \sum_\mu \left[f(x+a^\mu) - f(x-a^\mu)\right].$$

Transforming these two-point functions (7.7.27,7.7.28,7.7.29) into momentum space,

$$S_X(p) = \int d^4x e^{-ipx} S_X(x), \tag{7.7.33}$$

where $X = LL, ML, MM$ respectively, we obtain three recursion relations in momentum space

$$
\begin{aligned}
S_{LL}(p) &= \frac{1}{g}\left(\frac{i}{4a^3}\right) \sum_\mu \sin(p^\mu a) S_{ML}(p)\gamma_\mu, && (7.7.34)\\
S_{ML}(p) &= \frac{1}{2g} + \frac{i}{ga} \sum_\mu \sin(p^\mu a) S_{LL}(p)\gamma_\mu. && (7.7.35)\\
S_{MM}(p) &= \frac{1}{g}\left(\frac{i}{a}\right) \sum_\mu \sin(p^\mu a)\gamma_\mu\gamma_0 S^\dagger_{ML}(p)\gamma_0. && (7.7.36)
\end{aligned}
$$

We solve these recursion relations (7.7.34,7.7.35,7.7.36) and obtain

$$
\begin{aligned}
S_{LL}(p) &= \frac{\frac{i}{2a}\sum_\mu \sin(p^\mu a)\gamma_\mu}{\frac{1}{a^2}\sum_\mu \sin^2(p_\mu a) + M^2}, && (7.7.37)\\
\frac{1}{2a} S_{ML}(p) &= \frac{\frac{1}{2}M(p)}{\frac{1}{a^2}\sum_\mu \sin^2(p_\mu a) + M^2}, && (7.7.38)\\
\left(\frac{1}{2a}\right)^2 S_{MM}(p) &= \frac{\frac{i}{2a}\sum_\mu \sin(p^\mu a)\gamma_\mu}{\frac{1}{a^2}\sum_\mu \sin^2(p_\mu a) + M^2}, && (7.7.39)
\end{aligned}
$$

where the chiral-invariant mass is

$$M = 2ga. \tag{7.7.40}$$

In addition, the two-point function,

$$\langle \Psi(x), \bar{\psi}(0) \rangle = \frac{1}{2a} \gamma_0 S^{\dagger}_{ML}(x) \gamma_0. \tag{7.7.41}$$

As a result, in the lowest non-trivial order of the strong-coupling expansion we obtain the massive propagator of the composite Dirac fermions,

$$S(p) = \frac{\frac{i}{a} \sum_\mu \sin(p^\mu a) \gamma_\mu + M}{\frac{1}{a^2} \sum_\mu \sin^2(p_\mu a) + M^2} \simeq \frac{i p^\mu \gamma_\mu + M}{p^2 + M^2}, \tag{7.7.42}$$

for modes $p^\mu a \ll 1$. Eq. (7.7.42) corresponds to the bilinear term of massive fermion fields preserving chiral symmetries in the effective action, which can be written as,

$$S^{\rm eff}_F(e, \omega, \Psi, \bar{\Psi}) = \frac{1}{2} \int_{\mathcal{M}} d^4x \, \det(e) \left[\bar{\Psi} e^\mu \mathcal{D}_\mu \Psi + M \bar{\Psi} \Psi \right] + \text{h.c.}, \tag{7.7.43}$$

and its variation with respect to the tetrat field e^a_μ gives rise to the energy-momentum tensor that contributes to the right-handed side of the Einstein equation (7.7.4). This is the "strong-coupling symmetric phase", where fermion fields are massive.

The critical value $g_{\rm crit}$ that separates the " strong-coupling symmetric phase" from the "broken phase" can be qualitatively determined by considering the complex composite scalar field,

$$\mathcal{A} = \bar{\psi} \cdot \psi, \tag{7.7.44}$$

and its propagator, i.e., the two-point function:

$$G(x) \quad = \quad \langle \mathcal{A}(0), \mathcal{A}^\dagger(x) \rangle. \tag{7.7.45}$$

Analogously, using the strong-coupling expansion in powers of $1/g$ ($g/m_p^2 \gg 1$), we obtain the following recursion relation in the lowest order,

$$G(x) = \frac{\delta(x)}{g} + \frac{1}{g} \left(\frac{1}{2a} \right)^2 \sum_{\pm\mu} G(x + a^\mu). \tag{7.7.46}$$

Going to momentum space,

$$G(q) = \int d^4x e^{-iqx} G(x),$$

where q is the momentum of the composite scalar $\mathcal{A}$, we obtain the recursion relation (7.7.46) in momentum space

$$G(q) = \frac{1}{g} + \left(\frac{1}{2a^2}\right)\frac{1}{g}\sum_{\pm\mu}\cos(q_\mu a)G(q). \qquad (7.7.47)$$

As a result, we find the propagator of the massive composite scalar field $\mathcal{A}$,

$$G(q) = \frac{4}{\frac{4}{a^2}\sum_\mu \sin^2\frac{(q_\mu a)}{2} + \mu^2} \simeq \frac{4}{q^2+\mu^2}; \qquad (7.7.48)$$

$$\mu^2 = 4\left(g - \frac{2}{a^2}\right), \qquad (7.7.49)$$

where the factor 4 is due to the four components of the composite scalar field $\mathcal{A}$. Thus, $\mu^2\mathcal{A}\mathcal{A}^\dagger$ gives the mass term of the composite scalar field $\mathcal{A}$ in the effective Lagrangian. We assume that the 1PI vertex $\mathcal{A}\mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger$ is positive and the energy of ground states of the theory is bound from the bellow. Then, we can qualitatively discuss the second order phase transition (threshold) from the "strong-coupling symmetric phase" to the "broken phase" by examining the mass term of these composite scalars $\mu^2\mathcal{A}\mathcal{A}^\dagger$. Spontaneous symmetry breaking $SU(2) \to U(1)$ occurs, where $\mu^2 > 0$ turns to $\mu^2 < 0$. Eq. (7.7.49) for $\mu^2 = 0$ gives rise to the critical value $g_{\rm crit}$:

$$g_{\rm crit}a^2 = 2, \qquad (7.7.50)$$

where a phase transition takes place between the "strong-coupling symmetric phase" and the "broken phase".

Some discussions. As already mentioned, high-dimensional operators of massless fermion fields containing high-order derivatives are expected if the quantum gravity is included. In this case the four-fermion coupling (7.7.16) and fermion-mass (7.7.40) should be functions of fermion's momentum p^μ. Both the phase-structure and critical points for phase-transition characterized by the coupling g (7.7.16) depend clearly also on the Immirzi parameter γ. Although three different phases have been differentiated, we have not been able to identify the scaling-invariant region for the *low-energy limit* where some of high-dimensional operators receive anomalous dimensions become relevant operators of effective dimension-4, others are non-relevant and suppressed. We expect that the scaling-invariant regime be probably near to the critical point (7.7.50) so that the low-energy effective theory preserves chiral-gauge symmetry in high-energies and has a soft symmetry-breaking for fermion masses in low-energies. In this Letter we discuss the phase-structure of the Einstein-Cartan theory and a theoretical possibility to understand how fermion fields become massive and couple to torsion-less gravitational field.

7.8. Quantum Regge Calculus of Einstein-Cartan theory

Introduction. Since the Regge Calculus was proposed for the discretization of gravity theory in 1961, many progresses have been made in the approach of Quantum Regge Calculus and its variant dynamical triangulations. In particular, the renormalization group treatment is applied to discuss any possible scale dependence of gravity. In Lagrangian formalism, gauge-theoretic formulation of quantum gravity using connection variables on a flat hypercubic lattice of the space-time was inspired by the success of lattice regularization of non-Abelian gauge theories. A locally finite model for gravity has been recently proposed. In this Letter, based on the scenario of Quantum Regge Calculus, we present a diffeomorphism and *local* gauge-invariant invariant regularization and quantization of Euclidean Einstein-Cartan (EC) theory, invariant holonomies of tetrad and spin-connection fields $\omega_\mu(x)$ along large loops in 4-simplices complex, and some calculations in 2-dimensional case.

Euclidean Einstein-Cartan gravity. The basic gravitational variables in the Einstein-Cartan gravity constitute a pair of tetrad and spin-connection fields $(e^a_\mu, \omega^{ab}_\mu)$, whose Dirac-matrix values $e_\mu = e^a_\mu \gamma_a$ and $\omega_\mu = \omega^{ab}_\mu \sigma_{ab}$. The space-time metric of 4-dimensional Euclidean manifold $\mathcal{M}$ is $g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \delta_{ab}$, where $\delta^{ab} = (+,+,+,+)$. The diffeomorphism invariance under general coordinate transformations $x \to x'(x)$ is preserved by all derivatives and d-form fields on $\mathcal{M}$ made to be coordinate scalars with the help of tetrad fields $e^a_\mu = \partial\xi^a/\partial x^\mu$. Under the local Lorentz coordinate transformation $\xi'^a(x) = [\Lambda(x)]^a_b \xi^b(x)$, the *local* (w.r.t ξ) gauge transformations are:

$$e'_\mu(\xi) = \mathcal{V}(\xi) e_\mu(\xi) \mathcal{V}^\dagger(\xi), \tag{7.8.1}$$
$$\omega'_\mu(\xi) = \mathcal{V}(\xi)\omega_\mu(\xi)\mathcal{V}^\dagger(\xi) + \mathcal{V}(\xi)\partial_\mu \mathcal{V}^\dagger(\xi); \tag{7.8.2}$$

and fermion field $\psi'(\xi) = \mathcal{V}(\xi)\psi(\xi)$, the covariant derivative $\mathcal{D}'_\mu = \mathcal{V}(\xi)\mathcal{D}_\mu \mathcal{V}^\dagger(\xi)$, $\mathcal{D}_\mu = \partial_\mu - ig\omega_\mu(\xi)$ where g is the gauge coupling, $\partial_\mu = e^a_\mu(\partial/\partial\xi^a)$, $\mathcal{V}(\xi) = \exp i[\theta^{ab}(\xi)\sigma_{ab}] \in SO(4)$, and $\theta^{ab}(\xi)$ is an arbitrary function of ξ. In an $SU(2)$ gauge theory, gauge field $A_a(\xi_E)$ can be viewed as a connection $\int A_a(\xi_E) d\xi^a_E$ on the global flat manifold. On a locally flat manifold, the spin-connection $\omega_\mu dx^\mu = \omega_a(\xi) d\xi^a$, where $\omega_a(\xi) = e^\mu_a \omega_\mu$, one can identify that the spin-connection field $\omega_\mu(x)$ or $\omega_a(\xi)$ is the gravity analog of gauge field and its *local* curvature is given by

$$R^{ab} = d\omega^{ab} - g\omega^{ae} \wedge \omega^b{}_e, \tag{7.8.3}$$

and $R'^{ab} = \mathcal{V}(\xi) R^{ab}(\xi) \mathcal{V}^\dagger(\xi)$ under the transformation (7.8.1,7.8.2). The diffeomorphism and *local* gauge-invariant EC action for gravity is given by the

Palatini action S_P and Host modification S_H

$$S_{EC}(e,\omega) = S_P(e,\omega) + S_P(e,\omega) \quad (7.8.4)$$

$$S_P(e,\omega) = \frac{1}{4\kappa}\int_{\mathcal{M}} d^4x \det(e)\epsilon_{abcd}e^a \wedge e^b \wedge R^{cd}, \quad (7.8.5)$$

$$S_H(e,\omega) = \frac{1}{2\kappa\tilde{\gamma}}\int_{\mathcal{M}} d^4x \det(e)e_a \wedge e_b \wedge R^{ab} \quad (7.8.6)$$

where $\kappa \equiv 8\pi G$, the Newton constant $G = 1/m^2_{\rm Planck}$, and $\det(e)$ is the Jacobi of mapping $x \to \xi(x)$.

In addition, the diffeomorphism invariance under the general coordinate transformation $x \to x'(x)$ is preserved by all fields in Eqs. (7.7.5-7.7.6) made to be coordinate scalars by using tetrad fields. The derivatives represents the propagation of fields in coordinate space is related to the connection field in local Lorentz frame. spin-connection is the gravity analog of gauge field, how ever it is constructed by the general coordinate derivatives of tetrad field relating to general connection.

The complex Ashtekar connection with reality condition and the real Barbero connection are linked by a canonical transformation of the connection with a finite complex Immirzi parameter $\tilde{\gamma} \neq 0$, which is crucial for *Loop Quantum Gravity*.

A quantum theory of gravity in Hamiltonian formalism, where intrinsic discrete eigenvalues of invariant area and volume operators are obtained in the diffeomorphism invariant Hilbert space, as results, the space-time is discretized with the Planck length and the black-hole entropy is obtained.

Classical equations can be obtained by the invariance of the EC action (7.8.4) under the transformation (7.8.1-7.8.2),

$$\delta S_{EC} = \frac{\delta S_{EC}}{\delta e_\mu}\delta e_\mu + \frac{\delta S_{EC}}{\delta \omega_\mu}\delta \omega_\mu = 0, \quad (7.8.7)$$

where δe_μ and $\delta\omega_\mu$ are infinitesimal variations, which can be expressed in terms of independent Dirac matrix bases γ_5 and γ_μ. Therefore, for an arbitrary function θ_{ab}, we have $\delta S_{EC}/\delta e_\mu = 0$ and $\delta S_{EC}/\delta\omega_\mu = 0$, respectively leading to Einstein equation and Cartan's structure equation (torsion-free)

$$de^a - \omega^{ab} \wedge e_b = 0. \quad (7.8.8)$$

Regularized EC action. The four-dimensional Euclidean manifold $\mathcal{M}$ is discretized as an ensemble of $\mathcal{N}_0$ space-time points "x" and $\mathcal{N}_1$ links (edges) "$l_\mu(x)$" connecting two neighboring points, which is a simplicial manifold. The way to construct a simplicial manifold depends also on the assumed topology of the manifold, which gives geometric constrains on the numbers of sub-simplices ($\mathcal{N}_0, \mathcal{N}_1, \cdots$. In this Letter, analogously to the simplicial man-

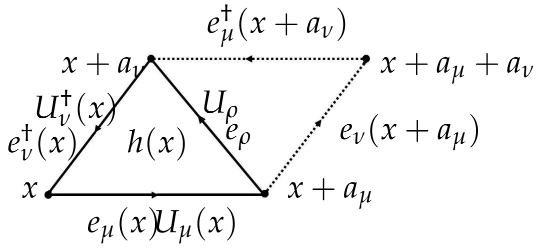

Figure 7.2.: Assuming edge spacing $a_{\mu,\nu}(x)$ is so small that the geometry of the interior of 4-simplex and its sub-simplex (3- and 2-simplex) is approximately flat, we assign a local Lorentz frame to each 4-simplex. On a local Lorentz manifold $\xi^a(x)$ at a space-time point "x", we sketch a closed parallelogram $\mathcal{C}_P(x)$ lying in the 2-simplex $h(x)$. Its edges $e_\mu(x)$ and $e^\dagger_\nu(x) = e_\nu(x+a_\nu)$ are two edges of the 2-simplex $h(x)$, and other edges (dashed lines) $e^\dagger_\mu(x+a_\nu)$ and $e_\nu(x+a_\mu)$ are parallel transports of $e_\mu(x)$ and $e^\dagger_\nu(x)$ along ν- and μ-directions respectively. Each 2-simplex in the 4-simplices complex has a closed parallelogram lying in it. Group-valued gauge fields $U_\mu(x)$ and $U^\dagger_\nu(x) = U_\nu(x+a_\nu)$ are respectively associated to edges $e_\mu(x)$ and $e^\dagger_\nu(x)$ of the 2-simplex $h(x)$, as indicated. The fields $e_\rho(x+a_\mu)$ and $U_\rho(x+a_\mu)$ are associated to the third edge $(x+a_\mu, x+a_\nu)$ of the 2-simplex $h(x)$.

ifold adopted by Regge Calculus we consider a 4-simplices complex, whose elementary building block is a 4-simplex (pentachoron). The 4-simplex has 5 vertexes – 0-simplex (a space-time point "x"), 5 "faces" – 3-simplex (a tetrahedron), and each 3-simplex has 4 faces – 2-simplex (a triangle), and each 2-simplex has three faces – 1-simplex (an edge or a link "$l_\mu(x)$"). Different configurations of 4-simplices complex correspond to variations of relative vertex-positions $\{x\}$, edges "$\{l_\mu(x)\}$" and "deficit angle" around each vertex x. These configurations will be described by the configurations of dynamical fields $e_\mu(x)$ and $\omega_\mu(x)$ (its group-valued $U_\mu(x)$) in a regularized EC-theory.

To illustrate how to construct a regularized EC theory describing dynamics of 4-simplices complex, we consider a 2-simplex (triangle) $h(x)$ (see Fig. 7.2). The fundamental tetrad field $e_\mu(x)$ and "gauge" field $\omega_\mu(x)$ are assigned to each 1-simplex (edge) of the 4-simplices complex. The values of $e_\mu(x)$-field characterize edge spacings $a_\mu(x) \equiv |l_\mu(x)|$, where $l_\mu(x) = ae_\mu(x)$ and the Planck length $a = (8\pi G)^{1/2}$. The fundamental area operator $S^{\rm h}_{\mu\nu} \equiv l_\mu(x) \wedge l_\nu(x)/2$, where $\mu \neq \nu$ indicates edges of the 2-simplex. The 2-simplex area $S_{\rm h}(x) = |S^{\rm h}_{\mu\nu}(x)|$.

The Cartan equation (7.8.8) is actually an equation for infinitesimal parallel transports of $e_\nu(x)$ fields. Applying this equation to the 2-simplex $h(x)$, as shown in Fig. 7.2, we show that $e_\nu(x)$ $[e_\mu(x)]$ undergoes its parallel transport to $e_\nu(x+a_\mu)$ $[e_\mu(x+a_\nu)]$ along the μ $[\nu]$-direction for an edge spacing $a_\mu(x)$

$[a_\nu(x)]$, following the discretized Cartan equation

$$e^a_\nu(x+a_\mu) - e^a_\nu(x) - a_\mu \omega^{ab}_\mu(x) \wedge e_{\nu b}(x) \quad = \quad 0, \tag{7.8.9}$$

and $\mu \leftrightarrow \nu$. The parallel transports $e^a_\nu(x+a_\mu)$ and $e^a_\mu(x+a_\nu)$ are neither independent fields, nor assigned to any edges of the 4-simplices complex. They are related to $e_\mu(x)$ and $\omega_\mu(x)$ fields assigned to edges of the 2-simplex $h(x)$ by the Cartan equation (7.8.9). Because of torsion-free, $e_\mu(x), e_\nu(x)$ and their parallel transports $e_\mu(x+a_\nu), e_\nu(x+a_\mu)$ form a *closed* parallelogram $\mathcal{C}_P(x)$ (Fig. 7.2). Otherwise this would means the curved space-time could not be approximated locally by a flat space-time.

Thus, for each 2-simplex, there is a closed parallelogram, whose two edges lying in the 2-simplex and other two edges of parallel transports not lying in any 2-simplex.

We define $\omega_\mu(x+a_\nu)$ and $\omega_\nu(x+a_\mu)$ by using the discretized equation for curvature (7.8.3),

$$\omega^{ab}_\nu(x+a_\mu) - \omega^{ab}_\nu(x) - a_\mu \omega^{ae}_\mu(x) \wedge \omega^b_{e\nu}(x) \;=\; a_\mu R^{ab}_{\mu\nu}(x), \tag{7.8.10}$$

and $\mu \leftrightarrow \nu$. For zero curvature case, analogously to (7.8.9), parallel transports $\bar{\omega}^{ab}_\nu(x+a_\mu)$ $[\bar{\omega}^{ab}_\mu(x+a_\nu)]$ can be defined as

$$\bar{\omega}^{ab}_\nu(x+a_\mu) - \omega^{ab}_\nu(x) - a_\mu \omega^{ae}_\mu(x) \wedge \omega^b_{e\nu}(x) \quad = \quad 0, \tag{7.8.11}$$

and $\mu \leftrightarrow \nu$. The difference ("deficit angle") between $\omega^{ab}_\nu(x+a_\mu)$ and $\bar{\omega}^{ab}_\nu(x+a_\mu)$ is the curvature $a_\mu R^{ab}_{\mu\nu}(x)$.

Instead of $\omega_\mu(x)$ field, we assign a group-valued field $U_\mu(x)$ to each 1-simplex of 4-simplices complex. For example, at edges (x,μ) and (x,ν) of the 2-simplex $h(x)$ ($\mu \neq \nu$ see Fig. 7.2), we define $SO(4)$ group-valued spin-connection fields,

$$U_\mu(x) = e^{iga\omega_\mu(x)}, \quad U_\nu(x) = e^{iga\omega_\nu(x)}, \tag{7.8.12}$$

which take value of fundamental representation of the compact group $SO(4)$, and their local gauge transformations,

$$U_\mu(x) \quad \rightarrow \quad \mathcal{V}(x) U_\mu(x) \mathcal{V}^\dagger(x+a_\mu), \tag{7.8.13}$$

and $\mu \leftrightarrow \nu$ in accordance with (7.8.2). Actually, these group-valued fields (7.8.12) can be viewed as unitary operators for finite parallel transportations. Eq. (7.8.9) can be generalized to

$$e_\nu(x+a_\mu) \quad = \quad U_\mu(x) e_\nu(x) U^\dagger_\mu(x), \tag{7.8.14}$$

and $\mu \leftrightarrow \nu$. While, corresponding to (7.8.10) for the field $\omega_\nu(x+a_\mu)$, we define

$$U_\nu(x+a_\mu) \equiv U_\mu(x)U_\nu(x)U^\dagger_\mu(x), \tag{7.8.15}$$
$$U_\nu(x+a_\mu) \equiv e^{iga\omega_\nu(x+a_\mu)}, \tag{7.8.16}$$
$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x) \equiv U_\nu(x+a_\mu)U_\mu(x), \tag{7.8.17}$$

and $\mu \leftrightarrow \nu$. Eq. (7.8.17) characterizes relative angles $\theta_{\mu\nu}(x)$ between two neighboring edges $e_\mu(x)$ and $e_\nu(x)$ (see Fig. 7.2). In the *naive continuum limit*: $ag\omega_\mu \ll 1$ (small coupling or weak-field), indicating that the wavelengths of weak and slow-varying fields $\omega_\mu(x)$ are much larger than the edge spacing $a_{\mu,\nu}$, we have

$$\begin{aligned} U_{\mu\nu}(x) &= \exp\Big\{ig[a\omega_\nu(x)+a\omega_\mu(x)] + iga^2\partial_\mu\omega_\nu(x) \\ &- \frac{1}{2}(ga)^2\left[\omega_\nu(x),\omega_\mu(x)\right] + \mathcal{O}(a^3)\Big\}, \end{aligned} \tag{7.8.18}$$

where $\mathcal{O}(a^3)$ indicates high-order powers of $ag\omega_\mu$.

