Early Cosmology and Fundamental General Relativity

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2 Participants

2.1 ICRANet participants

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- Riccardo Benini
- Massimiliano Lattanzi
- Giovanni Montani

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- Nicola Nescatelli

2.3 Ongoing Collaborations

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3 Brief Description

3.1 Highlights in Early Cosmology and Fundamental General Relativity

3.1.1 Dissipative Cosmology

In section "Dissipative Cosmology" we analyze the dynamics of the gravitational collapses (both in the Newtonian approach and in the pure relativistic limit) including dissipative effects. The physical interest in dealing with dissipative dynamics is related to the thermodynamical properties of the system: both the analyzed regimes are characterized by a thermal history which can not be regarded as settled down into the equilibrium. At sufficiently high temperatures, micro-physical processes are no longer able to restore the thermodynamical equilibrium and stages where the expansion and collapse induce non-equilibrium phenomena are generated. The average effect of having such kind of micro-physics results into dissipative processes appropriately described by the presence of *bulk viscosity*, phenomenologically described by a power-law of the energy density. With respect to dissipative dynamics, we also study the early singularity proposed in the scheme of matter creation. The attention is focused on those scenarios for which it is expected that the Universe has been created as a vacuum fluctuation, thus the study of the particle creation should be added for a complete analysis of its dynamics (Montani and Nescatelli, 2008). We can conclude (Carlevaro and Montani, 2008) that the Universe cannot be created like an isotropic system and only after a certain time it becomes close to our usual conception of isotropy. In this respect, this analysis encourages the idea of an early Universe as characterized by a certain degree of anisotropy and inhomogeneity.

The people involved in this line of research are Giovanni Montani and Nakia Carlevaro (Nicola Nescatelli as past collaborator).

3.1.2 On the Jeans instability of gravitational perturbations

In section "On the Jeans instability of gravitational perturbations" we focus on the non-relativistic analysis of gravitational collapses. The standard Jeans Mechanism and the generalization in treating the Universe expansion are both analyzed when bulk viscosity affects only the first order Newtonian dynamics. Since we deal with homogeneous model the so-called shear viscosity is neglected. In fact, no displacement of matter layer occurs and the dissipative effects are described by volume changes, *i.e.*, by the presence of bulk viscosity. As results (Carlevaro and Montani, 2009), the perturbations are founded to be dumped by dissipative processes and the top-down mechanism of structure formation is strongly suppressed. In such a scheme the Jeans mass remain unchanged also in presence of viscosity.

The people involved in this line of research are Nakia Carlevaro and Giovanni Montani

3.1.3 Extended Theories of Gravity

In section "Extended Theories of Gravity", we analyze the dynamical implications of an exponential Lagrangian density for the gravitational field, as referred to an isotropic FRW Universe (Lecian and Montani, 2008). Then, we discuss the features of the generalized deSitter phase, predicted by the new Friedmann equation. The existence of a consistent deSitter solution arises only if the ratio between the vacuum energy density and that associated with the fundamental length of the theory acquires a tantalizing negative character. This choice allows us to explain the present universe dark energy as a relic of the vacuum-energy cancellation due to the cosmological constant intrinsically contained in our scheme. The corresponding scalar-tensor description of the model is addressed too, and the behavior of the scalar field is analyzed for both negative and positive values of the cosmological term. In the first case, the Friedmann equation is studied both in vacuum and in presence of external matter, while, in the second case, the quantum regime is approached in the framework of repulsive properties of the gravitational interaction, as described in recent issues in Loop Quantum Cosmology. In particular, in the vacuum case, we find a pure non-Einsteinian effect, according to which a negative cosmological constant provides an accelerating deSitter dynamics, in the region where the series expansion of the exponential term does not hold.

Furthermore, we analyze the Solar-System constraints imposed on a non-analityc Lagrangian for the gravitational field, whose Taylor expansion does not hold. To this end, the weak-field limit of the model is considered. The parameter space of such a model is analyzed in both Jordan and Einstein frame. In the Einstein frame, those configurations are selected, according to which the potential of the scalar field behaves like an attractor for General Relativity. In the Jordan frame, the request that the effects of such a modified theory be a negligible correction to the Schwarzschild terms at Solar-System scales is fulfilled, as far as experimental evidence is concerned. As a result, we conclude that this kind of model is viable at Solar-System lengths.

The same form of a non-analytical polinomial function of the Ricci scalar is, moreover, analyzed in a cosmological context facing its influence on the dynamics of a pure isotropic Universe. The Riemann tensor is expressed, as usual, using a FRW metric and an ideal fluid is addressed as matter source. Two distinct regimes for the scale factor are analyzed: a power-law evolution and an exponential one. In the first case, the asymptotic Universe behaviors reach qualitatively the standard FRW ones with interesting new features. In particular, we get, as a result, a scale-factor asymptotic evolution like $t^{1/2}$ in the radiation dominated era and a $t^{2/3}$ growth for a matter dominated (flat) Universe. The most interesting feature which arises in this scenario is that, in the second case, an exponential (dark matter) evolution of the scale factor is found to be allowed for a matter dominated era and such a solution strongly depends on the $f(R)$ parameters.

The people involved in this research line are Orchidea Maria Lecian, Giovanni Montani and Nakia Carlevaro.

3.1.4 The interaction between relic neutrinos and primordial gravitational waves

In the section "The interaction between relic neutrinos and primordial gravitational waves" we study the effect of the anisotropic stress generated by free streaming relic neutrinos on the propagation of gravitational waves (Lattanzi and Montani, 2005). In the extremely low frequency region, this acts as an effective viscosity, absorbing gravitational waves and thus resulting in a damping of the B-modes of the Cosmic Microwave Background polarization . We have studied the generalization of this result to other regions of the frequency domain; in particular, we have considered GWs that enter the horizon before the electroweak phase transition (EWPT). This corresponds to an observable frequency today $\nu_0 \gtrsim 10^{-5}$ Hz, i.e., to all waves possibly detectable by interferometers.

In order to study this issue, one has to solve the Boltzmann equation for the phase space density f of cosmological neutrinos. It is found that the intensity of GWs is reduced to $\sim 90\%$ of its value in vacuum (see Fig H.4.1), its exact value depending only on one physical parameter, namely the density fraction of neutrinos. Neither the wave frequency nor the detail of neutrino interaction affect the value of the absorbed intensity, resulting in an universal behaviour in the frequency range considered.

The importance of our results relies in the fact that the damping affects GWs in the frequency range where the LISA space interferometer and future, second generation ground-based interferometers can possibly detect a signal of cosmological origin.

Recently we have refined our treatment of this problem, by using a different formalism, involving a multipole expansion of the Boltzmann equation, that makes the numerical integration of the coupled Einstein-Boltzmann system more feasible, and accounting for the precise thermal history of the neutrino fluid. We find that the abrupt change in the neutrino energy density fraction associated to the electron-positron annihilation occurring at $\sim$ 1 MeV can possibly give rise to a distinct spectral feature in the spectrum of primordial gravitational waves in the nano-hertz region (Benini et al., 2009). Interestingly, this region will be probed in the next years through the so-called Pulsar Timing Arrays (Verbiest et al., 2009).

The people involved in this research line are Riccardo Benini, Massimiliano Lattanzi and Giovanni Montani.

3.1.5 On the coupling between Spin and Cosmological Gravitational Waves

In the section "On the coupling between spin and cosmological gravitational waves" we study the influence of spin on the dynamics of particles and on their interaction with gravitational waves, in a cosmological framework . The equations of motion of spinning particles in the framework of general relativity were derived by Papapetrou in 1951.
We have considered a fluid of collisionless spinning particles in a Friedmann-Robertson-Walker (FRW) background. Considering only the unperturbed background, the distribution function of the fluid evolves in the same way as the spinless case, due to the symmetry proprieties of the metric tensor. Then we turned to consider the effect of small perturbations in the background spacetime. We added a small tensorial perturbation h_{ij} to the metric, looking for a coupling between the fluid and the perturbation itself, representing a gravitaional wave. The resulting Boltzmann equation gives a first order variation of the distribution function that is proportional to the product between the spin tensor and the time derivative of the metric perturbation. We find that, even if the spin alters some components of the anisotropic stress tensor, the final result is that these components are those that don't couple with the evolution of the metric perturbation. This implies that there is not coupling between spin and cosmological gravitational waves, if only tensor perturbations are present (Milillo et al., 2008). As a subsequent step, we have considered how the inclusion of scalar and vector perturbation as well alters the picture described here (Lattanzi et al., 2009). In this case we find that the presence of the spin couples the different kinds of perturbations between them. The next step will be to study if this modified cosmological evolution could possibly have any observational consequence.

The people involved in this line of research are Massimiliano Lattanzi, Irene Milillo and Giovanni Montani.

3.1.6 The role of Plasma Physics in Accretion disk morphology

In section "Accretion Disks", we analyse the two-dimensional MHD configurations characterising the steady state of the accretion disk on a highly magnetised neutron star. The model we describe has a local character and represents the extension of the crystalline structure outlined in Coppi (2005a), dealing with a local model too, when a specific accretion rate is taken into account. We limit our attention to the linearised MHD formulation of the electromagnetic back-reaction characterising the equilibrium, by fixing the structure of the radial, vertical and azimuthal profiles. Since we deal with toroidal currents only, the consistency of the model is ensured by the presence of a small collisional effect, phenomenologically described by a non-zero constant Nernst coefficient (thermal power of the plasma). Such an effect provides a proper balance of the electron force equation via non zero temperature gradients, related directly to the radial and vertical velocity components.

We show that the obtained profile has the typical oscillating feature of the crystalline structure, reconciled with the presence of viscosity, associated to the differential rotation of the disk, and with a net accretion rate. In fact, we provide a direct relation between the electromagnetic reaction of the disk and the (no longer zero) increasing of its mass per unit time. The radial accretion component of the velocity results to be few orders of magnitude below the equatorial sound velocity. Its oscillating-like character does not allow a real matter in-fall to the central object (an effect to be searched into non-linear MHD corrections), but it accounts for the out-coming of steady fluxes, favourable to the ring-like morphology of the disk.

The persons involved in this line of research are Giovanni Montani, Riccardo Benini, Massimiliano Lattanzi, Bruno Coppi (MIT) and Remo Ruffini.

3.1.7 On the gravitational polarizability of black holes.

The *gravitational polarizability* properties of black holes are compared and contrasted with their *electromagnetic polarizability* properties. The "shape" or "height" multipolar Love numbers h_l of a black hole are defined and computed. They are then compared to their electromagnetic analogs h_l^{EM}. The Love numbers h_l give the height of the l-th multipolar "tidal bulge" raised on the horizon of a black hole by faraway masses. We also discuss the shape of the tidal bulge raised by a test mass m, in the limit where m gets very close to the horizon.

The persons involved in this line of research are Thibault Damour (IHES and ICRANet) and Orchidea Maria Lecian (IHES and ICRANet).

3.2 Appendix: The Generic Cosmological Solution

In section "Birth and Development of the Generic Cosmological Solution" we propose a historical review about the generic cosmological solution, from its birth at the end of the 60's, up to the most advanced and recent developments. The review follows a chronological order discussing the most important papers by Vladimir A. Belinski et al, ending with three papers by Giovanni Montani which is the leader of the group of ICRANet people working now on this research line.

3.3 Appendix: Classical Mixmaster

In the section "Classical Mixmaster" the most important results achieved on the classical dynamics of homogeneous model of the type IX of the Bianchi classification are reviewed together with its generalization to the more important topic of the generic cosmological solution. The people involved in this line of research are Riccardo Benini, Giovanni Imponente and Giovanni Montani

Chaos covariance of the Mixmaster model

In "Chaos covariance of the Mixmaster model" we face the study of the subtle question concerning the covariance chaoticity of the Bianchi type VIII and IX model. We introduce the Arnowitt-Deser-Misner formalism for General Relativity, and adopt Misner-Chitre like variables. This way, the time evolution is that of a ball on a billiard characterized by a constant negative curvature. The statistical properties (Kirillov and Montani, 2002) are described using the ensemble representation of the dynamics, while the problem of a correct definition of the Lyapunov exponent in such a relativistic system is resolved adopting a generic time-variable (Imponente and Montani, 2001).

Chaos covariance of the generic cosmological solution

In "Chaos covariance of the generic cosmological solution" the question of covariance is extended to the more general frame of the generic cosmological solution (Benini and Montani, 2004). The problem is reformulated in terms of the Hamilton approach to General Relativity, and Misner-Chitre like variables are adopted. The problem of the dependence of the chaos on the choice of the gauge is solved with a quite general change of coordinates on the space-time manifold, allowing us to solve the super Hamiltonian constraint and the super-momentum one without fixing the forms of the lapse function and of the shift-vector. The analysis developed for the homogeneous Mixmaster model is then extended to this more generic case.

Inhomogeneous inflationary models

In "Inhomogeneous inflationary models" we consider the inflationary scenario as the possible way to interpolate the rich and variegate Kasner dynamics of the Very Early Universe (Imponente and Montani, 2004), in order to reach the present state observable FLRW Universe, via a *bridge solution*. Hence we show how it is possible to have a quasi-isotropic solution of the Einstein equations in presence of the ultrarelativistic matter and a real self-interacting scalar field.

The Role of a Vector Field

In "The Role of a Vector Field" we study the effects of an Abelian vector field on the dynamics of a generic $(n+1)$-dimensional homogeneous model in the BKL scheme; the chaos is restored for any number of dimensions, and a BKL-like map, exhibiting a peculiar dependence on the dimension number, is worked out (Benini et al., 2005). Within the same spirit of the Mixmaster analysis, an unstable n-dimensional Kasner-like evolution arises, nevertheless the potential term inhibits the solution to last up to the singularity and induces the BKL-like transition to another epoch. There are two most interesting features of the resulting dynamics: the map exhibits a *dimensional-dependence*, and it reduces to the standard BKL one for the four-dimensional case.

3.4 Appendix: Perturbation Theory in Macroscopic Gravity: On the Definition of Background

In section "Perturbation Theory in Macroscopic Gravity: On the Definition of Background" the notion of background metric adopted in the perturbation theory in general relativity is analysed and a new definition of background is proposed. An existence theorem for a metric tensor which serves as the background metric for a specific scale has been proven (Montani, 1995). It can be shown that the average value of a tensor field remains invariant under action of the averaging operator introduced in (Kirillov and Montani, 1997). Such an averaging procedure on a space-time manifold provides a natural criterium for a definition of background metric.
A background metric that is invariant with respect to the class of averagings can be introduced, and the following theorem considering the existence of such a metric tensor for a specific scale is proven:
Theorem. Given an averaging space-time procedure with an idempotent averaging kernel of the class of bounded and continuous functions on a space-time manifold $\mathcal{M}$, there always exists a continuous and bounded background metric $g_{\alpha\beta}(x)$ for a characteristic scale $d = V_\Sigma$ where Σ is a compact 4-region of $\mathcal{M}$.

The people involved in this line of research are Roustam Zalaletdinov and Giovanni Montani.

3.5 Appendix: On Schouten's Classification of the non-Riemannian Geometries with an Asymmetric Metric

In section ”On Schouten's Classification of the non-Riemannian Geometries with an Asymmetric Metric”, after reviewing the Schouten classification of non-Riemannian geometries with an asymmetric metric tensor, we find the inverse of the “structure matrix”, which links the generalized connection with all the metric objects, in the linear approximation (Casanova et al., 1999). By adopting this approach for affine-connection geometries with an asymmetric metric, the structure and variety of such geometries can be investigated in a fully-geometrical formalism without adopting any variational principle. The definition of autoparallel trajectories at different approximation orders has been established. Because of the first-order approximation, the asymmetricity object and the antisymmetric part of the non-metricity tensor do not contribute to the determination of the autoparallel trajectory. In this case, the role of torsion and of the antisymmetric part of the metric tensor has to be investigated according to the approximation order. As a physical field, if considered at zeroth order, torsion influences the dynamics by not allowing for a flat Minkowskian metric: in this as, the antisymmetric part of the metric tensor contributes to the determination of the solution only at first order. Contrastingly, if we require that torsion be of order 1, we find out that the antisymmetric part of the metric tensor contribute only at second order(Casanova et al., 2008). The persons involved in this research line are Sabrina Casanova, Orchidea Maria Lecian, Giovanni Montani, Remo Ruffini and Roustam Zalaledtinov.

3.6 Appendix: Approximate Symmetries, Inhomogeneous Spaces and Gravitational Entropy

In section “Approximate Symmetries, Inhomogeneous Spaces and Gravitational Entropy” we treat the problem of finding an appropriate geometrical/physical index for measuring a degree of inhomogeneity for a given spacetime manifold. Interrelations with the problem of understanding the gravitational/ informational entropy are also pointed out. We propose an approach based on the notion of approximate symmetry (Zalaletdinov, 2000): with this respect a definition of a Killing-like symmetry is given and we provide a classification theorem for all possible averaged space-times acquiring such symmetries upon averaging out a space-time with a homothetic Killing symmetry.

The main idea of the Killing-like symmetry is to consider the most general form of deviation from the Killing equations. The expression for such a deviation covers the cases of semi-Killing, almost-Killing and almost symmetries with additional equations. Also covered are standard generalizations of Killing symmetry such as conformal and homothetic Killing vectors. The algebraic classification of the deviation gives an invariant way to introduce a set of scalar indexes measuring the degree of inhomogeneity of the space-time compared with that isometries, or even weaker symmetry (*e.g.*, conformal Killing).

The person involved in this line of research is Roustdam Zalaletdinov.

3.7 Appendix: Gravitational Polarization in General Relativity: Solution to Szekeres' Model of Gravitational Quadrupole

In section "Gravitational Polarization in General Relativity: Solution to Szekeres' Model of Gravitational Quadrupole", we analyze a model for the static weak-field macroscopic medium. In this respect, the equation for the macroscopic gravitational potential is derived: such an equation is found to be a biharmonic equation which is a non-trivial generalization of the Poisson equation of Newtonian gravity (Montani et al., 2000).

In the case of the strong gravitational polarization the equation essentially holds inside a macroscopic matter source: the scheme is equivalent to a system of the Poisson equation and the nonhomogeneous modified Helmholtz equations. The general solution to this system is obtained by using Green's function method and it does not exist a limit to Newtonian gravity. In case of the insignificant gravitational quadrupole polarization, the equation for macroscopic gravitational potential becomes the Poisson equation with the matter density renormalized by the factor including the value of the quadrupole gravitational polarization of the source.

The persons involved in this line of research are Giovanni Montani, Remo Ruffini and Roustdam Zalaletdinov.

3.8 Appendix: Averaging Problem in Cosmology and Macroscopic Gravity

In section "Averaging Problem in Cosmology and Macroscopic Gravity", we discuss the averaging problem using the approach of macroscopic gravity. We start modifying the averaged Einstein equations of macroscopic gravity (*i.e.*, on cosmological scales) by the gravitational correlation tensor terms. Such a correlation tensor satisfies an additional set of structure and field equations. Then we focus on the cosmological solutions for spatially homogeneous and isotropic macroscopic space-times. As a result, we find that, for a flat geometry, the gravitational correlation tensor terms have the form of a spatial curvature term which can be either negative or positive. This scheme exhibits a very non-trivial phenomenon from the point of view of the general-relativistic cosmology: the macroscopic (averaged) cosmological evolution in a flat Universe is governed by the dynamical evolution equations for either a closed or an open Universe depending on the sign of the macroscopic energy-density with a dark spatial curvature term (Montani et al., 2002).

From the observational point of view, such a cosmological model gives a new paradigm to reconsider the standard cosmological interpretation and treatment of the observational data. Indeed, such model has the Riemannian geometry of a flat homogeneous, isotropic space-time and all measurements and data are to be considered and designed for this geometry. The dynamical interpretation of the obtained data should be considered and treated for the cosmological evolution of either a closed or an open spatially homogeneous, isotropic Riemannian space-time.

3.9 Appendix: Astrophysical Topics

In section "Astrophysical Topics" we propose a review of different astrophysical topics by a brief discussion of very important papers by V. A. Belinski et al.

4 Selected Publications Before 2005

4.1 Early Cosmology

1. G. Montani; "On the general behaviour of the universe near the cosmological singularity"; Classical and Quantum Gravity, 12, 2505 (1995).

   In this paper we discuss dynamical features characterizing the oscillatory regime near a spacelike singularity in a generic inhomogeneous cosmological model, the effect of which leads to a profound modification of the asymptotic behaviour toward that singularity, and creates conditions under which the system can evolve into a qualitatively turbulent regime. The well known pointwise 'chaotic' behaviour of the evolution of the gravitational field toward such a singularity is shown to lead to a similarly complicated spatial structure on the spacelike slices which approach it.

2. A.A Kirillov, G. Montani; "Description of statistical properties of the mixmaster universe"; Phys. Rev. D, 56, 6225 (1997).

   Stochastic properties of the homogeneous Bianchi type-VIII and -IX (the mixmaster) models near the cosmological singularity are more distinctive in the Hamiltonian formalism in the Misner-Chitre parametrization. We show how the simplest analysis of the dynamical evolution leads, in a natural way, to the construction of a stationary invariant measure distribution which provides the complete statistical description of the stochastic behavior of these systems. We also establish the difference between the statistical description in the framework of the Misner-Chitre approach and that one based on the BKL (BelinskiKhalatnikovLifshitz) map by means of an explicit reduction of the invariant measure in the continuous case to the measure on the map. It turns out that the invariant measure in the continuous case contains an explicit information about durations of Kasner eras, while the measure in the case of the BKL map does not.

3. G. Imponente, G. Montani; "Covariance of the mixmaster chaoticity"; Phys. Rev. D, 63, 103501 (2001).

We analyze the dynamics of the mixmaster universe on the basis of a standard Arnowitt-Deser-Misner Hamiltonian approach showing how its asymptotic evolution to the cosmological singularity is isomorphic to a billiard ball on the Lobachevsky plane. The key result of our study consists in the temporary gauge invariance of the billiard ball representation, once provided the use of very general Misner-Chitre-like variables.

4. Kirillov, A. A. and Montani, G.; "Quasi-isotropization of the inhomogeneous mixmaster universe induced by an inflationary process"; Phys. Rev. D, 66, 064010 (2002).

   We derive a generic inhomogeneous "bridge" solution for a cosmological model in the presence of a real self-interacting scalar field. This solution connects a Kasner-like regime to an inflationary stage of evolution and therefore provides a dynamical mechanism for the quasi-isotropization of the universe. In the framework of a standard Arnowitt-Deser-Misner Hamiltonian formulation of the dynamics and by adopting Misner-Chitre-like variables, we integrate the Einstein-Hamilton-Jacobi equation corresponding to a "generic" inhomogeneous cosmological model whose evolution is influenced by the coupling with a bosonic field, expected to be responsible for a spontaneous symmetry breaking configuration. The dependence of the detailed evolution of the universe on the initial conditions is then appropriately characterized.

5. Imponente, Giovanni and Montani, Giovanni; "Inhomogeneous de Sitter solution with scalar field and perturbations spectrum"; Mod. Phys. Lett., A19, 1281 (2004).

   We provide an inhomogeneous solution concerning the dynamics of a real self interacting scalar field minimally coupled to gravity in a region of the configuration space where it performs a slow rolling on a plateau of its potential. During the inhomogeneous de Sitter phase the scalar field dominant term is a function of the spatial coordinates only. This solution specialized nearby the FLRW model allows a classical origin for the inhomogeneous perturbations spectrum.

6. Riccardo Benini and Giovanni Montani; "Frame independence of the inhomogeneous mixmaster chaos via Misner-Chitré-like variables"; Physical Review D, 70, 103527 (2004).

   We outline the covariant nature, with respect to the choice of a reference frame, of the chaos characterizing the generic cosmological solution near the initial singularity, i.e., the so-called inhomogeneous mixmaster model. Our analysis is based on a gauge independent Arnowitt-Deser-Misner reduction of the dynamics to the physical degrees of freedom. The resulting picture shows how the inhomogeneous mixmaster model is isomorphic point by point in space to a billiard on a Lobachevsky plane. Indeed, the existence of an asymptotic (energylike) constant of the motion allows one to construct the Jacobi metric associated with the geodesic flow and to calculate a nonzero Lyapunov exponent in each space point. The chaos covariance emerges from the independence of our scheme with respect to the form of the lapse function and the shift vector; the origin of this result relies on the dynamical decoupling of the space points which takes place near the singularity, due to the asymptotic approach of the potential term to infinite walls. At the ground of the obtained dynamical scheme is the choice of Misner-Chitre-like variables which allows one to fix the billiard potential walls.

7. Imponente, G. and Montani, G.; "Bianchi IX chaoticity: BKL map and continuous flow"; Physica A, 338, 282 (2004).

   We analyze the Bianchi IX dynamics (Mixmaster) in view of its stochastic properties; in the present paper we address either the original approach due to Belinski, Khalatnikov and Lifshitz (BKL) as well as a Hamiltonian one relying on the ArnowittDeserMisner (ADM) reduction. We compare these two frameworks and show how the BKL map is related to the geodesic flow associated with the ADM dynamics. In particular, the link existing between the anisotropy parameters and the Kasner indices is outlined.

8. Imponente, G. and Montani, G.; "Covariant Feature of the Mixmaster Model Invariant Measure"; International Journal of Modern Physics D, 11, 1321 (2002).

   We provide a Hamiltonian analysis of the Mixmaster Universe dynamics showing the covariant nature of its chaotic behavior with respect to any choice of time variable. Asymptotically to the cosmological singularity, we construct the appropriate invariant measure for the system (which relies on the appearance of an "energy-like" constant of motion) in such a way that its existence is independent of fixing the time gauge, i.e. the corresponding lapse function. The key point in our analysis consists of introducing generic Misner-Chitr-like variables containing an arbitrary function, whose specification allows us to set up the same statistical scheme in any time gauge.

9. Imponente, Giovanni and Montani, Giovanni; "Covariant Mixmaster Dynamics".

   We provide a Hamiltonian analysis of the Mixmaster Universe dynamics on the base of a standard Arnowitt-Deser-Misner Hamiltonian approach, showing the covariant nature of its chaotic behaviour with respect to the choice of any time variable, from the point of view either of the dynamical systems theory, either of the statistical mechanics one.

10. Imponente, G. and Montani, G.; “On the Quasi-Isotropic Inflationary Solution”; International Journal of Modern Physics D, 12, 1845 (2003).

    In this paper we find a solution for a quasi-isotropic inflationary Universe which allows to introduce in the problem a certain degree of inhomogeneity. We consider a model which generalizes the (flat) FLRW one by introducing a first order inhomogeneous term, whose dynamics is induced by an effective cosmological constant. The 3-metric tensor is constituted by a dominant term, corresponding to an isotropic-like component, while the amplitude of the first order one is controlled by a ”small” function. In a Universe filled with ultra relativistic matter and a real self-interacting scalar field, we discuss the resulting dynamics, up to first order, when the scalar field performs a slow roll on a plateau of a symmetry breaking configuration and induces an effective cosmological constant. We show how the spatial distribution of the ultra relativistic matter and of the scalar field admits an arbitrary form but nevertheless, due to the required inflationary e-folding, it cannot play a serious dynamical role in tracing the process of structures formation (via the Harrison-Zeldovic spectrum). As a consequence, this paper reinforces the idea that the inflationary scenario is incompatible with a classical origin of the large scale structures.

11. Giovanni Montani; “Influence of particle creation on flat and negative curved FLRW universes”; Classical and Quantum Gravity, 18, 193 (2001).

    We present a dynamical analysis of (classical) spatially flat and negative curved Friedmann-Lameître-Robertson-Walker (FLRW) universes evolving (by assumption) close to the thermodynamic equilibrium in the presence of a particle creation process. This analysis is described by means of a realiable phenomenological approach, based on the application to the comoving volume (i.e. spatial volume of unit comoving coordinates) of the theory for open thermodynamic systems. In particular we show how, since the particle creation phenomenon induces a negative pressure term, then the choice of a well-grounded ansatz for the time variation of the particle number, leads to a deep modification of the very early standard FLRW dynamics. More precisely, for the considered FLRW models, we find (in addition to the limiting case of their standard behaviour) solutions corresponding to an early universe characterized respectively by an ‘eternal’ inflationary-like birth and a spatial curvature dominated singularity. In both these cases the so-called horizon problem finds a natural solution.

4.2 Fundamental General Relativity

1. G Montani, R Ruffini and R Zalaletdinov; "The gravitational polarization in general relativity: solution to Szekeres' model of quadrupole polarization"; Classical and Quantum Gravity, 20, 4195 (2003).

   A model for the static weak-field macroscopic medium is analysed and the equation for the macroscopic gravitational potential is derived. This is a biharmonic equation which is a non-trivial generalization of the Poisson equation of Newtonian gravity. In the case of strong gravitational quadrupole polarization, it essentially holds inside a macroscopic matter source. Outside the source the gravitational potential fades away exponentially. The equation is equivalent to a system of the Poisson equation and the non-homogeneous modified Helmholtz equations. The general solution to this system is obtained by using the Green function method and it is not limited to Newtonian gravity. In the case of insignificant gravitational quadrupole polarization, the equation for macroscopic gravitational potential becomes the Poisson equation with the matter density renormalized by a factor including the value of the quadrupole gravitational polarization of the source. The general solution to this equation obtained by using the Green function method is limited to Newtonian gravity.

2. Bisnovatyi-Kogan, G. S. and Lovelace, R. V. E. and Belinski, V. A.; "A Cosmic Battery Reconsidered"; ApJ, 580 (2002).

   We revisit the problem of magnetic field generation in accretion flows onto black holes owing to the excess radiation force on electrons. This excess force may arise from the Poynting-Robertson effect. Instead of a recent claim of the generation of dynamically important magnetic fields, we establish the validity of earlier results from 1977 that show that only small magnetic fields are generated. The radiative force causes the magnetic field to initially grow linearly with time. However, this linear growth holds for only a restricted time interval that is of the order of the accretion time of the matter. The large magnetic fields recently found result from the fact that the linear growth is unrestricted. A model of the Poynting-Robertson magnetic field generation close to the horizon of a Schwarzschild black hole is solved exactly using general relativity, and the field is also found to be dynamically insignificant. These weak magnetic fields may however be important as seed fields for dynamos.

3. Barkov, M. V. and Belinski, V. A. and Bisnovatyi-Kogan, G. S.; "Model of ejection of matter from non-stationary dense stellar clusters and chaotic motion of gravitating shells"; arXiv:astro-ph/0107051.

It is shown that during the motion of two initially gravitationally bound spherical shells, consisting of point particles moving along ballistic trajectories, one of the shells may be expelled to infinity at subrelativistic speed $v_{exp} \leq 0.25c$. The probelm is solved in Newtonian gravity. Motion of two intersecting shells in the case when they do not runaway shows a chaotic behaviour. We hope that this toy and oversimplified model can nevertheless give a qualitative idea on the nature of the mechanism of matter outbursts from dense stellar clusters.

4. Zalaletdinov, R. M.; "Averaging out the Einstein equations"; General Relativity and Gravitation, 24, 1015 (1992).

   A general scheme to average out an arbitrary 4-dimensional Riemannian space and to construct the geometry of the averaged space is proposed. It is shown that the averaged manifold has a metric and two equi-affine symmetric connections. The geometry of the space is characterized by the tensors of Riemannian and non-Riemannian curvatures, an affine deformation tensor being the result of non-metricity of one of the connections. To average out the differential Bianchi identities, correlation 2-form, 3-form and 4-form are introduced and the differential relations on these correlations tensors are derived, the relations being integrable on an arbitrary averaged manifold. Upon assuming a splitting rule for the average of the product including a covariantly constant tensor, an averaging out of the Einstein equations has been carried out which brings additional terms with the correlation tensors into them. As shown by averaging out the contracted Bianchi identities, the equations of motion for the averaged energy-momentum tensor do also include the geometric correction terms. Considering the gravitational induction tensor to be the Riemannian curvature tensor (then the non-Riemannian one is the macroscopic gravitational field), a theorem that relates the algebraic structure of the averaged microscopic metric with that of the induction tensor is proved. Due to the theorem the same field operator as in the Einstein equations is manifestly extracted from the averaged ones. Physical interpretation and application of the relations and equations obtained to treat macroscopic gravity are discussed.

5. Mars, M. and Zalaletdinov, R. M.; "Space-time averages in macroscopic gravity and volume-preserving coordinates"; Journal of Mathematical Physics, 38, 4741 (1997).

   The definition of the covariant space-time averaging scheme for the objects (tensors, geometric objects, etc.) on differentiable metric manifolds with a volume n-form, which has been proposed for the formulation of macroscopic gravity, is analyzed. An overview of the space-time averaging procedure in Minkowski space-time is given and comparison between this averaging scheme and that adopted in macroscopic gravity is carried out throughout the paper. Some new results concerning the algebraic structure of the averaging operator are precisely formulated and proved, the main one being that the averaging bilocal operator is idempotent iff it is factorized into a bilocal product of a matrix-valued function on the manifold, taken at a point, by its inverse at another point. The previously proved existence theorems for the averaging and coordination bilocal operators are revisited with more detailed proofs of related results. A number of new results concerning the structure of the volume-preserving averaging operators and the class of proper coordinate systems are given. It is shown, in particular, that such operators are defined on an arbitrary n-dimensional differentiable metric manifold with a volume n-form up to the freedom of (n1) arbitrary functions of n arguments and 1 arbitrary function of (n1) arguments. All the results given in this paper are also valid whenever appropriate for affine connection manifolds including (pseudo)-Riemannian manifolds.

6. Montani, G. and Ruffini, R. and Zalaletdinov, R.; "Gravitating macroscopic media in general relativity and macroscopic gravity"; Nuovo Cimento B, 115, 1343 (2000).

   The problem of construction of a continuous (macroscopic) matter model for a given point-like (microscopic) matter distribution in general relativity is formulated. The existing approaches are briefly reviewed and a physical analogy with the similar problem in classical macroscopic electrodynamics is pointed out. The procedure by Szekeres in the linearized general relativity on Minkowski background to construct a tensor of gravitational quadruple polarization by applying Kaufman's method of molecular moments for derivation of the polarization tensor in macroscopic electrodynamics and to derive an averaged field operator by utilizing an analogy between the linearized Bianchi identities and Maxwell equations, is analyzed. It is shown that the procedure has some inconsistencies, in particular, it has only provided the terms linear in perturbations for the averaged field operator which do not contribute into the dynamics of the averaged field, and the analogy between electromagnetism and gravitation does break upon averaging. A macroscopic gravity approach in the perturbation theory up to the second order on a particular background space-time taken to be a smooth weak gravitational field is applied to write down a system of macroscopic field equations: Isaacson's equations with a source incorporating the quadruple gravitational polarization tensor, Isaacson's energy-momentum tensor of gravitational waves and energy-momentum tensor of gravitational molecules and corresponding equations of motion. A suitable set of material relations which relate all the tensors is proposed.

7. Montani, G. and Ruffini, R. and Zalaletdinov, R.; "Modelling self-gravitating macroscopic media in general relativity: Solution to Szekeres' model of gravitational quadrupole"; Nuovo Cimento B, 118, 1109 (2003).

   A model for the static weak-field macroscopic medium is analyzed and the equation for the macroscopic gravitational potential is derived. This is a biharmonic equation which is a non-trivial generalization of the Poisson equation of Newtonian gravity. In case of the strong gravitational quadrupole polarization it essentially holds inside a macroscopic matter source. Outside the source the gravitational potential fades away exponentially. The equation is equivalent to a system of the Poisson equation and the nonhomogeneous modified Helmholtz equations. The general solution to this system is obtained by using Green's function method and it does not have a limit to Newtonian gravity. In case of the insignificant gravitational quadrupole polarization the equation for macroscopic gravitational potential becomes the Poisson equation with the matter density renormalized by the factor including the value of the quadrupole gravitational polarization of the source. The general solution to this equation obtained by using Green's function method has a limit to Newtonian gravity.

8. Montani, G. and Ruffini, R. and Zalaletdinov, R.; "Gravitating macroscopic media in general relativity and macroscopic gravity"; Nuovo Cimento B, 115, 1343 (2002).

   The problem of construction of a continuous (macroscopic) matter model for a given point-like (microscopic) matter distribution in general relativity is formulated. The existing approaches are briefly reviewed and a physical analogy with the similar problem in classical macroscopic electrodynamics is pointed out. The procedure by Szekeres in the linearized general relativity on Minkowski background to construct a tensor of gravitational quadruple polarization by applying Kaufman's method of molecular moments for derivation of the polarization tensor in macroscopic electrodynamics and to derive an averaged field operator by utilizing an analogy between the linearized Bianchi identities and Maxwell equations, is analyzed. It is shown that the procedure has some inconsistencies, in particular, it has only provided the terms linear in perturbations for the averaged field operator which do not contribute into the dynamics of the averaged field, and the analogy between electromagnetism and gravitation does break upon averaging. A macroscopic gravity approach in the perturbation theory up to the second order on a particular background space-time taken to be a smooth weak gravitational field is applied to write down a system of macroscopic field equations: Isaacson's equations with a source incorporating the quadruple gravitational polarization tensor, Isaacson's energy-momentum tensor of gravitational waves and energy-momentum tensor of gravitational molecules and corresponding equations of motion. A suitable set of material relations which relate all the tensors is proposed.

9. Zalaletdinov, R. M.; "Towards a theory of macroscopic gravity"; General Relativity and Gravitation, 25, 673 (1993).

   By averaging out Cartan's structure equations for a four-dimensional Riemannian space over space regions, the structure equations for the averaged space have been derived with the procedure being valid on an arbitrary Riemannian space. The averaged space is characterized by a metric, Riemannian and non-Rimannian curvature 2-forms, and correlation 2-, 3- and 4-forms, an affine deformation 1-form being due to the non-metricity of one of two connection 1-forms. Using the procedure for the space-time averaging of the Einstein equations produces the averaged ones with the terms of geometric correction by the correlation tensors. The equations of motion for averaged energy momentum, obtained by averaging out the contracted Bianchi identities, also include such terms. Considering the gravitational induction tensor to be the Riemannian curvature tensor (the non-Riemannian one is then the field tensor), a theorem is proved which relates the algebraic structure of the averaged microscopic metric to that of the induction tensor. It is shown that the averaged Einstein equations can be put in the form of the Einstein equations with the conserved macroscopic energy-momentum tensor of a definite structure including the correlation functions. By using the high-frequency approximation of Isaacson with second-order correction to the microscopic metric, the self-consistency and compatibility of the equations and relations obtained are shown. Macrovacuum turns out to be Ricci non-flat, the macrovacuum source being defined in terms of the correlation functions. In the high-frequency limit the equations are shown to become Isaacson's ones with the macrovauum source becoming Isaacson's stress tensor for gravitational waves.

5 Selected Publications (2005-2009)

1. R Benini, A A Kirillov and Giovanni Montani; "Oscillatory regime in the multidimensional homogeneous cosmological models induced by a vector field"; Classical and Quantum Gravity, 22, 1483 (2005).

   We show that in multidimensional gravity, vector fields completely determine the structure and properties of singularity. It turns out that in the presence of a vector field the oscillatory regime exists in all spatial dimensions and for all homogeneous models. By analyzing the Hamiltonian equations we derive the Poincaré return map associated with the Kasner indexes and fix the rules according to which the Kasner vectors rotate. In correspondence to a four-dimensional spacetime, the oscillatory regime here constructed overlaps the usual Belinski-Khalatnikov-Liftshitz one.

2. Nakia Carlevaro and Giovanni Montani; "On the gravitational collapse of a gas cloud in the presence of bulk viscosity; Classical and Quantum Gravity, 22, 4715 (2005).

   We analyse the effects induced by the bulk (or second) viscosity on the dynamics associated with the extreme gravitational collapse. The aim of the work is to investigate whether the presence of viscous corrections to the evolution of a collapsing gas cloud influences the top-down fragmentation process. To this end, we generalize the approach presented by Hunter (1962 Astrophys. J. 136 594) to include in the dynamics of the (uniform and spherically symmetric) cloud the negative pressure contribution associated with the bulk viscosity phenomenology. Within the framework of a Newtonian approach (whose range of validity is outlined), we extend to the viscous case either the Lagrangian or the Eulerian motion of the system addressed in Hunter (1962 Astrophys. J. 136 594) and we treat the asymptotic evolution. We show how the adiabatic-like behaviour of the gas is deeply influenced by viscous correction when its collapse reaches the extreme regime toward the singularity. In fact, for sufficiently large viscous contributions, density contrasts associated with a given scale of the fragmentation process acquire, asymptotically, a vanishing behaviour which prevents the formation of sub-structures. Since in the non-dissipative case density contrasts diverge (except for the purely adiabatic behaviour in which they remain constant), we can conclude that in the adiabatic-like collapse the top-down mechanism of structure formation is suppressed as soon as enough strong viscous effects are taken into account. Such a feature is not present in the isothermal-like collapse because the sub-structure formation is yet present and outlines the same behaviour as in the non-viscous case. We emphasize that in the adiabatic-like collapse the bulk viscosity is also responsible for the appearance of a threshold scale (dependent on the polytropic index) beyond which perturbations begin to increase; this issue, absent in the non-viscous case, is equivalent to dealing with a Jeans length. A discussion of the physical character that the choice n = 5/6 takes place in the present case is provided.

3. Carlevaro, N. and Montani, G.; "Bulk Viscosity Effects on the Early Universe Stability"; Modern Physics Letters A, 20, 1729 (2005).

   We present a discussion of the effects induced by the bulk viscosity on the very early Universe stability. The matter filling the cosmological (isotropic and homogeneous) background is described by a viscous fluid having an ultrarelativistic equation of state and whose viscosity coefficient is related to the energy density via a power-law of the form $\zeta = \zeta_0 \rho^\nu$. The analytic expression of the density contrast (obtained for $\nu = 1/2$) shows that, for small values of the constant ζ_0, its behavior is not significantly different from the non-viscous one derived by Lifshitz. But as soon as ζ_0 overcomes a critical value, the growth of the density contrast is suppressed forward in time by the viscosity and the stability of the Universe is favored in the expanding picture. On the other hand, in such a regime, the asymptotic approach to the initial singularity (taken at $t = 0$) is deeply modified by the apparency of significant viscosity in the primordial thermal bath, i.e. the isotropic and homogeneous Universe admits an unstable collapsing picture. In our model this feature also regards scalar perturbations while in the non-viscous case it appears only for tensor modes.

4. Lattanzi, M. and Montani, G.; "On the Interaction Between Thermalized Neutrinos and Cosmological Gravitational Waves above the Electroweak Unification Scale"; Modern Physics Letters A, 20, 2607 (2005).

   We investigate the interaction between the cosmological relic neutrinos, and primordial gravitational waves entering the horizon before the electroweak phase transition, corresponding to observable frequencies today $\nu_0 > 10^{-5} Hz$. We give an analytic formula for the traceless transverse part of the anisotropic stress tensor, due to weakly interacting neutrinos, and derive an integro-differential equation describing the propagation of cosmological gravitational waves at these conditions.

We find that this leads to a decrease of the wave intensity in the frequency region accessible to the LISA space interferometer, that is at the present the most promising way to obtain a direct detection of a cosmological gravitational wave. The absorbed intensity does not depend neither on the perturbation wavelength, nor on the details of neutrino interactions, and is affected only by the neutrino fraction f_ν. The transmitted intensity amounts to 88% for the standard value $f_\nu = 0.40523$. An approximate formula for non-standard values of f_ν is given.

5. Carlevaro, Nakia and Montani, Giovanni; "Study of the Quasi-isotropic Solution near the Cosmological Singularity in Presence of Bulk-Viscosity"; International Journal of Modern Physics D, 17(6), 881 (2008).

   We analyze the dynamical behavior of a quasi-isotropic Universe in the presence of a cosmological fluid endowed with bulk viscosity. We express the viscosity coefficient as a power-law of the fluid energy density: $\zeta = \zeta_0 \epsilon^s$. Then we fix $s = 1/2$ as the only case in which viscosity plays a significant role in the singularity physics but does not dominate the Universe dynamics (as requested by its microscopic perturbative origin). The parameter ζ_0 is left free to define the intensity of the viscous effects. Following the spirit of the work by E.M. Lifshitz and I.M. Khalatnikov on the quasi-isotropic solution, we analyze both Einstein and hydrodynamic equations up to first and second order in time. As a result, we get a power-law solution existing only in correspondence to a restricted domain of ζ_0.

6. O.M. Lecian, G. Montani; "Implications of non-analytical f(R) gravity at Solar-System scales"; Class.Quant.Grav.26:045014,2009. e-Print: arXiv:0807.4428 [gr-qc]

   We motivate and analyze the weak-field limit of a non-analytical Lagrangian for the gravitational field. After investigating the parameter space of the model, we impose constraints on the parameters characterizing this class of theories imposed by Solar-System data, i.e. we establish the validity range where this solution applies and refine the constraints by the comparison with planetary orbits. As a result, we claim that this class of models is viable within different astrophysical scales.

7. G. Montani, M.V. Battisti, R. Benini and G. Imponente; "Classical and Quantum Features of the Mixmaster Singularity"; International Journal of Modern Physics A,23, 2353 (2008).

   This review article is devoted to analyze the main properties characterizing the cosmological singularity associated to the homogeneous and inhomogeneous Mixmaster model. After the introduction of the main tools required to treat the cosmological issue, we review in details the main results got along the last forty years on the Mixmaster topic. We firstly assess the classical picture of the homogeneous chaotic cosmologies and, after a presentation of the canonical method for the quantization, we develop the quantum Mixmaster behavior. Finally, we extend both the classical and quantum features to the fully inhomogeneous case. Our survey analyzes the fundamental framework of the Mixmaster picture and completes it by accounting for recent and peculiar outstanding results.

8. Carlevaro N. and Montani G.; "Jeans instability in presence of dissipative effects"; International Journal of Modern Physics D, 18, 1257 (2009).