Using the tetrad fields $e_\mu(x)$ to construct coordinate and Lorentz scalars so as to obtain a regularized EC action preserving the diffeomorphism and *local* gauge-invariance, we define the smallest holonomy along closed triangle path of 2-simplex:

$$X_h(v,U) = \mathrm{tr}\left[v_{\nu\mu}(x)U_\mu(x)v_{\mu\rho}(x+a_\mu)U_\rho(x+a_\mu)v_{\rho\nu}(x+a_\nu)U_\nu(x+a_\nu)\right], \tag{7.8.19}$$

whose orientation is anti-clock-like, and $X^\dagger_h(e,U)$ is clock-like (see Fig. 7.2). We have following two possibilities for the vertex-field $v_{\nu\mu}(x)$. The first $v_{\mu\nu}(x) = e_{\mu\nu}(x)\gamma_5$:

$$\mathcal{A}_P(e,U) = \frac{1}{8g^2}\sum_h \{X_h(v,U) + \mathrm{h.c.}\}, \tag{7.8.20}$$
$$e_{\mu\nu}(x) \equiv (e^a \wedge e^b)\sigma_{ab}, \tag{7.8.21}$$

where $\sum_h$ is the sum over all 2-simplices $h(x)$. In the limit: $ag\omega_\mu \ll 1$, Eq. (7.8.20) becomes

$$\mathcal{A}_P(e,U_\mu) = \frac{1}{a^2}\sum_h S^2_{\mathrm{h}}(x)\epsilon_{cdab}\, e^c \wedge e^d \wedge R^{ab} + \mathcal{O}(a^4). \tag{7.8.22}$$

We define a 4-d volume element $V(x) = \sum_{h(x)} S^2_{\mathrm{h}}(x)$ around the vertex x. The interior of 4-simplex is approximately flat, leading to

$$\sum_x V(x) \Rightarrow \int d^4\xi(x) = \int d^4x \det[e(x)], \tag{7.8.23}$$

and Eq. (7.8.22) approaches to $S_P(e,\omega)$ (7.7.5) with an effective Newton constant $G_{\rm eff} = gG/4$. The second $v_{\mu\nu}(x) = e_{\mu\nu}(x)$:

$$\mathcal{A}_H(e, U_\mu) = \frac{1}{8g^2\gamma}\sum_h [X_h(v,U) + \text{h.c.}], \tag{7.8.24}$$

where the real parameter $\gamma = i\tilde{\gamma}$. Analogously, in the limit: $ag\omega_\mu \ll 1$, Eq. (7.8.24) approaches to $S_H(e,\omega)$ (7.8.6),

$$\mathcal{A}_H(e, U_\mu) = \frac{1}{2\kappa\tilde{\gamma}}\int d^4x \det[e(x)] e_a \wedge e_b \wedge R^{ab} + \mathcal{O}(a^4). \tag{7.8.25}$$

Under the gauge transformation (7.8.1),

$$v_{\mu\nu}(x) \rightarrow \mathcal{V}(x) v_{\mu\nu}(x) \mathcal{V}^\dagger(x). \tag{7.8.26}$$

The diffeomorphism and *local* gauge-invariant regularized EC action is then given by

$$\mathcal{A}_{EC} = \mathcal{A}_P + \mathcal{A}_H. \tag{7.8.27}$$

$a = \pi/\Lambda_{\rm cutoff}$, the momentum cutoff $\Lambda_{\rm cutoff} = m_p(\pi/8)^{1/2}$

Considering the following diffeomorphism and *local* gauge-invariant holonomies along a large loop $\mathcal{C}$ on the Euclidean manifold $\mathcal{M}$

$$X_{\mathcal{C}}(v,\omega) = \mathcal{P}_C \text{Tr} \exp\left\{ ig \oint_{\mathcal{C}} v_{\mu\nu}(x)\omega^\mu(x) dx^\nu \right\}, \tag{7.8.28}$$

where $\mathcal{P}_C$ is the path-ordering and "Tr" denotes the trace over spinor space, we attempt to regularize these holonomies on the 4-simplices complex. Suppose that an orientating closed path $\mathcal{C}$ passes space-time points $x_1, x_2, x_3, \cdots, x_N = x_1$ and edges connecting between neighboring points in the 4-simplices complex. At each point x_i two tetrad fields $e_\mu(x_i)$ and $e_{\mu'}(x_i)$ $(\mu \neq \mu')$ respectively orientating path incoming to $(i-1 \rightarrow i)$ and outgoing from $(i \rightarrow i+1)$ the point x_i, we have the vertex-field $v_{\mu\mu'}(x_i)$ defined by Eqs. (7.8.21,7.8.24). Link fields $U_\mu(x_i)$ are defined on edges lying in the loop $\mathcal{C}$, recalling the relationship $U_\mu(x_i) = U_{-\mu}(x_{i+1}) = U^\dagger_\mu(x_{i+1})$, we can write the regularization of the holonomies (7.8.28) as follows,

$$\begin{aligned} X_{\mathcal{C}}(v,U) &= \mathcal{P}_C \mathrm{Tr}\Big[v_{\mu\mu'}(x_1)U_{\mu'}(x_1)v_{\mu'\nu}(x_2)U_\nu(x_2) \\ &\cdots v_{\rho\rho'}(x_i)U_{\rho'}(x_i)v_{\rho'\sigma}(x_{i+1}) \\ &\cdots v_{\lambda\mu}(x_{N-1})U^\dagger_\mu(x_{N-1})\Big], \end{aligned} \tag{7.8.29}$$

preserving diffeomorphism and *local* gauge-invariances. Eq. (7.8.29) is consistent with Eq. (7.8.19).

Euclidean partition function. The partition function Z_{EC} and effective action $\mathcal{A}^{\mathrm{eff}}_{EC}$ are

$$Z_{EC} = \exp -\mathcal{A}^{\mathrm{eff}}_{EC} = \int \mathcal{D}e\mathcal{D}U \exp -\mathcal{A}_{EC}, \tag{7.8.30}$$

with the diffeomorphism and *local* gauge-invariant measure

$$\int \mathcal{D}e\mathcal{D}U \equiv \prod_{x,\mu} \int de_\mu(x) dU_\mu(x) \tag{7.8.31}$$

where $\prod_{x,\mu}$ indicates the product of overall edges, $dU_\mu(x)$ is the Haar measure of compact gauge group $SO(4)$ or $SU(2)$, and $de_\mu(x)$ is the measure of Dirac-matrix valued field $e_\mu(x) = \sum_a e^a_\mu(x)\gamma_a$, determined by the functional measure $de^a_\mu(x)$ of the bosonic field $e^a_\mu(x)$. It should be mentioned that the measure (7.8.31) is just a lattice form of the standard DeWitt functional measure over the continuum degrees, with the integral of the spin-connection field $\omega_\mu(x)$ replaced by the Haar integral over the $U_\mu(x)$'s, analytical integration or numerical simulations runs overall configuration space of continuum degrees and no gauge fixing is needed.

Note that the measure $\mathcal{D}U_\mu(x)$ includes all link fields lying in both edges (e_μ, e_ν) of 2-simplices and their parallel transports (e_μ, e_ν), as shown in Fig. 7.2.

$$[e^a_\mu(x), e^b_\nu(x')] = \delta_{\mu\nu}(x)\delta^{ab}\delta(x-x'), \tag{7.8.32}$$

and equivalently

$$\{e_\mu(x), e^\dagger_\nu(x')\} = \delta_{\mu\nu}(x)\delta(x-x'). \tag{7.8.33}$$

In this path-integral quantization formalism, values of the partition function (7.8.30) presents all dynamical configurations of 4-simplices complex, described by field configurations $e_\mu(x)$ and $U_\mu(x)$ in the weight $\exp -\mathcal{A}_{EC}$. The vacuum expectational values (v.e.v.) of diffeomorphism and *local* gauge-invariant quantities, for instance holonomies (7.8.29), are given by

$$\langle X_{\mathcal{C}}(e,U)\rangle = \frac{1}{Z_{EC}}\int \mathcal{D}e\mathcal{D}U\Big[X_{\mathcal{C}}(e,U)\Big]\exp -\mathcal{A}_{EC}\,. \tag{7.8.34}$$

In the action (7.8.20,7.8.24), $X_h(v,U)$ (7.8.19) contains the quadric term of $e_\mu(x)$-field associated to each edge (x,μ), the partition function Z_{EC} (7.8.30) and v.e.v. (7.8.34) are converge.

and we have the following formula:

$$\int \mathcal{D}e e^{-e_l\Delta^{lk}(U)e_k^\dagger} = \det[\Delta(U)], \tag{7.8.35}$$

$$\int \mathcal{D}e(e_i e_j^\dagger)e^{-e_l\Delta^{lk}(U)e_k^\dagger} = \Delta_{ij}(U), \tag{7.8.36}$$

$$\int \mathcal{D}e[e_i\Lambda^{ij}(U)e_j^\dagger]e^{-e_l\Delta^{lk}(U)e_k^\dagger} = \mathrm{Tr}[\Lambda(U)\Delta(U)], \tag{7.8.37}$$

where $\Delta(U)$ and $\Lambda(U)$ are operators in terms of links fields $\{U_\mu(x)\}$. Applying Eq. (7.8.35) to the partition function (7.8.30), we integrate over tetrad fields $e_\mu(x)$ and formally obtain,

$$Z_{EC} = \int \mathcal{D}U\det\Big[\frac{1}{8g}\gamma_5 U_{\mu\nu}\frac{i}{2} + \frac{1}{8g\gamma}U_{\mu\nu} + \text{h.c.}\Big]. \tag{7.8.38}$$

Analogously to Eq. (7.8.7), the *local* gauge-invariance of the partition function (7.8.30) ($\delta Z_{EC} = 0$) leads to

$$\langle\frac{\delta\mathcal{A}_{EC}}{\delta e_\mu}\delta e_\mu + U_\mu\frac{\delta\mathcal{A}_{EC}}{\delta U_\mu} + \text{h.c.}\rangle = 0, \tag{7.8.39}$$

which becomes "averaged" Einstein equation $\langle\delta\mathcal{A}_{EC}/\delta e_\mu\rangle + \text{h.c.} = 0$, and

$$\langle U_\mu\frac{\delta\mathcal{A}_{EC}}{\delta U_\mu} - U_\mu^\dagger\frac{\delta\mathcal{A}_{EC}}{\delta U_\mu^\dagger}\rangle = 0. \tag{7.8.40}$$

Eq. (7.8.40) is "averaged" torsion-free Cartan equation (7.8.8), which actually shows the impossibility of spontaneous breaking of *local* gauge symmetry. This should not be surprised, since the torsion-free (7.8.8) is a necessary condition to have a *local* Lorentz frame, therefore a *local* gauge-invariance.

The *local* gauge-invariance of (7.8.34) ($\delta\langle X\rangle = 0$) leads to dynamical equations for holonomies (7.8.29), which can be formally written as

$$\langle\frac{\delta X}{\delta e_\mu}\delta e_\mu + X\frac{\delta\mathcal{A}_{EC}}{\delta e_\mu}\delta e_\mu + X + XU_\mu\frac{\delta\mathcal{A}_{EC}}{\delta U_\mu} + \text{h.c.}\rangle = 0, \tag{7.8.41}$$

leading to $\langle \delta X/\delta e_\mu + X\delta \mathcal{A}_{EC}/\delta e_\mu\rangle + \text{h.c.} = 0$, and

$$\langle X\rangle + \langle X\Big(U_\mu \frac{\delta \mathcal{A}_{EC}}{\delta U_\mu} - U^\dagger_\mu \frac{\delta \mathcal{A}_{EC}}{\delta U^\dagger_\mu}\Big)\rangle = 0. \tag{7.8.42}$$

Eq. (7.8.42) has the same form as the Schwinger-Dyson equation for Wilson loops in lattice gauge theories.

The regularized EC theory (7.8.27) can be separated into left- and right-handed parts by replacing $U_\mu(x) = U^L_\mu(x) \otimes U^R_\mu(x)$, where $U^{L,R}_\mu(x) \in SU_{L,R}(2)$. In addition, we can generalize the link field $U_\mu(x)$ to be all irreducible representations $U^j_\mu(x)$ of the gauge group $SO(4)$. The regularized EC action (7.8.27) should be a sum over all representations $j \equiv j_{L,R} = 1/2, 3/2, \cdots$,

$$\mathcal{A}_{EC} = \sum_j \left[\mathcal{A}^j_P(e_\mu, U^j_\mu) + \mathcal{A}^j_H(e_\mu, U^j_\mu)\right], \tag{7.8.43}$$

and the measure (7.8.31) should include all representations of gauge group.

Some calculations in 2-dimensional case. We consider a 2-simplices complex, i.e., random simplicial surface, whose elementary building block is a triangle $h(x)$ (see Fig. 7.2). In this case, *local* gauge transformations (7.8.13,7.8.26) can be made so that all fields $v_{\mu\rho}(x+a_\mu)U_\rho(x+a_\mu)v_{\rho\nu}(x+a_\nu) = 1$ in Eq. (7.8.19), as if we choose a particular gauge. The partition function (7.8.30) can be calculated by integrating over $e_\mu(x)$- and $U_\mu(x)$-fields, using the Cayley-Hamilton formula for a determinant and the properties of invariant Haar measure: $\int dU^j_\mu(x) = 1$, $\int dU^j_\mu(x)U^j_\mu(x) = 0$ and

$$\int dU^j_\mu(x)U^{ab}_\mu(x)U^{\dagger cd}_\nu(x') = \frac{1}{d_j}\delta_{\mu\nu}\delta^{ac}\delta^{bd}\delta(x-x'), \tag{7.8.44}$$

where $d_j = n_{j_L}n_{j_R}$ ($n_{j_L,j_R} = 2j_{L,R}+1$), the dimension of irreducible representations $j = (j_L, j_R)$ of $SU_L(2)\otimes SU_R(2)$.

We calculate Eq. (7.8.30) for all representations j

$$Z_{EC} = \prod \Big[\frac{i}{2d_j g^2}\gamma_5 + \frac{2}{2d_j g^2\gamma}\Big]. \tag{7.8.45}$$

We obtain the entropy $\mathcal{S} = \ln Z_{EC}$

$$\mathcal{S} = \sum \text{Tr}\Big[\gamma_5\frac{i}{2d_j g^2} + \frac{2}{2d_j g^2\gamma}\Big] = \sum_j \frac{4}{d_j g^2\gamma a^2}S_{\text{surf}}, \tag{7.8.46}$$

where $\sum$ is the sum over all 2-simplices, degrees of freedom of gauge group representations and Dirac spinors. The 2-dimensional surface

$$S_{\rm surf} = \sum_h S_{\rm h}(x) = N_{\rm h} P_a, \quad P_a = \frac{1}{N_{\rm h}} \sum_h S_{\rm h}(x) \tag{7.8.47}$$

where $N_{\rm h}$ is the total number of 2-simplices and P_a averaged area of 2-simplices. The free energy $\mathcal{F} = -\frac{1}{\beta} \ln Z_{EC}$,

$$\mathcal{F} = -\frac{1}{\beta} \ln Z_{EC} = -\sum_j \frac{4}{d_j \gamma} S_{\rm surf}, \tag{7.8.48}$$

where the inverse "temperature" $\beta = 1/g^2$, see Eqs. (7.8.20,7.8.24). Selecting fundamental representation $d_j = 4$, we obtain $\mathcal{S} = S_{\rm surf}/(g^2\gamma a^2)$ and $\mathcal{F} = -S_{\rm surf}/(\gamma a^2)$.

In the same way, we calculate the average of regularized EC action $\mathcal{A}_{EC}$ (7.8.43),

a single 2-complex action

$$\mathcal{A}^j_{EC}[e_\mu(x), U^j_\mu(x)] = \frac{1}{8g^2} {\rm tr} \left\{ e_{\mu\nu}(x)\gamma_5 U^p_{\mu\nu}(x) + \frac{1}{\gamma}\tau_{\mu\nu}(x) U^p_{\mu\nu}(x) + {\rm h.c.} \right\}, \tag{7.8.49}$$

which is the regularized EC action $\mathcal{A}$ (7.8.43) at a single 2-simplex $h(x)$, i.e., Eqs. (7.8.20,7.8.24) without the sum $\sum_{x,\mu\nu} = \sum_{h(x)}$.

We integrate over tetrad fields $e_\mu(x)$ and obtain,

$$\begin{aligned} \langle \mathcal{A}^j_{EC}[e_\mu, U^j_\mu] \rangle &= \left(\frac{1}{8g^2}\right)^2 \frac{1}{Z_{EC}} \int \mathcal{D}U \sum_h \cdot \\ &\cdot \quad {\rm tr} \left\{ \gamma_5 U_{\mu\nu}(x) \left(\frac{i}{2}\right) + \frac{1}{\gamma} U_{\mu\nu}(x) + {\rm h.c.} \right\}^2, \end{aligned} \tag{7.8.50}$$

where Z_{EC} is given by Eq. (7.8.30). In the strong coupling (field) limit $g \gg 1$ or $ga\omega_\mu \sim \mathcal{O}(1)$, implying that ω_μ field's wavelength is comparable to the Planck length a, we expand Z_{EC} in powers of $1/g$ and use Eq. (7.8.44) to compute the average (7.8.50). As a result, the leading term is given by

$$\langle \mathcal{A}^j_{EC}[e_\mu, U^j_\mu] \rangle \simeq \frac{1}{d_j} \left(\frac{1}{8g^2}\right)^2 \left(1 + \frac{4}{\gamma^2}\right) N_{\rm h}, \tag{7.8.51}$$

in the strong coupling (field) limit $g \gg 1$ or $ga\omega_\mu \sim \mathcal{O}(1)$, which implies that ω_μ field's wavelength is comparable to the Planck length a, The average (7.8.51) of regularized EC action has discrete values corresponding to the fundamental state $d_j = 4$ and excitation states $d_j = 16$.

The average of total regularized EC action

$$\langle \mathcal{A}^j_{EC}[e_\mu, U^j_\mu]\rangle = \sum_{h(x)} \langle \mathcal{A}^j_{EC}[e_\mu(x), U^j_\mu(x)]\rangle \simeq \frac{1}{2d_j g^2\gamma} N_{\rm h}, \tag{7.8.52}$$

where $N_{\rm h}$ is the total number of 2-simplices.

Using the convexity inequality $\langle e^{-\mathcal{A}^j_{EC}}\rangle \geq e^{-\langle \mathcal{A}^j_{EC}\rangle}$, we have

$$\langle \mathcal{A}^j_{EC}[e_\mu, U^j_\mu]\rangle \leq \ln Z^j_{EC}(2/g^2) - \ln Z^j_{EC}(1/g^2). \tag{7.8.53}$$

Using Eqs. (7.8.46,7.8.47), we obtain

$$\frac{1}{d_j}\left(\frac{1}{8g^2}\right)^2\left(1+\frac{4}{\gamma^2}\right) N_{\rm h} \leq \frac{4}{d_j g^2\gamma a^2} S_{\rm surf}, \tag{7.8.54}$$

and averaged area of a 2-simplex

$$P_a \geq \frac{\pi}{32g^2}\left(1+\frac{4}{\gamma^2}\right)\frac{8\pi}{m^2_{\rm Planck}}, \tag{7.8.55}$$

implying that the Planck length is minimal separation between two space-time points.

$$P_a \geq \pi/m_p^2. \tag{7.8.56}$$

Using Eq. (7.8.51), we show the Planck area P_a has to be larger than π/m_p^2.

Some remarks.

The quantum dynamics of 4-simplices complex (space time) is described by quantum fields $e_\mu(x)$ and $\omega_\mu(x)$ of regularized and quantized EC theory (7.8.27-7.8.34). 4-simplex, an elementary building block of 4-simplices complex, has the size of order of the Planck length, which is probed by short wavelengths of quantum fluctuations of fields e_μ, ω_μ in strong gauge couplings g. The genuine violation of the diffeomorphism invariance at the size of a 4-simplex is negligible, when we consider large scales probed by long wavelengths of fields.

We have to point out that the regularization action (7.8.27) is not unique, it can possibly contain non-local high-dimensional ($d > 6$) operators of tetrad and link fields, permitted by diffeomorphism and *local* gauge-invariances.

Although the regularized EC action (7.8.27) approaches to the EC action (7.8.4) in the "*naive continuous limit*" $ag\omega_\mu \ll 1$, the regularized EC theory is physically sensible, provided it has a non-trivial continuum limit. It is crucial, on the basis of non-perturbative methods and renormalization group invariance, to find: (1) the scaling invariant regimes (ultraviolet fix points) g_c, where phase transition takes place and physical correlation length ξ is much larger than the Planck length a; (2) β-function $\beta(g)$ and renormalization-group invariant equation $\xi = \text{const.}\, a \exp \int^g dg'/\beta(g')$; (3) all relevant and renormalizable operators (one-particle irreducible (1PI) functions) with effective dimension-4 in these regimes to obtain effective low-energy theories. One may add by hand the cosmological Λ-term $\frac{\lambda}{4\cdot 4!}\epsilon^{\mu\nu\rho\sigma}\sum_x \text{tr}[e_\mu e_\nu e_\rho e_\sigma] + \text{h.c.}$, where $\lambda = \Lambda a^2$, into the regularized EC action (7.8.27). However, 1PI functions $\mathcal{A}^{\text{eff}}_{EC}$ (7.8.30) effectively contain this dimensional operator, which is related to the truncated Green function $\langle \mathcal{A}_{EC}\mathcal{A}_{EC}\rangle$. It is then a question what is the scaling property of this operator in terms of ξ^{-2}, where inverse correlation length ξ^{-1} gives the mass scale of low-energy excitations of the theory.

One can consider the following regularized fermion action,

$$\begin{aligned}\mathcal{A}_F(e_\mu, U_\mu, \psi) &= \frac{1}{2}\sum_{x\mu}\Big[\bar{\psi}(x)e^\mu(x)U_\mu(x)\psi(x+a_\mu) \\ &- \bar{\psi}(x+a_\mu)U^\dagger_\mu(x)e^\mu(x)\psi(x)\Big], \end{aligned} \tag{7.8.57}$$

where fermion fields $\psi(x)$ and $\psi(x+a_\mu)$ are defined at two neighboring points (vertexes) of 4-simplices complex, fields $U_\mu(x)$ and $e_\mu(x)$ are added to preserve *local* gauge and diffeomorphism invariances, and $\sum_{x\mu}$ is the sum over all edges (1-simplices) of 4-simplices complex. This bilinear fermion action (7.8.57) introduces a non-vanishing torsion field. We need to study whether the regularized EC action (7.8.27) with fermion action (7.8.57) can be effectively written in form of a torsion-free part and four fermion interactions, as the EC theory in continuum. In addition, the bilinear fermion action (7.8.57) has the problem of either fermion doubling or chiral (parity) gauge symmetry breaking, due to the No-Go theorem. Resultant four fermion interactions can possibly be resolution to this problem.

8. Unification Theories

8.1. Brown-Kuchar approach in 5D Kaluza-Klein model

The 5D Kaluza-Klein model in vacuum describes the coupling between gravity and electromagnetism plus a scalar field. It is interesting to check whether such a field could play the role of a relational time. In the scenario of a vanishing electromagnetic field the Hamiltonian density governing fields dynamics reads as follows :

$$H = NH^N + S_i H^i \,, \tag{8.1.1}$$

where Superhamiltonian and Supermomenta are:

$$\begin{aligned} H^N &= b\sqrt{\theta}\phi R - 2b\sqrt{\theta}D^i\partial_i\phi - \frac{T}{2b\sqrt{\theta}\phi} - \frac{\phi\pi^2}{6b\sqrt{\theta}} + \frac{\pi\Sigma}{3b\sqrt{\theta}} && (8.1.2) \\ H^i &= -2D_j\Sigma^{ij} + \pi\partial^i\phi \,. && (8.1.3) \end{aligned}$$

Here we have defined $T_{ijkl}\Sigma^{ij}\Sigma^{kl} = T$ and $\Sigma_{ij}\theta^{ij} = \Sigma$, where ϕ is the scalar field , θ_{ij} the spatial 3D induced metrics , π , Σ_{ij} are their conjugate momenta and T_{ijkl} is the supermetrics.

Via the eq. (8.1.3) we can rule out the spatial gradient of ϕ from eq. (8.1.2); then, multiplying by $b\sqrt{\theta}$ we get a new constraint equivalent to the superhamiltonian one $H^N = 0$, i.e.

$$\tilde{H}^N[x] = b^2\theta\phi R[x] - 2b^2\theta D_i\left(\frac{H_i}{\pi}\right)[x] - \frac{T}{2\phi}[x] - \frac{\pi^2\phi}{6}[x] + \frac{\pi\Sigma}{3}[x] = 0 \tag{8.1.4}$$

Now, we upgrade $\tilde{H}^N$ to an operator acting on some functions space; therefore the constraint becomes $\tilde{H}^N(f) = 0$ for any given function $f(x)$. Thereafter we consider its integral on a probe function f; in such a way, using integration by parts, we can shift the derivative operator D_i to such a function:

$$\tilde{H}^N(f) = \int d^3x \Bigg[\left(b^2\theta\phi R - \frac{T}{2\phi} - \frac{\pi^2\phi}{6} + \frac{\pi\Sigma}{3} \right) [x] f[x] + \\ +2b^2\theta \left(\frac{H_i}{\pi} \right) [x] D_i f[x] \Bigg] = \tilde{H}^N \cdot f = 0 \qquad (8.1.5)$$

Via some algebraic step we can solve the constraint with respect the momentum π. There exist three solutions, but only the following one does not contain imaginary parts:

$$\pi = \frac{1}{3\phi} \left(-2\Sigma + \frac{\Xi}{\phi \left(\Gamma \pm \sqrt{\Xi^3 + \Gamma^2} \right)^{1/3}} - \phi \left(\Gamma \pm \sqrt{\Xi^3 + \Gamma^2} \right)^{1/3} \right) \qquad (8.1.6)$$

where:

$$\begin{aligned} \Xi &= -9T - 4\Sigma^2 + 18b^2 R\theta\phi^2 = \\ &= 2\Sigma^2 - 18 \left(\Sigma_{ij}\Sigma^{ij} + b^2 R\theta\phi^2 \right) \\ \Gamma &= 27T\Sigma + 8\Sigma^3 - 54b^2\theta(3D^i H_i + R\Sigma)\phi^2 = \\ &= -10\Sigma^3 + 54 \left(\Sigma_{ij}\Sigma^{ij}\Sigma - b^2\theta\phi^2(3D^i H_i + R\Sigma^2) \right) \end{aligned} \qquad (8.1.7)$$

It is worth noting now that the new constraint (8.1.6) fulfils the first request needed by the BK procedure: the member on the right side does not depend on spatial derivatives of ϕ, which is indeed the field conjugate to π. Clearly, to claim that this procedure yields a successful BK scheme other checks are needed: it must be shown whether the constraint here derived satisfies the same algebraic properties of the Superhamiltonian and, together with the others constraints, act as a generator for diffeomorphisms (this is actually an expected result), and moreover we have to study the sign of the new Hamiltonian and the probability flow induced by π. The fact , however, that the field ϕ deparametrizes in such a way that it leads to the relation (8.1.6) - which was anyway a not so guaranteed result - is an interesting issue that deserves further investigations and represents, indeed, the first step in the definition of a well defined BK scheme in the Kaluza-Klein model.

8.2. Test Particle Dynamics

The problem of the matter coupling is a longstanding puzzle that affects KK models from the foundation. Indeed, while KK models are successful in vacuum, they show unsatisfactory features when the presence of matter is considered. The standard approach to the dynamics of test particles is to generalize to five dimensions the "geodesic" Action usually adopted in 4D, namely $S = m \int ds$. Therefore, starting from $S_5 = \hat{m} \int ds_5$, where $\hat{m}$ is the 5D mass parameter, it is shown via dimensional reduction, that the motion of a free 5D test particle is reduced into the motion of a 4D test particle interacting with the electromagnetic field, plus the extra scalar field. In such a scheme, the q/m ratio is defined in term of the fifth component of the 5D-velocity w_5, which is a constant of the motion. Even if electrodynamics is formally restored, setting $\phi = 1$, the q/m ratio results to be upper bounded in such a way that this bound cannot be satisfied by every known elementary particles. In the simple case $\phi = 1$, indeed we have:

$$\begin{cases} \frac{d}{ds} w_5 = 0 \\ \frac{D}{Ds} u^\mu = \sqrt{4G} F^{\mu\nu} u_\nu \left(\frac{w_5}{\sqrt{1+w_5^2}} \right) \end{cases} ,$$

where

$$q/m\sqrt{4G} = \frac{w_5}{\sqrt{1+w_5^2}} < 1 .$$

The problem of the geodesic approach relies in a bad definition of the rest mass of the particle. By studying the Hamiltonian formulation of the dynamics we can get the dispersion relation for the $4D$ reduced particle: such a relation is consistent with an interacting particle whose charge q and mass m arise defined as follows:

$$q = \sqrt{4G} P_5 \qquad m^2 = \hat{m}^2 + P_5^2/\phi^2$$

Given that $P_5 = \hat{m} w_5$, in the case $\phi = 1$, we recover the previous bound. These relations show that the physical mass m of the particle does not coincide with the mass parameter $\hat{m}$ we put in the Action; moreover, if we consider the compactification of the extra dimension, we get a quantized charge, as well as a tower of massive modes; but, fixing the length of the extra dimension using the value of the elementary charge, we get massive modes beyond Planck scale (which is indeed the order of magnitude of mass requested by the q/m bound). Hence, the 5D geodesic approach , within the compactified model, is not able to take into account the definition of the rest mass for a test particle. The weak point of the geodesic procedure relies in the assumption of the existence of a 5D point-like particle, which is actually not guaranteed due to the presence of a compactified extra dimension, whose size could be comparable to the size of the particle.

8.3. Coupling with matter: Papapetrou approach

We propose a new scheme , that allows us to deal rigourously with test particle, without giving up with the compactification scenario. Such a new scheme is based on the multipole expansion of Papapetrou and the particle turns out to be described as a localized source in $\mathcal{M}^4$ but still delocalized along the fifth dimension as a consequence of the compactification. Introducing a generic energy-momentum tensor T^{AB} associated to the body, governed by conservation laws and not depending on the fifth coordinate, like it happens for metric fields, the following equations are considered:

$$^{(5)}\nabla_A T^{AB} = 0 \quad \partial_5 T^{AB} = 0 \tag{8.3.1}$$

Performing a multipole expansion centred on a trajectory X^a, at the lowest order the procedure gives the motion equation for a test particle:

$$m\frac{Du^\nu}{Ds} = (u^\nu u^\rho - g^{\nu\rho})\left(\frac{\partial_\rho \phi}{\phi^3}\right) A + qF^{\nu\rho}u_\rho \tag{8.3.2}$$

Below the definitions for coupling factor m, q, A and the according definitions for the effective test-particle tensor component follow:

$$\begin{aligned} m &= \frac{1}{u^0}\int d^3x\sqrt{g}\phi T^{00}, \quad \phi\sqrt{g}T^{\mu\nu} = \int dsm\delta^4(x-X)\,u^\mu u^\nu \\ q &= ek\int d^3x\sqrt{g}\phi T^0_5, \quad ek\phi\sqrt{g}T^\mu_5 = \int dsq\delta^4(x-X)\,u^\mu = \sqrt{g}J^\mu \\ A &= u^0\int d^3x\sqrt{g}\phi T_{55}, \quad \phi\sqrt{g}T_{55} = \int dsA\delta^4(x-X) \end{aligned}$$

The parameter m correctly represents the mass of the particle, which turns out to be localized just in the ordinary 4D space, as it is envisaged by the presence of a 4D Dirac delta function in the above definitions. Charge q is still conserved, in consequence of the continuity equation $\nabla_\mu J^\mu = 0$, which arises from (8.3.1). Mass is in general not conserved and its behaviour is given by

$$\frac{\partial m}{\partial x^\mu} = -\frac{A}{\phi^3}\frac{\partial \phi}{\partial x^\mu}\,. \tag{8.3.3}$$

Therefore the behaviour of mass is related to the variation of the scalar field and the new coupling A (which has a pure extra-dimensional origin) along the path. The equation 8.3.2 admits an effective action, $S_5 = -\int m\,ds + q(A_\mu dx^\mu + \frac{dx^5}{\sqrt{4G}})$, where m is now a variable function whose derivatives are known, which does not coincides to the geodesic action given by the old procedure. Via an Hamiltonian analysis of such a revised action, it can be proved that the KK tower of massive modes is suppressed, due to the presence of a proper counterterm, and the $\frac{q}{m}$ ratio is no more upper bounded. Therefore such an approach allows to deal with test particle consistently without giving up with the compactification hypothesis.

Kakuza-Klein model with source Given that such a new scheme allows us to deal with matter we consider a generic model with matter: hence, starting from $5D$ Einstein equation ${}^5R^{AB} = 8\pi G_5 \tau^{AB}$, being G_5 the unknown $5D$ Newton constant, we get the following set:

$$G^{\mu\nu} = \frac{1}{\phi}\nabla^\mu\partial^\nu\phi - \frac{1}{\phi}g^{\mu\nu}g^{\alpha\beta}\nabla_\alpha\partial_\beta\phi + 8\pi G\phi^2 T^{\mu\nu}_{em} + 8\pi G\frac{T^{\mu\nu}}{\phi}\,, \quad (8.3.4)$$

$$\nabla_\nu\left(\phi^3 F^{\nu\mu}\right) = 4\pi j^\mu\,, \quad (8.3.5)$$

$$g^{\alpha\beta}\nabla_\alpha\partial_\beta\phi = -G\phi^3 F^{\mu\nu}F_{\mu\nu} + \frac{8}{3}\pi G\left(T + 2\frac{\vartheta}{\phi^2}\right)\,. \quad (8.3.6)$$

In the above equations $G^{\mu\nu}$ is the usual Einstein tensor, $F^{\mu\nu}$ the Faraday tensor, ϕ the extra scalar field governing the expansion of the extra dimension, and, given the coordinate length of the fifth dimension $l_5 = \int dx^5$, we have :

$$T^{\mu\nu} = l_5\phi\tau^{\mu\nu}, \qquad j^\mu = \sqrt{4G}\,l_5\phi\tau^\mu_5, \qquad \vartheta = l_5\phi\tau_{55}, \qquad G = G_5 l_5^{-1}\,. \quad (8.3.7)$$

We are now focusing just on some simple scenarios in absence of electromagnetic fields: equations of particular interest are 8.3.6, 8.3.2, 8.3.3 which in such a case read:

$$g^{\alpha\beta}\nabla_\alpha\partial_\beta\phi = \frac{8}{3}\pi G\left(T + 2\frac{\vartheta}{\phi^2}\right)\,, \quad (8.3.8)$$

$$\frac{dm}{ds} = -\frac{A}{\phi^3}\frac{d\phi}{ds}\,, \qquad m\frac{Du^\mu}{Ds} = A(u^\rho u^\mu - g^{\mu\rho})\frac{\partial_\rho\phi}{\phi^3}\,. \quad (8.3.9)$$

Interesting equations of state we would like to outline are the following:

- $2\vartheta = -\phi^2 T$
  Here $\phi = 1$ is a suitable solution and it yields $m = cost$, $\frac{Du^\mu}{Ds} = 0$; therefore the Free Falling Universality (FFU) of particles still holds and we just recover General Relativity.

- $\vartheta = 0$
  Now $m = cost$, being $A = 0$ and we have $\frac{Du^\mu}{Ds} = 0$. Therefore the FFU holds, but we can have ϕ variable, thus we have a modified theory.

- $A = \alpha m\phi^2$
  Now ϕ is variable as well as m but the equation of motion is: $\frac{Du^\mu}{Ds} = \alpha(u^\rho u^\mu - g^{\mu\rho})\frac{\partial_\rho\phi}{\phi}$. The mass is ruled out and then FFU still holds even if the theory is now modified by two additional degrees of freedom.

Noticeably, in the last scenario the equation for mass behaviour admits an easy integration and we get a scaling law for mass:

$$m = m_0 \left(\frac{\phi}{\phi_0} \right)^{-\alpha} . \tag{8.3.10}$$

Promising perspectives of this model deal with its developments within homogeneous background. In the last case a recent proposal under investigation concern the role of $T^{\mu\nu}$. It has been suggested that such a tensor depends in general on ordinary matter degrees of freedom plus extra-dimensional degrees of freedom. Requiring that it reduces to the ordinary matter tensor when the extra-dimensional source vanishes, a suitable parametrization appears $T^{\mu\nu} = T^{\mu\nu}_{matter} + \lambda \frac{\theta}{\phi^2}$. With such a choice there exist solutions, in an homogeneous background, characterized by an accelerating universe together with a collapsing extra dimension. Indeed such a term provides a pressure source, described by a dark energy equation of state. At the same time, assuming that particles we observe are given by localized matter distribution without pressure, effective particles turn out to be associated with distribution with vanishing θ ; therefore, being $A = 0$ we restore all the properties of our observed particles.

8.4. Geodesic deviation

In a work of Kerner et al. (2000) the problem of geodesic deviation in 5D KK is faced. The 4D space-time projection of the obtained equation is identical with the equations obtained by direct variation of the usual geodesic equation in the presence of the Lorenz force, provided that the fifth component of the deviation vector satisfies an extra constraint there derived . The analysis was performed taking $\phi = 1$ and it was developed within the scheme of the geodesic approach. Therefore, our research focused on the extension of this work to the model where the presence of the scalar field is considered. Our results coincide with those of Kerner et al. when the minimal case $\phi = 1$ is considered, while it shows some departures in the general case. The novelty due to the presence of ϕ is that the variation of the q/m between the two geodesic line is not conserved during the motion; an exact law for such a behavior has been derived. In principle such a results is interesting in order to check if it is possible to find a mark of the extra dimension via tidal effects due to the scalar field.

$$\frac{D^2\delta x^\alpha}{Ds^2} - \left(\frac{w_5^2}{1+\frac{w_5^2}{\phi^2}}\right)\frac{1}{\phi^3}\frac{d\phi}{ds}\frac{D\delta x^\alpha}{ds} = -R^\alpha_{\beta\gamma\lambda}u^\beta\delta x^\gamma u^\lambda +$$

$$+\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\delta x^\nu\nabla_\nu\left[F^{\alpha\beta}u_\beta\right] + \left(\frac{w_5^2}{1+\frac{w_5^2}{\phi^2}}\right)\delta x^\nu\nabla_\nu\left(\frac{\partial^\alpha\phi}{\phi^3}\right) -$$

$$-\left(F^\alpha_\nu u^\nu - 2\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\frac{\partial^\alpha\phi}{\phi^3}\right)\delta\left(\delta\frac{q}{m}\right). \qquad (8.4.1)$$

$$\frac{d}{ds}\left(\delta\frac{q}{m}\right) = \left[\frac{\phi}{w_5^2}\frac{d}{ds}\delta x_5 - \alpha\delta\alpha\frac{\phi^3}{w_5^2} - \frac{\delta Q}{\alpha}\right]\frac{d}{ds}\alpha. \qquad (8.4.2)$$

The above equation thus gives the deviation of the factor δQ in term of the re-parameterization factor α; in such a way we can link the problem of the not conservation of the charge-mass ratio with the projection factor from ds^5 to ds.

8.5. Massive test particles motion in Kaluza-Klein gravity

The metric family

$$^5ds^2 = \left(1-\frac{2M}{r}\right)^{\epsilon k}dt^2 - \left(1-\frac{2M}{r}\right)^{-\epsilon(k-1)}dr^2 -$$

$$-r^2\left(1-\frac{2M}{r}\right)^{1-\epsilon(k-1)}d\Omega^2 - \left(1-\frac{2M}{r}\right)^{-\epsilon}dx^{5\,2} \qquad (8.5.1)$$

in the 4D–spherical polar coordinate is a solution of 5D–Kaluza Klein equation in the vacuum $R_{AB} = 0$, with 4D–spherical symmetry. The free "metric parameters" (ϵ, k) are real constants related by $\epsilon^2\left(k^2-k+1\right) = 1$. Metric (8.5.1) reduces to the Schwarzschild solution on the surface $x^5 = cost$ as $\epsilon \to 0$ and $k \to \infty$. In this limit the parameter M is the central body mass. We explored $k \geq 0$ and $\epsilon \geq 0$, analyzing particle motion in the region $r > 2M$, as interesting regions in which to investigate the physical properties of solutions (8.5.1). For these values of the metric parameters, the Gross and Perry solution presents a naked singularity behavior: it is a black hole one only in the Schwarzschild's limit for (ϵ, k). Particles dynamic has been studied first by classical approach, considering particle motion as described by a geodesic in 5D-spacetime, then the analysis has been performed by approach an a la Papapetrou to the motion in Kaluza Klein, therefore considering a 5D-particle described by an energy momentum tensor picked along the particle 4D-world-tube. Finally a comparison of the results obtained into the two different approaches has been made. In both cases we find the effective potential for the circular polar orbits of charges as well as neutral test particles In the first case, assuming a geodesic motion in 5D manifold with a constant particle mass μ_5 the effective potential reads

$$ {}^5Veff \equiv \sqrt{\left(1-\frac{2M}{r}\right)^{\epsilon k}\left[1+r^2\left(1-\frac{2M}{r}\right)^{-1+\epsilon(k-1)}\frac{{}_5L^2}{\mu_5^2}+\left(1-\frac{2M}{r}\right)^{\epsilon}\frac{{}^5\Gamma^2}{\mu_5^2}\right]} \tag{8.5.2} $$

where Γ is the conserved fifth component of the particle momentum. The analysis shows that last circular orbits radius, $r_{co} = [1+\epsilon(2k-1)]M$ is always located under the expected values of $r_{co} = 3M$ of the Schwarzschild's limit: circular orbits (instable and stable) are possible also in a region $r < 3M$. The energy and angular momentum of circular orbits have also been found.

8.5.1. Papapetrou analysis

Here we investigated motion in the backgrounds 8.5.1 by an approach a la Papapetrou. Consider the following 4D–dispersion relation $P_\mu P^\mu = m^2$, where $P_\mu = m u_\mu$, where m is the particle mass and u^μ the 4-velocity of the particle; the followings constants of motion can be defined $E = p_0 = m g_{00} u^0 \quad L = p_\varphi = m g_{\varphi\varphi} u^\varphi$ An effective potential for a test particle of mass m can be defined[1] as

$$ V_{eff} \equiv E = \sqrt{g_{00}\left(m^2 - \frac{L^2}{g_{\varphi\varphi}}\right)} \tag{8.5.3} $$

We focus attention on particulary scenarios.

First we consider the case $A = 0$ where $u^a\, {}^4\nabla_a u^b = 0 \quad \text{and} \quad \partial_\mu m = 0$

they describe a geodetic motion in the ordinary 4D–spacetime for a test particle of constant mass m, where no scalar field coupling term appears. Last stable circular orbit is $r_{LSCO} = \left[1+\epsilon(3k-2)+\epsilon\sqrt{(-1+k)(-1+4k)}\right] M$ with $r_{LSCO} < 6M \quad \forall k > 0$ It is worthwhile noting that last stable circular orbit radius is located under its Schwarzschild' limit; this means that in principle there could be particles in stable orbits for values of radius orbit just less that $6M$, and this could represent a valid constraint to compare theory with experimental data. The energy and angular momentum of the last circular

[1] In this case the V_{eff} has unit of mass.

orbits are respectively:

$$\frac{L^{\pm}_{\epsilon k}}{Mm} = \pm \frac{r^{+}_{LSCO}}{M} \left(1 - \frac{2M}{r^{+}_{LSCO}}\right)^{\frac{1}{2}[(1-k)\epsilon+1]} \sqrt{\frac{-\epsilon k}{\epsilon\,(2k-1) + \left(1 - \frac{r^{+}_{LSCO}}{M}\right)}} \quad (8.5.4)$$

$$\frac{\mathcal{E}_{\epsilon k}}{m} = \left(1 - \frac{2M}{r^{+}_{LSCO}}\right)^{\frac{k\epsilon}{2}} \sqrt{1 + \frac{-\epsilon k}{\epsilon(2k-1) + \left(1 - \frac{r^{+}_{LSCO}}{M}\right)}} \quad (8.5.5)$$

The energy $\mathcal{E}_{\epsilon k}$ for all values of k–parameter is always under its Schwarzschild limit, the angular momentum $\mathcal{L}_{\epsilon k}$ is over the Schwarzschild limit for $k > 3.45644$. This fact should not be read as a direct consequence of a possible motion along a 5–dimension, since the equation of motion does not depend on it but neither on the g_{55}–metric component, but on the contrary it seems us more suitable interpret it a features related to deformation of the Schwarzschild metric as long as k is sufficiently small. This seems to be confirmed also by the fact that Eqs.(8.5.4, 8.5.5) are, as matter of fact, the same one can obtain from the geodesic approach with $\omega_5 = 0$. In the following analysis we choice different values of the dynamical parameter A where $\mathcal{E}_{LSCO}$ and $\mathcal{L}_{LSCO}$ have the same behavior.