   An analysis of the gravitational instability in presence of dissipative effects is addressed. In particular, the standard Jeans Mechanism and the generalization in treating the Universe expansion are both analyzed when bulk viscosity affects the first-order Newtonian dynamics. As results, the perturbations evolution is founded to be dumped by dissipative processes and the top-down mechanism of structure formation is suppressed. In such a scheme, the Jeans Mass remain unchanged also in presence of viscosity.

9. O. M. Lecian, G. Montani; "Exponential Lagrangian for the gravitational field and the problem of vacuum energy"; International Journal of Moder Physics A, 23(8), 1248 (2008).

   We will analyze two particular features of an exponential gravitational Lagrangian. On the one hand, while this choice of the Lagrangian density allows for two free parameters, only one scale, the cosmological constant, arises as fundamental when the proper Einsteinian limit is to be recovered. On the other hand, the vacuum energy arising from $f(R)$ theories such that $f(0) \neq 0$ needs a cancellation mechanism, by which the present value of the cosmological constant can be recast.

10. O. M. Lecian, G. Montani; "Dark energy as a relic of the vacuum-energy cancellation?" International Journal of Moder Physics D, 17, 111-133,(2008).

    We analyze the dynamical implications of an exponential Lagrangian density for the gravitational field, as referred to an isotropic FRW Universe. Then, we discuss the features of the generalized deSitter phase, predicted by the new Friedmann equation. The existence of a consistent deSitter solution arises only if the ratio between the vacuum-energy density and that associated with the fundamental length of the theory acquires a tantalizing negative character. This choice allows us to explain the present universe dark energy as a relic of the vacuum-energy cancellation due to the cosmological constant intrinsically contained in our scheme. The corresponding scalar-tensor description of the model is addressed too, and the behavior of the scalar field is analyzed for both negative and positive values of the cosmological term. In the first case, the Friedmann equation is studied both in vacuum and in presence of external matter, while, in the second case, the quantum regime is approached in the framework of ”repulsive” properties of the gravitational interaction, as described in recent issues in Loop Quantum Cosmology. In particular, in the vacuum case, we find a pure non-Einsteinian effect, according to which a negative cosmological constant provides an accelerating deSitter dynamics, in the region where the series expansion of the exponential term does not hold

11. I. Milillo, M. Lattanzi and G. Montani; “On the coupling between spinning particles and cosmological gravitational waves”; International Journal of Moder Physics A, 23(8), 1248 (2008).

    The influence of spin in a system of classical particles on the propagation of gravitational waves is analyzed in the cosmological context of primordial thermal equilibrium. On a flat Friedmann-Robertson-Walker metric, when the precession is neglected, there is no contribution due to the spin to the distribution function of the particles. Adding a small tensor perturbation to the background metric, we study if a coupling between gravitational waves and spin exists that can modify the evolution of the distribution function, leading to new terms in the anisotropic stress, and then to a new source for gravitational waves. In the chosen gauge, the final result is that, in the absence of other kind of perturbations, there is no coupling between spin and gravitational waves.

12. R. Benini, M. Lattanzi and G. Montani; “Signatures of the neutrino thermal history in the spectrum of primordial gravitational waves’; Gen. Rel. Grav., in press

    The stochastic background of primordial gravitational waves is a target for several experiments, spanning many orders of magnitude in frequency. At large wavelengths, it can be detected in the B-modes of the cosmic microwave background. At small scales, it is the target of detectors like LISA and can be probed by Pulsar Timing Arrays. Gravitational waves interact with the anisotropic stress of the cosmological fluid, mainly given by its neutrino component. While at the large scales this is effectively collisionless, in the small-scale limit, the collisions between neutrinos should be properly taken into account for the computation of the anisotropic stress. In this talk we will present the results of the integration of the Einstein-Boltzmann system of differential equations for the coupled gravitational wave-neutrino system, and we will discuss how this will affect the expected signal of stochastic gravitational waves.

13. G. Montani and R. Benini; "Linear Two-Dimensional MHD of Accretion Disks: Crystalline structure and Nernst coefficient"; to appear on Modern Physics Letters A (2009)

    We analyse the two-dimensional MHD configurations characterising the steady state of the accretion disk on a highly magnetised neutron star. The model we describe has a local character and represents the extension of the crystalline structure outlined in Coppi (2005), dealing with a local model too, when a specific accretion rate is taken into account. We limit our attention to the linearised MHD formulation of the electromagnetic back-reaction characterising the equilibrium, by fixing the structure of the radial, vertical and azimuthal profiles. Since we deal with toroidal currents only, the consistency of the model is ensured by the presence of a small collisional effect, phenomenologically described by a non-zero constant Nernst coefficient (thermal power of the plasma). Such an effect provides a proper balance of the electron force equation via non zero temperature gradients, related directly to the radial and vertical velocity components. We show that the obtained profile has the typical oscillating feature of the crystalline structure, reconciled with the presence of viscosity, associated to the differential rotation of the disk, and with a net accretion rate. In fact, we provide a direct relation between the electromagnetic reaction of the disk and the (no longer zero) increasing of its mass per unit time. The radial accretion component of the velocity results to be few orders of magnitude below the equatorial sound velocity. Its oscillating-like character does not allow a real matter in-fall to the central object (an effect to be searched into non-linear MHD corrections), but it accounts for the outcoming of steady fluxes, favourable to the ring-like morphology of the disk.

H.1 Dissipative Cosmology

With respect to this line of research, peculiar topics concerning the dynamics of the gravitational collapses are developed both in the Newtonian approach and in the pure relativistic limit, including dissipative effects mainly reassumed by the presence of viscosity. The physical interest in dealing with dissipative dynamics is related to thermodynamical properties of the analyzed system. In fact, both the extreme regime of a gravitational collapse and the very early stages of the Universe evolution are characterized by a thermal history which can not be regarded as settled down into the equilibrium. At sufficiently high temperatures, the cross sections of the micro-physical processes are no longer able to restore the thermodynamical equilibrium. Thus, stages where the expansion and collapse induce non-equilibrium phenomena are generated. The average effect of having such kind of micro-physics results into dissipative processes appropriately described by the presence of *bulk viscosity* ζ, phenomenologically described as a function of the energy density ρ in terms of a power-law as

$$\zeta = \zeta_0 \rho^s \,, \qquad \zeta_0,\, s = const \,. \tag{H.1.0.1}$$

In this approach, this kind of viscosity affects the form of the energy-momentum tensor with a corrective term:

$$T_{\mu\nu} = (\tilde{p} + \rho) u_\mu u_\nu - \tilde{p}\, g_{\mu\nu} \,, \qquad \tilde{p} = p - \zeta\, u^\rho_{;\rho} \,, \tag{H.1.0.2}$$

where p denotes the usual thermostatic pressure.
The analysis is focused on three main models:
(i) *Perturbed FRW-Universe*
We present a discussion of the effects induced by the bulk viscosity on the very early Universe stability Carlevaro and Montani (2005a), Carlevaro and Montani (2007). The matter filling the cosmological isotropic and homogeneous background is described by a viscous fluid having an ultra-relativistic equation of state (*i.e.*, $p = \rho/3$). The analytic expression of the density contrast, obtained for $s = 1/2$ (*i.e.*, in order to deal with the maximum effect that bulk viscosity can have without dominating the dynamics), shows two different dynamical regimes characterized by intensity of the viscous effects related to the critical value

$$\zeta_0^* = \frac{2}{9\sqrt{3}} \,. \tag{H.1.0.3}$$

In the case $0 \leqslant \zeta_0 < \zeta_0^*$, perturbations increase forward in time. This behavior corresponds qualitatively to the same picture of the non-viscous Universe (obtained setting $\zeta_0 = 0$) in which the expansion can not imply the gravitational instability. In the case $\zeta_0^* < \zeta_0$, the density contrast is suppressed since it behaves like negative powers of the time variable. When the density contrast results to be increasing, the presence of viscosity induces a *damping* of the perturbation evolution in the direction of the expanding Universe. In this regime, density fluctuations decrease forward in time, but the most interesting result is the instability that the isotropic and homogeneous Universe acquires in the direction of the collapse toward the Big-Bang: the density contrast diverges approaching the cosmological singularity, thus scalar perturbations destroy asymptotically the primordial Universe symmetry.

The dynamical implication of these issues is that an isotropic and homogeneous stage of the Universe can not be generated, from generic initial conditions, as far as the viscosity becomes smaller than the critical value, *i.e.*, $\zeta_0 < \zeta_0^*$.

(ii) *Quasi-isotropic model*
In 1963, E.M. Lifshitz and I.M. Khalatnikov first proposed this model which is based on the idea that, as a function of time, the 3-metric is expandable in powers of t, *i.e.*, a Taylor expansion of the spatial metric is addressed. In Carlevaro and Montani (2008) we propose a generalization of the line element in order to include dissipative effects:

$$\gamma_{\alpha\beta} = t^x\, a_{\alpha\beta} + t^y\, b_{\alpha\beta}\,, \qquad \gamma^{\alpha\beta} = t^{-x}\, a^{\alpha\beta} - t^{y-2x}\, b^{\alpha\beta}\,, \tag{H.1.0.4}$$

where $x > 0$ (constraint for the space contraction) and $y > x$ (consistence of the perturbation scheme). In this approach, the pure Friedmann model becomes a particular case of a larger class of solutions existing only for space filled with matter. In the analysis, the viscous exponent is fixed $s = 1/2$ as the only case in which viscosity plays a significant role in the singularity physics. The parameter ζ_0 is left free to define the intensity of the viscous effects.

Following the spirit of the LK's work, both Einstein and hydrodynamic equations, up to first- and second-order in time, are analyzed. A power-law solution exists only in correspondence to a restricted domain of ζ_0. In fact, the consistence of the perturbation scheme, *i.e.*, $y > x$, yields the validity constraint

$$\zeta_0 < 3\,\zeta_0^*\,, \tag{H.1.0.5}$$

in agreement with the results obtained for the pure isotropic model.

(iii) *Extreme gravitational collapse of a gas cloud*
Aim of this analysis Carlevaro and Montani (2007), Carlevaro and Montani (2005b) is to investigate whether the presence of viscous corrections to the evolution of a collapsing gas cloud can influence the top-down fragmentation process. To this end, a generalization of the approach firstly presented by C. Hunter is developed in order to include the negative pressure contri-

bution associated to the bulk viscosity phenomenology in the dynamics of the (uniform and spherically symmetric) cloud. Within the framework of a Newtonian approach, both the Lagrangian, and the Eulerian equation of motion of the system are extended to the viscous case. We construct such an extension requiring that the asymptotic dynamics of the collapsing cloud is not qualitatively affected by the presence of viscosity: in this respect, we can assume the viscous exponent as $s = 5/6$.

The adiabatic-like behavior of the gas (*i.e.*, when the politropic index γ takes values $4/3 < \gamma \leqslant 5/3$) is deeply influenced by viscous corrections when its collapse reaches the extreme regime towards the singularity. In fact, for sufficiently large viscous contributions, density contrasts acquire, asymptotically, a vanishing behavior that prevents the formation of sub-structures. Since, in the non-dissipative case, density contrasts diverge (except for the purely adiabatic behavior $\gamma = 5/3$ in which they remain constant), in the adiabatic-like collapse the top-down mechanism of the structure formation is suppressed as soon as enough strong viscous effects are taken into account. Such a feature is not present in the isothermal-like case (*i.e.*, $1 \leqslant \gamma < 4/3$).

In the adiabatic-like collapse the bulk viscosity is also responsible for the appearance of a threshold scale (dependent on the politropic index),

$$k_C^2 = f(\gamma)\, \rho_0^{1/6} \;/\; \zeta_0 \,, \tag{H.1.0.6}$$

beyond which perturbations begin to increase; this issue, absent in the non-viscous case, is equivalent to deal with a Jeans length.

(iv) *Lemaitre-Tolman-Bondi Solution*
If the Lemaitre-Tolman-Bondi Solution is addressed, we deal with anisotropic corrections to the pure FRW model:

$$ds^2 = dt^2 - e^{2\alpha(r,t)}\, dr^2 - e^{2\beta(r,t)}\, (d\theta^2 - \sin^2\theta\, d\varphi^2)\,. \tag{H.1.0.7}$$

In this picture, using comoving coordinates, the energy-momentum tensor assumes a more complicated form since we are also in presence of shear viscosity related to such anisotropies:

$$\begin{aligned} T_\mu^\nu = {} & \epsilon\,(w+1)\, u_\mu u^\nu - w\,\epsilon\, \delta_\mu^\nu + (\zeta - \tfrac{2}{3}\,\eta)\, u^\rho_{;\rho}\, (\delta_\mu^\nu - u_\mu u^\nu) + \\ & + \eta\, (u_\mu^{;\nu} + u^\nu_{;\mu} - u^\nu u^\rho u_{\mu;\rho} - u_\mu u^\rho u^\nu_{;\rho})\,, \end{aligned} \tag{H.1.0.8}$$

where $w = p/\epsilon$ and p is the thermostatic pressure while ϵ the energy density. The viscosity coefficients are expressed by: $\zeta = \zeta_0\, \epsilon^s$ and $\eta = \eta_0\, \epsilon^q$. A system for the Einstein equations can be found and our purpose is to integrate such e system in correspondence of the asymptotic limit:

$$\epsilon \to \infty : \quad 0 \,\leq\, s \,<\, 1/2\,, \qquad q \,\geq\, 1/2 \,+\, s \tag{H.1.0.9}$$

and

$$\epsilon \to 0: \quad s \geq 1 \qquad q \geq 1 \,. \tag{H.1.0.10}$$

Effects of matter creation Another specific research line deals with the study of the early cosmological singularity proposed in the scheme of matter creation (Montani and Nescatelli, 2008). A wide number of different proposals exist about the extreme physics characterizing early cosmology. Among such proposals, the attention is focused on those scenarios for which it is expected that the Universe has been created as a vacuum fluctuation, thus the study of the particle creation should be added for a complete analysis of its dynamics. The aim of the research line is to include in the work by E.M. Lifshitz and I.M. Khalatnikov on the gravitational stability a term of matter creation to study how it influences the Universe dynamics and its stability near the cosmological singularity.

A reliable framework to describe such a phenomenon was provided by Y. Prigogine, who proposed to apply the thermodynamics of the open systems to deal with the variation of particle number. Successively, this theory was extended to the case of flat or negative FLRW Universe by fixing a suitable *ansatz* for the particle creation rate (Montani, 2001). In this scheme, the effect of dealing with a time varying particle number is summarized by an additional negative pressure term, having the form of a power-law in the energy density. This negative pressure term leads to a re-interpretation of the stress-energy tensor. Particle creation, which comes out from the rapid time variation of the gravitational field, can explain both an increase of the entropy of the Universe and a remarkable stability compared with the one of the Cosmological Standard Model.

In order to analyze the Universe stability, it is necessary to start studying a cosmological fluid with an "*ad hoc*" choice of the parameter β, which controls the rate of particle creation (*i.e.*, $\beta = 1/2$) and this way a complete phenomenological scheme can be addressed. The advantage to use such special value of β consists in the analytical integrability of the (zeroth-order) Friedmann equation. The general case for the *ansatz* is furthermore considered, retaining only the zeroth-order term of an expansion in the energy density. Both cases indicate that the Universe is clearly stable in the direction of expansion as in the Standard Model. On the other hand, we find, as a crucial result, an instability backward in time which does not appear in the Lifshitz model.

We can conclude that the Universe cannot be created like an isotropic system and only after a certain time it becomes close to our usual conception of isotropy. In this respect, this analysis encourages the idea of an early Universe as characterized by a certain degree of anisotropy and inhomogeneity. The natural backward evolution of the model here presented is expected to be that of the so called Mixmaster Universe. Such a homogeneous model is characterized by an oscillatory regime which, on the horizon scale, survives also in the generic inhomogeneous solution.

H.2 On the Jeans instability of gravitational perturbations

The Universe is uniform at big scales but many concentrations at small scales are presented, *i.e.*, galaxies and clusters, where the mass density is larger than the Universe mean density. These mass agglomerates are due to the gravitational instability: if density perturbations are generated in a certain volume, the gravitational forces act contracting this volume allowing a gravitational collapse. The only forces which contrast such gravitational contraction are the pressure ones which act in order to maintain uniform the energy density. The Jeans Mechanism analyzes what are the conditions for which perturbations become unstable to the gravitational collapse.

The Jeans Model describes the time evolution of small fluctuation in a static homogeneous and isotropic fluid. This model is based on a Newtonian approach and the effects of the expanding Universe are neglected. As a result, density perturbations are found to follow an exponential collapse or a pure oscillatory regime depending on their initial scale (or their mass), if an ideal fluid is addressed. The transition between such two different regimes is regulated by a threshold value of the perturbation mass: the *Jeans Mass*.

We are aimed Carlevaro and Montani (2009) to consider dissipative effect in the fluid dynamics. In particular we introduce in the first-order analysis the so-called bulk viscosity (we neglect the shear viscosity since we are dealing with homogeneous model and no internal frictions arise). Such a viscosity can be expressed in terms of the thermodynamical parameters of the fluid. In the homogeneous models, this quantity depends only on time, and therefore we may consider it as a function of the Universe energy density ρ: $\zeta = \zeta_o \, \rho^{\,s}$ where $s = const$ and ζ_o is a parameter which defines the intensity of viscous effects.

If homogeneous matter is treated, a viscous fluid is described by the system

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \,, \tag{H.2.0.1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{\rho} + \nabla \phi - \frac{\zeta}{\rho} \nabla (\nabla \cdot \mathbf{v}) = 0 \,, \tag{H.2.0.2}$$

$$\nabla^2 \phi = 4\pi G \rho \,. \tag{H.2.0.3}$$

This is the starting point to analyze the evolution of the density perturbations and the gravitational instability. Zeroth-order solutions are supposed uniform and static ($\mathbf{v} = 0,\ \rho = cost,\ p = cost,\ \phi = cost,$ "Jeans swindle") and such solutions are not affected by viscosity.

Let us now perform a perturbation theory by adding small fluctuations ($\rho + \delta\rho,\ p + \delta p,\ \phi + \delta\phi,\ \mathbf{v} + \delta\mathbf{v}$) to the zeroth-order solutions and for the bulk viscosity perturbations we use the expansion $\zeta \to \zeta + \delta\zeta$ where

$$\zeta = \zeta(\rho) = const., \qquad \delta\zeta = \delta\rho\ (\partial\zeta/\partial\rho) + ... = \zeta_o\, s\, \rho^{s-1}\, \delta\rho\ + ... \,. \tag{H.2.0.4}$$

With some little algebra, one can obtain an unique equation for density perturbations

$$\frac{\partial^2}{\partial t^2}\,\delta\rho - v_s^2\nabla^2\,\delta\rho - \frac{\zeta}{\rho}\frac{\partial}{\partial t}\nabla^2\,\delta\rho = 4\pi G\rho\,\delta\rho\,, \tag{H.2.0.5}$$

where the adiabatic sound speed is defined as $v_s^2 = \delta p/\delta\rho$. Using the linearity of the equation above, a decomposition in Fourier expansion can be performed and we can address plane waves solutions: $\delta\rho\,(\mathbf{r}, t) = Ae^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}$. Substituting this expression in Eq (H.2.0.5), a dispersion relation for the angular frequency ω and the wave number $k =\mid \boldsymbol{k} \mid$ is obtained:

$$\omega^2 - i\,\frac{\zeta\,k^2}{\rho}\,\omega + (4\pi G\rho - v_s^2k^2) = 0\,. \tag{H.2.0.6}$$

The nature of the angular frequency is responsible of two regimes for $\delta\rho$: if ω is pure imaginary, we deal with an exponential behavior of the perturbations and the collapse is addressed. On the other hand, if ω is complex, an oscillatory regime occurs and we do not obtain structure formation. The dispersion relation has the solution

$$\omega = i\,\frac{\zeta k^2}{2\rho} \pm \sqrt{\bar\omega}\,, \qquad \bar\omega = -\frac{k^4\zeta^2}{4\rho^2} + v_s^2k^2 - 4\pi G\rho\,, \tag{H.2.0.7}$$

thus we obtain the exponential regime for $\bar\omega \leqslant 0$: $\delta\rho \sim e^{-zt}$ and a dumped oscillatory regime for $\bar\omega > 0$: $\delta\rho \sim e^{-yt}\cos x$. It's worth noting that the pure oscillatory regime of the ideal fluid Jeans Model is lost.

The solutions of the equation $\bar\omega = 0$ are

$$K_1 = \sqrt{2}\left(1 - \sqrt{1 - K_J^2\, z^2}\,\right)^{\frac{1}{2}}\,/\,z\,, \quad K_2 = \sqrt{2}\left(1 + \sqrt{1 - K_J^2\, z^2}\,\right)^{\frac{1}{2}}\,/\,z\,, \tag{H.2.0.8}$$

where $z = \zeta\ /\ \rho v_s$, and $K_J = \sqrt{4\pi G\rho/v_s^2}$ is the well-known Jeans length and the relations $K_1, K_2 > 0$, $K_1 < K_2$ holds. The existence of the square root in such solutions give rise to a constraint on the viscosity coefficient: $\zeta \leqslant \sqrt{v_s^4\rho/4\pi G} = \rho v_s/K_J = \zeta_c$. An estimation in the recombination era, after decoupling yields to the value $\zeta_c = 7.38 \cdot 10^4 \; g \, cm^{-1} \, s^{-1}$ and confronting this threshold with usual viscosity (*e.g.* $Hydr. = 8.4 \cdot 10^{-7} g \, cm^{-1} \, s^{-1}$) we can conclude that the range $\zeta \leqslant \zeta_c$ is the only of physical interest.

We study now the $\delta\rho$ exponential solutions in correspondence of $\bar{\omega} \leqslant 0$

$$\delta\rho = A e^{-ik\cdot x} \, e^{w\,t} \;, \quad w = -\frac{\zeta k^2}{2\rho} \mp \sqrt{\frac{k^4 \zeta^2}{4\rho^2} - v_s^2 k^2 + 4\pi G \rho} \;, \tag{H.2.0.9}$$

obtaining the structure formation, *i.e.*, an exponential collapse for $t \to \infty$ for

$$w > 0 \qquad \text{iff} \qquad k < K_J = \sqrt{\frac{4\pi G \rho}{v_s^2}} \;, \qquad (K_J < K_1 < K_2) \;. \tag{H.2.0.10}$$

As result we show how the structure formation occurs if

$$M > M_J = \lambda_J^3 \rho = \left(\frac{2\pi}{K_J}\right)^3 \rho = \pi^{\frac{3}{2}} \frac{v_s^3}{\sqrt{G^3 \rho}} \tag{H.2.0.11}$$

thus the viscous effects do not alter the threshold value of the Jeans Mass, but they change the behavior of the perturbation, in particular the pure oscillatory regime is lost. In fact, in the standard Jeans Model for $\zeta = 0$, we obtain $K_1, K_2 \to \infty$ and, in the case $k > K_J$, $\delta\rho$ behave like two progressive sound waves, of constant amplitude, propagate in the directions $\pm\mathbf{k}$ with velocity $c_s = v_s \sqrt{1 - (\lambda/\lambda_J)^2}$.