As a simplest generalization of the previous case we are going to consider $A = cost$, with $u^a\, {}^{(4)}\nabla_a u^b = (u^b u^c - g^{bc}) \left(\frac{\partial_c \phi}{2\phi^5}\right)$ we set $A = 2\phi_0^2 m_0$, therefore $m = A/2\phi^2$ where in the Schwarzschild's limit $m = m_0$. Last circular orbit is located at $r = r_{LCO} \equiv M\,[1 + \epsilon\,(2k+1)]$ Last stable circular orbit is in

$$r = r_{LSCO} \equiv \frac{\sqrt{M^2\,[4 + (15k-8)\epsilon^2 + 5(8-3k)\epsilon^4]} + M\,[3 + \epsilon(2 + k - 11\epsilon + 5k\epsilon)]}{(2+k)\epsilon} \quad (8.5.6)$$

this is a free–A quantity, but it is a function of the only metric parameter. Also in this case $r_{LSCO} < 6M$ and in the Schwarzschild's limit $r_{LSCO} = 6M$.

In the case $A = \beta m \phi^2$ where β is a real number. the mass m is no more a constant but integrating along a curve $\gamma = \gamma(s)$ between the points $P = \gamma(s)$ and $P_0 = \gamma(s_0)$ follows the scaling law $m = \left(m_0 \phi_0^{\beta}\right) \phi^{-\beta}$ and the equations of motion became $u^a\, {}^{(4)}\nabla_a u^b = \left(u^b u^c - g^{bc}\right) \frac{\partial_c \phi}{\phi} \beta$ where ϕ_0 is a constant. This equation does not depend on m but on the constant β. Introducing the parameter $B^2 \equiv m_0^2 \phi_0^{2\beta}$, and the notation $\bar{g}_{55} \equiv -g_{55} = \phi^2$. The angular momentum L and the energy E of timelike circular orbits are respectively

$$L^2 = (-1)^{\beta} \left(1 - \frac{2M}{r}\right)^{(\beta+1-k)\epsilon} \frac{(2M-r)(k+\beta)\epsilon B^2 M r}{M - r + (2k-1)M\epsilon} \quad (8.5.7)$$

and

$$E = \sqrt{\frac{B^2 \left(1 - \frac{2M}{r}\right)^{(k+\beta)\epsilon} (-1)^{\beta} \left[M - r + M(-1 + k - \beta)\epsilon\right]}{M - r + (2k-1)M\epsilon}} \tag{8.5.8}$$

In the Schwarzschild's limit thy became respectively:

$$L^2 = (-1)^{\beta} \frac{r^2 B^2 M}{r - 3M}, \quad E = \sqrt{-\frac{(-1)^{\beta} B^2 (r - 2M)^2}{(3M - r)r}} \tag{8.5.9}$$

where $B = m_0$ and $\beta = 2n$ with $n \in Z$. In general for $k > -\beta$ last circular orbit is located at $r = r_c \equiv M\left[1 + \epsilon\,(2k+1)\right]$ and $r_c < 3M$. Last stable circular orbit is in

$$r = r_{LSCO} \equiv M \frac{3 + \epsilon[k + \beta + (-3 + k + 2k\beta - \beta(2+\beta))\epsilon]}{(k+\beta)\epsilon} +$$

$$+ M \frac{\sqrt{4 + \epsilon^2 \left[-3k(1+2\beta)\left(\epsilon^2 - 1\right) + (2+\beta)\left(\beta - 4 + \left(\beta^3 + 2\right)\epsilon^2\right)\right]}}{(k+\beta)\epsilon}$$

note that in the Schwarzschild's limit $r_{LSCO} = 6M$. Radius of last stable circular orbit depends on two free parameters, k as the independent metric parameter and β as a "dynamical" one. Moreover $r_{LSCO} < 6M$ for $\beta > 0$, meanwhile for $\beta < 0$ and $k > -\beta$, $r_{LSCO} > 6M$ is possible. For $\beta = 2$ we recover the same physical situations sketched in the case $A = 0$. More generally it is possible to see that at an increase of $\beta > 0$ for fixed values of the parameter k, an increase of the difference $\Delta r_{LSCO} = |r - 6M|$ occurs.

9. Activities

This group lives within the Relativistic Astrophysics Center at the Physics Department of "Sapienza" University of Rome (**Prof. Remo Ruffini** - 2^{nd} Chair in Theoretical Physics). It deals with three main research lines, each of them aimed to specific topics, according to the following scheme:

- ***Early Cosmology***:

  Chaotic Universes, Dissipative cosmologies

- ***Quantum Gravity***:

  Quantum cosmology, The problem of time

- ***Multidimensional Physics***:

  Particle and Field dynamics in Kaluza-Klein theories,
  Geometrization of the gauge connection (the electroweak model)

The group is directed by **Dr. Giovanni Montani** and it is composed of about ten members, undergraduate students, PhD students and post-docs. The main goal of this investigation paradigm is to find, through different aspects of the gravitational field, markers for a unification picture of the fundamental interactions. In this respect, the Cosmological framework is the natural arena of this expected scenario.

9.1. Seminars and Workshops

9.1.1. Seminar at University of Trento, Italy.

Trento, May 15 2009.

- Title: "The Taub Universe: Polymer Quantum Dynamics"

  Authors: O. M. Lecian and G. Montani

  Abstract: After briefly comparing difference operators with differential operators, I will review the main features of the so-called polymer representation of quantum mechanics, as far as the kynematics, the dynamics and the continuum limit are concerned. I will then recall the key points of Loop Quantum Gravity as a quantum theory of gravity based on holonomies of the connections and fluxes of the densitized triads, and the fundamental aspects of Loop Quantum Cosmology, based on the strong simplification arising from considering isotropic and homogeneous geometries. I will eventually apply these tools to the analysis of the Taub cosmological model in the ADM splitting. The behaviour of the gaussian wave-packets will show that, despite the appearence of 'polymer' modifications, the cosmological singularity is not probabilistically suppressed for the polymer Taub universe.

9.1.2. "The directions of moderm cosmologies" meeting

Barcellona, March 2 marzo 2009.

- Title: "An example of f(R) model passing Solar-System tests"

  Authors: O. M. Lecian and G. Montani

  Abstract: The weak-field limit of a f(R) model consisting in the Ricci scalar plus a non-analytic function of it will be proposed, and the parameter space of the model will be constrained by means of the validity range of the weak-field limit approximation and of the planetary motion. These results will be compared with those obtained for the analytical case.

9.1.3. 6th Italian-Sino Workshop on Relativistic Astrophysics

Pescara, June 29-July 1, 2009.

- Title: "Higgs Field from a Scalar-Tensor Theory with Barbero-Immirzi Variables"

  Authors: F. Cianfrani and G. Montani

  Abstract: The Hamiltonian formulation of a scalar field non-minimally coupled to gravity is performed in a first-order approach. It is shown how the scalar field itself enters into the definition of the discrete spatial structure proper of Loop Quantum Gravity. This result suggests to work in the Einstein frame, where the scalar field is minimally-coupled to a fictitious metric. Within this scheme the Higgs potential naturally arises and a non-vanishing vacuum expectation value is predicted for the scalar field.

9.1.4. 2nd Italian-Pakistani Workshop on Relativistic Astrophysics

Pescara, July 8-10, 2009.

- Title: "Massive test particles motion in Kaluza-Klein gravity"

  Authors: V. Lacquaniti, G. Montani, D. Pugliese, R. Ruffini

  Abstract: A class of static, vacuum solutions of (free-electromagnetic) Kaluza-Klein equations with three-dimensional spherical symmetry is studied. In order to explore the dynamic in such spacetimes, geodesic equations are obtained and the effective potential for massive test particles is analyzed. Particular attention is devoted to the properties of the four-dimensional counterpart of these solutions in their Schwarzschild's limit. A modification of the circular stable orbits compared with the Schwarzschild's case is investigated.

9.1.5. XII Marcel Grossman meetings

Paris, July 12-18, 2009.

- Title: "On the Removal of Time-Gauge in Loop Quantum Gravity, with and without Matter".

  Authors: F. Cianfrani and G. Montani

  Abstract: We perform the Hamiltonian formulation of gravity at the first order without fixing the local Lorentz frame. We demonstrate that the Gauss constraints of the Lorentz group reduce to SU(2)-Gauss constraints plus the vanishing of some momenta. This result definitively clarifies the peculiar role played by the SU(2) gauge symmetry in the phase space of gravity. Hence the Loop Quantum Gravity quantization procedure can be safely applied and no gauge condition has to be fixed for the local Lorentz frame.

- Title: "Quantum Suppression of Weak Universe Anisotropy".

  Authors: R. Belvedere, M. V. Battisti and G. Montani

  Abstract: An exact solution of the quantum quasi-isotropic Mixmaster model is described through a semi-classical mechanism. The volume of the Universe is regarded as an external-like observer and a probabilistic interpretation of the wave function naturally arises. We show that near the cosmological singularity all values of the anisotropies are almost equally favored but, once large volume regions are investigated, the closed FRW Universe configuration is deeply privileged.

- Title: "Comparison of the Cosmological Singularity in the Polymer and GUP Frameworks".

  Authors: O. M. Lecian and G. Montani

  Abstract: We investigate the role played by quantization schemes and the choice of the variables to be quantized in the removal of the cosmological singularity. In particular, we analyze the Taub cosmological model, for which an internal time variable can be found, such that the only degree of freedom left is the space-like variable describing the anisotropy. The quantization of this remaining degree of freedom is performed within the frameworks of a generalized uncertainty principle (GUP) and of the polymer representation of quantum mechanics. As a result, the cosmological singularity is probabilistically removed in the Gup Taub Universe, while the polymer effects do not suppress the singularity. We compare these two approaches and results with other proposals, outline the main differences, which can be interpreted as responsible for different results, and consider possible generalizations.

- Title: "Bianchi IX in the GUP approach".

  Authors: M. V. Battisti and G. Montani

  Abstract: We describe the dynamics of the Bianchi I, II and IX cosmological models in the generalized uncertainty principle framework. We show that the Mixmaster Universe is still a chaotic system.

- Title: "Big-bounce from a deformed Heisenberg algebra".

  Authors: M. V. Battisti

  Abstract: The implementation of the Snyder non-commutative geomery in the FRW minisuperspace is analyzed. We show that a big-bounce a la LQC is obtained.

- Title: "Effective potential approach to the motion of massive test particles in Kaluza-Klein gravity"

  Authors: V. Lacquaniti, G. Montani, D. Pugliese, R. Ruffini

  Abstract: Effective potential for a class of static solutions of Kaluza-Klein equations with three-dimensional spherical symm etry is studied. Test particles motion is analyzed. In attempts to read the obtained results with the experimental data, particular attention is devoted to the Schwarzschild's limit of the four–dimensional counterpart of these (free-electromagnetic) solutions. Massive particles stable circular orbits in particular are studied, and a comparison between the well–known results if the Schwarzschild's case and ones found for the static higher-dimensional case is performed. A modification of the circular stable orbits is investigated in agreement with the experimental constraints.

- Title: "Restated Dynamics for Particles and Fields in a 5-D Framework: solution of the q/m problem"

  Authors: V. Lacquaniti and G Montani

  Abstract: In this work we present a revised approach to the problem of matter in the framework of the 5D compactified Kaluza-Klein model. We introduce a 5D external matter tensor and perform a simultaneous reduction of matter and geometry, facing the test-particle motion via an appropriate multipole expansion. Within this scheme the q/m puzzle is solved and the tower of huge massive modes is removed, without giving up with the compactification hypothesis. The model looks like a consistent modified gravity theory, where an extra scalar source term appears. Interesting scenarios and perspectives related to dark energy are discussed.

9.1.6. Invited talk at the Rudjer Boskovic Institute

Zagreb, October 2009.

- Title: "Loop quantum gravity"

  Author: M. V. Battisti

  Abstract: We review the main aspects of the loop approach to quantum gravity. Particular attention will be paid at the connection formulation of GR and at the kinemetic sector of the quantum theory.

9.1.7. ICRA Seminars

Roma, October 29 2009.

- Title: SU(2) gauge structure in Quantum Gravity and the Immirzi field.

  Authors: F. Cianfrani and G. Montani

  Abstract: It is outlined the relevance for Quantum Gravity of inferring a kinematical SU(2) gauge structure in a generic Lorentz frame. Then, the analysis of Hamiltonian constraints is performed in vacuum and in presence of matter fields. SU(2) Gauss constraints are shown to arise even though no restriction of the local Lorentz frame takes place. Finally, it is presented the application of the proposed scheme to an external Immirzi scalar field, for which a dynamical relaxation to a fixed vacuum expectation value is proposed.

9.1.8. 19th International Conference on General Relativity and Gravitation (GR19)

Mexico City, July 5-9, 2010.

Poster

- Title: "On the SU(2) gauge symmetry in the Holst formulation of gravity"
  Author: F. Cianfrani

9.2. Review Work

9.2.1. Fundamentals and recent developments in non-perturbative canonical Quantum Gravity

- ***Authors***: F. Cianfrani, O.M. Lecian and G. Montani

In this work fundamental and recent aspects of canonical quantum gravity are reviewed. The aim of the presentation is to provide a pedagogical approach to the problem of quantizing the gravitational field which provides the tools for a proper understanding of recent issues in this research line.

After a detailed discussion of some relevant features concerning the classical and quantum field dynamics, the Wheeler-DeWitt formulation of canonical quantum gravity is presented with a careful discussion of its main shortcomings. Then a detailed analysis of the Loop Quantum Gravity approach is given starting from the basic mathematical notions at the ground of this modern formulation. Finally the full paradigm is developed giving emphasis on the successes and the open questions concerning the loop representation of space-time.

Contents:

- **2. Hamiltonian formulation of the geometrodynamics**
  - 2.1 The action for the gravitational field
  - 2.2 The space-time slicing
  - 2.3 The Hamiltonian structure
  - 2.4 The Hamilton-Jacobi equation
  - 2.5 Reduction to the canonical form
- **3. Gravity as a gauge theory**
  - 3.1 Gauge theories
  - 3.2 First-order formulation for the gravitational field
  - 3.3 Gravity as a gauge theory of the Lorentz group?
  - 3.4 Poincaré gauge theory
  - 3.5 The Holst formulation
  - 3.6 The Kodama state
- **4. Quantization of the gravitational field**
  - 4.1 The WDW equation
  - 4.2 The problem of time
  - 4.3 Interpretation of the wave function
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- **5. Loop Quantum Gravity**
  - 5.1 Holonomies and Fluxes
    - 5.1.1 Why a reformulation in terms of Wilson loops?
    - 5.1.2 Lattice gauge theories
    - 5.1.3 Holonomies and fluxes in Quantum Gravity
  - 5.2 Spectrum of space-time operators
  - 5.3 Quantum dynamics in LQG
  - 5.4 Open issues in Loop Quantum Gravity
    - 5.4.1 Master Constraint and Algebraic QG
  - 5.5 Loop Quantum Cosmology
  - 5.6 On the physical meaning of the Immirzi parameter
  - 5.7 Time gauge and boost invariance
  - 5.8 The picture of the space-time

A. Brief description of Quantum Gravity

A.1. The time gauge problem in the path integral formalism

In section "The time gauge problem in the path integral formalism" we turn our attention to general relativity expressed in first-order formalism, in order to investigate [10] the physicality condition for the states of the gravitational field arising from BRST invariance of the theory, following the same procedure employed for non-Abelian gauge theories. In this procedure we will intentionally avoid to use canonical quantization methods. We are to determine a physical state condition on quantum states without thinking of classical Hamiltonian constraints in order to compare, at the end of our calculation, our physicality condition required by BRST symmetry and derived with path-integral methods with the one obtained using the Dirac quantization method employed within Ashtekar's canonical formulation.

The people involved in this line of research are Michele Castellana and Giovanni Montani.

A.2. Minisuperspace and Generalized Uncertainty Principle

In section "Minisuperspace and Generalized Uncertainty Principle" we explain some results obtained in a recent approach to quantum cosmology, in which the notion of a fundamental scale naturally appears. This scheme realizes in quantizing a cosmological model by using a deformed Heisenberg algebra, which reproduces a Generalized Uncertainty Principle as arises from studies on string theory. We find that the classical cosmological singularity of the Taub model is solved by this approach in the sense that the quantum Universe can be regarded as probabilistically singularity-free [4], [12]. Moreover, the Taub GUP wave packets provide the right behavior in the establishment of a quasi-isotropic configuration for the Universe. The Bianchi I, II and IX cosmological models are also analyzed in the GUP framework [27] and the ordinary dynamics appears to be deeply modified and, in particular, the

Mixmaster Universe can be still considered a chaotic system. Furthermore, in the context of a deformed Heisenberg minisuperspace algebra framework, a deep phenomenological relation between loop quantum cosmology, brane cosmology and the κ-Poincaré scheme is obtained.

The people involved in this line of research are Marco Valerio Battisti and Giovanni Montani.

A.3. Evolutionary Quantum Gravity

In section "Evolutionary Quantum Gravity" we review the fundamental aspects of the so-called evolutionary quantum gravity.

An evolutionary paradigm is inferred by restricting the covariance principle within a Gaussian gauge and the corresponding implications for a generic cosmological scenario are investigated both on a classical and a quantum level [13]. A dualism between time and the reference frame fixing is then inferred.

The people involved in this line of research are M.Valerio Battisti, Francesco Cianfrani and Giovanni Montani (past collaborator: Simone Mercuri).

A.4. Polymer quantum cosmology

In section "Polymer quantum cosmology" we explain some results obtained applying the polymer quantization paradigm to the Taub Universe. The polymer approach is based on a inequivalent representation of the Weyl algebra and its physical relevance arises from consideration on the mechanical-system-limit of the loop quantum gravity theory. As a result of our analysis, the cosmological singularity is not probabilistically removed, as in the GUP approach, since the dynamics of the wave packets is not able to stop the evolution toward the classical singularity.

The people involved in this line of research are Marco Valerio Battisti, Orchidea Maria Lecian and Giovanni Montani.

A.5. Lorentz Gauge Theory

In section "Lorentz Gauge Theory" we implement a non-standard gauge theory of the local Lorentz group both in flat and in curved space-time, based on diffeomorphism induced Lorentz transformation and the ambiguity which emerges in the transformation laws of the usual spin connection and spinors.

We propose a model [8][22] to analyze the interaction of a 4-spinor with the new connections of the Lorentz group (addressed in flat space). This scheme exhibits strong analogies with the electro-magnetic case and the so-called Pauli equation. The analysis of this interaction is devoted to find out anomalous selection rules for a hydrogen-like model and, of course, energy-level splits. According to standard quantum mechanics new energy levels are present, but no new transitions arise.

The peoples involved in this line of research are Giovanni Montani, Nakia Carlevaro and Orchidea M. Lecian (past collaborator: Simone Mercuri).

A.6. Quantum Mixmaster

In section "Quantum Mixmaster" we propose a semiclassical treatment and a Schrödinger quantization scheme applied to the Mixmaster dynamics; the associated eigenvalue problem is solved. This approach gives a set of eigenfunctions (here we assume an ordering for the position and momentum operators such that $v^2 p_v^2 \rightarrow \hat{v}\hat{p_v}^2\hat{v}$, which is the only one able to reproduce the proper statistical dynamics). As soon as (approximated) Dirichlet boundary condition are taken into account, the energy spectrum is obtained. This spectrum is a discrete one, and it admits a minimum value given by $E_0^2 = 19.831\hbar^2$. In the figures in the section the wave function of the ground state and its probability distribution are plotted [7]. The persons working on this topic are Riccardo Benini and Giovanni Montani.

A.7. Loop Quantum Cosmology

In section "Loop Quantum Cosmology" we perform a general analysis of the equations governing the evolution of the Universe within semi-classical Loop Quantum Cosmology by using qualitative methods of the theory of dynamical systems. Specifically, two cases are considered with different type of corrections to the Friedmann equations [8], [3]. Quadratic terms on the energy density correction to the Friedmann equation, coming from effective Hamiltonian of Loop Quantum Cosmology, and corrections due to the inverse scale factor operator (both in the gravitational and the matter part of the effective Hamiltonian) were analyzed, respectively.

Our general conclusion, considering both types of corrections, is the absence of cosmic singularity, so in all solutions the usual expansion stage follows after the generic bounce. Moreover, we have shown that in both cases there exist successful mechanisms for generation of initial conditions suitable for inflation.

This work is relevant for the development of the theory of Early Universe. In particular, better understanding of background solutions and their properties should be reached to study of the cosmological perturbations.The dynamics of these perturbations, in turn, is crucial in view of verifications of predictions of the theory, confronted with observational data.

The peoples involved in this line of research are Giovanni Montani and Gregory V. Vereshchagin.

A.8. Lorentz gauge connection

The Yang-Mills picture of the local Lorentz transformations is approached in a second-order formalism. For the Lagrangian approach to reproduce the second Cartan structure equation, as soon as the Lorentz gauge connections are identified with the contortion tensor, an interaction term between the Lorentz gauge fields and the spin connections ω has to be postulated. This interaction term induces a Riemannian source to the Yang-Mills equations; thus, the real vacuum dynamics of the Lorentz gauge connection takes place on a Minkowski space only, when the Riemannian curvature and the spin currents provide negligible effects. In fact, it is the geometrical interpretation of the torsion field as a gauge field that generates the non-vanishing part of the Lorentz connection on flat space-time. The full picture involving gravity, torsion and spinors is described by a coupled set of field equations, which allows one to interpret both gravitational spin connections and matter spin density as the source term for the Yang-Mills equations. The contortion tensor acquires a propagating character, because of its non-Abelian feature, and the pure contact interaction is restored in the limit of vanishing Lorentz connections [8].

B. Brief description of Quantum Fields on Classical Background

B.1. Dirac equation on a curved spaces and classical trajectories

In section "Dirac equation on a curved space-time and classical trajectories", the interaction between geometry and internal spinor-like degrees of freedom has been investigated with the aim to infer the analogous of Papapetrou equations for a quantum spin. This task has been approached by an eikonal approximation, and a localization hypothesis along the integral curve of the momentum [34]. Hence, a dispersion relation has been recovered starting from the squared Dirac equation and by virtue of an integration on spatial coordinates. It is worth noting the emergence of a Papapetrou-like interaction between the Riemann tensor and a tensor, which characterizes the internal structure of spinors.

The persons involved in this research line are Giovanni Montani and Francesco Cianfrani.

C. Brief description of Unification Theories

C.1. Classical and Quantum spinning particles in Kaluza-Klein space-times

In the section "Classical and Quantum spinning particles in Kaluza-Klein space-times", we analyze the introduction of spinor fields in a KK model. The dynamics of a classical spinning particle, in a KK space-time, is inferred from the extension of Papapetrou equations to the 5-dimensional case, with Pirani conditions. This way, the system reproduces exactly equations of motion of a spinning particle, endowed with a charge and an electro-magnetic moment. This result demonstrates that the geometrization of electro-dynamics does not modify the dynamics of spinning objects [43].

The introduction of spinor fields in a KK model is the main open point of such an approach. The standard way to deal with them is to extend the Dirac equation to the multi-dimensional case and to try to identify extra-dimensional quantum numbers with internal ones. However this procedure fails, because of the emergence of mass terms of the compactification scale order and because quantum numbers of Standard model particles cannot be inferred. In this respect, our investigation has been focused on a more phenomenological approach, based on recovering 4-dimensional properties by an averaging procedure on the extra-dimensional manifold. This average is motivated by the undetectability of the extra-space and the need for it is not restricted to the case spinors are present. In fact, we showed that it is required in order to reproduce non-Abelian gauge transformations from extra-dimensional isometries and to get the equations of motion, proper of the 4-dimensional picture, starting from multi-dimensional ones. As far as spinors are concerned, the average produces a non-trivial effect on extra-dimensional symmetries, such that some of the above mentioned issues can be solved [37], [38], [47].

The people involved in this research line are Francesco Cianfrani, Irene Milillo, Andrea Marrocco and Giovanni Montani.

C.2. Generalized 5-Dimensional Theories

In section "Generalized 5-Dimensional Theories", we analyze possible generalizations of the 5D Kaluza-Klein model. The introduction of torsion has been shown to produce interesting structures after dimensional reduction. In a 5D scenario, the geometrization of the Electro-weak model has been worked out on the ground of the broken 5D Lorentz group and the properties of torsion [39], and proposal for the introduction of Ashtekar variables within this scheme has been evaluated.On the other hand, the truncation of the infinite tower that characterizes KK theories has been evaluated within the framework of polymer representation and generalized uncertainty principle: in the first case, compactification is illustrated to occur because of the truncation, while, in the second case, compactification is illustrated to be compatible with the main hypotheses of the scheme.

The people involved in this research line are Orchidea M. Lecian and Giovanni Montani.

D. Quantum Gravity

D.1. The time gauge problem in the path integral formalism

The problem of quantization of constrained systems arises in many contexts of physical interest. The presence of constraints at a classical level avoids us to threat all the dynamical variables as independent ones, and entails several difficulties when we are to construct the quantum theory. In a program of canonical quantization which promotes all classical canonical variables to quantum operators one has to deal with the problem of quantum-mechanically imposing the constraints. In the procedure *à la Dirac*, the constraint operators are imposed to annihilate physical states. This procedure stems from the observation that in the classical theory, the constraint functions are generators of infinitesimal canonical transformations which don't alter the physical state of the system.
The Dirac procedure is widely used in different contexts, including quantization of general relativity. Nevertheless this procedure of quantization encounters several difficulties when we require the Dirac conditions on physical states to be consistent with each other and the physical states selected by constraint operators to posses a finite scalar product allowing a probabilistic interpretation: moreover, in some cases this procedure can lead to a physical subspace of the entire Hilbert space that is curiously empty. Other difficulties arise when one tries to implement the Dirac procedure, which are not properly to be ascribed to the Dirac theory for constrained systems, but to the canonical quantization framework this procedure is developed in. As a matter of fact, our experience on quantum field theory in special relativity showed us how canonical quantization methods, when applied to systems with infinite degrees of freedom, lead to several inconsistencies: for example, it is a remarkable fact that the Glashow - Weinberg - Salam theory for electroweak interactions cannot be consistently formulated by canonical quantization methods, while the only way by which can be coherently written by is the Feynman path integral technique. Even if Feynman's path integral can be derived after constructing the quantum theory by means of canonical quantization methods, such inconsistencies need to postulate the path-integral approach as a founding element of the quantum theory when we deal with systems with infinite degrees of freedom. It is for these reasons that we developed all of our work (Castellana and Montani, 2008) avoiding using the

Dirac procedure for constrained systems and canonical quantization methods at all, employing a method to derive conditions on physical states based on BRST symmetry and path=integral methods uniquely.

BRST symmetry was conceived at first within non-Abelian gauge theories and shown to apply to a really wide class of systems of physical interest. Anyway, in the literature, there are different formulations for the BRST formalism, with substantial differences from each other. First of all, there exists a formulation of BRST symmetry for constrained systems based on canonical quantization methods which is widely diffused, being also employed in quantization of general relativity. Another approach, the one we followed in this work, is to derive BRST symmetry, based entirely on path integral methods, and it is applicable to systems with infinite degrees of freedom, avoiding those inconsistencies proper of canonical quantization methods we discussed above.

We start with an enlightening and more or less known example, considering BRST symmetry for a non-Abelian gauge theory. In order to compare path integral methods with canonical quantization ones, one can consider the Nöether charge following from BRST symmetry of the action and, taking an appropriate choice for the gauge fixing functionals in the DeWitt - Fadeev - Popov method, show it to be the generator of quantum BRST transformation within a canonical quantization framework.
Otherwise, using solely path integral methods, we show the BRST Nöether charge

$$Q \equiv \int d^3x J^0(x) \tag{D.1.1}$$

related to the BRST current J^μ to generate quantum BRST transformation by means of Ward's identities for the ensemble of gauge fields, ghost and antighost fields and Nakanishi - Lautrup fields, designed by $\psi_i(x)$, i. e.

$$\begin{aligned} 0 &= \partial^x_\mu \left\langle \psi_{i_k}\left(x_k\right) \cdots \psi_{i_1}\left(x_1\right) J^\mu(x) \right\rangle_{j=0} - i \sum_{l=1}^{k} \sigma^{i_1} \cdots \sigma^{i_l} \langle \psi_{i_k}\left(x_k\right) \cdots \\ & \cdots \psi_{i_{l+1}}\left(x_{l+1}\right) s\psi_{i_l}(x) \psi_{i_{l-1}}\left(x_{l-1}\right) \cdots \psi_{i_1}\left(x_1\right) \rangle_{j=0} \delta^{(4)}\left(x - x_l\right). \end{aligned} \tag{D.1.2}$$

where $\sigma^i = \pm 1$ for ψ_i bosonic or fermionic respectively. The fact that in (D.1.2) the gauge fixing functionals are completely arbitrary allows us to infer a physical-state condition on states of the gauge fields following from BRST invariance, given by the usual Gauss'

$$\mathcal{D}_a F^{0a\alpha}(x)\left|\psi\right\rangle = 0. \tag{D.1.3}$$

Afterward, we turn our attention to general relativity expressed in first-order formalism, in order to investigate the physicality condition for the states of the gravitational field arising from BRST invariance of the theory, following the same procedure employed for non-Abelian gauge theories. In this procedure we will intentionally avoid to use canonical quantization methods. We are to determine a physical state condition on quantum states without thinking of classical Hamiltonian constraints in order to compare, at the end of our calculation, our physicality condition required by BRST symmetry and derived with path-integral methods with the one obtained using the Dirac quantization method employed within Ashtekar's canonical formulation. Employing the same method leading us to the usual Gauss' constraint for non-Abelian gauge theories, we arrive at the following physical state condition for the densitized triad E_i^a

$$\mathcal{D}_a \left[E_j^a(x) + i e_{jb}(x) e_{0c}(x) \epsilon^{abc} \right] |\psi\rangle = 0. \tag{D.1.4}$$

Comparing our physicality condition with the one used in loop quantum gravity, we find they differ by an additional non-vanishing term. We think the origin of this discrepancy is in the choice of a particular gauge in the classical theory which is made within Ashtekar's approach and which was intentionally avoided in our work. Finally, we show how we recover the Dirac canonical condition in our BRST quantization only by a suitable choice of gauge fixing functionals within the DeWitt - Fadeev - Popov method.

D.2. Minisuperspace and Generalized Uncertainty Principle

This section is devoted to explain some results obtained in a recent approach to quantum cosmology, in which the notion of a minimal length naturally appears. In particular, this scheme realizes in quantizing a cosmological model by using a modified Heisenberg algebra, which reproduces a Generalized Uncertainty Principle (GUP)

$$\Delta q \Delta p \geq \frac{1}{2} \left(1 + \beta (\Delta p)^2 + \beta \langle \mathbf{p} \rangle^2 \right), \tag{D.2.1}$$

where β is a "deformation" parameter. The above uncertainty principle (D.2.1) can be obtained by considering an algebra generated by $\mathbf{q}$ and $\mathbf{p}$ obeying the commutation relation

$$[\mathbf{q}, \mathbf{p}] = i(1 + \beta \mathbf{p}^2). \tag{D.2.2}$$

Such a deformed Heisenberg uncertainty principle was appeared in studies on string theory and leads to a fundamental minimal scale. More precisely, from the string theory point of view, a minimal observable length it is a consequence of the fact that strings can not probe distance below the string scale. However, we have to stress that the minimal scale predicted by the GUP is, by its nature, different from the minimal length predicted by other approaches. In fact, the equation (D.2.1) implies a finite minimal uncertainty in position $\Delta q_{min} = \sqrt{\beta}$. This way, we will introduce a minimal scale in the quantum dynamics of a cosmological model.

Of course the appearance of a nonzero uncertainty in position pose some difficulty in the construction of an Hilbert space. In fact, as well-known, no physical state which is a position eigenstate can be constructed. An eigenstate of an observable necessarily has to have vanishing uncertainty on it. Although it is possible to construct position eigenvectors, they are only formal eigenvectors but not physical states. In order to recover information on position, we have to study the so-called *quasiposition wave functions*

$$\psi(\zeta) \sim \int_{-\infty}^{+\infty} \frac{dp}{(1+\beta p^2)^{3/2}} \exp\left(i\frac{\zeta}{\sqrt{\beta}} \tan^{-1}(\sqrt{\beta}p) \right) \psi(p), \qquad \text{(D.2.3)}$$

where ζ is the *quasiposition* defined by the main value of the position $\mathbf{q}$ on certain functions, *i.e.*, $\langle \mathbf{q} \rangle = \mathbf{1}$. The quasiposition wave function (D.2.3) represent the probability amplitude to find a particle being maximally localized around the position ζ (*i.e.*, with standard deviation Δq_{min}).

It is notable to stress how, the GUP approach relies on a modification of the canonical prescription for quantization, and therefore it can be reliable applied to any dynamical system. Moreover, the application of such a formalism in quantizing a cosmological model allows us to analyze some peculiar features of string theory in the minisuperspace dynamics.

Let us now extend the above framework to the Taub general cosmological model, discussing its quantization in the GUP scheme. The Taub model is a particular case of the Bianchi IX model which line element (in the Misner parametrization) reads

$$ds^2 = N^2 dt^2 - e^{2\alpha} \left(e^{2\gamma} \right)_{ij} \omega^i \otimes \omega^j, \qquad \text{(D.2.4)}$$

where $N = N(t)$ is the lapse function, the variable $\alpha = \alpha(t)$ describes the isotropic expansion of the Universe and $\gamma_{ij} = \gamma_{ij}(t)$ is a traceless symmetric matrix which determines the shape change (the anisotropy) *via* $\gamma_\pm$. Since the determinant of the 3-metric is given by $h = \det e^{\alpha+\gamma_{ij}} = e^{3\alpha}$, it is easy to recognize that the classical singularity appears for $\alpha \to -\infty$.The Taub model is the Bianchi IX model in the $\gamma_- = 0$ case and thus its dynamics is equivalent to the motion of a particle in a one-dimensional closed domain. Its ADM Hamiltonian in the Poincaré-plane framework is

$$H^T_{ADM} = p_x \equiv p, \qquad x \in [x_0 \equiv \ln(1/2), \infty), \qquad \text{(D.2.5)}$$

where $x = \ln v$ and the classical singularity now appears for $\tau \to \infty$.

The canonical quantization of this model is not able to solve the classical singularity problem. In fact, the incoming Universe ($\tau < 0$) bounces at the potential wall at $x = x_0$ and then falls toward the classical singularity ($\tau \to \infty$). Such situation is drastically changed in the GUP scheme and two main conclusions can be inferred: (i) The probability amplitude to find the Universe is peaked near the potential wall. In other words, the GUP Taub Universe exhibits a singularity-free behavior. (ii) The large anisotropy states, i.e. those for $|\gamma_+| \gg 1$, are probabilistically suppressed. In fact the Universe wave function appears to be peaked at values of anisotropy $|\gamma_+| \simeq \mathcal{O}(10^{-1})$. In this respect, the GUP wave packets *predict the establishment of a quantum isotropic Universe* differently from what happens in the WDW theory.

When this approach is applied to the Bianchi IX cosmological model we show that three important features. i) The velocity of the anasotropy-particle (Universe) inside the allowed domain of the Mixmaster model grows with respect to the undeformed case. Furthermore, although the dynamics is still Kasner-like, two negative Kasner indices are now allowed. Therefore, during each Kasner era, the volume of the Universe can contracts in one direction while expands in the other two. ii) The velocity $\dot{\gamma}_w$ of the potential walls, bounding the triangular domain of Bianchi IX, is increased by the deformation terms. However, it no rises so much to avoid the bounces of the γ-particle against the walls, i.e. the particle bounces are not stopped by the GUP effects. As matter of fact, when the ultra-deformed regime is reached the dynamics is that of a particle which bounces against stationary walls (no maximum incidence angle appears). iii) No BKL map (reflection law $\theta_f = \theta_f(\theta_i)$) can be in general analytically computed. In fact, such a map arises from the analysis of the Bianchi II model which is no longer analytically integrable in the deformed scheme. Thus, a non-vanishing minimal uncertainty in the anisotropies complicates so much the Mixmaster dynamics in such a way that each its wall-side is no longer an integrable system. This way, we can conclude that the chaoticity of the Bianchi IX model is not tamed by the GUP effects on the Universe anisotropies.

A relation between the effective dynamics of loop quantum cosmology and the Randall-Sundrum braneworlds scenario can by obtained quantizing the FRW models with the use of the following deformed algebra

$$[\mathbf{q}, \mathbf{p}] = i\sqrt{1 \pm \alpha \mathbf{p}^2}, \tag{D.2.6}$$

where $\alpha > 0$ is a deformation parameter such that for $\alpha = 0$ the ordinary Heisenberg algebra is recovered. In particular, such an algebra is related to the κ-Poincaré one which is the mathematical structure which describes the so-called doubly special relativity, where an other invariant, observer independent, scale (the Planck scale) is included ab initio in the theory. From this approach the deformed Friedmann equation

$$H_{k=0}^2 = \frac{8\pi G}{3}\rho\left(1 \pm \frac{\rho}{\rho_P}\right), \tag{D.2.7}$$

for the flat case is obtained. The most interesting point to be stressed is the equivalence, at phenomenological level, between the $(-)$-deformed Friedmann equation (D.2.7) and the one obtained considering the effective dynamics in loop quantum cosmology. On the other hand, the string inspired Randall-Sundrum braneworlds scenario leads to a modified Friedmenn equation as in (D.2.7) with the positive sign. The opposite sign of the ρ^2-term in such an equation, is the well-known key difference between the effective loop quantum cosmology and the Randall-Sundrum framework. In fact, the former approach leads to a non-singular bouncing cosmology while in the latter, because of the positive sign, $\dot{a}$ can not vanish and there is not place for a big-bounce.

D.3. Evolutionary Quantum Gravity

We establish a fundamental link between the identification of a reference and the appearance of a matter term from the point of view of Lagrangian symmetries. In particular, by fixing a synchronous frame of reference, which is characterized by a metric tensor having the following fixed components $g_{00} = 1$ and $g_{0i} = 0$, general covariance is restricted to the invariance under the following set of coordinate transformations

$$t' = t + \xi(x^l), \quad x^{i'} = x^i + \partial_j \xi \int h^{ij} dt + \phi^i(x^l), \tag{D.3.1}$$

ξ and ϕ^i being three generic space functions.

This feature implies replacing the super-Hamiltonian and the super-momentum constraints with the following ones,

$$H^* \equiv H - \mathcal{E}(x^l) = 0, \qquad H_i = 0, \tag{D.3.2}$$

$\mathcal{E}$ being a scalar density of weight 1/2, hence it can be written as $\mathcal{E} \equiv -2\sqrt{h}\rho(t,\ x^i)$, with ρ a scalar function.

Hence the super-momentum still vanishes, while the super-Hamiltonian acquires a non-vanishing eigen-value, which can be interpreted as the emergence of a dust fluid co-moving with the slicing.

One can think at this contribution as the physical realization of the synchronous reference. However, it is clear that we are not dealing with an external matter field since its energy density ρ is not always positive and $\mathcal{E}(x^i)$ is fixed, once initial conditions are assigned on a non-singular hypersurface.

We perform quantization of the synchronous gravitational field in a canonical way and we implemented according with the Dirac prescription, so fixing an evolutionary character for wave functional, which can be described by the Schrödinger equation

$$i\hbar\partial_t\chi = \int_{\Sigma_t^3} \hat{H} d^3x\chi\,. \tag{D.3.3}$$

Therefore, the quantum features of the dust contribution outline its behavior as a clock-like matter. The next task is to find out a negative portion of the super-Hamiltonian spectrum, which allows to interpret the additional contribution as a physical matter field.

This can be done in a generic inhomogeneous cosmological setting, where the 3-metric is given by

$$h_{ij} = e^{q_a}\delta_{ad}O_b^a O_c^d \partial_i y^b \partial_j y^c, \quad a, b, c, d, \alpha, \beta = 1, 2, 3, \tag{D.3.4}$$

with $q^a = q^a(x^l, t)$ and $y^b = y^b(x^l, t)$ six scalar functions and $O_b^a = O_b^a(x^l)$ a $SO(3)$ matrix.

The dynamics of different points decouples near the singularity and the Schrödinger functional equation splits to the sum of ∞^3 independent point-like contributions as follows (we denote by the subscript x any minisuperspace quantity)

$$i\hbar\partial_t\psi_x = \hat{H}_x\psi_x = \frac{c^2\hbar^2 k}{3}\left[\partial_\alpha e^{-3\alpha}\partial_\alpha - e^{-3\alpha}\left(\partial_+^2 + \partial_-^2\right)\right]\psi_x - \\ -\frac{3\hbar^2}{8\pi}e^{-3\alpha}\partial_\varphi^2\psi_x - \left(\frac{1}{2k\mid J\mid^2}e^\alpha V(\beta_\pm) - \frac{\Lambda}{k}e^{3\alpha}\right)\psi_x \tag{D.3.5}$$

$$\psi_x = \psi_x(t,\ \alpha,\ \beta_\pm,\ \varphi)\,, \tag{D.3.6}$$

where a cosmological constant Λ and a scalar field φ have been added to the dynamical description.

If an integral representation is taken for the wave function ψ_x

$$\psi_x = \int d\mathcal{E}_x \mathcal{B}(\mathcal{E}_x)\sigma_x(\alpha,\ \beta_\pm,\ \varphi,\ \mathcal{E}_x) exp\left\{-\frac{i}{\hbar}\int_{t_0}^t N_x\mathcal{E}_x dt'\right\} \tag{D.3.7}$$

$$\sigma_x = \xi_x(\alpha,\ \mathcal{E}_x)\pi_x(\alpha,\ \beta_\pm,\ \varphi)\,, \tag{D.3.8}$$

where $\mathcal{B}$ is fixed by the initial conditions at t_0, the dynamics is given by

$$\hat{H}\sigma_x = \mathcal{E}_x\sigma_x \quad \text{(D.3.9)}$$

$$\left(-\partial_+^2 - \partial_-^2 - \frac{9\hbar^2}{8\pi c^2 k}\partial_\varphi^2\right)\pi_x - \frac{3e^{4\alpha}}{2c^2\hbar^2 k^2 \mid J \mid^2} V(\beta_\pm)\pi_x = v^2(\alpha)\pi_x \quad \text{(D.3.10)}$$

$$\left[\frac{c^2\hbar^2 k}{3}\left(\partial_\alpha e^{-3\alpha}\partial_\alpha \xi_x + e^{-3\alpha}v^2(\alpha)\right) + \frac{\Lambda}{k}e^{3\alpha}\right]\xi_x = \mathcal{E}_x\xi_x \,. \quad \text{(D.3.11)}$$

Let us now consider wave packets which are flat over the width $\Delta\beta \sim 1/\Delta v_\beta \gg 1$ (Δv_β being the standard deviation in the momenta space).

In the new variable $\tau = e^{3\alpha}$, the equation (D.3.10) reads

$$\frac{c^2\hbar^2 k}{3}\left(9\frac{d^2}{d\tau^2} + \frac{v^2}{\tau^2}\right)\xi_x + \frac{\Lambda}{k}\xi_x = \frac{\mathcal{E}_x}{\tau}\xi_x \,. \quad \text{(D.3.12)}$$

A solution to equation (D.3.12) is provided by

$$\xi_x = \tau^\delta f_x(\tau), \qquad \delta = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{9}v^2}\right) \quad \text{(D.3.13)}$$

$$f = \mathcal{C}e^{-\beta^2\tau^2 + \gamma\tau}, \qquad \gamma = 2 \mid \beta \mid \sqrt{\delta + \frac{1}{2} - \frac{1}{12L_\Lambda^2 l_P^4 \beta^2}}, \qquad \frac{1}{L_\mathcal{E} l_P^2} = 6\delta(\gamma). \quad \text{(D.3.14)}$$

$L_\mathcal{E} = \frac{\hbar c}{\mathcal{E}}$ being the characteristic length associated to the Universe "energy", while $l_P \equiv \sqrt{\hbar c k}$ denotes the Planck scale length. However, the validity of the solution above requires the condition $\beta^2\tau \ll \gamma = 2\sqrt{\delta + \frac{1}{2} - \frac{1}{12L_\Lambda^2 l_P^4\beta^2}} \mid \beta \mid$.

Hence, the quantum dynamics in a fixed space point (*i.e.* over a causal portion of the Universe) is described, in the considered approximation ($\tau \ll 1$), by a free wave-packet for the variables $\beta_\pm$ and φ and by a profile in τ which has a maximum in $\tau = (\gamma + \sqrt{\gamma^2 + 8\delta\beta^2})/4\beta^2$.