As shown above, since the pure oscillatory regime does not occurs, we deal with a decreasing exponential or a damped oscillatory evolution of perturbations. This allows to perform a qualitative analysis of the top-down fragmentation scheme, *i.e.*, the comparison between the evolution of two structures: one collapsing agglomerate with $M \gg M_J$ and an internal non-collapsing sub-structure with $M < M_J$. If this picture is addressed, the sub-structure mass must be compared with a decreasing Jeans Mass since the latter is inversely proportional to the collapsing agglomerate background mass. This way, as soon as such a Jeans Mass reaches the sub-structure one, the latter begins to condense implying the fragmentation. In the standard Jeans Model, this mechanism is always allowed since the amplitude for perturbations characterized by $M < M_J$ remains constant in time. On the other hand, the presence of decreasing fluctuations in the viscous model, requires a discussion on the effective damping and an the efficacy of the top-down mechanism. Of course, such an analysis contrasts the hypothesis of a constant background density, but it can be useful to estimate the strength of the dissipative effects. In this scheme, a perturbation validity limit has to be set: we suppose $\delta\rho/\rho \sim 0.01$ as the limit of the model and we use recombination era parameters with no expansion $\rho = const$. As a result, in correspondence of a very small viscosity coefficient, we show how the sub-structure survives in the oscillatory regime until the end of the approximation scheme, since the viscous dumping is small. On the other hand, if we deal with consistent viscous effects, the top-down mechanism is strongly suppressed. In fact, the dumping becomes very strong and the sub-structure vanish during the agglomerate evolution.

Expanding Universe generalization We here calculate Carlevaro and Montani (2009) the behavior of small fluctuations, using Newtonian equations, but now taking into account the expansion of the Universe. In this case, no "Jeans swindle" must be addressed: the zeroth-order solutions are derived by the motion equations of the isotropic and homogeneous Universe and we don not treat the static and constant solution. In particular, we deal with the matter-dominated era where very small energy density are involved: this way, we are able to neglect the bulk viscosity effects in the unperturbed dynamics since $\zeta = \zeta_o\, \rho^s$. It must be underline that we can safely employ Newtonian mechanic (perturbation theory) to deal with astronomical problems in which the energy density is dominated by non-relativistic particles and in which the linear scales involved are small compared with the characteristic scale of the Universe.

The zeroth-order dynamics is described by the Friedmann Equations for an homogeneous and isotropic Universe. In the matter-dominated era we assume $p \ll \rho$ obtaining the following background solutions:

$$\rho = \rho_0 \left(\frac{a_0^3}{a^3} \right) , \qquad \mathbf{v} = \mathbf{r}\, \frac{\dot{a}}{a} , \qquad \nabla\phi = \mathbf{r} \frac{4\pi G\rho}{3} , \tag{H.2.0.12}$$

where $a(t)$ is the scale factor. Using these expressions, we are now able to perform a perturbation theory starting from equations (H.2.0.1), (H.2.0.2) and (H.2.0.3). Neglecting second order terms, we get the perturbed system

$$\frac{\partial}{\partial t}\, \delta\rho \,+\, 3\, \frac{\dot{a}}{a}\, \delta\rho + \frac{\dot{a}}{a}\, (\mathbf{r}\cdot\nabla)\delta\rho + \rho\; \nabla\cdot\delta\mathbf{v} \;= 0 \,, \tag{H.2.0.13}$$

$$\frac{\partial}{\partial t}\, \delta\mathbf{v} + \frac{\dot{a}}{a}\, \delta\mathbf{v} + \frac{\dot{a}}{a}\, (\mathbf{r}\cdot\nabla)\delta\mathbf{v} + \frac{v_s^2}{\rho} \nabla\, \delta\rho + \nabla\, \delta\phi - \frac{\zeta}{\rho} \nabla\, (\nabla\cdot\delta\mathbf{v}) \;= 0 \,, \tag{H.2.0.14}$$

$$\nabla^2 \delta\phi = 4\pi G \delta\rho \,. \tag{H.2.0.15}$$

The equations above are spatially homogeneous so we expect to find plane waves solutions: $\delta\rho(\mathbf{r}, t) \to \delta\rho(t)\, e^{i\mathbf{r}\cdot\mathbf{q}/a(t)}$ and likewise for $\delta\mathbf{v}$ and $\delta\phi$. Assuming now the condition for the validity of the Newtonian approximation, *i.e.*, $r \ll a\,,\; r/a \sim 0$, it is now convenient to decompose $\delta\mathbf{v}$ into parts normal and parallel to $\mathbf{q}$:

$$\delta\mathbf{v} = \delta\mathbf{v}_\perp + i\mathbf{q}\,\epsilon\,, \qquad \text{with} \qquad \mathbf{q}\cdot\delta\mathbf{v}_\perp = 0\,, \quad \epsilon = -\tfrac{i}{q^2}(\mathbf{q}\cdot\delta\mathbf{v})\,. \tag{H.2.0.16}$$

With these assumption, we finally get (setting $\delta\rho = \rho(t)\,\delta$)

$$\frac{\partial}{\partial t}\delta\mathbf{v}_\perp + \frac{\dot a}{a}\delta\mathbf{v}_\perp = 0\,, \quad \dot\epsilon + \Big(\frac{\dot a}{a} + \frac{\zeta\,q^2}{\rho\,a^2}\Big)\,\epsilon = \Big(\frac{4\pi G\rho a}{q^2} - \frac{v_s^2}{a}\Big)\,\delta\,, \quad \dot\delta = \frac{q^2}{a}\,\epsilon\,. \tag{H.2.0.17}$$

Inspection of these equations shows that there are two quite different types of normal mode in the scheme. The *rotational modes*, governed by the first of the equations above, are not affected by the presence of viscosity:

$$\delta\mathbf{v}_\perp(t) \sim a^{-1}(t)\,, \tag{H.2.0.18}$$

i.e., the velocity perturbations normal to $\mathbf{q}$ decay as $1/a$ during the Universe expansion.

On the other hand, the *compressional modes* are governed by the equation

$$\ddot\delta + \Big(2\frac{\dot a}{a} + \frac{\zeta q^2}{\rho\,a^2}\Big)\,\dot\delta + \Big(\frac{v_s^2\,q^2}{a^2} - 4\pi G\rho\Big)\,\delta = 0\,, \tag{H.2.0.19}$$

where we indicate the physical wave vector as $\mathbf{k} = \mathbf{q}/a$. In our scheme, we assume $a(t) \ll a_0$ ($\dot a^2\,, 8\pi\rho a^2/3 \gg 1$) and we can use the zero curvature solution of the Friedmann equation, *i.e.*,

$$a \sim t^{\frac{2}{3}}\,, \quad \rho = \frac{1}{6\pi G t^2}\,, \quad p \sim \rho^\gamma\,, \; v_s = \Big(\frac{\gamma p}{\rho}\Big)^{\frac{1}{2}} \Rightarrow \quad v_s \sim t^{1-\gamma}\,, \tag{H.2.0.20}$$

and we obtain $\zeta = \bar\zeta_o\,t^{-2s}$, $\bar\zeta_o = \zeta_o/(6\pi G)^s$ for the bulk viscosity coefficient. The main equation rewrites now

$$\ddot\delta + \Big[\frac{4}{3\,t} + \frac{\chi}{t^{2(s-1/3)}}\Big]\,\dot\delta + \Big[\frac{\Lambda^2}{t^{2\gamma-2/3}} - \frac{2}{3\,t^2}\Big]\,\delta = 0\,, \tag{H.2.0.21}$$

where χ and Λ are two constants:

$$\chi = \frac{t^{2(s-1/3)}\,\zeta q^2}{\rho\,a^2}\,, \quad \Lambda^2 = \frac{t^{2\gamma-2/3}\,v_s^2 q^2}{a^2}\,. \tag{H.2.0.22}$$

This equation can not be analytically solved in general. But setting $s = 5/6$ we can get the solutions

$$\delta(t) = t^{-\frac{1}{6}-\frac{\chi}{2}}\,\Big[C_1\,J_n\Big(\frac{\Lambda t^{-\bar\gamma}}{\bar\gamma}\Big) + C_2\,Y_n\Big(\frac{\Lambda t^{-\bar\gamma}}{\bar\gamma}\Big)\Big]\,, \tag{H.2.0.23}$$

where J e Y denotes the Bessel functions of first and second species respectively and

$$n = -\sqrt{25 + 6\chi + 9\chi^2}\,/\,(6\,\bar{\gamma})\,, \qquad \bar{\gamma} = \gamma - \tfrac{4}{3}\,. \tag{H.2.0.24}$$

Bessel functions behave like power-laws or oscillate in the asymptotic limits for their argument. In particular the threshold value which characterize the transition between the two regimes is determined by

$$\Lambda\, t^{-\bar{\gamma}}/\bar{\gamma} < 1 \qquad \Rightarrow \qquad t < \Lambda^{1/\bar{\gamma}}/\bar{\gamma}^{1/\bar{\gamma}}\,. \tag{H.2.0.25}$$

In correspondence of an adiabatic Universe (*i.e.*, $\gamma > 4/3$), we get the threshold value

$$k < \bar{K}_J = \sqrt{\frac{6\pi G\rho}{\bar{\gamma}^2\, v_s^2}}\,, \tag{H.2.0.26}$$

which is substantially the same as the Jeans condition: $K_J = \sqrt{4\pi G\rho/v_s^2}$. This solution will apply for a matter-dominated Universe after recombination and, in the pure adiabatic case for $\gamma = 5/3$, the density contrast behaves like

$$\delta \sim t^{-1/6\,-\,\chi/2\,\mp\,n/3}\,, \tag{H.2.0.27}$$

and it can be shown that the exponent of such an expression is positive $\forall\chi$, yielding gravitational collapse.

Confronting now this result with the non-viscous case, $\delta \sim t^{2/3}$ $(\chi = 0)$ we can conclude that the viscous effects are summarized by a damping of the density contrast evolution while the threshold Jeans mass can be addressed also in presence of viscosity.

H.3 Extended Theories of Gravity

The main interesting proposals to interpret the presence of Dark Energy can be divided into two classes: those theories, that make explicitly presence of matter and the other ones, which relay on modifications of the Friedmann dynamics. We address a mixture of these two points of view, with the aim of clarifying how the "non-gravitational" vacuum energy affects so weakly the present Universe dynamics. In particular, we study (Lecian and Montani, 2008) the modified gravitational action

$$S_G = -\frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\, f(R). \tag{H.3.0.1}$$

It is possible to demonstrate that the non-linear gravitational Lagrangian (H.3.0.1), in the Jordan frame, can be cast in a dynamically-equivalent form, *i.e.*, the action for a scalar field ϕ in GR (with a rescaled metric), in the Einstein frame, by means of the conformal transformation $g_{\mu\nu} \rightarrow e^{\phi} g_{\mu\nu}$, which provides the on-shell condition $\phi \equiv -\ln f'(R)$.
Within the scheme of modified gravity, an exponential Lagrangian density was considered, *i.e.*, $f(R) = 2\Lambda \exp{(R/2\Lambda)}$, and the corresponding scalar-tensor description was addressed for both positive and negative values of the cosmological constant.
We determined the Friedmann equation corresponding to an exponential form for the gravi-tational-field Lagrangian density. The peculiar feature of our model is that the geometrical components contain a cosmological term too, whose existence can be recognized as soon as we expand the exponential form in Taylor series of its argument. An important feature of our model arises when taking a Planckian value for the fundamental parameter of the theory (as requested by the cancellation of the vacuum-energy density). In fact, as far as the Universe leaves the Planckian era and its curvature has a characteristic length much greater than the Planckian one, then the corresponding exponential Lagrangian is expandible in series, reproducing General Relativity to a high degree of approximation. As a consequence of this natural Einsteinian limit (which is reached in the early history of the Universe), most of the thermal history of the Universe is unaffected by the generalized theory. The only late-time effect of the generalized framework consists of the relic cosmological term actually accelerating the Universe. Indeed, our model is not aimed at showing that the present Universe acceleration is a consequence of non-Einsteinian dynamics of the gravitational field, but at out-

lining how it can be recognized from a vacuum-energy cancellation. Such a cancellation must take place in order to deal with an expandable Lagrangian term and must concern the vacuum-energy density as far as we build up the geometrical action only by means of fundamental units. The really surprising issue fixed by our analysis is that the deSitter solution exists in presence of matter only for a negative ratio between the vacuum-energy density and the intrinsic cosmological term, $\epsilon_{vac}/\epsilon_{\Lambda}$. We can take the choice of a negative value of the intrinsic cosmological constant, which predicts an accelerating deSitter dynamics. Nevertheless, in this case, we would get a vacuum-energy density greater than the modulus of the intrinsic term. This fact looks like a fine-tuning, especially if we take a Planckian cosmological constant. The vacuum-energy density is expected to be smaller than the Planckian one by a factor $\mathcal{O}(1) \times \alpha^4$, where $\alpha < 1$ is a parameter appearing in non-commutative formulations of the relativistic particle, and , in particular, it is linked to the modified commutation relations

$$[\, x, p \,] = i\hbar \left(1 + \frac{1}{\alpha^2} \frac{G}{c^3 \hbar}\, p^2 \right). \tag{H.3.0.2}$$

The analysis of the corresponding scalar-tensor model helped us to shed light on the physical meaning of the sign of the cosmological term. In fact, for negative values of the cosmological term, the potential of the scalar field exhibits a minimum, around which scalar-field equations can be linearized. The study of the deSitter regime shows that a comparison with the modified-gravity description is possible in an off-shell region, *i.e.*, in a region where the classical equivalence between the two formulations is not fulfilled.

The small value of the present curvature of the Universe leads us to believe that, independently of its specific functional form, the $f(\mathcal{R})$ term must be regarded as a lower-order expansion in the Ricci scalar. On the other hand, it is easily understood that the peculiarities of such an expansion will be extremely sensitive of the morphology of the deformed Lagrangian.
The most immediate generalization is of course to deal with a function of the Ricci scalar analytical in the point $R = 0$, so that its Taylor expansion holds. This approach is equivalent to deal with a polynomial form, whose free parameters are available to fit the observed phenomena on different sectors of investigation. Despite the appealing profile of such a choice, it is extremely important to observe that it could not be the most general case, since real (non-integer) exponent of the Ricci scalar are in principle on the same footing as the simplest case. We concentrate our attention on such an open issue,and we will develop a modified theory of the form

$$f(\mathcal{R}) = \mathcal{R} + \gamma \mathcal{R}^{\beta}, \tag{H.3.0.3}$$

where γ and β are two free parameters to be constrained on a physical level.

In particular, β is dimensionless and γ has the dimensions of *length* $^{2\beta-2}$. We can define the characteristic length scale of our model as $L_\gamma \equiv \gamma^{1/(2\beta-2)}$. It is straightforward to verify that (H.3.0.3) is non-analytical in $\mathcal{R} = 0$ for rational, non-integer β. Furthermore, for $2 < \beta < 3$, Einstein equations are solved in the weak-field limit up to the proper approximation order.
For this model, the parameter space is constrained according to several criteria.
In the Einstein frame, those configurations are selected, for which the potential of the scalar field admits a minimum, since such a minimum becomes, sooner or later, an attractive stable configuration for the system.
In the Jordan frame, the solution of Einstein equations is illustrated to consist of a Newtonian part and a post-Newtonian one. The most suitable arena where to evaluate the reliability and the validity range of the weak-field solution is, of course, the Solar System. To this end, we impose that the post-Newtonian term be a small correction with respect to the Newtonian one at Solar-System scales, and then evaluate the maximum distance at which the weak-field approximation holds. As a result, the validity range of the model is found. Further constraints are obtained by the request that Solar-System data be reproduced within experimental errors. As a compelling example, we evaluate the correction to the Keplerian period of a given planet, compare it with experimental data and uncertainties, and then impose that the correction be smaller than the experimental uncertainty.
To provide proper estimations, we choose a typical (non-peculiar) value of the parameter β, say $\beta = 8/3$ and then collect all the constraints together. High-precision measurements are nowadays available for the distances between Solar-System planets and the Sun, so that the relative error in the orbital period is extremely small. According to this fact, we specify our analysis for example for the Earth. We find a lower bound for L_γ, $L_\gamma > 1.147466382 \cdot 10^{11} Km$, according to which the post-Newtonian term is a small correction tot he Newtonian one up to $\sim 1.6 \cdot 10^{10} km$, and that the weak-field approximation holds up to $\sim 1.5 \cdot 10^{12} km$, in perfect agreement with Solar-System scales. Even though Solar-System experiments take place in vacuum, also an interior solution (for the model of a dust central mass). Matching the exterior solution (and its derivatives) with the interior solution (and its derivatives) allows one to establish the relation between the matter density of the central mass, its mass (through the Schwarzschild radius), its radius and the parameter β.

H.3.1 Cosmological expansion and the effects of a non-analytic f(R)

The same form (H.3.0.3) of $f(R)$ is, moreover, analyzed in a pure cosmological context facing its influence on the dynamics of a FRW isotropic Universe. We start from the usual gravitational action (H.3.0.1) and we set $\chi = 8\pi G/c^3$ (the signature is $[+,-,-,-]$). Furthermore, we use the following notation

$$f(R) = R + qR^n \, . \tag{H.3.1.1}$$

Variations of the total action $\mathcal{S}_{tot} = \mathcal{S} + \mathcal{S}_M$ (where $\mathcal{S}_M$ denotes the matter term) wrt the metric give, after manipulations and modulo surface terms:

$$\begin{aligned} &f' R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} f - f''[\nabla_\mu \nabla_\nu R - g_{\mu\nu} \nabla^\rho \nabla_\rho R] - \\ &\qquad - f'''[\nabla_\mu R \nabla_\nu R - g_{\mu\nu} \nabla^\rho R \nabla_\rho R] = \tfrac{1}{c}\, \chi\, T_{\mu\nu} \, . \end{aligned} \tag{H.3.1.2}$$

where $(')$ indicates the derivative wrt R and $T_{\mu\nu}$ is the Energy-Momentum Tensor (EMT).

We assume now the standard FLRW line element in the synchronous reference system and we indicate with $a(t)$ the scale factor and with K the curvature constant. In this scheme, the 00-component of Eq.(H.3.1.2) results, for symmetry, the only independent one and it writes

$$f' R_{00} - \tfrac{1}{2} f + 3(\dot{a}/a)\, f''\, \dot{R} = \chi\, T_{00}/c \, , \tag{H.3.1.3}$$

where $(\dot{\ })$ denotes the time derivative. As matter source, we assume a perfect-fluid EMT which writes usually as:

$$T_{\mu\nu} = (p + \rho)\, u_\mu u_\nu - p g_{\mu\nu} \, , \tag{H.3.1.4}$$

in a comoving reference system ($T_{00} = \rho$), where p is the thermostatic pressure, ρ the energy density and u_μ denotes the 4-velocity. The Continuity Equation, *i.e.*, $T^\nu_{\mu;\nu} = 0$, assuming the Equation of State (EoS) $p = w\rho$, gives the following expression for the energy density

$$\rho = \rho_0\, a^{-3(1+\mathrm{W})} \, . \tag{H.3.1.5}$$

We are now able to explicitly write the 00-component of gravitational equations, *i.e.* Eq.(H.3.1.3), using the expressions set above. It reads as follows

$$
\begin{aligned}
&2\,\tilde{\chi}\,a^{1-3w} \;+\; 6\,a^3\,\ddot{a}\left[1+6^{n-1}\,n\,q\,a^{2(1-n)}\,(-K-\dot{a}^2-a\,\ddot{a})^{n-1}\right]+\\
&\quad +a^2\left[-6K-6\,\dot{a}^2-6\,a\,\ddot{a}+6^n\,q\,a^{2(1-n)}\,(-K-\dot{a}^2-a\,\ddot{a})^n\right]+\\
&\qquad +6^n(n-1)n\,q\,\dot{a}\,a^{2(2-n)}\,(-K-\dot{a}^2-a\,\ddot{a})^{n-2}\times\\
&\qquad\quad \times\left[-2\,\dot{a}^3-2K\,\dot{a}+a\,\dot{a}\,\ddot{a}\,+a^2\,\dddot{a}\right]=0\,,
\end{aligned}
\tag{H.3.1.6}
$$

where $\tilde{\chi}=\chi\rho_0/c$.

H.3.1.1 Power-law scale factor

Using a power-law like: $a=a_0\,t^x$, for the scale factor, Eq.(H.3.1.6) rewrites

$$
\begin{aligned}
&A\,t^{4x-2}\left[1+nq\left(B\,t^{-2x}+D\,t^{-2}\right)^{n-1}\right]+\\
&\quad +t^{2x-2}\left[E\,t^2-a_0^4\,t^{2x}\left(F-qt^2\left(B\,t^{-2x}+D\,t^{-2}\right)^n\right)\right]+\\
&\qquad -G\,t^{6x}\left(B\,t^{-2x}+D\,t^{-2}\right)^n\frac{(K\,t^2+L\,t^{2x})}{(K\,t^2+xL\,t^{2x})^2}+\\
&\qquad\qquad +2\tilde{\chi}a_0^{1-3w}\,t^{x(1-3w)}\;=0\,,
\end{aligned}
\tag{H.3.1.7}
$$

with the following identifications

$$
A=6\,a_0^4\,x\,(x-1)\,,\quad B=-6\,K/a_0^2\,,\quad D=6x\,(1-2x)\,,
\tag{H.3.1.8a}
$$

$$
L=-Da_0^2/6x\,,\quad E=-6\,K\,a_0^2\,,\quad F=-D\,,\quad G=2\,a_0^6\,(n-1)\,n\,q\,x^2\,.
\tag{H.3.1.8b}
$$

A. Radiation-dominated Universe Let us now assume the usual radiation-dominated Universe EoS as: $w=1/3$ ($\rho=\rho_0\,a^{-4}$). In the following, we study the two distinct regimes for $x<1$ and $x>1$ separately, in the asymptotic limit for $t\to 0$. We then discuss the case $x=1$.

(i) Case $x<1$. In Eq.(H.3.1.7), all terms containing explicitly the curvature K results to be negligible for $t\to 0$ and $x<1$ ($x\neq 1$). In such an asymptotic limit, Eq.(H.3.1.7) reduces as follows:

$$
[\,nqAD^{n-1}+qa_0^4D^n\,-\,6GD^{n-1}/x\,]\,t^{4x-2n}=0\,.
\tag{H.3.1.9}
$$

This equation admits asymptotic solutions if and only if $x\leqslant n/2$ which, in our case, is always satisfied by the condition $x<1$, since the following constraint $2<n<3$ holds. Furthermore, the existence of solutions leads to the condition $n=(2m+1)/\ell$ (where, here and in the following, m and

ℓ denotes integer numbers) if $x > 1/2$, while, for $x \leqslant 1/2$ all n-values are allowed.

The solutions (for $x \neq 0$) are given by $(nqA + qa_0^2 D + 6G/x) = 0$ and furthermore $D = 0$. Solving these equations, we get, in the limit $t \to 0$ and for $x < 1$ ($x \neq 1$ and $x \neq 0$),

$$x = 1/2\,, \qquad x = (n + 2a_0^2 n - 2a_0^2 n^2 - 1)/(n-2)\,. \tag{H.3.1.10}$$

The second solution give us a constraint on the a_0 parameter in correspondence of the restricted range $2 < n < 3$. In fact, to obtain $0 < x < 1$, a_0 must satisfy the inequality:

$$\sqrt{1/(2n(n-1))} < a_0 < \sqrt{1/(2n)}\,. \tag{H.3.1.11}$$

(ii) Case $x > 1$. In this regime, the consistence of the leading order (for $t \to 0$ and $K \neq 0$) of Eq.(H.3.1.7) leads to the condition $n = (2m+1)/\ell$ for positive curvature K, while, if $K < 0$ all n-values are allowed. Eq.(H.3.1.7) asymptotically rewrites $t^{4x-2xn}(a_0^4 q B^n) = 0$, and power-law solutions are not allowed.

If we now set, in the asymptotic limit as $t \to 0$, the value $x = 1$, Eq.(H.3.1.7) reads

$$\left(\frac{a_0^6 q(12n(n-1)-1)}{6(K+a_0^2)}\right)\left(-\frac{6K}{a_0^2} - 6\right)^n t^{4-2n} = 0\,, \tag{H.3.1.12}$$

which does not admits power-law solution for the scale factor (for permitted n-values), also if we set $K = 0$: in fact, we get $a_0 = 0$ and $n < 1$.

Analysis of the case K=0: In the case $x > 1$, Eq.(H.3.1.7) reduces to the same Eq.(H.3.1.9), *i.e.*,

$$[\,nqAD^{n-1} + qa_0^4 D^n - 6GD^{n-1}/x\,]\, t^{4x-2n} + 2\tilde{\chi} = 0\,, \tag{H.3.1.13}$$

and the existence of solutions leads to the same constraint $n = (2m+1)/\ell$ for $x > 1/2$ and $\forall n$ for $x \leqslant 1/2$. Three distinct regimes have to be discussed. For $x > n/2$, the leading order of the equation above does not admit solutions since it writes simply $2\tilde{\chi} = 0$. For $x < n/2$, the solutions of Eq.(H.3.1.13) are given by $(nqA + qa_0^2 D + 6G/x) = 0$ and, as the case of Eq.(H.3.1.10), we get $x = (n + 2a_0^2 n - 2a_0^2 n^2 - 1)/(n-2)$. Indeed, the condition $x < n/2$ leads to the unphysical one $a_0^2 < (n-2)/4(1-n)$ and we can conclude that power-law solutions are not allowed in this case. For $x = n/2$, Eq.(H.3.1.13) rewrites

$$[\,nqA + qa_0^4 D - 6GD^{n-1}/x\,]\, D^{n-1} = -2\tilde{\chi}\,, \tag{H.3.1.14}$$

and, also in this regime, power-law solutions are not allowed.

It is worth noting that, if we assume $K = 0$ in Eq.(H.3.1.7), we get that the radiation-dominated solution for $w = 1/3$ and $x = 1/2$ is an exact (non-asymptotic and allowed for all n-values) solution setting the a_0 parameter as

$$3a_0^4 = 4\tilde{\chi} \quad \Rightarrow \quad a_0 = \sqrt{2}\, a_0^{R(FLRW)} \,, \qquad a_0^{R(FLRW)} = (\frac{\chi}{3c}\,\rho_0)^{1/4} \,. \quad \text{(H.3.1.15)}$$

Furthermore, we have seen that all terms in K can be neglected for $t \to 0$ (if $x < 1$) and we get the asymptotic solutions (H.3.1.10) for $0 < x < 1$.

B. Matter-dominated Universe We assume now the matter-dominated Universe EoS: $w = 0$ ($\rho = \rho_0\, a^{-3}$). Let us study the two distinct regimes for $x < 1$ and $x > 1$ separately, in the asymptotic limit for $t \to \infty$. We then discuss the case $x = 1$.

(i) Case $x < 1$. If we assume $K \neq 0$, the leading-order of Eq.(H.3.1.7) in the limit $t \to \infty$ results to be

$$(6\, K\, a_0^2)\; t^{2x} = 0 \,, \qquad \text{(H.3.1.16)}$$

i.e., the curvature term dominates the dynamics and the consistence of the leading order leads to the condition $n = (2m+1)/\ell$ for positive curvature K, while, if $K < 0$ all n-values are allowed. In the regime $x < 1$, power-law solutions for the scale factor do not exist since the equation above admits the only unphysical solution $a_0 = 0$.

(ii) Case $x > 1$. In this case, all terms proportional to K of Eq.(H.3.1.7) are negligible in the limit $t \to \infty$ and, furthermore, the existence of solutions leads to the condition $n = (2m+1)/\ell$, since $D < 0$. In this regime, we get the following lading-order equation

$$[A + a_0^4\, D]\; t^{4x-2} = 0 \,. \qquad \text{(H.3.1.17)}$$

This equation can be rewritten as $(6\, x^2\, a_0^4)\; t^{4x-2} = 0$ and, like in the previous case, power-law solutions for the scale factor are not admitted.

Let us now briefly discuss the case $x = 1$. For all K-value, Eq.(H.3.1.7) reduces, in the asymptotic limit as $t \to \infty$, to $(6\, a_0^4)\; t^2 = 0$ and, as the previous regimes, it does not admit any power-law solutions (also for the permitted n-values).

Analysis of the case K=0: In this case, the $x \geqslant 1$ regimes admit only the already obtained results and power-law solutions do not exist.

On the other hand, for $x < 1$ and assuming zero curvature in Eq.(H.3.1.7), we get the following asymptotic expression for $t \to \infty$:

$$[\, A + a_0^4 D\,]\; t^{4x-2} + 2\tilde{\chi} a_0\; t^x = 0 \,, \qquad \text{(H.3.1.18)}$$

and the existence of solutions leads to the condition $n = (2m+1)/\ell$ in the case $x > 1/2$, while, for $x \leqslant 1/2$ all n-values are allowed. The equation above admits three distinct regimes: $x < 2/3$, $x > 2/3$ and $x = 2/3$. Both the cases for $x \neq 2/3$ do not admit solutions. In fact, for $x < 2/3$, Eq.(H.3.1.18) asymptotically reduces to $2\tilde{\chi}a_0 = 0$ and for $x > 2/3$, the same results of Eq.(H.3.1.17) are obtained.

The case $x = 2/3$ is the only one that admit an asymptotic solution for $t \to \infty$. In fact, Eq.(H.3.1.18) reduces to

$$-8a_0^3 + 6\tilde{\chi} = 0\,, \tag{H.3.1.19}$$

and the FLWR matter-dominated power-law $a = a_0\, t^{2/3}$ is reached setting

$$x = 2/3\,, \qquad a_0 = (\tfrac{3}{2})^{2/3}\, a_0^{M(FLRW)}\,, \qquad a_0^{M(FLRW)} = (\frac{\chi}{3c}\,\rho_0)^{1/3}\,. \tag{H.3.1.20}$$

Concluding, we can infer that for $f(R) = R + qR^n$ the FRW behavior of the scale factor $a = a_0\, t^{2/3}$ is the only asymptotic (as $t \to \infty$) power-law solution with the constraint $n = (2m+1)/\ell$.

H.3.1.2 Exponential scale factor

Using an exponential-law like: $a = a_0\, e^{st}$, for the scale factor (where $s > 0$) Eq.(H.3.1.6) rewrites as

$$\begin{aligned} & a_0^4\, e^{4st} \times \\ & \times \Big[q\Big(-\frac{6K}{a_0^2}\, e^{-2st} - 12\, s^2\Big)^n + s^2\Big(-6 + 6nq\Big(-\frac{6K}{a_0^2}\, e^{-2st} - 12\, s^2\Big)^{n-1}\Big)\Big] + \\ & - 2Ka_0^2\, e^{2st}\Big[3 + \frac{a_0^4\,(n-1)\, nqs^2}{(K + 2a_0^2 s^2\, e^{2st})^2}\, e^{4st}\Big(-\frac{6K}{a_0^2}\, e^{-2st} - 12\, s^2\Big)^n\Big] = \\ & = -2\chi(a_0\, e^{st})^{1-3w}\,. \end{aligned} \tag{H.3.1.21}$$

In the equation above, all terms containing explicitly the curvature K results to be negligible for $t \to \infty$. In such an asymptotic limit, Eq.(H.3.1.21) reduces as follows:

$$a_0^4\, e^{4st}\Big[q(-12)^n\, s^{2n}(1 - n/2) - 6s^2\Big] + 2\chi(a_0\, e^{st})^{1-3w} = 0\,. \tag{H.3.1.22}$$

It easy to see that, in the limit $t \to \infty$, the second term of the equation above is negligible for all $w > -1$ and, furthermore, for phantom matter $w < -1$ no exponential scale-factor solutions are allowed.

For all type of ordinary matter, in particular in the case of matter-dominated

Universe (*i.e.*, $w = 0$), we get the asymptotic solution for $q \neq 0$

$$s = \left[\frac{(-12)^n}{6}\, q(1-n/2)\right]^{1/(2-2n)}, \tag{H.3.1.23}$$

and the solution for $q = 0$ results to be, of course, $s = 0$. The solution above exist if and only if n can be written as $n = (2p+1)/(2l+1)$ and it can be plotted as follow

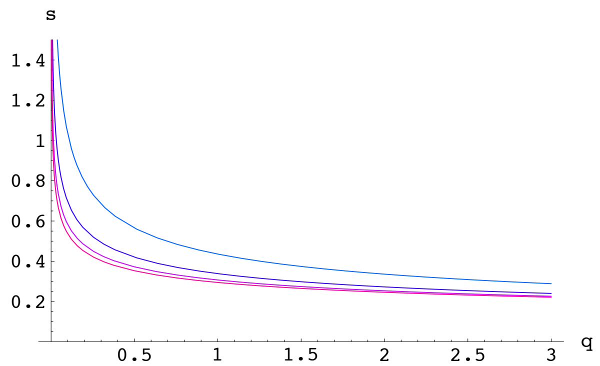

Figure H.3.1: The lines represent the values $n = 7/3$, $n = 13/5$, $n = 14/5$ and $n = 38/13$, in a decreasing order, respectively.

H.4 The interaction between relic neutrinos and primordial gravitational waves

The presence in the Universe today of a stochastic background of gravitational waves (GWs) is a quite general prediction of several early cosmology scenarios. In fact, the production of gravitons is the outcome of many processes that could have occurred in the early phases of the cosmological evolution. Notable examples of this kind of processes include the amplification of vacuum fluctuations in inflationaryand pre-big-bang cosmology scenarios, phase transitions, and finally the oscillation of cosmic strings loops. In most of these cases, the predicted spectrum of gravitational waves extends over a very large range of frequencies; for example, inflationary expansion produces a flat spectrum that spans more than 20 orders of magnitude in frequency, going from 10^{-18} to 10^{9} Hz.

The detection of such primordial gravitational waves, produced in the early Universe, would be a major breakthrough in cosmology and high energy physics. This is because gravitons decouple from the cosmological plasma at very early times, when the temperature of the Universe is of the order of the Planck energy. In this way, relic gravitational waves provide us a "snapshot" of the Universe near the Planck time, in a similar way as the cosmic microwave background radiation (CMBR) images the Universe at the time of recombination.

The extremely low frequency region ($\nu_0 \lesssim 10^{-15}$Hz) in the spectrum of primordial gravitational waves can be probed through the anisotropies of the CMBR. In particular, gravitational waves leave a distinct imprint in the so-called magnetic or B-modes of its polarization field. The amplitude of the primordial spectrum of gravitational waves is usually parameterized through the tensor-to-scalar ratio r, i.e., the ratio between the amplitudes of the initial spectra of the tensor and scalar perturbations in the metric. The Planck satellite, scheduled for launch in July 2008, is expected to be sensitive to $r \geq 0.05$. The lower limit corresponds to a density parameter $\Omega_{GW}(\nu) \equiv (1/\rho_c) d\rho_{GW}/d\log\nu$ as faint as $\sim 3 \times 10^{-16} h^{-2}$ (h is the dimensionless Hubble constant) in the low frequency range. Although this value looks incredibly small, it should be noted that, in order to produce such an amount in the framework of inflationary models (that at present time represent the most promising way to produce a signal in the region under consideration), a very early (starting at $t \sim 10^{-38}$ sec) inflation is required, and this possibility looks, from a theoretical point of view, quite unnatural. Polarization dedicated experiments will enhance the sensitivity of one and maybe two orders of magnitude .

On the other hand the planned large scale interferometric GW detectors, although designed with the aim to detect astrophysical signals, can possibly also detect signals of cosmological origin. They give complementary information with respect of the CMBR polarization field since, even if their sensitivity is by no means comparable to the one than can be reached by CMBR polarization experiments, nevertheless they probe a different region in the frequency domain that would not be accessible to those ones. In particular the ground-based interferometers, such as the LIGO, VIRGO, GEO600and TAMA300experiments, operate in the range $1\,\text{Hz} < \nu_0 < 10^4\,\text{Hz}$, and are expected to be sensitive to $\Omega_{GW}h^2 \geq 10^{-2}$. Even more interesting is the LISA space interferometer, that will probably operate from 2013 to 2018. Not being hampered by the Earth seismic noise, it will probe the frequency region between 10^{-4} and 1 Hz and will in principle be able to detect $\Omega_{GW}h^2 \geq 10^{-12}$ at $\nu_0 = 10^{-3}\,\text{Hz}$. According to theoretical predictions, a large enough GW signal at this frequencies can be produced, with the appropriate choice of parameters, by a pre-big-bang accelerated expansion, by the oscillation of cosmic strings, or by the electroweak phase transition occurring at $T = 300\,\text{GeV}$.

In order to compare the theoretical predictions with the expected instrument sensitivities, one needs to evolve the GWs from the time of their production to the present. It is usually assumed that gravitons propagate in vacuum, i.e., they freely stream across the Universe. In this case, the only effect on a propagating GW is a change in frequency (corresponding to the usual redshift of the graviton energy caused by the expansion of the Universe), while the intensity of the wave remains the same. However, GWs are sourced by the anisotropic stress part of the energy-momentum tensor of matter, so that the vacuum approximation is well-motivated only when this can be neglected. The relevant equation describing a GW propagating on a Friedmann-Robertson-Walker metric is :

$$\partial_t^2 h_{ij} + \left(\frac{3}{a}\frac{da}{dt}\right)\partial_t h_{ij} - \left(\frac{\nabla^2}{a^2}\right) h_{ij} = 16\pi G \pi_{ij}\,, \tag{H.4.0.1}$$

where $a(t)$ is the cosmological scale factor, h_{ij} is a small tensor perturbation representing the GW, and π_{ij} is the anisotropic stress part of the energy-momentum tensor $T^{\mu}{}_{\nu}$.

It is already known that the anisotropic stress of free streaming relic neutrinos acts as an effective viscosity, absorbing gravitational waves in the extremely low frequency region, thus resulting in a damping of the B-modes of CMBR. We have studied the generalization of this phenomenon to other regions of the frequency domain (Lattanzi and Montani, 2005). In particular, we have considered GWs that enter the horizon before the electroweak phase transition (EWPT). This corresponds to an observable frequency today $\nu_0 \gtrsim 10^{-5}$ Hz, i.e., to all waves possibly detectable by interferometers.

In order to study this issue, one has to solve the Boltzmann equation for the phase space density f of cosmological neutrinos:

$$\hat{L}[f] \equiv \frac{df}{d\lambda} = \hat{C}[f], \tag{H.4.0.2}$$

where λ is some affine parameter over the neutrino word line, and the collision operator $\hat{C}$ takes into account the interaction between neutrinos and other particles. The two equations (H.4.0.1) and (H.4.0.2) are coupled by the following expression relating the energy momentum tensor and the phase space density:

$$T^i_j = \frac{1}{\sqrt{-g}} \int f(x^i, p_j, t) \frac{p^i p_j}{p^0} \, dp_1 \, dp_2 \, dp_3. \tag{H.4.0.3}$$

Manipulation of the above equation leads to an integro-differential equation for the normalized amplitude $\chi(t) \equiv h_{ij}(t)/h_{ij}(t=0)$ of the gravitational wave. In the limit of very short neutrino mean free path, valid in the very early Universe and relevant for waves well below a frequency of 10^8 Hz, this equation can be cast in purely differential form:

$$\ddot{\chi} + \frac{2}{u}\dot{\chi} + \chi = -\frac{8f_\nu}{5u^2}(\chi - 1) \tag{H.4.0.4}$$

where u is a time variable related to conformal time, and f_ν is the fraction of the total density of the Universe provided by neutrinos. In the standard cosmological scenario, $f\nu \simeq 0.4$, although non-standard processes can change this value. Thus, a numerical solution to Eq. (H.4.0.4) can be sought with standard methods. It is found that the intensity of GWs is reduced to $\sim 90\%$ of its value in vacuum (see Fig H.4.1), its exact value depending only on one physical parameter, namely the density fraction of neutrinos. Neither the wave frequency nor the detail of neutrino interaction affect the value of the absorbed intensity, resulting in an universal behaviour in the frequency range considered. A fitting formula for the transmitted intensity $\mathcal{T}_\infty$ given by:

$$\mathcal{T}_\infty = 1 - 0.32 f_\nu + 0.05 f_\nu^2 \tag{H.4.0.5}$$

The importance of our results relies in the fact that the damping affects GWs in the frequency range where the LISA space interferometer and future, second generation ground-based interferometers can possibly detect a signal

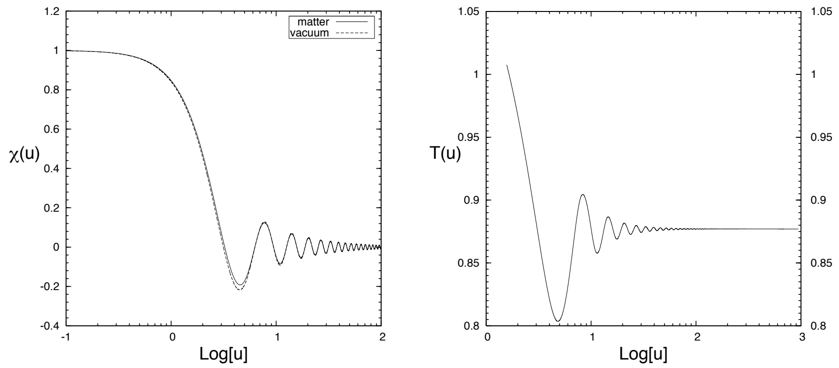

Figure H.4.1: (*Left panel*) Time evolution of the gravitational wave amplitude $\chi(u)$. Solid line represents a GW propagating in neutrino matter. Dashed line represents a GW propagating in vacuum. (*Right panel*) Time evolution of the transmitted wave intensity $\mathcal{T}$.

of cosmological origin. This effect is roughly of the same order of magnitude as the one affecting GWs detectable through the B-modes of CMB polarization. The damping is not so severe to make the detection of cosmological waves unfeasible by interferometers. However it should be taken into account when testing the theoretical predictions of early Universe scenarios against observations. Moreover, the dependence of T_∞ on f_ν can be exploited to measure the latter, and to constrain models of non-standard physics. This even more important in view of the fact that in this way we could measure the value of f_ν at very early times, while available constraints regard the neutrino fraction at the time of cosmological nucleosynthesis or at the time of matter-radiation decoupling.

In fact, recently we have extended this work in order to take into account the thermal history of the neutrino fluid Benini et al. (2009). First of all we have used a different formalism that is more suitable for numerical integration, namely we have multipole expanded the perturbation to the distribution function .

Let us write the Boltzmann equation in the form:

$$\dot{F}_\nu + i(\vec{k}\cdot\hat{n})F_\nu + 2\dot{h}_{ij}n_i n_j = \frac{4\pi}{a^4\bar{\rho}}\int dqq^3 \left(\frac{\partial f}{\partial \tau}\right)_C, \tag{H.4.0.6}$$

having defined

$$F_\nu(\vec{k},\,\hat{n},\,\tau) \equiv \frac{\int dqq^3 f_0(q)\Psi(\vec{k},\,q,\,\hat{n},\,\tau)}{\int dqq^3 f_0(q)}, \tag{H.4.0.7}$$

where Ψ is the fractional perturbation to the distribution function, i.e. $f =$

$f_0(1+\Psi)$. In general we follow the notation of Ma and Bertschinger (1995). Since we are interested in tensor modes, we take h_{ij} to be transverse and traceless, i.e. $h_{ii} = 0$ and $h_{ij}k_j = 0$. The perturbation h_{ij} evolves according to the Einstein equation:

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} + k^2 h_{ij} = 16\pi G a^2 \Pi_{ij}, \tag{H.4.0.8}$$

where $\mathcal{H}$ is the conformal Hubble constant' $\mathcal{H} = \dot{a}/a$.
The anistropic stress tensor is given by Π_{ij}:

$$\Pi_{ij} \equiv T^i_j - \bar{P}\delta_{ij} = a^{-4}\left[\int d\Omega n_i n_j F_\nu\right]\left[\int dq q^3 f_0\right] = \frac{\bar{\rho}_\nu}{4\pi}\int d\Omega n_i n_j F_\nu. \tag{H.4.0.9}$$

Then we define

$$G_{ij}(\vec{k}, \mu, \tau) = \int_0^{2\pi} n_i n_j F_\nu d\phi, \tag{H.4.0.10}$$

where ϕ is the polar angle, i.e. $d\Omega = \sin\theta d\theta\, d\phi$. By multiplying Eq.(H.4.0.6) by $n_i n_j$ and integrating in $d\phi$, we obtain ($\mu \equiv \hat{k}\cdot\hat{n}$):

$$\dot{G}_{ij} + ik\mu\, G_{ij} + 2\dot{h}_{\ell m}\int_0^{2\pi} n_\ell n_m n_i n_j d\phi = \int_0^{2\pi} n_i n_j (RHS) d\phi. \tag{H.4.0.11}$$

For the moment let us consider a collisionless gas: $RHS \propto (\partial f/\partial\tau)_C = 0$. Then we expand G_{ij} in in Legendre polynomials:

$$G_{ij}(\vec{k}, \mu, \tau) = \sum_{\ell=0}^{\infty}(-i)^\ell(2\ell+1)G^{(\ell)}_{ij}(\vec{k}, \tau)P_\ell(\mu). \tag{H.4.0.12}$$

With this definition, it can be shown that the Boltzmann Eq. (H.4.0.11) with vanishing RHS is equivalent to the following infinite "tower" of first order differential equations for the moments $G_{ij}^{(\ell)}$:

$$\dot{G}_{ij}^{(0)} = -k\, G_{ij}^{(1)} - \frac{8\pi}{15}\dot{h}_{ij} - \frac{G_{ij}^{(0)}}{\tau}, \qquad \text{(H.4.0.13)}$$

$$\dot{G}_{ij}^{(2)} = -\frac{k}{5}\left[3G_{ij}^{(3)} - 2G_{ij}^{(1)}\right] - \frac{16\pi}{105}\dot{h}_{ij} - \frac{G_{ij}^{(2)}}{\tau}, \qquad \text{(H.4.0.14)}$$

$$\dot{G}_{ij}^{(4)} = -\frac{k}{9}\left[5G_{ij}^{(5)} - 4G_{ij}^{(3)}\right] - \frac{8\pi}{315}\dot{h}_{ij} - \frac{G_{ij}^{(4)}}{\tau}, \qquad \text{(H.4.0.15)}$$

$$\dot{G}_{ij}^{(\ell)} = -\frac{k}{2\ell+1}\left[(\ell+1)G_{ij}^{(\ell+1)} - \ell\, G_{ij}^{(\ell-1)}\right] - \frac{G_{ij}^{(\ell)}}{\tau} \qquad (\ell \neq 0, 2, 4). \qquad \text{(H.4.0.16)}$$

The evolutive equation for h_{ij} [Eq. (H.4.0.8)] reduces to:

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} + k^2 h_{ij} = 4Ga^2\bar{\rho}_\nu G_{ij}^{(0)}. \qquad \text{(H.4.0.17)}$$

Numerically solving the above system of differential equations, we can calculate how much a gravitational wave with a given frequency is damped due to the interaction with neutrinos [we remember that in the absence of anisotropic stress $h_{ij} \propto \sin(k\tau)/(k\tau)$.] and thus obtain a "transfer function" for primordial gravitational waves. It can be shown that, for waves entering the horizon during the radiation dominated era, the amount of damping does not depend on the frequency of the wave. It is found that in this regime the damping actually depends just on the neutrino fraction $f_\nu = \rho_\nu/(\rho_{rad} + \rho_\nu)$, that basically controls the magnitude of the source term on the right-hand side of Einstein equation for h_{ij}. An important feature is that f_ν has a sharp decrease at a temperature of ~ 1 MeV due to creation of photons following the annihilation of relic electrons and positrons. This leads to a feature' in the transfer function at a frequency $f \sim 10^{-9}$ Hz, corresponding to a wavelength equal to the comoving size of the horizon at $T \sim 1$ MeV. This is shown in Fig H.4.2. This feature is potentially very interesting from the observational point of view because the nano-Hertz region will be probed by Pulsar Timing Arrays (Verbiest et al., 2009).