If a lattice structure for the space-time is assumed on the Planckian scale, to preserve the reality of $\mathcal{E}$ we have to impose some inequalities, leading to

$$\mid \mathcal{E}_x \mid \ll \frac{c^2 k\hbar^2}{l_{Pl}^3} \sim \mathcal{O}(M_{Pl}c^2) \to L_\mathcal{E} \gg l_P \,, \quad \text{(D.3.15)}$$

$M_{Pl} \equiv \hbar/(l_{Pl}c)$ being the Planck mass.

Therefore, the existence of a cut-off implies that a ground state exists for the evolutionary approach. Hence it is a natural request to assume the Universe to approach this state during its evolution.

The associated critical parameter turns out to be

$$\Omega_{\mathcal{E}} \equiv \frac{\rho_{\mathcal{E}}}{\rho_c} \ll \mathcal{O}\left(\frac{10^{-2}GM_{Pl}}{c^2 R_0}\right) \sim \mathcal{O}\left(\frac{10^{-2}l_{Pl}}{R_0}\right) \sim \mathcal{O}\left(10^{-60}\right) . \qquad \text{(D.3.16)}$$

Therefore, the dust contribution cannot play the role of dark matter.

Within this scheme a proper quantum to classical transition for the Universe volume can also be described.

D.4. Polymer Quantum Cosmology

The polymer representation of quantum mechanics is based on a non-standard representation of the canonical commutation relations. In particular, in a two-dimensional phase space, it is possible to choose a discretized operator, whose conjugate variable cannot be promoted as an operator directly. From a physical point of view, this scheme can be interpreted as the quantum-mechanical framework for the introduction of a cutoff. Its continuum limit, which corresponds to the removal of the cutoff, has to be understood as the equivalence of microscopically-modified theories at different scales. This approach is relevant in treating the quantum-mechanical properties of a background-independent canonical quantization of gravity. In fact, the holonomy-flux algebra used in Loop Quantum Gravity reduces to a polymer-likealgebra, when a system with a finite number of degrees of freedom is taken into account. From a quantum-field theoretical point of view, this is substantially equivalent to introducing a lattice structure on the space. Loop Quantum Cosmology can be regarded as the implementation of this quantization technique in the minisuperspace dynamics.

The Taub model is approached in the scheme of an Arnowitt-Deser-Misner (ADM) reduction of the dynamics in the Poincare plane. As a result, a time variable naturally emerges, and the Universe is described by an anisotropy-like variable. The anisotropy variable and its conjugate momentum are quantized within the framework of the polymer representation. More precisely, the former appears as discretized, while the latter cannot be implemented as an operator in an appropriate Hilbert space directly, but only its exponentiated version exists. The analysis is performed at both classical and quantum levels. The modifications induced by the cutoff scale on ordinary trajectories are analyzed from a classical point of view. On the other hand, the quantum regime is explored in detail by the investigation of the evolution of the wave packets of the universe (Battisti et al., 2008).

From a classical point of view, in the ordinary case, the model can be interpreted as a photon in the Lorentzian minisuperspace, and the classical trajectory is its light-cone. More precisely, the incoming particle bounces on the wall and falls into the classical cosmological singularity. Contrastingly, in

the discretized case, the one-parameter family of trajectories flattens, i.e. the angle between the incoming trajectory and the outgoing one is greater than $\pi/2$.
From a quantum point of view, the modified Schroedinger equation is solved. As a result, a modified dispersion relation is found, and wave functions depend on this modified dispersion relation.
The analysis of the corresponding wavepackets shows the implications of the polymer representation of quantum mechanics mostly when a spread weighting function is taken into account. In fact, in this case, as a result, a strong interference phenomenon appears between the incoming (outgoing) wave and the wall. However, as a matter of fact, such an interference phenomenon is not able to localize the wave packet in a determined region of the configuration space, so that the probability density to find the Universe far away the singularity is not peaked, i.e. the cosmological singularity of this model is not tamed by the polymer representation from a probabilistic point of view. Consequently, the incoming particle (Universe) is initially localized around the classical polymer trajectory. It then bounces against the wall, where the wave packet spreads in the 'outer' region, regains the classical polymer trajectory and eventually falls into the cosmological singularity. This way, we claim that the classical singularity is not solved by this quantization of the model.
The result can be also discussed as compared with the application of the polymer representation of quantum mechanics to other cosmological models, as well as with the implementation of a generalized uncertainty principle to the Taub model itself. In these cases, the peculiarity of this scheme are clarified.

D.5. Lorentz Gauge Theory

General Relativity admits two different symmetries, namely the diffeomorphism invariance, defined in the real space-time, and the local Lorentz invariance, associated to the tangent fiber. Such two symmetries reflect the different behavior of tensors and spinors, respectively, when global Lorentz transformations become local, *i.e.*, while tensors do not experience the difference between the two transformations, spinors do. In our proposal, the diffeomorphism invariance concerns the metric structure of the space-time and it finds in the vier-bein fields the natural gauge counterpart, though the gauge picture holds on a qualitative framework. On the other hand, the real gauge symmetry corresponds to the local rotations in the tangent fiber and admits a geometrical gauge field induced by the space-time torsion and its properties.

This picture has led us to infer the existence of (metric-independent) gauge fields of the Lorentz group, identified with $A_\mu^{\ ab}$, which interacts with spinors. The Ricci spin connection $\omega_\mu^{\ ab}$ could not be identified with the suitable gauge field, for it is not a primitive object (it depends on bein vectors) and defines local Lorentz transformations on the tangent bundle.

Perspectives on observability We propose here a model to analyze the interaction of a *4*-spinor ψ with the gauge field A_μ of the Lorentz group (addressed in flat space) (Carlevaro et al., 2009). Using the tetrad formalism, the implementation of the local Lorentz symmetry leads to the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}\,, \qquad \mathcal{L}_0 = \tfrac{i}{2}\,\bar{\psi}\gamma^a e^\mu_a \partial_\mu \psi - \tfrac{i}{2}\,e^\mu_a \partial_\mu \bar{\psi}\gamma^a \psi \,-\, m\,\bar{\psi}\psi\,, \tag{D.5.1}$$

$$\mathcal{L}_{int} = \tfrac{1}{8}\,e^\mu_c\,\bar{\psi}\{\gamma^c, \tau_{ab}\}\,A^{ab}_\mu\,\psi = \tfrac{1}{8}\,e^\mu_c\,\bar{\psi}\;\,2\epsilon^c_{abd}\,\gamma_5\,\gamma^d\;A^{ab}_\mu\,\psi\,. \tag{D.5.2}$$

To study the interaction terms, we perform a *3+1* splitting of the gauge field and impose the time-gauge condition associated to this picture (*i.e.*, $A^{ij}_0 = 0$). Using variational principles, we are able to write down the motion equations for the spinor field. In this scheme, it is convenient to express the Lorentz gauge field trough the fields $C_0 = \frac{1}{4}\,\epsilon^k_{ij0}A^i_k$, and $C_i = \frac{1}{4}\,\epsilon^k_{0ji}A^{0j}_k$, describing rotations and Lorentz boosts respectively.

Our purpose is the analysis of corrections, due to the implementation of the Lorentz gauge theory, and to a one-electron-atom model. In this respect, we look for stationary solutions of the Dirac equation and we express the *4*-component spinor $\psi(t, \mathbf{x})$ in terms of two stationary *2*-spinors $\chi(\mathbf{x})$ and $\phi(\mathbf{x})$, assuming standard-representation Dirac matrices. To investigate the low-energy limit, we can write the spinor-field total energy in the form $\mathcal{E} = E + m$, obtaining the expression

$$\phi \;=\; \tfrac{1}{2m}\,(\sigma^i p_i + C_0)\,\chi\,. \tag{D.5.3}$$

It is immediate to see that ϕ is smaller than χ by a factor of order $\frac{p}{m}$ (*i.e.*, $\frac{v}{c}$ where v is the magnitude of the velocity): the *2*-component spinors ϕ and χ form the so-called *small* and *large components*, respectively.

Using standard Pauli relations, we finally get the following equation for the large components

$$E\,\chi \;=\; \tfrac{1}{2m}\left[p^2 \,+\, C_0^2 \,+\, 2C_0\,\sigma^i p_i \,+\, \sigma^i C_i\right]\chi\,. \tag{D.5.4}$$

This equation exhibits strong analogies with the electro-magnetic case and the so-called Pauli equation

$$E\,\chi(\mathbf{x}) \;=\; \tfrac{1}{2m}\left[(\mathbf{p} + \mathbf{A})^2 \,+\, \mu_B\,\boldsymbol{\sigma}\cdot\mathbf{B} \,+\, \Phi\right]\chi(\mathbf{x})\,, \tag{D.5.5}$$

where $\mu_B = e/2m$ is the Bohr magneton and $\mathbf{A}$ denotes the vector potential ($\mathbf{B}$ and Φ are the external magnetic and electric field respectively). These eqs can be used in the analysis of the energy levels as in the Zeeman effect.

Let us now neglect the term C_0^2 in eq. (D.5.4) and implement the symmetry

$$\partial_\mu \to \partial_\mu + A_\mu^{U(1)} + A_\mu^{ab}\Sigma_{ab} \,, \tag{D.5.6}$$

with a vanishing electromagnetic potential $\mathbf{A} = 0$. This way, we can introduce a Coulomb central potential $V(r)$ ($E \to E - V(r)$), obtaining the expressions

$$H_0 = \frac{p^2}{2m} - \frac{Ze^2}{(4\pi\epsilon_0)r} \,, \tag{D.5.7}$$

$$H' = \tfrac{1}{2m}\left[\, 2C_0\,(\sigma_i \cdot p^i) + \sigma \cdot C^i \,\right] \,, \tag{D.5.8}$$

which characterize the electron dynamics in a hydrogen-like atom in presence of a gauge field of the LG. It is worth noting the presence of a term related to the helicity of the 2-spinor: this coupling is controlled by the rotation-like component associated to C_0. A Zeeman-like coupling associated to the boost-like component C_i is also present.

D.6. Boost invariance in a second order formulation

Given an hyperbolic space-time manifold V, endowed with a metric $g_{\mu\nu}$, a $3+1$ splitting consists in a map $V \to \Sigma \otimes R$, Σ being spatial 3-hypersurfaces (in the following x^i $(i = 1,2,3)$ indicate spatial coordinates, while t is the coordinate on the real time-like axis). The crucial choice consists in introducing an arbitrary vier-bein, *i.e.*,

$$e^0 = Ndt + \chi_a E^a_i dx^i \,, \qquad e^a = E^a_i N^i dt + E^a_i dx^i \,, \qquad (a = 1,2,3) \,, \tag{D.6.1}$$

where the time-gauge is obtained for $\chi_a = 0$.
From a physical point of view, χ_a gives the velocity components of the e^A frame with respect to one at rest, *i.e.*, adapted to the spatial splitting.
The standard variables of the ADM formulation (the lapse function $\tilde{N}$, the shift vector $\tilde{N}^i$ and the 3-geometry h_{ij}) read as follows in terms of e^A_μ components

$$\tilde{N} = \frac{1}{\sqrt{1-\chi^2}}(N - N^i E^a_i \chi_a) \,, \qquad \tilde{N}^i = N^i + \tfrac{E^c_l \chi_c N^l - N}{1-\chi^2} E^i_a \chi^a \,, \quad \chi^a = \chi_b \delta^{ab} \,,$$
$$h_{ij} = E^a_i E^b_j (\delta_{ab} - \chi_a \chi_b) \,. \tag{D.6.2}$$

Once the $3+1$ splitting of the Einstein-Hilbert action has been performed, by taking as configuration variables $\tilde{N}$, $\tilde{N}^i$, E^a_i and χ_a, the full Hamiltonian density turns out to be

$$\mathcal{H} = \tilde{N}' H + \tilde{N}^i H_i + \lambda^{\tilde{N}} \pi_{\tilde{N}} + \lambda^i \pi_i + \lambda^{ab} \Phi_{ab} + \lambda_a \Phi^a \ , \tag{D.6.3}$$

$\tilde{N}'$ being $\sqrt{h}\tilde{N}$, while the super-Hamiltonian and the super-momentum, H and H_i, respectively, take the following forms

$$H = \pi^i_a \pi^j_b \left(\frac{1}{2} E^a_i E^b_j - E^b_i E^a_j \right) + h^3 R \ , \tag{D.6.4}$$

$$H_i = D_j(\pi^j_a E^a_i) \ , \tag{D.6.5}$$

D_i being the covariant derivative built up from h_{ij}.
Lagrangian multipliers $\lambda^{\tilde{N}}$, λ^i, λ_a and $\lambda^{ab} = -\lambda^{ba}$ ensure the standard first-class constraints

$$\pi_{\tilde{N}} = 0 \ , \qquad \pi_i = 0 \ , \tag{D.6.6}$$

and new conditions, coming out as a consequence of variables adopted,

$$\Phi^a = \pi^a - \pi^b \chi_b \chi^a + \delta^{ab} \pi^i_b \chi_c E^c_i = 0 \ , \tag{D.6.7}$$

$$\Phi_{ab} = \pi^c \delta_{c[a} \chi_{b]} - \delta_{c[a} \pi^i_{b]} E^c_i = 0 \ . \tag{D.6.8}$$

The investigation on these new constraints is performed by analyzing their action on the phase space, once a canonical symplectic structure is introduced. It outlines that Φ_{ab} and Φ^a generate rotations and boosts, modulo a time re-parametrization, respectively, on the phase space. Therefore, they arise because General Relativity is a Lorentz-invariant theory.
We also probe that the algebra of constraints is first-class.
Before performing the quantization, a formal fixing of the boost symmetry is performed, such that transformations between χ-sectors can be studied. In this respect, we set $\chi_a = \bar{\chi}_a(t;x)$, $\bar{\chi}_a(t;x)$ being arbitrary functions of space-time coordinates. The boost constraint can be solved classically, so finding

$$\pi^a = -\left(\delta^{ab} + \frac{\chi^a \chi^b}{1-\chi^2} \right) \pi^i_b \chi_c E^c_i \ . \tag{D.6.9}$$

Hence the action becomes

$$S = -\tfrac{1}{16\pi G} \int [\pi^i_a \partial_t E^a_i + \pi_{\tilde{N}'} \partial_t \tilde{N}' + \pi_i \partial_t \tilde{N}^i - \tilde{N}' H^{\bar{\chi}} - \tilde{N}^i H^{\bar{\chi}}_i - \lambda^{ab} \Phi'_{ab} + \\ - \lambda^{\tilde{N}} \pi_{\tilde{N}} - \lambda^i \pi_i] dt d^3x \ , \tag{D.6.10}$$

where the new constraint for rotations is

$$\Phi'_{ab} = \bar{\chi}_{[a}\pi^i_{b]}E^d_i\bar{\chi}_d - \delta_{c[a}\pi^i_{b]}E^c_i \;, \tag{D.6.11}$$

while, in $H^{\bar{\chi}}$ and in $H^{\bar{\chi}}_i$, χ are replaced by functions $\bar{\chi}$.
In this picture, we have completely fixed the gauge associated with the boost symmetry, because $\bar{\chi}_a$ are three functions to be assigned explicitly together with the Cauchy data.

The canonical quantization consists in promoting to operators $\tilde{N}$, $\tilde{N}^i$, E^a_i and the corresponding conjugated momenta, then Poisson brackets are replaced by commutators in a canonical way. Once an Hilbert space has been defined to which wave functionals $\psi = \psi_{\bar{\chi}}(\tilde{N}, \tilde{N}^i, E^a_i)$ belong, according with the Dirac prescription for constrained systems, physical states are defined as states annihilated by quantum constraints.
In order to investigate if the transformation between different $\bar{\chi}$-sectors can be implemented in a quantum setting, an operator connecting Hilbert spaces with different forms of $\bar{\chi}$ must be defined.
Let us now consider a wave functional ψ_0 in the time gauge: it is a solution of the following system of constraints (we do not consider primary constraints (D.6.6), since they are not affected by transformations changing $\bar{\chi}_a$)

$$H^0\psi_0 = 0\,, \quad H^0_i\psi_0 = 0\,, \quad -\delta_{c[a}\pi^i_{b]}E^c_i\psi_0 = 0\,, \tag{D.6.12}$$

H^0 and H^0_i being the super-Hamiltonian and super-momentum built up from the metric tensor $h_{ij} = \delta_{ab}E^a_iE^b_j$, *i.e.*, in the case $\bar{\chi} \equiv 0$, respectively.
The action of the boost constraint Φ^a, restricted to the hypersurface $\chi_a = 0$, is reproduced by the unitary operator U_ϵ

$$U_\epsilon = I - \frac{i}{4}\int \epsilon^a\epsilon_b(E^b_i\pi^i_a + \pi^i_aE^b_i)d^3x + O(\epsilon^4)\;, \tag{D.6.13}$$

which maps the metric h_{ij} from $\bar{\chi} = 0$ to $\bar{\chi}_a = \epsilon_a \ll 1$. The new state $\psi' = U_\epsilon\psi$ satisfies, at the ϵ^2 order,

$$U_\epsilon H^0U_\epsilon^{-1}\psi' = H^\epsilon = \psi'0\,, \qquad U_\epsilon H^0_iU_\epsilon^{-1}\psi' = H^\epsilon_i\psi' = 0\,, \tag{D.6.14}$$

$$U_\epsilon(-\delta_{c[a}\pi^i_{b]}E^c_i)U_\epsilon^{-1}\psi' = -\left[\delta_{c[a}\pi^i_{b]}E^c_i + \frac{1}{2}\delta_{c[a}\epsilon_{b]}\epsilon^dE^c_i\pi^i_d - \frac{1}{2}\epsilon_d\epsilon_{[a}\pi^i_{b]}E^d_i\chi_d\right]\psi' = 0\,. \tag{D.6.15}$$

While the first two relations reproduce the vanishing of the super-Hamiltonian and of the super-momentum in the ϵ-sector, the last condition can be shown to be equivalent to $\Phi'_{ab}\psi' = 0$ for $\bar{\chi}_a = \epsilon_a$.
Therefore, since the unitary operator U_ϵ maps physical states corresponding to $\bar{\chi} = 0$ and $\bar{\chi} = \epsilon$, the transformation between a frame at rest and one moving with respect to Σ can be implemented as a symmetry on a quantum level. This provides us with an explanation for the use of the time-gauge condition,

because any other choice for the Lorentz frame gives the same expectation values for observables.

D.7. Quantum Mixmaster

The quantization of the Bianchi IX geometry is investigated in the approximation of a squared potential well, after an ADM reduction of the dynamics with respect to the super-momentum constraint only. A functional representation of the quantum dynamics, equivalent to the Misner-like one, was extended point by point, since the Hilbert space factorizes into ∞^3 independent components, due to the parametric role that the three-coordinates assume in the asymptotic potential term. Finally, we obtain the conditions for a semiclassical behavior of the dynamics, equivalent to mean occupation numbers $n = \mathcal{O}(10^2)$ [Imponente and Montani (2006)].

A physical link between the chaoticity characterizing the system at a classical level and the quantum indeterminism appearing in the Planckian era was constructed through the canonical quantization of the model via a Schrödinger approach (equivalent to the Wheeler-DeWitt scheme) and then developed the WKB semiclassical limit to be compared with the classical dynamics [Imponente and Montani (2003a)], [Imponente and Montani (2003b)]. We found a correspondence between the continuity equation of the microcanonical distribution function and that one describing the dynamics of the first-order corrections in the wave function for $\hbar \to 0$ [Imponente and Montani (2002)].

The dynamics of the homogeneous model of the type IX of the Bianchi classification (the Mixmaster model) exhibits an oscillatory like behavior while approaching the Big Bang; furthermore, Belinskii et al. showed in the 70's how this model can be used to construct a generic cosmological solution in the neighborhood of a time-like singular point, in the sense of the correct number of physically-arbitrary functions.

However, this classical description is in conflict with the requirement of a quantum behavior of the Universe through the Planck era; there are reliable indications that the Mixmaster dynamics overlaps the quantum Universe evolution, requiring an appropriate analysis of the transition between these two different regimes. Indeed, the dynamics of the very early Universe corresponds to a very peculiar situation, with respect to the link existing between the classical and quantum regimes. The expansion of the Universe is the crucial phenomenon which maps into each other these two stages of the evolution. The appearance of a classical background takes place essentially at the end of the Mixmaster phase, when the anisotropy degrees of freedom can be treated as small perturbations; this result indicates that the oscillatory regime takes place almost during the Planck era and therefore it is a problem

$$
\begin{aligned}
Q_1(u,v) &= -u/\delta \geq 0\\
Q_2(u,v) &= (1+u)/\delta \geq 0\\
Q_3(u,v) &= (u^2+u+v^2)/\delta \geq 0\\
\delta &= u^2+u+1+v^2
\end{aligned}
\tag{D.7.3}
$$

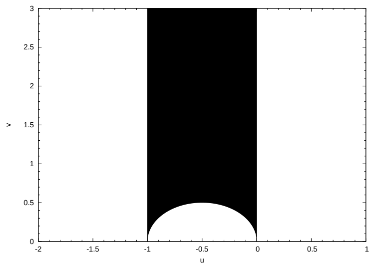

Figure D.1.: The billiard where the Mixmaster Universe moves.

of quantum dynamics. However the end of the Mixmaster (and in principle the *quantum to classical* transition phase) is fixed by the initial conditions on the system and, in particular, it takes place when the cosmological horizon reaches the inhomogeneity scale of the model; therefore the question of an appropriate treatment for the semiclassical behavior arises when the inhomogeneity scale is so larger than the Planck scale, so that the horizon can approach it only in the classical limit.

In the Arnowitt-Deser-Misner (ADM) formalism, the classical dynamics of the Mixmaster can be reduced to the physical degrees of freedom: the evolution resembles that one of a billiard ball on a constant negative curved 2-dimensional surface, described in the Poincaré half-plane by the following action principle:

$$I = \int_{\Gamma_Q} (p_u \partial_t u + p_v \partial_t v - H_{ADM})\, dt\,, \tag{D.7.1}$$

$$H_{ADM} = \epsilon = v\sqrt{p_u^2 + p_v^2}\,, \tag{D.7.2}$$

where Γ_Q is a portion of the full Poincarè plane described by the inequalities above.

A Schrödinger quantization scheme can be applied to the squared Hamiltonian operator, and the associated eigenvalue problem is solved. This approach gives a set of eigenfunctions (here we assume an ordering for the position and momentum operators such that $v^2 p_v^2 \rightarrow \hat{v}\hat{p_v}^2\hat{v}$, which is the only one able to reproduce the proper statistical dynamics). As soon as (approximated) Dirichlet boundary condition are taken into account, the energy spectrum results to be given by

$$(E/\hbar)^2 = t^2 + 1/4\,. \tag{D.7.4}$$

where the values of the parameter t have to be evaluated solving $K_{it}(2n) = 0$ for a generic integer n. This spectrum is a discrete one, and it admits a minimum value given by $E_0^2 = 19.831\hbar^2$. In the figures below the wave function of the ground state and its probability distribution are plotted.