The next step will be to account for the weak interactions between neutrinos, electrons and positros at $T \simeq 1$ MeV, by adding a suitable collision term in the right-hand side of the Boltzmann equation.

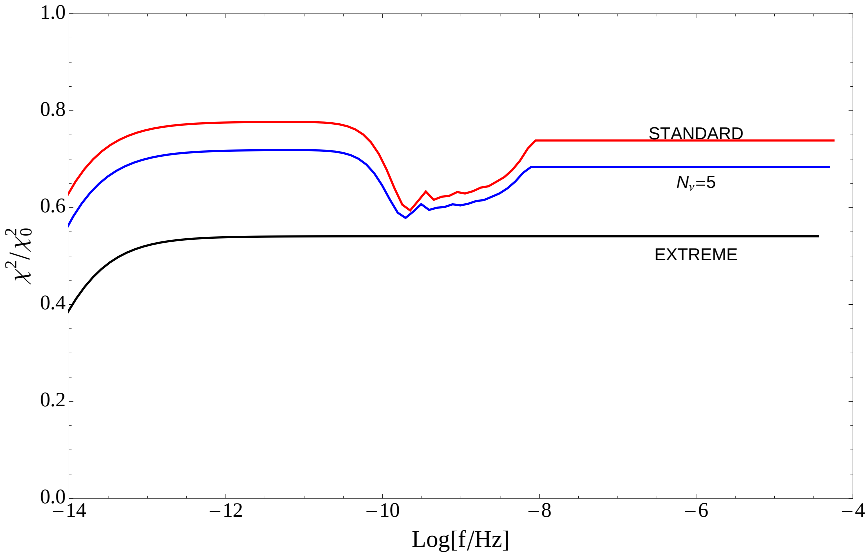

Figure H.4.2: Damping in the wave intensity χ^2/χ_0^2 as a function of frequency. The curve labeled "standard" refers to the usual case of 3 neutrino families. The curve labeled "$N_\nu = 5$" refers to the case of an additional energy density of decoupled species, equivalent to two more neutrino specise. The curve labeled "Extreme" refers to the case where $\rho_\nu \gg \rho_\gamma$ and thus f_ν is constant and equal to1.

H.5 On the coupling between Spin and Cosmological Gravitational Waves

We are interested in studying the influence of spin on the dynamics of particles and on their interaction with gravitational waves, in a cosmological framework (Milillo et al., 2008). The equations of motion of spinning particles in the framework of general relativity were derived by Papapetrou in 1951: through a multipole expansion for the energy-momentum tensor he found that, at dipole order, a deviation from geodetic motion and an equation describing spin precession are obtained. These equations are:

$$\frac{D}{Ds}p^{\mu} = -\frac{1}{2}R^{\mu}_{\ \nu\rho\sigma}S^{\rho\sigma}u^{\nu} \quad ; \tag{H.5.0.1}$$

$$\frac{D}{Ds}S^{\mu\nu} = p^{\mu}u^{\nu} - p^{\nu}u^{\mu}, \tag{H.5.0.2}$$

where ds is the affine parameter, the vector p^{μ} is the generalized momentum, the antisymmetric tensor $S^{\mu\nu}$ is the angular momentum (Spin) and $u^{\mu} = dx^{\mu}/ds$. In order to close the system we impose a supplementary condition which determines the center of mass of the spinning particles: $S^{\mu\nu}u_{\nu} = 0$ (Pirani condition).
In our work we consider the case of absence of precession, so that the right-hand side of H.5.0.2 is zero and the generalized momentum is equal to the standard momentum. In this case, solving the Papapetrou equations we obtain the temporal dependence of S^{ij} through the cosmological scale factor a:

$$S^{ij} = \frac{1}{a^2}\Sigma^{ij}, \tag{H.5.0.3}$$

in which Σ^{ij} is a constant.

Since we are interested in the application of this formalism in a cosmological framework, we consider a fluid of collisionless spinning particles in a Friedmann-Robertson-Walker (FRW) background. We find that, due to the symmetry proprieties of the metric tensor, the Boltzmann equation for the evolution of the distribution function of the spinning particles, remains unchanged by the presence of the spin. Then we add a small tensor perturbation

h_{ij} in the metric, representing a gravitational wave, looking for a coupling between it and the spin of the particles in the fluid. The resulting Boltzmann equation gives a first order variation of the distribution function that is proportional to the product between the spin tensor and the time derivative of h_{ij}. The form of the distribution function up to the first order in metric perturbation allows us to calculate the anisotropic stress arising by the presence of spin:

$$\pi_{ij}^{(S)} = \frac{i}{2} n \int_0^u du' K(u - u') k_m \Sigma^{lm} (k_i \dot{h}_{jl} + k_j \dot{h}_{il}), \tag{H.5.0.4}$$

where u is the conformal time, k_i are the components of the wave vector of the gravitational wave, n is the number density of the spinning particles and the integral kernel K is defined as:

$$K(s) \equiv \frac{1}{64} \int_{-1}^{1} e^{ixs} (1 - x^2) x^2 dx. \tag{H.5.0.5}$$

Even if this shows that the spin alters some components of the anisotropic stress tensor, the final result is that these components are those that don't couple directly with the evolution of h_{ij}. This is easily understood considering the differential equation for tensorial perturbations:

$$\ddot{h}_{ij}(u) + \frac{2\dot{a}(a)}{a(u)} \dot{h}_{ij} = 16\pi G (\pi_{ij}^{(0)}(u) + \pi_{ij}^{(S)}(u)) \tag{H.5.0.6}$$

and fixing the gauge such that $h_{i3} = 0, \vec{k} = (0, 0, 1).(\pi_{ij}^{(0)}(u)$ is the part of the anisotropic stress that does not depend on the spin).

The final result is that there is no coupling between spin and cosmological gravitational waves, if only tensor perturbations are present.

We have also studied the generalization of this result when also scalar and vector perturbations are present (Lattanzi et al., 2009). In this case we find that the presence of the spin couples the different kinds of perturbations between them. The next step will be to study if this modified cosmological evolution could possibly have any observational consequence.

H.6 The role of Plasma Physics in Accretion disk morphology

Let us fix here the basic statements concerning the steady MHD regime for the specific case of an accretion disk configuration around a compact astrophysical object (a mass over the critical value to have a neutron star within a radius of few kilometres), which is also strongly magnetised (a dipole-like field of about 10^{12}*Gauss*). Thus, we deal with a typical pulsar, accreting material from a binary companion via a thin disk structure made up of rotating plasma in equilibrium.

To adapt the MHD equations to the disk symmetry, we introduce cylindrical coordinates $\{r, \phi, z\}$. The gravitational potential of the pulsar takes, in this coordinates, the form

$$\chi(r, z) = -\frac{GM}{\sqrt{r^2 + z^2}}, \tag{H.6.0.1}$$

where G denotes the Newton's constant and M the mass of the spherical central object, while the self-gravitation of the disk is regarded as a negligible effect.

To describe the magnetic field, we take the potential vector in the form

$$\vec{A} = \partial_r \Pi \vec{e}_r + \psi \vec{e}_\phi + \partial_z \Pi \vec{e}_z, \tag{H.6.0.2}$$

where $\Pi(r, z)$ and $\psi(r, z^2)$ are arbitrary functions, but only the last one is a physical degree of freedom, since the magnetic field reads

$$\vec{B} = -\frac{1}{r}\partial_z \psi \vec{e}_r + \frac{1}{r}\partial_r \psi \vec{e}_z. \tag{H.6.0.3}$$

The flux function has to be decomposed as $\psi = \psi_0 + \psi_1$, where ψ_0 accounts for the dipole-like magnetic field of the pulsar (i.e. $\psi_0 = \mu r^2 (r^2 + z^2)^{-3/2}$, with $\mu = const.$), while ψ_1 is a contribution due to the toroidal currents rising in the disk configuration. Recalling that the axial symmetry prevents any dependence on the azimuthal angle ϕ of all the quantities involved in the problem, the continuity equation, associated to the mass density ϵ and to the velocity field $\vec{v}$, takes the explicit form

$$\frac{1}{r}\partial_r\left(r\epsilon v_r\right)+\partial_z\left(\epsilon v_z\right)=0\,, \tag{H.6.0.4}$$

which admits the solution

$$\epsilon\vec{v}=-\frac{1}{r}\partial_z\Theta\vec{e}_r+\epsilon\omega(r\,,z)r\vec{e}_\phi+\frac{1}{r}\partial_r\Theta\vec{e}_z\,, \tag{H.6.0.5}$$

$\Theta=\Theta(r\,,z)$ being a generic function, but for its odd symmetry in the z-coordinate. This property has to be required in order to ensure that the accretion rate of the disk, as averaged over the vertical direction, be non-vanishing, i.e.

$$\dot{M}_d=-2\pi r\int_{-z_0}^{z_0}\epsilon v_r dz=4\pi\Theta(r\,,z_0)\equiv 2\pi I\neq 0\,, \tag{H.6.0.6}$$

where z_0 is the half depth of the disk and we recall that $v_r<0$ on the vertical average, to ensure a real accretion of the disk.

The different symmetry of the functions ψ and Θ with respect to the z-dependence, prevents to fix *a priori* a relation between them. The angular velocity $\omega(r\,,z^2)$, describing the differential rotation of the disk, is an even function of z and, by virtue of the corotation theoremFerraro (1937), can be taken as a function of the flux surfaces, i.e. $\omega=\omega(\psi)$.

The momentum transfer through the disk structure is characterised by an azimuthal friction between the radial layers, properly described by a viscosity coefficient η, responsible for the corresponding non-vanishing stress tensor components. The coefficient η can be decomposed as $\eta=\eta_0+\eta_1$, where the η_1 component is associated with the additional effects arising from the toroidal currents. The necessity to include the electromagnetic back-reaction terms in the viscosity coefficient, is a consequence of the specific coupling such back-reaction establishes. In fact, it was demonstratedCoppi (2005b) that the currents induced in the plasma of the disk, provide an intrinsic local coupling between the radial and the vertical equilibrium, absent in the lowest-order approximation.

Local configuration of the thin disk Let us now specialise our model to the case of a thin disk, i.e. $z_0/r\ll 1$ over the whole configuration. On average, the dominant contribution to the angular velocity of the plasma particles, is then the (equatorial) Keplerian value $\omega_K\equiv(GM/r^3)^{\frac{1}{2}}$.

For a thin disk, the z-component of the velocity can be properly identified (see Shakura (1973) Shakura (1973)) with the sound velocity $v_s\equiv\sqrt{2K_BT/m_i}$ (K_B denoting the Boltzmann constant, T the plasma temperature, and m_i the ion mass), entering the fundamental inequality

$$\frac{z_0}{r} \sim \frac{v_s}{v_\phi} \ll 1\,, \tag{H.6.0.7}$$

which ensures the possibility to neglect the vertical motion with respect to the azimuthal one.

To estimate on average the ratio between the radial velocity and the azimuthal one, we can follow the simple equilibrium condition to fix the asymptotic radial velocity of a plasma element falling into the disk

$$\nu_c v_r \sim \omega^2 r = \omega v_\phi \quad \Rightarrow \quad \frac{v_r}{v_\phi} \sim \frac{\omega}{\nu_c}\,, \tag{H.6.0.8}$$

ν_c being a characteristic frequency of particle collision. A rough estimation provides $\nu_c \sim v_s n^{1/3}$, where n denotes the number density. Taking $v_s \sim 10^{-3}c$ and $n \sim 10^{10} cm^{-3}$, we get $\nu_c \sim 10^9 Hz$, a value much greater than the typical Keplerian frequency. Thus, we conclude that $v_\phi \sim \omega_K r$ is, in practice, responsible for the main matter flow within the disk, despite its stationary accretion takes place along the radial direction.

In this limit of approximation, the MHD condition on the balance of the Lorenz force has dominant radial and vertical components, providing the electric field in the form expected by the corotation theorem, i.e.

$$\vec{E} = -\frac{\vec{v}}{c} \wedge \vec{B} = -\frac{d\Phi}{d\psi}\vec{\nabla}\psi = -\frac{\omega}{c}\left(\partial_r \psi \vec{e}_r + \partial_z \psi \vec{e}_z\right)\,, \tag{H.6.0.9}$$

Φ denoting the electrostatic potential. However, the axial symmetry prevents a dependence of Φ on the azimuthal angle and hence, the corresponding ϕ-component of the electric field vanishes identically. This fact requires an additional effect to be included into the problem, able to balance the non-vanishing Lorenz force in the azimuthal direction, due to the velocity components v_r and v_z. We will address this crucial question in Section 6, in the limit of a linear theory.

In the thin disk approximation, the strength of the gravitational field $\vec{G}$ reads as

$$\vec{G} = -\omega_K^2 r \vec{e}_r - \omega_K^2 z \vec{e}_z\,. \tag{H.6.0.10}$$

We see that a thin disk configuration is justified only in the presence of a sufficiently high rotation to deal with a confining vertical gravitational force (for the linear scheme, here addressed, the Lorenz force can not affect this statementCoppi and Rousseau (2006)).

According to the analysis in Coppi (2005b), we now develop the disk configuration around a given value of the radial coordinate r_0, limiting our attention to a narrow enough interval to express the local angular velocity field in the form

$$\omega \simeq \omega_K + \delta\omega \simeq \omega_K + \frac{d\omega}{d\psi_0}\psi_1 \equiv \omega_K + \omega_0'\psi_1 \,, \tag{H.6.0.11}$$

where $\psi_1 = \psi_1(r_0\,, z^2\,, r - r_0)$. We approximate the dipole surface function, at r_0, as $\psi_0(r_0) \simeq \frac{\mu}{r_0}$. We drop the relic z-dependence in view of the thin nature of the disk.

In what follows, the analysis is performed by addressing the drift ordering approximation that fixes the dominant character of the second spatial derivatives of ψ_1. In particular, the profile of the disk, so outlined, will include viscous features, as in Shakura (1973), but making account for the crystalline structure derived in Coppi (2005b). Indeed, reconciling the accretion feature of the disk with the plasma effects, as described in a bidimensional MHD approach, we will need a significant deviation from the standard mechanism of angular momentum transfer, in the sense that the azimuthal equation holds at the level of electromagnetic back-reaction only.

Configuration Equations In order to fix the profile of the disk around the configuration at r_0, we have to provide the equations governing the radial, the vertical and the tangential equilibrium in the presence of the toroidal currents. The radial and the vertical configurations are not significantly affected by the viscosity, since its presence mainly concerns the differential rotation of the diskShakura (1973). Therefore, these systems stand in the same form as in Coppi (2005b), while the implications due to a non-zero viscous stress are summarised by the azimuthal equilibrium.

In order to cast the whole system, we split the energy density and the pressure in the form

$$\epsilon = \bar{\epsilon}(r_0,\, z^2) + \hat{\epsilon}(r_0,\, z^2,\, r - r_0) \quad , \quad p = \bar{p}(r_0,\, z^2) + \hat{p}(r_0,\, z^2,\, r - r_0)\,, \tag{H.6.0.12}$$

where the barred quantities are the contributions existing in absence of the toroidal currents, while the terms denoted with a hat are induced by such an electromagnetic reaction. According to the framework we outlined, the vertical equilibrium is governed by the two relations

$$D(z^2) \equiv \frac{\bar{\epsilon}}{\epsilon_0(r_0)} = \exp^{-\frac{z^2}{H_0^2}}\,,\ \epsilon_0(r_0) \equiv \epsilon(r_0,\, 0)\,,\ H_0^2 \equiv \frac{2K_B\bar{T}}{m_i\omega_K^2}\,, \tag{H.6.0.13}$$

$$\partial_z\hat{p} + \omega_K^2 z\hat{\epsilon} + \frac{1}{4\pi r_0^2}\left(\partial_z^2\psi_1 + \partial_r^2\psi_1\right)\partial_z\psi_1 = 0 \tag{H.6.0.14}$$

The radial equation underlying the equilibrium of the rotating layers of the disk, takes the form

$$\omega \simeq \omega_K + \delta\omega \simeq \omega_0(\psi_0) + \frac{d\omega_0}{d\psi_0}\psi_1 \,, \tag{H.6.0.15}$$

$$\begin{aligned} 2\omega_K r_0(\bar{\epsilon} + \hat{\epsilon})\frac{d\omega_0}{d\psi_0}\psi_1 - \frac{1}{4\pi r_0^2}\left(\partial_z^2\psi_1 + \partial_r^2\psi_1\right)\partial_r\psi_1 = \\ = \partial_r\left[\hat{p} + \frac{1}{8\pi r_0^2}\left(\partial_r\psi_1\right)^2\right] + \frac{1}{4\pi r_0^2}\partial_r\psi_1\partial_z^2\psi_1 \end{aligned} \tag{H.6.0.16}$$

The azimuthal equation exactly reads

$$\epsilon v_r\partial_r(\omega r) + \epsilon v_z\partial_z(\omega r) + \epsilon\omega v_r = \frac{1}{r^2}\partial_r\left(\eta r^3\partial_r\omega\right) + \partial_z\left[\eta\partial_z(\omega r)\right] \,. \tag{H.6.0.17}$$

Accounting for the corotation theorem $\omega = \omega(\psi)$ we can restate the spatial derivatives of ω, in terms of the corresponding ones taken on ψ. Recalling the expression $\eta = \eta_0 + \eta_1$, we can now make the reasonable assumption that the viscosity correction η_1 be written as

$$\eta_1(r_0\,,\psi_1) \sim \left(\frac{d\eta_1}{d\psi_1}\right)_{\psi_1=0\,,r=r_0}\psi_1 \equiv \eta_0'\psi_1\,, \tag{H.6.0.18}$$

we recall that for vanishing ψ_1, the correction η_1 vanishes too. In the work Shakura (1973), a valuable proposal for the expression describing the viscosity coefficient $\eta_0(r_0)$ is provided, i.e.

$$\eta_0(r_0) \equiv \frac{2}{3}\alpha\epsilon_0 v_{s0} z_0\,, \tag{H.6.0.19}$$

where v_{s0} denotes the sound velocity on the equatorial plane and α is a parameter, whose value must be assigned. Since we have $\partial_{r_0}\eta_0 \sim \eta_0/r_0$, the first term on the right-hand-side is negligible in comparison to the term containing second derivatives. Furthermore, the request $\eta_1 \ll \eta_0$ implies that $\eta_0' \ll \eta_0/\psi_1$, ensuring that also the quadratic terms in the derivatives of ψ_1 are much smaller than the remaining ones.

Therefore the form taken by the azimuthal equation for the steady state of the disk reads as

$$\begin{aligned} -\partial_z\Theta\left(\partial_{r_0}\psi_0 + \partial_r\psi_1\right) + \partial_r\Theta\partial_z\psi_1 - \frac{2}{r_0}\partial_z\Theta\frac{\omega_0}{\omega_0'} = \\ r_0\left[\partial_r\psi_1\eta_0'\partial_{r_0}\psi_0 + \eta_0(r_0)\left(\partial_r^2\psi_1 + \partial_z^2\psi_1\right)\right] \,. \end{aligned} \tag{H.6.0.20}$$

This equation provides a differential relation between the flux surface ψ_1 and the function Θ characterising the radial and vertical matter fluxes.

Configuration in the linear approximation Let us analyse the compatibility between the radial and the azimuthal equations, in the limit when the induced z-field B_z^1 is much smaller than the source one B_{z0}, i.e.

$$\frac{B_z^1}{B_{0z}} \sim k_0 r_0 \frac{\psi_1}{\psi_0} \ll 1\,, \tag{H.6.0.21}$$

where the magnetic surface ψ_1 admits the explicit dependence

$$\psi_1 = \psi_1 \left(r_0,\, k_0(r - r_0),\, \frac{z^2}{\Delta^2} \right) . \tag{H.6.0.22}$$

Here, we set $k_0^2 = 3\omega_K^2/v_{A0}^2$, with $v_{A0}^2 = B_{z0}^2/4\pi\epsilon_0$ being the Alfvén velocity in the plasma. Furthermore, Δ denotes a narrow interval for the localisation of the z-dependence. Under these hypotheses, the radial equation rewrites

$$\partial_r^2\psi_1 + \partial_z^2\psi_1 = -k_0^2 D(z^2)\psi_1\,. \tag{H.6.0.23}$$

At lowest orders in z, we take the following expansion $D(z^2) \simeq 1 - z^2/H_n^2$, H_n being of the same order of magnitude of H_0. In this limit, the radial equation admits the solution discussed in Coppi (2005b) which oscillates as a sin function along the radial direction, while it exponentially decays in z^2 along the vertical configuration, i. e. ψ_1 reads as

$$\psi_1 = \psi_1^0 \sin\left[k(r - r_0)\right] \exp\{-\frac{k_0 z^2}{H_n}\}\,, \tag{H.6.0.24}$$

where we set $k \equiv k_0\sqrt{1 - \frac{1}{k_0 H_n}}$ and we obtained the identification $\Delta^2 \equiv H_n/k_0$. Such a structure is the linear feature of the crystalline morphology induced in the disk by the toroidal currents.