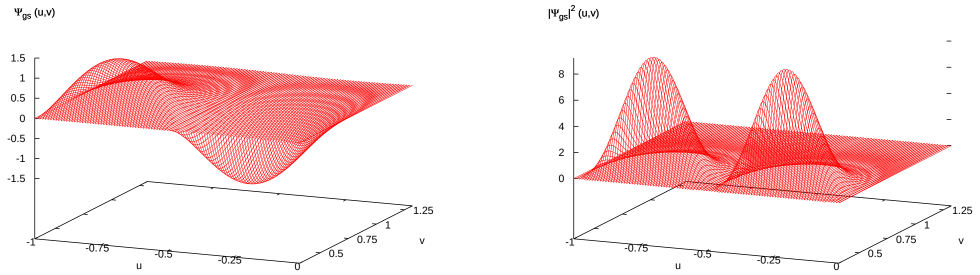

Figure D.2.: The ground state wave function and the probability distribution.

D.8. Dualism between time evolution and matter fields

In this section we review the fundamental aspects of the so-called evolutionary quantum gravity as presented in (Montani, 2002), (Mercuri and Montani, 2004). First we analyze the implication of a Schrödinger formulation of the quantum dynamics for the gravitational field and then we establish a dualism between time evolution and matter fields. Finally, we stress how an evolutionary paradigm can be fixed by restricting the admissible set of coordinate transformations to synchronous ones (Montani and Cianfrani, 2008).

Let us assume that the quantum evolution of the gravitational field is governed by the smeared Schrödinger equation

$$i\partial_t \Psi = \hat{\mathcal{H}}\Psi \equiv \int_\Sigma d^3x \left(N\hat{H}\right)\Psi \,, \qquad \text{(D.8.1)}$$

being $\hat{H}$ the super-Hamiltonian operator, N the lapse function and the wave functional Ψ is defined on the Wheeler superspace, *i.e.*, it is annihilated by the super-momentum operator $\hat{H}_\alpha$. Let us now take the following expansion for the wave functional

$$\Psi = \int D\epsilon \chi(\epsilon, \{h_{\alpha\beta}\}) \exp\left\{-i\int_{t_0}^{t} dt' \int_\Sigma d^3x (N\epsilon)\right\}, \qquad \text{(D.8.2)}$$

$D\epsilon$ being the Lebesgue measure in the space of the functions $\epsilon(x^\rho)$. Such an expansion reduces the Schrödinger dynamics to an eigenvalues problem of the form

$$\hat{H}\chi = \epsilon\chi, \qquad \hat{H}_\alpha\chi = 0, \qquad \text{(D.8.3)}$$

which outlines the appearance of a non zero super-Hamiltonian eigenvalue.

In order to reconstruct the classical limit of the above dynamical constraints, we address the limit $\hbar \to 0$ and replace the wave functional χ by its corresponding zero-order WKB approximation $\chi \sim e^{iS/\hbar}$. Under these restrictions, the eigenvalues problem (D.8.3) reduces to the following classical counterpart

$$\hat{H}JS = \epsilon \equiv -2\sqrt{h}T_{00}, \qquad \hat{H}J_\alpha S = 0 \,, \tag{D.8.4}$$

where $\hat{H}J$ and $\hat{H}J_\alpha$ denote operators which, acting on the phase S, reproduce the super-Hamiltonian and super-momentum Hamilton-Jacobi equations respectively. We see that the classical limit of the adopted Schrödinger quantum dynamics is characterized by the appearance of a new matter contribution (associated with the non zero eigenvalue ϵ) whose energy density reads

$$\rho \equiv T_{00} = -\frac{\epsilon(x^\rho)}{2\sqrt{h}}, \tag{D.8.5}$$

where by T_{ij} we refer to the new matter energy-momentum tensor.

Since the spectrum of the super-Hamiltonian has, in general, a negative component, we can then infer that, when the gravitational field is in the ground state, this matter out-coming in the classical limit has a positive energy density. The explicit form of (D.8.5) is that of a dust fluid co-moving with the slicing 3-hypersurfaces, *i.e.*, the field n^i begin the 4-velocity normal to the 3-hypersurfaces (in other words, we deal with an energy-momentum tensor $T_{ij} = \rho n_i n_j$).

We stress that in this approach, it is possible to turn the solution space into Hilbert one and therefore a notion of probability density naturally arises, from the squared modulus of the wave-functional.

Let us now consider the opposite sector, *i.e.*, a gravitational system in the presence of a macroscopic matter source. In particular, we choice a perfect fluid having a generic equation of state $p = (\xi - 1)\rho$ (p being the pressure and ξ the polytropic index). The energy-momentum tensor, associated to this system reads

$$T_{ij} = \xi\rho u_i u_j - (\xi - 1)\rho g_{ij} \,. \tag{D.8.6}$$

To fix the constraints when matter is included in the dynamics, let us make use of the relations

$$G_{ij}n^i n^j = -\kappa\frac{H}{2\sqrt{h}} \,, \tag{D.8.7}$$

$$G_{ij}n^i \partial_\alpha y^j = \kappa\frac{H_i}{2\sqrt{h}}, \tag{D.8.8}$$

where $\partial_\alpha y^i$ are the tangent vectors to the 3-hypersurfaces, *i.e.*, $n_i \partial_\alpha y^i = 0$. Equations (D.8.7) and (D.8.8), by (D.8.6) and identifying u_i with n_i (*i.e.*, the physical space is filled by the fluid), rewrite

$$\rho = -\frac{H}{2\sqrt{h}}, \qquad H_i = 0\,; \tag{D.8.9}$$

furthermore, we get the equations

$$G_{ij}\partial_\alpha y^i \partial_\beta y^j \equiv G_{\alpha\beta} = \kappa(\xi - 1)\rho h_{\alpha\beta}\,. \tag{D.8.10}$$

We now observe that the conservation law $\nabla_j T_i^j = 0$ implies the following two conditions

$$\xi \nabla_i \left(\rho u^i\right) = (\xi - 1) u^i \partial_i \rho\,, \tag{D.8.11}$$

$$u^j \nabla_j u_i = \left(1 - \frac{1}{\xi}\right)\left(\partial_i \ln\rho - u_i u^j \partial_j \ln\rho\right)\,. \tag{D.8.12}$$

If we now adapt the spacetime slicing, looking the dynamics into the fluid frame (*i.e.*, $n^i = \delta^i_0$), then, by the relation $n^i = (1/N,\ -N^\alpha/N)$, we see that the co-moving constraint implies the synchronous nature of the reference frame. As it is well-known that a synchronous reference is also a geodesic one, the right-hand-side of equation (D.8.12) must vanish identically and, for a generic inhomogeneous case, this means to require $\xi \equiv 1$. Hence, equations (D.8.11) yields $\rho = -\bar{\epsilon}(x^\rho)/2\sqrt{h}$; substituting the last expression into (D.8.9), we get the same Hamiltonian constraints associated to the Evolutionary Quantum Gravity at the point i), as soon as the function $\bar{\epsilon}$ is turned into the eigenvalue ϵ. In this respect, we stress that, while $\bar{\epsilon}$ is positive by definition, the corresponding eigenvalue can also take negative values because of the H-structure.

Thus, we conclude that a dust fluid is a good choice to realize a clock in Quantum Gravity, because it induces a non-zero super-Hamiltonian eigenvalue into the dynamics; furthermore, for vanishing pressure ($\xi = 1$), the equations (D.8.10) reduces to the right vacuum evolution for $h_{\alpha\beta}$. Moreover, we stress how the above two points outline, in quantum gravity, a real dualism between time evolution and the presence of a dust fluid.

The approach above was applied to a generic cosmological model in (Battisti and Montani, 2006c) where is shown how, from a phenomenological point of view, an evolutionary quantum cosmology overlaps the Wheeler-DeWitt framework.

In particular, for such a model, the eigenvalues problem (D.8.3) rewrite as

$$\left\{\kappa\left[\partial_R \frac{1}{R}\partial_R - \frac{1}{R^3}\left(\partial_+^2 + \partial_-^2\right)\right] - \frac{3}{8\pi R^3}\partial_\phi^2 - \frac{R^3}{4\kappa l_{in}^2}V(\beta_\pm) + R^3(\rho_{ur} + \rho_{pg})\right\}\chi = \epsilon\chi. \tag{D.8.13}$$

where $\kappa = 8\pi l_P^2$ and we have added to the dynamics of the system an ultrarelativistic energy density ($\rho_{ur} = \mu^2/R^4$), a perfect gas contribution ($\rho_{pg} = \sigma^2/R^5$) and a scalar field ϕ (a free inflaton field). Such a problem can be analytically solved and the spectrum of the super-Hamiltonian reads as

$$\epsilon_{n,\gamma} = \frac{\sigma^2}{l_P^2(n+\gamma-1/2)}. \quad \text{(D.8.14)}$$

Therefore the ground state $n = 0$ eigenvalue, for $\gamma < 1/2$, is negative and so it is associated via (D.8.5) to a positive dust energy density.

In order to analyze the cosmological implication of the new matter contribution, we have to impose a cut-off length in our model, requiring that the Planck length l_P is the minimal physical length accessible by an observer ($l \geq l_P$). This way, we get $\sigma^2 \leq \mathcal{O}(l_P)$ and so $|\epsilon_0| \leq (1/l_P)$: *the spectrum is limited by below*. Moreover the contribution of such a dust fluid to the actual critical parameter is

$$\Omega_{dust} \sim \frac{\rho_{dust}}{\rho_{Today}} \sim \mathcal{O}\left(10^{-60}\right). \quad \text{(D.8.15)}$$

As matter of fact, such a parameter is much less then unity and so no phenomenology can came out (today) from our dust fluid. In this sense we claim that an evolutionary quantum cosmology overlaps the Wheeler-DeWitt approach and therefore it can be inferred as appropriate to describe early stages of the Universe without significant traces on the later evolution.

D.9. Loop Quantum Cosmology

Standard cosmological model raises several fundamental issues such as initial singularity and the problem of horizon. We analyze these well known problems within the framework of cosmological models based on Loop Quantum Gravity.

One of the fundamental issues of the theory of Early Universe is cosmic singularity. Many researchers, such as J.A. Wheeler, believed that appearance of initial singularity in Friedmann Equations marks a breakdown of General Relativity theory and searched for a possible solution in quantization of gravity. The well known Wheeler-de Witt equation is one example of such an approach, although unsuccessful. At the same time, it is clear that attempts to construct viable nonsingular cosmologies within classical theories of gravitation did not succeed, as discussed by (Vereshchagin, 2004a, 2005).

Loop Quantum Gravity is at present the main background independent and nonperturbative candidate for a quantum theory of gravity; Loop Quantum Cosmology is the application of Loop Quantum Gravity to a homogeneous minisuperspace environment. The underlying geometry in LQG is

discrete and the continuum spacetime is obtained from quantum geometry in a large eigenvalue limit. Numerical calculations performed within Loop Quantum Gravity theory established the possibility of resolution of singularities in various situations.
The underlying dynamics in LQC is governed by a discrete quantum difference equation in quantum geometry. However, using semiclassical states one can construct an effective Hamiltonian description on a continuum spacetime which has been shown to very well approximate the quantum dynamics.

We have performed a general analysis of equations governing evolution of the Universe within semiclassical Loop Quantum Cosmology by using qualitative methods of the theory of dynamical systems. Specifically, two cases were considered with different type of corrections to the Friedmann equations.
In the work by (Singh et al., 2006) quadratic on the energy density correction to the Friedmann equation, coming from effective Hamiltonian of Loop Quantum Cosmology was studied. The modified Friedmann equation takes the form

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\,\rho\left(1 - \frac{\rho}{\rho_{\rm crit}}\right), \tag{D.9.1}$$

where a is the scale factor, k denotes spatial curvature, c and G are the speed of light and the gravitational constant respectively. The energy density of the scalar field is

$$\rho = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V. \tag{D.9.2}$$

The critical energy density is

$$\rho_{\rm crit} = \frac{\sqrt{3}}{16\pi^2\gamma^3}\rho_{\rm pl}, \tag{D.9.3}$$

where γ is Barbero-Immirzi parameter, $\rho_{\rm pl}$ is the Planckian density. The usual continuity equation for the real scalar field ϕ with effective potential $V(\phi)$ takes the form

$$\frac{d^2\phi}{dt^2} + 3\frac{1}{a}\frac{da}{dt}\frac{d\phi}{dt} + \frac{\partial V}{\partial \phi} = 0. \tag{D.9.4}$$

Equations (D.9.1) and (D.9.4) can be analysed by means of qualitative theory of dynamical systems. First of all, the derivative of the scale factor can be expressed from (D.9.1) and substituted into (D.9.4) thus reducing the phase space to two dimensions. The corresponding phase space variables are the scalar field ϕ and its time derivative $\dot{\phi} \equiv \frac{d\phi}{dt}$. Expample of the phase portrait is represented in Fig. D.3 The boundary of the phase space, defined as

$$\rho = \rho_{\rm crit}, \tag{D.9.5}$$

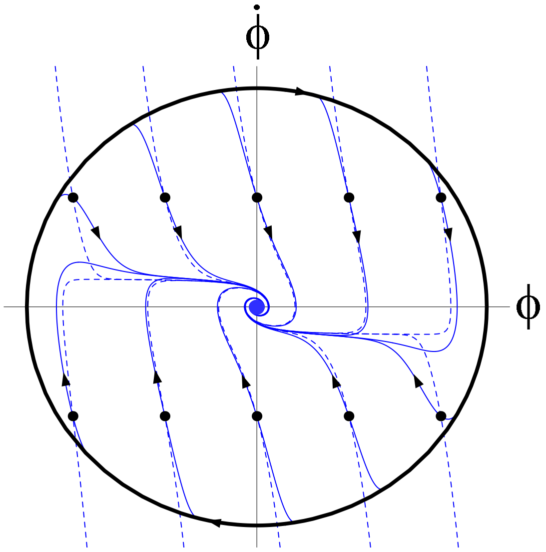

Figure D.3.: Phase portrait for massive scalar field $V = m^2\phi^2/2$ potential. Dashed curves represent GR case and solid curves shown LQC case.

prevents appearance of singularities for positive energy density, unlike the case of General Relativity, where the boundary is absent. Details see in (Singh et al., 2006).

In the work of (Vereshchagin, 2004b) corrections due to the inverse scale factor operator both in the gravitational and the matter part of the effective Hamiltonian were analyzed. These corrections appear both in the energy density (D.9.2) and in the continuity equation as

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\left[\frac{1}{2D}\left(\frac{d\phi}{dt}\right)^2 + V\right], \tag{D.9.6}$$

$$\frac{d^2\phi}{dt^2} + 3\frac{1}{a}\frac{da}{dt}\frac{d\phi}{dt} - \frac{1}{D}\frac{dD}{dt}\frac{d\phi}{dt} + D\frac{\partial V}{\partial \varphi} = 0, \tag{D.9.7}$$

where the function D is defined as

$$D(q) = \left(\frac{8}{77}\right)^6 q^{3/2} 7\left[(q+1)^{11/4} - |q-1|^{11/4}\right] - \\ - 11q\left[(q+1)^{7/4} - \mathrm{sign}(q-1)|q-1|^{7/4}\right]^6, \tag{D.9.8}$$

with $q = (a/a_*)^2$ and $a_*^2 = \frac{j\ln 2}{3\sqrt{3}\pi} l_P^2$ being the scale where quantum corrections become essential. The latter can be larger than the planckian length l_P, since the quantization parameter j, which must take half integer values, but is arbitrary. Equation (D.9.7) can be substituted into (D.9.6) and role of dynamical variables is then played by the scale factor and its derivative $H \equiv \frac{1}{a}\frac{da}{dt}$. Examples of phase portraits are shown in Fig. D.4. The left figure represents the case of General Relativity, while central and right figures correspond to Loop Quantum Cosmology. Due to different structure of the phase space, again singular solutions do not appear.

Our general conclusion, considering both types of corrections, is the absence of cosmic singularity, so in all solutions the usual expansion stage follows after the generic bounce. Moreover, we have shown that in both cases there exist successful mechanisms for generation of initial conditions suitable for inflation.

This work is relevant for the development of the theory of Early Universe. In particular, better understanding of background solutions and their properties should be reached prior to study of the cosmological perturbations. Dynamics of these perturbations, in turn, is crucial in view of verification of predictions of the theory, confronted with observational data.

D.10. FRW cosmological model in the GUP framework

Let us now investigate the consequences of the an Heisenberg deformed algebra (D.2.2) of the quantum dynamics of the flat ($k = 0$) FRW model in the presence of a massless scalar field ϕ. In particular, we will interested about the fate of the classical singularity in this framework. In which follows we summarize the discussion and results reported in [1]. The Hamiltonian constraint for this model has the form

$$H_{grav} + H_\phi \equiv -9\kappa p_x^2 x + \frac{3}{8\pi}\frac{p_\phi^2}{x} \approx 0\,, \qquad x \equiv a^3, \tag{D.10.1}$$

where a is the scale factor. In the classical theory, the phase space is 4- dimensional, with coordinates $(x, p_x; \phi, p_\phi)$ and at $x = 0$ the physical volume of the Universe goes to zero and the singularity appears. Moreover, it is not difficult to see that each classical trajectory can be specified in the (x, ϕ)-plane, *i.e.*, ϕ can be considered as a relational time for the dynamics. In particular, the dynamical trajectories read as

$$\phi = \pm\frac{1}{\sqrt{24\pi\kappa}}\ln\left|\frac{x}{x_0}\right| + \phi_0\,, \tag{D.10.2}$$

where x_0 and ϕ_0 are integration constants. In this equation, the plus sign describes an expanding Universe from the Big-Bang, while the minus sign a contracting one into the Big-Crunch. We now stress that the classical cosmological singularity is reached at $\phi = \pm\infty$ and every classical solution, in this model, reaches the singularity.

As well-known the canonical approach (the WDW theory) to this problem does not solve the singularity problem. More precisely, it is possible to construct a state localized at some initial time. Then, in the backward evolution, its peak will moves along the classical trajectory (D.10.2) and thus it falls into the classical singularity. This way, the classical singularity is not tamed by quantum effects.

This picture is radically changed in the GUP framework and the modifications can be realized in two different steps. At first, it is possible to show how the probability density $|\Psi(\zeta,t)|^2$ to find the Universe around $\zeta \simeq 0$ (around the Planckian region) can be expanded as

$$|\Psi(\zeta,t)|^2 \simeq |A(t)|^2 + \zeta^2|B(t)|^2. \tag{D.10.3}$$

Here t is a dimensionless time $t = \sqrt{24\pi\kappa}\phi$ and the wave packets

$$\Psi(\zeta,t) = \int_0^\infty d\epsilon g(\epsilon)\Psi_\epsilon(\zeta)e^{i\epsilon t} \ , \tag{D.10.4}$$

are such that the state is initially packed at late time, *i.e.*, the weight function $g(\epsilon)$ is a Gaussian distribution peaked at some $\epsilon^* \ll 1$ (at energy much less then the Plank energy $1/l_P$). Of course, $\Psi_\epsilon(\zeta)$ rapresent the *quasiposition eigenfunctions* (D.2.3) of this problem.

Therefore, near the Planckian region, the probability density to find the Universe is $|A(t)|^2$, which is very well approximated by a Lorentzian function packed in $t = 0$. This value corresponds to the classical time for which $x(t) = x_0$. Thus, for $x_0 \sim \mathcal{O}(l_P^3)$, the probability density to find the Universe in a Planckian volume is peaked around the corresponding classical time. As a matter of fact this probability density vanishes for $t \to -\infty$, where the classical singularity appears. This is the meaning when we claim that the classical cosmological singularity is solved by this model.

Of course the more interesting differences between the WDW and the GUP approaches can be recognized in the wave packets dynamics. In particular, we consider a wave packet initially peaked at late times and let it evolve numerically "backward in time". The result of the integration is that the probability density, at different fixed values of ζ, is very well approximated by a Lorentzian function yet. Moreover, the width of this function remains, actually, the same as the states evolves from large ζ (10^3) to $\zeta = 0$. The peaks of Lorentzian functions, at different ζ values, move along the classically expanding trajectory (D.10.2) for values of ζ larger then ~ 4. Near the Planckian region, *i.e.*, when $\zeta \in [0,4]$, we observe a modification of the trajectory of the peaks. In fact they follow a power-law up to $\zeta = 0$, reached in a finite time interval and "escape" from the classical trajectory toward the classical singularity. The peaks of the Lorentzian at fixed time t, evolves very slowly remaining close to the Planckian region. Such behavior outlines that the Universe has a stationary approach to the cutoff volume.

An important fact has now to be stressed. The peculiar behavior of our quantum Universe is different from other approaches to the same problem. In fact, recently, it was shown how the classical Big-Bang is replaced by a Big-Bounce in the framework of Loop Quantum Cosmology (LQC). Intuitively, one can expect that the bounce and so the consequently repulsive features of the gravitational field in the Planck regime are consequences of a Planckian cut-off length. But this is not the case. As matter of fact that there is not a bounce for our quantum Universe. The main differences between the two approaches resides in the quantum modification of the classical trajectory. In fact, in the LQC framework we observe a "quantum bridge" between the expanding and contracting Universes; in our approach, contrarily, the probability density of finding the Universe reaches the Planckian region in a stationary way.

D.11. Gauge potential of a Lorentz gauge theory

A gauge theory of the local Lorentz group has been implemented both in flat and in curved space-time, and the resulting dynamics is analyzed in view of the geometrical interpretation of the gauge potential. The Yang-Mills picture of the local Lorentz transformations in curved space-time is first approached in a first-order formalism. For the Lagrangian approach to reproduce the II Cartan Structure Equation as soon as the Lorentz gauge connections are identified with the contortion tensor, an interaction term between the new Lorentz gauge fields A_{μ}^{ab} and the spin connections ω_{μ}^{ab},has to be postulated, *i.e.*,

$$S_{int} = 2\int \det(e)\, d^4x\ \ e^{\mu}{}_{a} e^{\nu}{}_{b}\, \omega_{\mu c}^{[a}\, A_{\nu}{}^{bc]}\ . \tag{D.11.1}$$

This interaction term induces a Riemannian source to the Yang-Mills equations; thus, the real vacuum dynamics of the Lorentz gauge connections takes place on a Minkowski space only, when the Riemannian curvature and the spin currents provide negligible effects. In fact, it is the geometrical interpretation of the torsion field as a gauge field that generates the non-vanishing part of the Lorentz connection on flat space-time. The full picture involving gravity, torsion and spinors is described by a coupled set of field equations, which allows one to interpret both gravitational spin connections and matter spin density as the source term for the Yang-Mills equations. The contortion tensor acquires a propagating character, because of its non-Abelian feature, and the pure contact interaction is restored in the limit of vanishing Lorentz connections (Carlevaro et al., 2007).