In the approximation where $k_0 r_0 \gg 1$, we can require that the contribution due to the currents on the viscosity coefficient be sufficiently small to neglect the term in η_0' of (H.6.0.20), i. e. by virtue of the inequality

$$\frac{\eta_0'\psi_0}{\eta_0} \ll k_0 r_0\,. \tag{H.6.0.25}$$

Thus, the final form we address for the azimuthal equation is as follows

$$-\partial_z\Theta\left(\partial_{r_0}\psi_0 + \frac{2\omega_0}{r_0\omega_0'} + \partial_r\psi_1\right) + \partial_r\Theta\partial_z\psi_1 = r_0\eta_0(r_0)\left(\partial_r^2\psi_1 + \partial_z^2\psi_1\right) . \tag{H.6.0.26}$$

We use this equation to complete the configuration scheme of our thin disk equilibrium. Let us now assume that the term containing the derivative $\partial_r\Theta$ be negligible (this is natural because it is multiplied by the small r-component of the magnetic field) and then divide the equation by $\partial_{r_0}\psi_0$. Observing that $\omega_0'\partial_{r_0}\psi_0 = \partial_{r_0}\omega_0 = -3\omega_0/2$ (where $\omega_0 \equiv \omega_K$), we arrive to the relation

$$\epsilon_0(r_0)v_r = -\frac{1}{r_0}\partial_z\Theta = 3\eta_0 k_0 Y\,, \qquad \text{(H.6.0.27)}$$

in which we made use of the definition of the dimensionless function $Y \equiv k_0\psi_1/\partial_{r_0}\psi_0$. Comparing the two equations, we neglected the factor $D(z^2)$ with respect to the z-dependence of ψ_1, because $H_n \gg \delta$, i.e. $k_0 H_n \gg 1$.

Such an accreting profile provides a net increasing of the disk mass, according to the relation

$$\dot{M}_d = -2\pi r\int_{-\infty}^{\infty}\epsilon v_r dz \simeq 6\sqrt{2}\pi^{3/2}\eta_0(k_0r_0)^2\frac{\Delta\psi_1^0}{\mu}r\sin k(r-r_0)\,, \qquad \text{(H.6.0.28)}$$

where we made use of the dipole-like expression for ψ_0 and we extended the integration over the vertical coordinate up to infinity.

It is worth noting that, despite the radial velocity oscillates, the radial matter flux is, on average, an in-falling one. In fact, averaging the expression above between two nodes around r_0, we get

$$\langle\dot{M}_d\rangle = 12\sqrt{2}\pi^{5/2}\eta_0(k_0r_0)^2\frac{\Delta\psi_1^0}{\mu k^2}\,. \qquad \text{(H.6.0.29)}$$

Equation (H.6.0.27) can be easily solved for Θ in agreement with the adopted approximation of neglecting the term containing $\partial_r\Theta$, as follows

$$\Theta = -3\eta_0 k_0 r_0\int Y dz\,. \qquad \text{(H.6.0.30)}$$

Hence we get the vertical velocity in the form

$$v_z = \frac{1}{\epsilon_0 r_0}\partial_r\Theta = -\frac{3\eta_0 k_0 k}{\epsilon_0}\int Y dz\,. \qquad \text{(H.6.0.31)}$$

We see that, in the linear approximation, the azimuthal equation ensures the existence of a non-zero accretion rate, which is modulated by the oscillating profile of the disk configuration.

The Electron Force Balance Equation Despite we have properly argued that the radial and vertical components of the fluid velocity are significantly smaller than the ϕ-component of the velocity, nevertheless, the MHD electron force balance, restated here

$$\vec{E} + \frac{\vec{v}}{c}\wedge\vec{B} = 0\,, \qquad \text{(H.6.0.32)}$$

requires $(\vec{v} \wedge \vec{B})_\phi = 0$, because of $E_\phi \equiv 0$, by virtue of the axial symmetry. For instance, this feature arises naturally if the corotation theorem is addressed. More simply, this request comes out from the independence of ϕ characterising the electrostatic potential. Thus, on the left-hand side of equation (H.6.0.32) an additional term has to appear in order to save the model consistence. In the limit of the linear theory, the most natural term to be included in the analysis, preserving the MHD scenario we addressed, is a weak collisional effect, as phenomenologically described by a non-zero constant Nernst coefficient. However, to introduce this hypothesis we need to give up the idea, pursued above, that the plasma in the disk be perfectly isothermal. Thus, we now introduce temperature gradientsCoppi (2008) in the spirit that they do not affect significantly the three configuration equations developed in the previous sections. We discuss only the implications for the electron force balance when a Nernst coefficient is present and then we fix the restriction on the model parameters to get full consistence.

According to this idea, we split the temperature as $T = T_0 + T_1(r, z^2)$, with $T_0 = const.$ and $T_1 \ll T_0$. In the presence of a non-zero Nernst coefficient $\mathcal{N}$, the equation (H.6.0.32) takes the form

$$\vec{E} + \frac{\vec{v}}{c} \wedge \vec{B} = \mathcal{N}\vec{B} \wedge \vec{\nabla}T \, . \tag{H.6.0.33}$$

Assuming that the gradients of the temperature are negligible in the radial and vertical component of this equation, its ϕ-component provides

$$v_r = -\mathcal{N}c\partial_r T_1 \quad , \quad v_z = -\mathcal{N}c\partial_z T_1 \, . \tag{H.6.0.34}$$

We now have to observe that the temperature of the plasma in the static MHD framework, must obey the equation

$$\vec{v} \cdot \vec{\nabla}T + \frac{2}{3}T\vec{\nabla} \cdot \vec{v} = 0 \, . \tag{H.6.0.35}$$

Expressing the gradients of the temperature by the relation (H.6.0.34), the equation above takes the form

$$\frac{1}{r}\partial_r(rv_r) + \partial_z v_z = \frac{3}{2\mathcal{N}T_0 c}\left(v_r^2 + v_z^2\right) \, . \tag{H.6.0.36}$$

It is easy to see that this equation is automatically satisfied in the linear regime, by virtue of the expressions for the radial and vertical velocity (H.6.0.27) and (H.6.0.31) respectively.

The complete consistence of our linear model is then guaranteed by requiring that the radial temperature gradient be negligible in the corresponding configuration equation. Such gradients arise in the radial equation from the spatial variation of the pressure $p = 2\epsilon K_B T/m_i$ (from which we get

$\hat{p} = 2K_BT_0\hat{\epsilon}/m_i + 2\epsilon_0K_BT_1/m_i)$. The condition for the negligibility of this term, can be stated, by virtue of equations (H.6.0.27) and (H.6.0.34), in the form

$$\partial_{r_0}\psi_0 \gg r_0\sqrt{\frac{6\eta_0K_B}{m_i\mathcal{N}c}}\,, \tag{H.6.0.37}$$

To get this result, we made account for the fact that obtaining equation (H.6.0.23), we divided the original radial equation by the term $\partial_{r_0}\psi_0/r_0^2$. By making use of the expression (H.6.0.19) for η_0, we can rewrite the condition above in terms of the Alfvén and equatorial sound velocities as $v_A \gg \delta v_{s0}$, with $\delta \equiv \sqrt{\frac{\alpha H_0 v_{s0}}{4\pi\mathcal{N}T_0c}}$ (we approximated everywhere the sound velocity with its value on the equatorial plane and we have taken $z_0 \equiv H_0$). Furthermore we implicitly required that the radial variation of $\hat{\epsilon}$ be smaller or, at most, equal to the term with the temperature gradient.

Finally, the vertical equation, in the linear approximation reads as

$$\partial_z\hat{p} + \omega_K^2 z\hat{\epsilon} = 0\,, \tag{H.6.0.38}$$

which easily rewrites as an equation for $\hat{\epsilon}$, i. e.

$$\frac{K_BT_0}{m_i}\partial_z\hat{\epsilon} + \frac{\omega_K^2}{2}z\hat{\epsilon} + \frac{3K_B\eta_0k_0k}{m_i\mathcal{N}c}\int Y dz = 0\,. \tag{H.6.0.39}$$

Such an equation fixes the following form for the perturbed mass density $\hat{\epsilon} = \bar{\epsilon}(z^2)\sin k(r - r_0)$, where the z-dependence comes from the ordinary differential equation

$$\frac{K_BT_0}{m_i}\frac{d\bar{\epsilon}}{dz} + \frac{\omega_K^2(r_0)}{2}z\bar{\epsilon} + \frac{3K_B\eta_0k_0k}{m_i\mathcal{N}c}Y^0\int \exp\left\{-\frac{k_0z^2}{H_n}\right\}dz = 0\,, \tag{H.6.0.40}$$

having defined $Y^0 \equiv k_0\psi_1^0/\partial_{r_0}\psi_0$. The above equation, using the expression (H.6.0.19) for η_0, can be rewritten in the dimensionless form

$$\xi_1\frac{d\bar{D}}{d\bar{z}} + \frac{\bar{z}}{2}\bar{D} + \xi_2\,\mathrm{erf}\left(\frac{\bar{z}}{\sqrt{2}}\right) = 0\,, \tag{H.6.0.41}$$

$$\xi_1 = \frac{K_BT_0}{m_i\omega_K^2\Delta^2} > 1\,, \quad \xi_2 = \frac{2K_B\alpha v_{s0}z_0k_0k}{m_i\mathcal{N}c\omega_K^2}Y^0\sqrt{\frac{\pi}{2}}$$

where we defined $\bar{z} \equiv z/\Delta$, $\bar{D} \equiv \bar{\epsilon}/\epsilon_0$ and erf is the error function: $\sqrt{\pi}\mathrm{erf}(z) \equiv$

$2\int \exp(-z^2)dz$. This equation admits the formal solution

$$\bar{D} = e^{-\frac{\bar{z}^2}{4\xi_1}} \left(\bar{D}(\bar{z}=0) - \xi_2 \int_0^{\bar{z}} \frac{e^{\frac{u^2}{4\xi_1}} \mathrm{erf}(\frac{u}{\sqrt{2}})}{\xi_1} du \right) \qquad \text{(H.6.0.42)}$$

which is plotted in fig H.6.1 for several values of ξ_1 and ξ_2

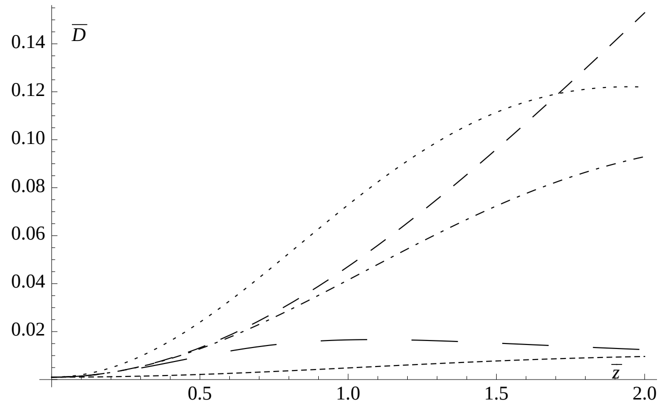

Figure H.6.1: Behaviour of $\hat{D}$ for several values of ξ_1 and ξ_2. The figure outlines an increasing behaviour of the function $\bar{D}$, and a maximum sometimes arises in the range of validity for the linear approximation.

We conclude this analysis, by stressing that, because of (H.6.0.19), the expression (H.6.0.27) provides the following estimation for v_r

$$v_r \sim 2\alpha v_{s0} k_0 H_0 \mathcal{O}(Y^0) \,. \qquad \text{(H.6.0.43)}$$

We see that the accretion velocity is few orders of magnitude less than the sound velocity on the equatorial plane. In this respect, we observe that the parameter α can be taken in the range $10^{-3} - 1$, while the dimensionless term $k_0 H_0 \sim k_0 H_n \sim \sqrt{\beta}$ can be even much greater than unity and the linear approximation requires $Y^0 \ll 1$.

Despite the picture we fixed above is well-grounded in the linear approximation, it is well-known that the Nernst coefficient is obtained in a kinetic theory via an expansion in the inverse powers of the cyclotronic frequency and it corresponds to a first order approximation scheme. Thus, this effect is, in principle, small and we can expect that it be the proper explanation to the accretion features of a crystalline disk, in the weak field limit only, i.e. we have to require $\beta > 1$, to ensure the model reliability.

H.7 On the gravitational polarizability of black holes

In Newtonian gravity, the quantitative theory of the "gravitational polarizability" of elastic, self-gravitating bodies was pioneered by Love Love (1911), who introduced two dimensionless measures of the response of an elastic body to an external tidal solicitation. To define them, let us first decompose the external tidal potential into multipolar components, say $U_{\rm ext} = \sum U_l = \sum T_l r^l P_l(\cos\theta)$, where the coefficient T_l measures the strength of the l-th multipolar component of the external tidal field. The first "Love number", h_l, measures, essentially, the ratio between the l-th multipolar component of the distortion of the shape of the considered elastic body and T_l, while the second "Love number", k_l, measures the ratio between the l-th order multipole moment induced in the elastic body and T_l.

In the membrane approach to black holes (BH's) Damour (1978); Znajek (1978); Damour (1979, 1982); Macdonald and Suen (1985); Thorne et al. (1986), BH's are treated as elastic objects, endowed with usual physical properties. This raises the issue of defining and determining the BH analog of the Love numbers h_l and k_l. However, there are subtleties inherent in any definition of the multipole moments of BH's, so that there is currently no unambiguous determination of the k_l Love number of BH's (see, e.g., Fang and Lovelace (2005) and Damour and Nagar (2009) for discussions of the various ambiguities in the definition of the multipole moments of BH's). We shall not try to address these subtleties, and we shall, instead, focus on the computation of h_l, i.e. on the quantitative measure of the tidal distortion of the shape of a BH. By contrast to k_l, we shall see that there is no ambiguity in the computation of the first Love number h_l of a BH.

We compute (in the linear approximation to tidal effects) the "shape distortion" Love numbers h_l of a BH, for all values of $l \geq 2$. The sequence of h_l's is a way of parameterizing the tidal effects due to a disturbing mass m located *far away* from the considered BH. We are also interested in studying the distortion of the shape of a BH when a disturbing *test* mass gets *very near* the surface of the BH. In this respect, it is useful to compare and contrast the gravitational polarizability of BH's to their electric polarizability properties. Let us recall that Hanni and Ruffini Hanni and Ruffini (1973) pioneered the study of the electric polarizability properties of BH's, and introduced the notion of the charge density, σ, induced on the BH horizon by an external charge q. [This concept of induced charge density was one of the origins of the membrane approach to BH physics Damour (1978); Znajek (1978); Damour (1979, 1982); Macdonald and Suen (1985); Thorne et al. (1986).] We use the concept of induced charge density to define electric analogs of the h_l Love numbers, which we compare to the gravitational ones. Then, we also compare the evolution of the gravitational, or electric, induced effects as the external tidally-influencing (test) mass m, or charge q, approach the horizon.

H.8 Activities

H.8.1 Book

H.8.1.1 Primordial Cosmology

- ***Authors***: G. Montani, M.V. Battisti, R. Benini and G. Imponente
- ***Editors***: World Scientific (Singapore)

This book faces fundamental topics in very Early Cosmology. After a brief review of the fundamental tools associated with the General Relativity framework, we provide a general analysis of the Friedmann-Robertson-Walker Universe, presenting both its kinematical and dynamical properties. The Inflation paradigm is treated at the ground level of its fundamental ideas, analyzing those components of the model which appear as the most well grounded ones. After discussing the quasi isotropic solution in some details, the main properties characterizing the cosmological singularity associated to the homogeneous and inhomogeneous Mixmaster model are presented. We firstly assess the classical picture of the homogeneous chaotic cosmologies and, after a presentation of the canonical method for the quantization, we develop the quantum Mixmaster behavior. Our survey analyzes the fundamental framework of the Mixmaster picture and completes it by accounting for recent and peculiar outstanding results.

Contents:

Tetradic Formalism

Gauge-like Formulation of GR

Singularity Theorems

- The structure and dynamics of the Isotropic Universe

  Features of the observed Universe

  The Robertson-Walker geometry

  The Robertson-Walker Cosmology

  Dissipative cosmology

  Inhomogeneous fluctuations in the Universe

- The theory of Inflation

  The shortcomings of the Standard Cosmology

  The Inflation Paradigm

  Presence of a Self-interacting scalar Field

  Inflationary Dynamics

  Solution to the Horizon and Flatness Paradoxes

  The generation of density perturbations: the problem of classical limit

- Inhomogeneous Quasi-isotropic Cosmologies

  Quasi-isotropic solution

  The presence of ultra-relativistic matter

  The role of a massless scalar field

  The role of an electromagnetic field

  Quasi-isotropic Inflation

  Quasi-isotropic viscous solution

- Homogeneous Universes

  Homogeneous Space-Times

  Kasner Solution

  The Dynamics of the Bianchi Models

  Bianchi types VIII and IX models

  Dynamical System approach

  Multidimensional Homogeneous Universes

- Hamiltonian formulation of the Mixmaster
  - Hamiltonian Formulation of the Dynamics
  - Misner variables and the Mixmaster model
  - Misner-Chitrélike variables
  - The Invariant Liouville Measure
  - Invariant Lyapunov Exponent
  - Chaos covariance
  - Cosmological Chaos as a dimensional and matter dependent phenomenon
  - Isotropization Mechanisms
- The Generic Cosmological Solution
  - Formulation of the generic cosmological problem
  - The fragmentation process
  - Hamiltonian formulation
  - Dynamics of inhomogeneities of metric in the vicinity of a singular point
  - Multidimensional oscillatory regime
- Quantum Cosmology
  - Quantum Geometrodynamics
  - The Problem of Time
  - What is quantum cosmology?
  - Path integral in minisuperspace
  - Scalar field as a relational time
  - Wave function of the Universe
  - Boundary conditions
  - Quantization of the FRW model filled with a scalar field
  - Quantum dynamics of the Taub Universe
  - Quantization of the Mixmaster model in the Misner picture
  - The Spectrum of the Mixmaster
  - Loop Quantum Gravity
  - Loop Quantum Cosmology
  - Mixmaster Universe in LQC
  - Snyder-deformed quantum cosmology

GUP and polymer quantum cosmology: the Taub Universe

Mixmaster Universe in the GUP approach

H.8.2 Seminars and Workshops

H.8.2.1 "The directions of moderm cosmologies" meeting

Barcellona, March 2 marzo 2009.

- Title: "An example of f(R) model passing Solar-System tests"

  Authors: O. M. Lecian and G. Montani

  Abstract: The wak-field limit of a f(R) model consisting in the Ricci scalar plus a non-analytic function of it will be proposed, and the parameter space of the model will be constrained by means of the validity range of the weak-field limit approximantion and of the planetary motion. These results will be compared with those obtained for the analytical case.

6th Italian-Sino Workshop

Pescara, June 29th - July 1st 2009.

- Title: "MHD Features of Stellar Accretion Disks"
- Authors: G. Montani and R. Benini
- Abstract: We present a discussion on the accretion disk configurations around compact magnetized stars, as described in a two-dimensional MHD framework. The problem of balancing the azimuthal Lorentz force, acting on the electrons, is properly addressed, both in terms of a non-zero Nernst coefficient, as well as when the resistivity of the plasma in the disk is taken into account. The morphology of the accretion mechanism in the linear regime and the phenomenological implications of the model are eventually analyzed.

H.8.2.2 II Italian-Pakistani Workshop on Relativistic Astrophysics

Pescara, July 8 - 10, 2009

- Title: "The role of plasma resistivity in MHD Configurations of Accretion Disks"

  Authors: G. Montani and R. Benini

  Abstract: We present a discussion of two-dimensional MHD configurations concerning the equilibrium of accretion disks of a strongly magnetized astrophysical object. The role of the plasma resistivity in solving the puzzle of the electron force balance along the azimuthal direction is investigated. The linear features of the addressed model are analytically developed and the non-linear configuration problem is addressed by studying the plasma instabilities associated to the resistivity term.

- Title: "On the Propagation of Gravitational Waves across the Universe: Interaction with the Neutrino Component"

  Authors: R. Benini, M. Lattanzi and G. Montani

  Abstract: The stochastic background of primordial gravitational waves is a target for several experiments, spanning many orders of magnitude in frequency. At large wavelengths, it can be detected in the B-modes of the cosmic microwave background. At small scales, it is the target of detectors like LISA. Gravitational waves interact with the anisotropic stress of the cosmological fluid, mainly given by its neutrino component. While at the large scales this is effectively collisionless, in the small-scale limit, the collisions between neutrinos should be properly taken into account for the computation of the anisotropic stress. In this talk we will present the results of the integration of the Einstein-Boltzmann system of differential equations for the coupled gravitational wave-neutrino system, and we will discuss how this will affect the expected signal of stochastic gravitational waves.

H.8.2.3 XII Marcel Grossman meetings

Paris, July 12-18, 2009.

- Title: "Gravitational instability in presence of bulk viscosity: the Jeans Mass and the Quiasi-Isotropic Solution".

  Authors: N. Carlevaro and G. Montani

  Abstract: The talk focuses on the analysis of the gravitational instability in presence of dissipative effects both in the Newtonian regime and in the fully-relativistic approach. In particular, the standard Jeans Mechanism and the so-called Quasi-Isotropic Solution are treated when bulk viscosity enters the dynamics. We express the viscosity coefficient ζ as a power-law of the fluid energy density ρ: $\zeta = z_0\rho^s$ and we analyze the dynamics up to first- and second-order in time. In the Newtonian regime, the perturbation evolution is founded to be damped by bulk viscosity and the top-down mechanism of structure fragmentation is suppressed. In such a scheme, the value of the Jeans Mass remains unchanged also in presence of viscosity. On the other hand, in the relativistic approach, we get, as a result, a power-law solution existing only in correspondence to a restricted domain of z_0.

- Title: "2-D MHD Configurations for Accretion Disks around Magnetized Stars".

  Authors: R. Benini, B. Coppi and G. Montani

  Abstract: The description of an accretion disk around a compact and strongly magnetized star within the framework of two-dimensional local MHD profiles is presented. After a brief review of the crystalline structure the disk acquires because of the coupling of the radial and vertical equilibria (as shown by B. Coppi in 2005), we face the question of making account for a non-zero accretion rate. In particular, the puzzle of a self-consistent configuration, for which the electron force balance holds in the azimuthal direction, is addressed. In the limit of a linearized perturbation theory, the accretion characteristics, as coming out of the two-dimensional MHD, are discussed either in the case of a non-zero Nernst coefficient of the plasma, and when resistivity is involved in the model. In both these cases, we will deal with an oscillating radial matter in-fall. The possible implication of non-linear resistivity features, in view of the growth of plasma instabilities are eventually presented within the ring-morphology preservation. Results and perspectives are given. In particular, we stress that one of us (B.C.) is investigating an accretion scenario involving very low relative values of the plasma resistivity.

- Title: "On the Propagation of Gravitational Waves across the Universe: Interaction with the Neutrino Component"

  Authors: R. Benini, M. Lattanzi and G. Montani

  Abstract: The stochastic background of primordial gravitational waves is a target for several experiments, spanning many orders of magnitude in frequency. At large wavelengths, it can be detected in the B-modes of the cosmic microwave background. At small scales, it is the target of detectors like LISA. Gravitational waves interact with the anisotropic stress of the cosmological fluid, mainly given by its neutrino component. While at the large scales this is effectively collisionless, in the small-scale limit, the collisions between neutrinos should be properly taken into account for the computation of the anisotropic stress. In this talk we will present the results of the integration of the Einstein-Boltzmann system of differential equations for the coupled gravitational wave-neutrino system, and we will discuss how this will affect the expected signal of stochastic gravitational waves.