To better understand the physical implications of first- and second-order approaches, a comparison between field equations has been accomplished in the linearized regime, by considering the case of small perturbations $h_{\mu\nu}$ of a flat Minkowskian metric $\eta_{\mu\nu}$. Because of the interaction term (D.11.1) postulated in the first-order approach, it is possible to solve the structure equation and to express the connection as a sum of the pure gravitational (Ricci) connection plus other contributions, both in absence and in presence of spinor matter. The Ricci connection $\omega_{\mu}{}^{ab} = e^{b\nu}\nabla_{\mu}e_{\nu}{}^{a}$ rewrites, because of the linearization,

$$\omega_{\mu}{}^{ab} = \delta^{b\nu}\left(\partial_{\nu}\zeta^{a}_{\nu} - \tilde{\Gamma}(\zeta)^{\rho}_{\mu\nu}\delta^{b}_{\rho}\right) , \tag{D.11.2}$$

where $\tilde{\Gamma}(\zeta)^{\rho}_{\mu\nu}$ are the linearized Christoffel symbols. Since it acquires the physical meaning of a source for torsion, it can be interpreted as a spin-current density. Nevertheless, it is linear in ζ, since the interaction term (D.11.1) is linear itself; as suggested by the comparison with gauge theories, and with the current

$$M^{\tau\ \beta}_{\alpha} = \frac{\partial L}{\partial h_{\mu\nu,\tau}}\Sigma^{\rho\beta\sigma}_{\alpha\mu\nu}h_{\rho\sigma} = \left(\delta^{c\mu}\zeta^{\nu,\tau}_{c} + \delta^{c\nu}\zeta^{\mu,\tau}_{c}\right)\Sigma^{\rho\beta\sigma}_{\alpha\mu\nu}\left(\delta_{f\rho}\zeta^{f}_{\sigma} + \delta_{f\sigma}\zeta^{f}_{\rho}\right) , \tag{D.11.3}$$

(where $\Sigma^{\rho\alpha\beta\sigma}_{\mu\nu} = \eta^{\gamma[\alpha}(\delta^{\rho}_{\gamma}\delta^{\beta]}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\rho}_{\mu}\delta^{\sigma}_{\gamma}\delta^{\beta]}_{\nu}))$, the interaction term is quadratic. In this case, however, it would be very difficult to split up the solution of the structure equation as the sum of the pure gravitational connection plus other contributions.

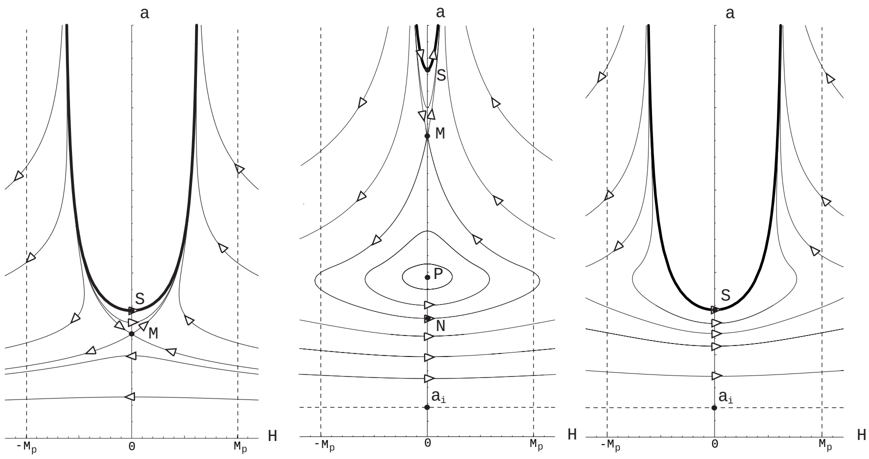

Figure D.4.: Phase portraits of cosmological models with the scalar field with flat effective potential. Left figure corresponds to the case of GR, $V_0 = 0.05\rho_{\rm pl}$. Central figure represents the phase space with focus and saddle for $V_0 = 0.006\rho_{\rm pl}$. Right figure represents the phase space with $V_0 = 0.05\rho_{\rm pl}$. Thick curves surround regions where the derivative of the scalar field $\dot{\phi}$ is complex and there are no solutions. For central and right figure $j = 100$. Dashed lines again surround the region where semi-classical approach is valid.

E. Quantum Fields on Classical Background

E.1. Dirac equation on a curved spaces and classical trajectories

The interaction between geometry and internal spinor-like degrees of freedom has been investigated with the aim to infer the analogous of Papapetrou equations for a quantum spin (Cianfrani and Montani, 2008a). This task has been approached by an eikonal approximation, *i.e.*, $\psi = e^{iS}u$, and a localization hypothesis for u along the integral curve of the momentum K_μ. Hence, a dispersion relation has been recovered starting from the squared Dirac equation and by virtue of an integration on spatial coordinates. This way, the following relation has been obtained

$$(K_\mu K^\mu - K^\mu S_\mu)(1 + O(\lambda^2)) + \mu^2 = 0, \tag{E.1.1}$$

λ and μ being the Compton length of the particle and the mass, respectively, while the quantity S_μ reads as

$$S_\mu = 2i\frac{\bar{u}_0\gamma^{\bar{0}}D_\mu u_0 - D_\mu \bar{u}_0\gamma^{\bar{0}}u_0}{\bar{u}_0\gamma^{\bar{0}}u_0}. \tag{E.1.2}$$

Hence the dynamics of K_μ is obtained by acting on the relation (E.1.1) with the derivative operator ∇_ν and we have

$$\begin{cases} U^\mu\nabla_\mu P_\nu - \frac{\hbar}{2}R_{\rho\sigma\mu\nu}U^\mu S^{\rho\sigma} - \hbar\nabla_\nu U^\mu S_\mu - \\ \quad -2i\hbar U^\mu D_{[\nu}\bar{u}_0\gamma^{\bar{0}}D_{\mu]}u_0 + O(\lambda^2) = 0 \\ \quad P_\nu = K_\nu - S_\nu \end{cases}. \tag{E.1.3}$$

Here the quantity $S^{\mu\nu}$ is given by the expression

$$S^{\mu\nu} = \frac{\int d^3x\sqrt{h}\bar{u}\{\gamma^{\bar{0}},\Sigma^{\mu\nu}\}u}{2\int d^3x\sqrt{h}\bar{u}\gamma^{\bar{0}}u} = \frac{\bar{u}_0\{\gamma^{\bar{0}},\Sigma^{\mu\nu}\}u_0}{2\bar{u}_0\gamma^{\bar{0}}u_0} + O(\lambda^2), \tag{E.1.4}$$

for which we have

$$S^{\nu\mu}U_\nu = 0, \tag{E.1.5}$$

Since we are performing a multi-pole expansion, it is possible to assume that

$$D_\mu u_0 = iU_\mu v, \tag{E.1.6}$$

v being an arbitrary spinor. It can be shown that this hypotheses is well-grounded by an analysis on the dynamics of the wave-function.

This way, the following equations are obtained

$$U^\mu \nabla_\mu U_\nu - \frac{\hbar}{2} R_{\rho\sigma\mu\nu} U^\mu S^{\rho\sigma} = 0 \tag{E.1.7}$$

Therefore, Dirac particles follow the trajectory of classical spinning ones (according with the Mathisson-Papapetrou formulation), whose spin tensor is given by $S_{\mu\nu}$ (E.1.4).

F. Unification theories

F.1. Hamiltonian Formulation of the 5-dimensional Kaluza-Klein model

A first line of research is the analysis of the ADM splitting of the 5D KK model, to achieve the Hamiltonian formulation of the dynamics and get insights onto the gauge-symmetry generation. The ADM slicing of KK model, and its physical meaning, is not obvious, due to the existence of two possible procedures; we refer to these as KK-ADM and ADM-KK procedures. In KK-ADM we firstly perform the usual KK reduction of the metrics, and then a *3+1* ADM splitting of the gravitational tensor and the abelian gauge vector. The 5D metric j_{AB} splits as follows

$$j_{AB} \Rightarrow \begin{cases} g_{\mu\nu} \to \vartheta_{ij}, S_i, N \\ A_\mu \to A_i, A_0 \\ \phi \to \phi \end{cases} \to \begin{pmatrix} N^2 - S_i S^i - \phi^2 A_0^2 & -S_i - \phi^2 A_0 A_i & -\phi^2 A_0 \\ -S_i^2 \phi^2 A_0 A_i & -\vartheta_{ij} - \phi^2 A_i A_j & -\phi^2 A_i \\ -\phi^2 A_0 & -\phi^2 A_i & -\phi^2 \end{pmatrix} .$$

Here $N,\ S_i,\ \vartheta_{ij}$ are the *lapse* function, the 3D *shift* vector and the 3D induced metrics ($A, B = 0,1,2,3,5;\ \mu,\nu = 0,1,2,3;\ i,j = 1,2,3$). This way, we have a non-complete space-time slicing, due to the fact that we are doing a *3+1* splitting of a 5-D background, so that the extra-dimension is not included. In the ADM-KK procedure we firstly deal with a *4+1* splitting that includes the extra-dimension and then we consider the KK reduction related to the pure spatial manifold:

$$j_{AB} \Rightarrow \begin{cases} h_{\hat{I},\hat{J}} \to A_i, \vartheta_{ij}, \phi \\ N_{\hat{I}} \to N_i, N_5 \\ N \to N \end{cases} \Rightarrow \begin{pmatrix} N^2 - h_{\hat{I}\hat{J}} N^{\hat{I}} N^{\hat{J}} & -N_i & -N_5 \\ -N_i & -\vartheta_{ij} -^2 \phi^2 A_i A_j & -\phi^2 A_i \\ -N_5 & -\phi^2 A_i & -\phi^2 \end{pmatrix} .$$

Here $N_{\hat{I}}$ and $h_{\hat{I}\hat{J}}$ are the 4D *shift* vector and the 4D spatial induced metric ($\hat{I}, \hat{J} = 1,2,3,5$). Now we have a complete slicing but, in this set of variables, the component A_0 is missing. Hence, both procedures are unsatisfactory and it must be checked if they commute. Despite the outcoming metric seem to be different, we are dealing with objects that must show well defined properties under pure spatial KK diffeomorphisms. This allow us to look for "conversion formulas" between this two metrics. Indeed, we can implement the KK reduction on $N_{\hat{I}}$; it is possible to recognize that $N_{\hat{I}}$ is not a pure 4D spatial vector neither simple gauge vector but a mixture of them. A detailed study of the 5-bein structure yields the following formulas for $N_{\hat{I}}$

$$\begin{cases} N_i = S_i + \phi^2 A_0 A_i \\ N_5 = \phi^2 A_0 \end{cases} , \qquad \begin{cases} N^i = S^i \\ N^5 = N^2 A^0 \end{cases} .$$

As soon as the Lagrangians resulting from these two procedures are recasted in the same set of variables, it is possible to recognize that they differ only for surface terms. Then, we conclude that we are dealing with equivalent dynamics and with a unique well defined Hamiltonian. Hence, ADM splitting is provided to commute with KK reduction and this allows us to compute the Hamiltonian. Moreover, the Hamiltonian formulation, together with conversion formulas, clearly shows how the time component of the electromagnetic field is given by a combination of the geometrical Lagrangian multipliers coming out in a 5D scheme.
People involved in this topic are Valentino Lacquaniti and Giovanni Montani (Lacquaniti and Montani, 2006a).

F.2. Classical and Quantum spinning particles in Kaluza-Klein space-times

The dynamics of a classical spinning particle, in a KK space-time, is inferred from the extension of Papapetrou equations to the 5-dimensional case, with Pirani conditions, *i.e.*,

$$\begin{cases} \frac{D}{^{(5)}Ds} {}^{(5)}P^A = \frac{1}{2} {}^{(5)}R_{BCD}{}^{A} \Sigma^{BC} {}^{(5)}u^D \\ \frac{D}{^{(5)}Ds} \Sigma^{AB} = {}^{(5)}P^A {}^{(5)}u^B - {}^{(5)}P^B {}^{(5)}u^A \\ {}^{(5)}P^A = {}^{(5)}m {}^{(5)}u^A - \frac{D\Sigma^{AB}}{^{(5)}Ds} {}^{(5)}u_B \\ \Sigma^{AB} {}^{(5)}u_A = 0 \end{cases} . \tag{F.2.1}$$

The main new feature is the 4 additional components of the spin-tensor Σ_{AB}, whose physical meaning is going to be clarified by our analysis. At first, under coordinate transformations, proper of a KK model, $\Sigma^{\mu\nu}$ and $\Sigma_{5\mu}$ behave like 4-dimensional quantities, in particular a tensor $S^{\mu\nu}$ and a vector S_μ, respectively.
By rewriting the full system above in terms of 4-dimensional quantities, $S^{\mu\nu}$,

S_μ, A_μ and $g_{\mu\nu}$, one finds, after some manipulations,

$$\begin{cases} \frac{D}{Ds}\hat{P}^\mu = \frac{1}{2}R_{\alpha\beta\gamma}{}^{\mu}S^{\alpha\beta}u^\gamma + qF^\mu{}_\nu u^\nu + \frac{1}{2}\nabla^\mu F^{\nu\rho}M_{\nu\rho} \\ \frac{Dq}{Ds} = \frac{D}{Ds}(\alpha^2\widetilde{P}_5 + \frac{1}{4}ekF_{\alpha\beta}S^{\alpha\beta}) \\ \frac{DS^{\mu\nu}}{Ds} = \hat{P}^\mu u^\nu - \hat{P}^\nu u^\mu + F^\mu{}_\rho M^{\rho\nu} - F^\nu{}_\rho M^{\rho\mu} \\ \hat{P}^\mu = \alpha^2 P^\mu + u_5\frac{DS^\mu}{Ds} - ekF_{\rho\nu}u^\rho S^{\nu\mu}u_5 + \frac{1}{2}ekF^\mu{}_\rho S^\rho \\ S^{\nu\mu}u_\nu + S^\mu u_5 = 0 \end{cases} . \qquad \text{(F.2.2)}$$

The quantity $M^{\mu\nu}$ has the following expression

$$M^{\mu\nu} = \frac{1}{2}ek(S^{\mu\nu}u_5 + u^\mu S^\nu - u^\nu S^\mu) , \qquad \text{(F.2.3)}$$

and becouse of its coupling into equations of motion it has to be identified with the electro-magnetic moment.
This way, it is worth nothing that the system (F.2.2) reproduces exactly equations of motion of a spinning particle, endowed with a charge q and an electro-magnetic moment $M_{\mu\nu}$. This result demonstrates that the geometrization of the electro-dynamics does not modify the dynamics of spinning objects.
In this scenario, from the expression (F.2.3), the quantity S_μ is recognized as describing an electric dipole moment. The emergence of an electric dipole moment term seems to be a proper feature of a KK approach, since it arises also for spinors, in the Riemannian case.

The introduction of spinor fields in a KK model is the main open point of such an approach. The standard way to deal with them is to extend the Dirac equation to the multi-dimensional case and to try to identify extra-dimensional quantum numbers with internal ones. However this procedure fails, because of the emergence of mass terms of the compactification scale order and because quantum numbers of Standard model particles cannot be inferred.
In this respect, our investigation has been focused on a more phenomenological approach, based on recovering 4-dimensional properties by an averaging procedure on the extra-dimensional manifold. This average is motivated by the undetectability of the extra-space and the need for it is not restricted to the case spinors are present. In fact, we showed that it is required in order to reproduce non-Abelian gauge transformations from extra-dimensional isometries and to get the equations of motion, proper of the 4-dimensional picture, starting from multi-dimensional ones. As far as spinors are concerned, the average produces a non-trivial effect on extra-dimensional symmetries, such that some of the above mentioned issues can be solved.
For instance, we considered the case of a 3-sphere in view of performing the geometrization of an $SU(2)$ gauge theory. We look for a solution of the Dirac equation integrated over the sphere. Even though we do not find an exact

solution, an approximated one, with corrections controlled by an order parameter, is inferred. This is given by

$$\chi_r = \frac{1}{\sqrt{V}} e^{-\frac{i}{2}\sigma_{(p)rs}\lambda^{(p)}_{(q)}\Theta^{(q)}(y^m)} \,, \tag{F.2.4}$$

V being the volume of S^3, $\sigma_{(p)}$ $SU(2)$ generators, while the constant matrix λ satisfies

$$(\lambda^{-1})^{(p)}_{(q)} = \frac{1}{V}\int_{S^3}\sqrt{-\gamma}e^m_{(q)}\partial_m\Theta^{(p)}d^3y \,. \tag{F.2.5}$$

Θ functions are fixed as having the following form

$$\Theta^{(p)} = \frac{1}{\beta}c^{(p)}e^{-\beta\eta} \,, \qquad \eta > 0 \,, \tag{F.2.6}$$

with $c^{(p)}$ and η some arbitrary functions, while β is the order parameter, such that corrections to the Dirac equation are of the β^{-1} order.
This form for the spinor is able to geometrize the $SU(2)$ gauge connection at the leading order in β^{-1}, while, at next orders, gauge-violating terms come out. Hence, this procedure can be used to geometrize the electro-weak model and infer a lower bound for β from current limits on gauge-violating processes. Moreover, the introduction of the Higgs field in such a scenario succeeds in stabilizing its mass and in reproducing mass terms for neutrinos, too.

F.3. Generalized 5-Dimensional Theories

5D KK models provide an interesting toy-model for the analysis of compactification schemes, and the features of generalized 5D models has been investigated, and the symmetries arising after dimensional reduction have been considered. In particular, alternative mechanisms that can imply compactification have been proposed, and broken 5D symmetries have been explored.
On the one hand, the presence of torsion in a 5D model has been shown to produce interesting structures after dimensional reduction. In a 5D scenario, the geometrization of the Electro-weak model has been worked out on the ground of the broken 5D Lorentz group and the properties of torsion [Lecian and Montani (2006)], and proposal for the introduction of Ashtekar variables within this scheme has been evaluated [Lecian and Montani (2007)]. Starting from the 5D Gauss-Codacci formula, and making sure that the residual symmetry of the metric components does not violate the Frobenius-Geroch requirements, evolutionary variables have been proposed.
On the other hand, a truncation of the KK towers has proposed from both

theoretical and phenomenological points of views. In particular, the simplest toy model of a scalar field in 5 dimensions has been analyzed: the truncation of such a tower has been considered as the hint of a modification of the extraD geometrical structures and related symmetries and compactification scenarios.
In the simplest toy model, i.e., a scalar field in a 5-dimensional (5D) space-time, described as the Dirac product of a 4D manifold plus a ring, $M^5 = M^4 \otimes S^1$, the Kaluza-Klein (KK) tower is defined as

$$\Psi^5(x^\rho, x^5) = \sum_{-\infty}^{+\infty} \psi_n(x^\rho) e^{ix^5 m/L}, \quad L \equiv 2\pi R, \tag{F.3.1}$$

that is the infinite sum of the Fourier harmonics, labeled by m. In this compactification scheme, because of the periodic (boundary) condition on the modes of the tower,

$$\psi(x^5) = \psi(x^5 + L), \quad L = 2\pi R,$$

i.e., of the identification of the points $0 \leftrightarrow 2\pi R$, $\psi(x_5)$ is defined on $S_1/\mathbb{Z} \sim \mathbb{R}$.
The scalar-field wavefunction obeys the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + {}^5M_m^2)\Psi = 0 \Rightarrow {}^5M_m^2 \equiv {}^4M^2 + (m/L)^2, \tag{F.3.2}$$

and it expression in the momentum representation reads

$$\tilde{\psi}^m(P_5) = \delta(P_5 - m/L).$$

From F.3.1, it is easy to understand the the structure of the extraD geometry can be described by means of the extraD projection of physical objects.
The analysis of truncated Kaluza-Klein (KK) tower can be performed on the ground of several considerations.
In fact, as it can be easily seen in (F.3.2), the label of the mode is deeply connected both with mass and extraD momentum, which can also be identified with the quantum number of a geometrized interaction, thus allowing for supposing a strict connection between the extraD and the internal structure. From theoretical point of view, the truncation of the tower would correspond to the introduction of a cutoff in the extra D, based on the fact that it would make little sense to specify the localization of a particle below its Schwartzschield radius. The exact localization of a particle in the extraD geometry would yield interpretative difficulties, such that a more general description of the internal structure, which does not automatically allow for an exact notion of point, should be looked for. Furthermore, an infinite spectrum of particles brings field-theoretical as well as algebraic difficulties. From

a phenomenological point of view, possible indications of the existence of an extraD would be provided both by geodesic deviation and scattering amplitudes. In the second case, the truncation would simplify the calculation of scattering amplitudes and would anyhow account for the impossibility of reaching an infinite energy in experiments.
As a result, the symmetries that characterize KK theories can be compared in the cases of infinite and truncated series. Since, in this toy model, the extraD expansion of the wavefunction of the scalar field is the only feature that accounts for the extraD, the truncation of the series would correspond to some modifications of the extraD geometry, as remarked for F.3.1. This way, it will be possible to analyze KK symmetries in both cases and possible compactification scenarios.
It has already been proposed to gain insight into the geometrical interpretation of truncated harmonics expansion on a circle by considering it as worked out from a higher-dimensional structure, thus obtaining a "fuzzy circle", in a "matrix-manifold" scheme. As a result, the ultraviolet cutoff of the model implies a minimal wavelength. As a first attempt, we propose a finite set of approximating wave functions, whose finite sum should reproduce the periodicity on the extraD coordinate, with the aim of pointing out the main difficulties of the problem.This preliminary speculation will be aimed at pointing out the main difficulties of the problem.
As a second strategy, we have considered the truncated wavefunction as a quasi-periodic function, projected on a finite set of Fourier modes. For this purpose, we have analyzed different representations of the standard operator algebra, given by the canonical commutation relations of the extraD operators $\hat{x}_5$ and $\hat{P}_5$, within the framework of the polymer representation. In this case, compactification has been illustrated to occur because of the truncation. We have then established generalized commutation relations; this way, the occurrence of compactification has been investigated through the fundamental wavelength of the theory.
The investigation of the role of the operators $\hat{x}_5$ and $\hat{P}_5$ in the extra-D symmetry and the different compactification mechanisms that arise from these scheme have motivated the comparison between the different approaches from a mathematical point of view [Cianfrani and Montani (2008b)].

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