- Title: "The last ten years of the Mixmaster model"

  Authors: "R. Benini and G. Montani"

  Abstract: From the first analyses of BKL and Misner, the Mixmaster model has attracted a lot of interest because of its intrinsic complexity and physical relevance. After a brief description of the ideas and of formalisms of these classical works, we present a review of the latest developments and results obtained mainly facing the dynamical properties of such a cosmological model. In particular we will discuss some crucial questions concerning the definition of chaos with respect to the Mixmaster model. The main results we obtained over the last ten years in this field are presented and compared with the relevant issues of the complementary literature. Finally we will focus on the covariance of the inhomogeneous Mixmaster chaos and on the open issues concerning the final fate of this model toward the initial singularity.

A.1 Birth and Development of the Generic Cosmological Solution

In 1968-1975 the question of existence of the cosmological singularity in the general solution of Einstein equations have been solved and the theory of the chaotic oscillatory behaviour of gravitational field and matter in vicinity of this singularity have been created by V. Belinski, I. Khalatnikov and E. Lifschitz (BKL).

This problem appeared around 85 years ago when the first exactly solvable cosmological models revealed the presence of the Big Bang singularity. Since that time the fundamental question has arisen whether this phenomenon is due to the special simplifying assumptions underlying the exactly solvable models or if a singularity is a general property of the Einstein equations. The BKL showed that a singularity is an unavoidable property of the general cosmological solution of the gravitational equations and not a consequence of the special symmetric structure of exact models. Most importantly they were able to find the analytical structure of this generic solution and showed that its behaviour is of a complex oscillatory character of chaotic type.

The detailed theory of the oscillatory cosmological regime can be found in the following papers:

- V. Belinski and I. Khalatnikov On the Nature of the singularities in the General Solution of the Gravitational Equations,*Sov. Phys. JETP*, **29**, 911, (1969).
  This was the first investigation of the homogeneous cosmological model of Bianchi IX type and it was the first discovery of the new type of cosmological singularity - oscillating cosmological regime. In the subsequent literature this model has been given the second name "Mixmaster Universe".

- V. Belinski and I. Khalatnikov General Solution of the Gravitational Equations with a physical Singularity,*Sov. Phys. JETP*,**30**, 1174, (1970).
  In this paper was made the first statement that the oscillating cosmological regime of Bianchi IX model is the paradigm of the behaviour of the General cosmological Solution near singularity and that the General Solution with singularity really exists. Paper investigated a number of analytical properties of this Solution.

- V. Belinski, I. Khalatnikov and E. Lifshitz Oscillatory Approach to a Singular Point in the Relativistic Cosmology,*Adv. in Phys.*, **19**, 525, (1970).

The properties of the General Cosmological Solution near singularity was described. It was constructed the method for qualitative description of the oscillating cosmological evolution in terms of successively changing "Kasner epochs". It was described the statistical properties of the chaotic oscillating regime in ultra asymptotic region near singularity.

- V. Belinski and I. Khalatnikov Effect of scalar and Vector Fields on the nature of the cosmological singularity, *Sov. Phys. JETP*, **36**, 591, (1973).
  The effect of scalar and vector fields on the character of the cosmological singularity is investigated. The fields may either be gravitational (in the sense of the Brans-Dicke ideas) or extraneous physical fields which are sources of an ordinary gravitational field. It is shown that in the presence of only a scalar field the gravitational equations possess a monotonic power-law asymptotic for the general solution near the singular point in place of an oscillating form. However, if a vector field is included on the basis of five-dimension geometry concepts, the general solution becomes oscillatory again.

- V. Belinski and I. Khalatnikov, On the influence of matter and Physical Fields upon the Nature of Cosmological Singularities,*Soviet Physics Reviews, Harwood Acad. Publ.*, **3**, 555, (1981).
  It was investigated the influence of Yang-Mills fields and perfect liquid matter with unusual equations of state on cosmological singularities. It was shown that Yang-Mills fields do not change qualitatively the oscillating regime near singular point. The same is correct for perfect liquid in a wide range of equations of state with only one exception, namely, the stiff matter equation of state. In this case the asymptotic near singularity changes to the smooth Kasner-like (similar to the scalar field case) behaviour. For this case we constructed the general Cosmological Solution near the singularity in analytical form.

- V. Belinski, I. Khalatnikov and E. Lifshitz, A general solution of the Einstein equations with a time singularity, *Adv. in Phys.*, **31**, 639, (1982).
  This paper is a concluding review exposition of the investigations aimed at the construction of a general cosmological solution of the Einstein equations with a singularity in time (including the description of the new phenomenon of the rotations of Kasner axes). Thus it is a direct continuation of the previous (1970) paper by the authors in this Journal. A detailed description is given of the analysis which leads to the construction of such a solution, and of its properties.

These results have a fundamental significance not only for Cosmology but also for evolution of collapsing matter forming a black hole. The last stage of collapsing matter in general will follow the BKL regime.

The BKL analysis provides the description of intrinsic properties of the Einstein equations which can be relevant also in the quantum context. Recently (T.Damour, M.Henneaux, H. Nicolai et al., 2000-2007) it has been shown that the BKL regime is inherent not only to General Relativity but also to more general physical theories, such as the string models. This discovery has created an important field of research which has been continuously active. During the last three decades the BKL theory of the cosmological singularity has attracted the active attention of the scientific community. The developments of this theory made by many researches between 1980 and 2007 (among them Ya. Sinai, J. Barrow, B.K. Berger, A.A. Kirillov, V. Moncrief, G. Montani, J. Wainwright, D. Garfinkle, H. Ringstrom, L. Andersson, A. Rendall, C. Uggla, M. Henneaux, T. Damour, H. Nicolai) was dedicated to the foundation of its rigorous statistical description, to the numerical confirmation of its principal statements, to the quest of its more deep hidden mathematical structure and to its extension to the multidimensional space and to the string theories. The last reviews are:

- J.M.Heinzle, C.Uggla, N.Rohr, The cosmological billiard attractor, gr-qc/0702141
- L.Andersson On the relation between mathematical and numerical relativity,*Class. Quant. Grav.* **23**, S307 (2006), gr-qc/0607065
- A.Rendall, The nature of spacetime singularities,*100 Years of Relativity, Space-Time Structure: Einstein and Beyond*, A. Ashtekar (ed.); gr-qc/0503112.
- T. Damour, M. Henneaux, H. Nicolai, Cosmological billiards, *Class. Quant. Grav.* **20**, R145 (2003), hep-th/0212256
- T. Damour and S. de Buyl, Describing general cosmological singularities in Iwasawa variables, gr-qc/0710.5692.

Early Cosmology. ICRANet Activity In the ICRANET group the research on the oscillatory regime near the cosmological singularity has been one of the principal research field starting from 1992 . The most important papers made in this group are

- G.Montani On the general behaviour of the Universe near the cosmological singularity, *Class. Quant. Grav.* **12**, 2505 (1995)
- G.P. Imponente and G.Montani, On the Covariance of the Mixmaster Chaoticity, *Phys. Rev.* **D63**, 103501 (2001)
- R. Benini and G. Montani,Frame independence of the inhomogeneous mixmaster chaos via Misner-Chitre-like variables, *Phys. Rev.* **D70**, 103527-1 (2004).

- G. Montani, M.V. Battisti, R. Benini and G. Imponente, Classical and Quantum Features of the Mixmaster Singularity, to appear on *Int. J. Mod. Phys. A* (2008)

The research on the properties of oscillatory behaviour of the gravitational field and matter near the cosmological singularity is still in progress in this group, the main topics are: the multidimensional generalization, influence of viscosity, influence of quantum effects. The group is working under the leadership of G. Montani.

A.2 Appendix: Classical Mixmaster

A.2.1 Chaos covariance of the Mixmaster model

The study of the subtle question concerning the covariance chaoticity of the Bianchi type VIII and IX model, led to important issues favourable to the independence of the "chaos" with respect to the choice of the temporal gauge in terms of positive Lyapunov numbers. Such analysis found its basis either on the standard approach using the Jacobi metric (a scheme allowed by the existence of an energy-like constant of motion), either by a Statistical Mechanics approach in which the Mixmaster evolution is represented as a billiard on a Lobatchevski plane and therefore admitting a Microcanonical ensemble associated to such an energy-like constant.

A detailed discussion was pursued in view of clarifying the peculiarity existing to characterize chaos in General Relativity; in particular, we critically analyzed the predictability allowed by the fractal basin boundary approach in qualifying the nature of the Mixmaster dynamics, getting the numerical approximations limits when treating iterations of irrational numbers and overall on the potential methods commonly adopted in the dynamical systems approach. The description of chaos finds its ambiguity also in terms of geodesic deviation when the background metric is a pseudo-Riemannian one; a correct characterization of the Lyapunov exponents required a projection of the connecting vector over a Fermi basis.

We develop the Hamiltonian formulation of the cosmological problem showing how it can be reduced to the dynamics of a billiard-ball (IMPONENTE and MONTANI, 2005).
In particular an original reformulation of the Bianchi type IX dynamics is studied by using a set of Misner–Chitre-like variables with a generic function of one coordinate, thus overcoming the ambiguities of many assessments found in the literature, due to the dependence of the choice of the time parameter (Imponente and Montani, 2001).

Our reformulation is not affected by such a possibility and permits to discuss the dynamics via a standard Arnowitt-Deser-Misner (ADM) approach in the reduced phase space. The Jacobi metric obtained induces the derivation of an invariant formulation of the Liouville measure (Imponente and Montani, 2002) within the *microcanonical ensemble* framework (Imponente and Montani, 2005b).
This new approach permits to derive, within the potential approximation, an

analytic expression for the Lyapunov exponents (Imponente and Montani, 2001), independently of the choice of the temporal gauge and a discussion about a correct formulation of the same problem in General Relativity (Imponente and Montani, 2004).

A.2.2 Chaos covariance of the generic cosmological solution

In the homogeneous Mixmaster model, it was shown that chaos is a property of the Einsteinian dynamics because it is not induced by particular choices of the temporal variables as previously argued in literature. This result was extended to the more general case of the generic cosmological solution (Benini and Montani, 2004).

The complex dynamics of the generic cosmological solution was analyzed by means of the Hamiltonian formulation of General Relativity; in this framework, the gravitational degrees of freedom are twelve, the six components of the three dimensional metric tensor h_{ij} and their conjugate momenta Π^{ij}. Among these variables, only four are physical, while the remaining concern with the diffeomorphism invariance of the theory. The ″embedding variables“ can be eliminated solving the four constraints, the super-Hamiltonian and the supermomentum ones, that emerge when the Legendre transformation is performed to pass from the Lagrangian to the Hamiltonian framework.

The analysis of the Ricci scalar (that in vacuum behaves as a potential term that couples the space points) showed how the time evolution of the space points dynamically decouple from each other while reaching the Big Bang (in accordance with the previous results of Belisnkii et al. in the field equations framework); in each space point, a Mixmaster like evolution takes place. Here, the physical meaning of ″space point“ is that of a cosmological horizon, and the obtained decoupling corresponds to deal with ″super-horizon“ sized perturbation. This fact is also known as long wave length approximation, that mathematically corresponds to the result that the spatial gradients in the Ricci scalar grow slower in time than the time derivatives.

We succeeded in applying the ADM technique to the embedding variables without choosing any particular form for the lapse function N or for the shift vector N^i; this was done with a particular but quite general choice of the coordinates for the space-time, and using an infinite potential well structure for the Ricci scalar. The resulting dynamics consists of the sum of infinite Mixmaster model, and the previous discussion on the covariance of the chaos in the homogeneous case was extended to the generic cosmological solution.

A.2.3 Inhomogeneous inflationary models

The investigation performed about a quasi-isotropic inflationary solution (Imponente and Montani, 2005c) showed how there is no chance for classical inhomogeneous perturbations to survive after the de Sitter phase, strongly supporting the idea that only quantum fluctuations of the scalar field can provide a satisfactory explanation for the observed spectrum of inhomogeneous perturbations, when requiring the matter to dominate the first order of the solution (Imponente and Montani, 2005a).

We consider the inflationary scenario as the possible way to interpolate the rich and variegate Kasner dynamics of the Very Early Universe discussed so far with an inflationary scenario (Imponente and Montani, 2004), in order to reach the present state observable FLRW Universe, via a *bridge solution*. The Einstein-Hamilton-Jacobi equation is solved in presence of a real self-interacting scalar field.

Hence we show how it is possible to have a quasi-isotropic solution of the Einstein equations in presence of the ultrarelativistic matter and a real self-interacting scalar field. In this case, the spatial distributions of both admit an arbitrary form but such a small inhomogeneity is incompatible with structures formation of classical origin (Imponente and Montani, 2003).
Furthermore, a generic inhomogeneous solution has been provided concerning the dynamics of a real self interacting scalar field minimally coupled to gravity in a region of the configuration space where it performs a slow rolling on a plateau of its potential. During the generic inhomogeneous deSitter phase the scalar field which dominates zero- and first-order of approximation is a function of the spatial coordinates only. This solution specialized nearby the Friedmann-Lemaitre-Robertson-Walker (FLRW) model allows a classical origin for the inhomogeneous perturbation spectrum.

A.2.4 The Role of a Vector Field

The effects of an Abelian vector field on the dynamics of a generic $(n+1)$-dimensional homogeneous model has been investigated in the BKL scheme; the chaos is restored for any number of dimensions, and a BKL-like map, exhibiting a peculiar dependence on the dimension number, is worked out (Benini et al., 2005). These results have also been inserted in more general treatment by Damour and Hennaux.

A generic $(n+1)$-dimensional space-time coupled to an Abelian vector field $A_\mu = (\varphi, A_\alpha)$, with $\alpha = (1, 2, \ldots, n)$ in the ADM framework is described by the action

$$S = \int d^n x dt \left(\Pi^{\alpha\beta} \frac{\partial}{\partial t} h_{\alpha\beta} + \Pi^\alpha \frac{\partial}{\partial t} A_\alpha + \varphi D_\alpha \Pi^\alpha - NH - N^\alpha H_\alpha \right) , \quad \text{(A.2.4.1)}$$

where

$$H = \frac{1}{\sqrt{h}} \left[\Pi^{\alpha}_{\beta}\Pi^{\beta}_{\alpha} - \frac{1}{n-1} (\Pi^{\alpha}_{\alpha})^2 + \frac{1}{2} h_{\alpha\beta}\Pi^{\alpha}\Pi^{\beta} + h \left(\frac{1}{4} F_{\alpha\beta}F^{\alpha\beta} - {}^{(N)}R \right) \right] , \tag{A.2.4.2a}$$

$$H_\alpha = -\nabla_\beta \Pi^{\beta}_{\alpha} + \Pi^{\beta} F_{\alpha\beta} , \tag{A.2.4.2b}$$

denote the super-Hamiltonian and the super-momentum respectively, while $F_{\alpha\beta}$ is the spatial electromagnetic tensor, and the relation $D_\alpha \equiv \partial_\alpha + A_\alpha$ holds. Moreover, Π^α and $\Pi^{\alpha\beta}$ are the conjugate momenta to the electromagnetic field and to the n-metric, respectively, which result to be a vector and a tensorial density of weight $1/2$, since their explicit expressions contain the square root of the spatial metric determinant. The variation with respect to the lapse function N yields the super-Hamiltonian constraint $H = 0$, while with respect to φ it provides the constraint $\partial_\alpha \Pi^\alpha = 0$.
We will deal with a source-less Abelian vector field and in this case one can consider the transverse (or Lorentz) components for A_α and Π^α only. Therefore, we choose the gauge conditions $\varphi = 0$ and $D_\alpha \Pi^\alpha = 0$, enough to prevent the longitudinal parts of the vector field from taking part to the action.
It is worth noting how, in the general case, *i.e.* either in presence of the sources, or in the case of non-Abelian vector fields, this simplification can no longer take place in such explicit form and the terms $\varphi(\partial_\alpha + A_\alpha)\Pi^\alpha$ must be considered in the action principle.

A BKL-like analysis can be developed Benini et al. (2005) as well as done previously, following some steps: after introducing a set of Kasner vectors $\vec{l}_a$ and the Kasner-like expanding factors $\exp(q^a)$, the dynamics is dominated by a potential of the form $\sum e^{q_a} \widetilde{\lambda}_a^2$, where $\widetilde{\lambda}_a$ are the projection of the momenta of the Abelian field along the Kasner vectors. With the same spirit of the Mixmaster analysis, an unstable n-dimensional Kasner-like evolution arises, nevertheless the potential term inhibits the solution to last up to the singularity and, as usual, induces the BKL-like transition to another epoch. Given the relation $\exp(q^a) = t^{p_a}$, the map that links two consecutive epochs is

$$p'_1 = \frac{-p_1}{1 + \frac{2}{n-2} p_1} , \qquad p'_a = \frac{p_a + \frac{2}{n-2} p_1}{1 + \frac{2}{n-2} p_1} , \tag{A.2.4.3a}$$

$$\widetilde{\lambda}'_1 = \widetilde{\lambda}_1 , \qquad \widetilde{\lambda}'_a = \widetilde{\lambda}_a \left(1 - 2 \frac{(n-1)\, p_1}{(n-2)\, p_a + n p_1} \right) . \tag{A.2.4.3b}$$

An interesting new feature, resembling that of the inhomogeneous Mixmaster (as we will discuss later), is the *rotation of the Kasner vectors*,

$$\vec{\ell}'_a = \vec{\ell}_a + \sigma_a \vec{\ell}_1 \,, \tag{A.2.4.4a}$$

$$\sigma_a = \frac{\widetilde{\lambda}'_a - \widetilde{\lambda}_a}{\widetilde{\lambda}_1} = -2 \frac{(n-1)\, p_1}{(n-2)\, p_a + n p_1} \frac{\widetilde{\lambda}_a}{\widetilde{\lambda}_1} \,. \tag{A.2.4.4b}$$

which completes our dynamical scheme.

The homogeneous Universe in this case approaches the initial singularity described by a metric tensor with oscillating scale factors and rotating Kasner vectors. Passing from one Kasner epoch to another, the negative Kasner index p_1 is exchanged between different directions (for istance $\vec{\ell}_1$ and $\vec{\ell}_2$) and, at the same time, these directions rotate in the space according to the rule (A.2.4.4b). The presence of a vector field is crucial because, independently of the considered model, it induces a dynamically closed domain on the configuration space.

In correspondence to these oscillations of the scale factors, the Kasner vectors $\vec{\ell}_a$ rotate and the quantities σ_a remain constant during a Kasner epoch to lowest order in q^a; thus, the vanishing of the determinant h approaching the singularity does not significantly affect the rotation law (A.2.4.4b).

There are two most interesting features of the resulting dynamics: the map exhibits a *dimensional-dependence*, and it reduces to the standard BKL one for the four-dimensional case.

A.3 Perturbation Theory in Macroscopic Gravity: On the Definition of Background

1. The notion of background metric adopted in the perturbation theory in general relativity is analysed. A new definition of background is proposed. An existence theorem for a metric tensor which serves as the background metric for a specific scale has been proven. (G Montani and Zalaletdinov, 2003). Let us consider the covariant volume averaging procedure adopted in macroscopic gravity (Zalaletdinov, 1992)⁻ (Mars and Zalaletdinov, 1997). The average value of a metric tensor is defined

$$\overline{g}_{\alpha\beta}(x) = \frac{1}{V_\Sigma}\int_\Sigma g_{\mu'\nu'}(x')\mathcal{A}^{\mu'}_{\alpha}(x',x)\mathcal{A}^{\nu'}_{\beta}(x',x)\sqrt{-g'}d^4x' \,. \qquad \text{(A.3.0.1)}$$

Here V_Σ is 4-volume of a compact 4-region Σ, $\mathcal{A}^{\mu'}_{\alpha}(x',x)$ is the averaging operator which is idempotent, $\mathcal{A}^{\alpha}_{\beta'}(x,x')\mathcal{A}^{\beta'}_{\gamma''}(x',x'') = \mathcal{A}^{\alpha}_{\gamma''}(x,x'')$, and hence factorized (Zalaletdinov, 1997),(Mars and Zalaletdinov, 1997) in general as $\mathcal{A}^{\mu'}_{\alpha}(x',x) = e^{\mu'}_{i}(x')e^{-1}{}^{i}_{\alpha}(x)$ where $e^{\mu}_{i}(x)$ is a vector basis with constant anholonomicity coefficients C^k_{ij}, $i = 1,2,3,4$. Note that the Brill-Hartle procedure belongs to the same class of linear averagings under some additional restrictions on the structure of space-time (Zalaletdinov, 1992). The volume averages (A.3.0.1) possess the property of idempotency (Zalaletdinov, 1992)⁻ (Mars and Zalaletdinov, 1997), that is $\overline{\overline{g}}_{\alpha\beta}(x) = \overline{g}_{\alpha\beta}(x)$. This is a fundamental property which means geometrically that the average value of a tensor field remains invariant under action of the same averaging operator. Such an averaging procedure on a space-time manifold provides a natural criterium for a definition of background metric.

Definition. Given an averaging space-time procedure (A.3.0.1) with an idempotent averaging kernel, a metric tensor $g_{\alpha\beta}(x)$ is called a background metric if

$$g_{\alpha\beta}(x) = \overline{g}_{\alpha\beta}(x). \qquad \text{(A.3.0.2)}$$

Such a background metric is invariant with respect to the class of averagings, and it works in the framework of the perturbation theory as described above. An averaged metric is always the background one according to the definition (A.3.0.2).

The following important theorem considers the existence of a metric tensor which serves as the background metric for a specific scale.

Theorem. Given an averaging space-time procedure (A.3.0.1) with an idempotent averaging kernel of the class of bounded and continuous functions on a space-time manifold $\mathcal{M}$, there always exists a continuous and bounded background metric $g_{\alpha\beta}(x)$ (A.3.0.2) for a characteristic scale $d = V_\Sigma$ where Σ is a compact 4-region of $\mathcal{M}$.

A.4 On Schouten's Classification of the non-Riemannian Geometries with an Asymmetric Metric

Application of non-Riemannian geometries with an asymmetric metric tensor to the problem of geometric unification is discussed. An approach to a classification for such kind of geometries in spirit of Schouten is proposed (Casanova et al., 1999). By adopting Schouten's classification approach to the affine connection geometries with an asymmetric metric the structure and variety of such geometries can be investigated in a fully geometrical formalism without adopting a variational principle. It may also give the possibility to generalize the scheme to more general geometries including spinor fields on manifolds.

In the case of an asymmetric metric tensor $g_{\mu\nu}$, $g_{\mu\nu} \neq g_{\nu\mu}$, similar to the case of the symmetric metric, analysis of the incompatibility between metric and connection $g_{\mu\nu|\rho} = N_{\mu\nu\rho}$ brings about the following expression for the connection $\Pi^{\theta}_{\kappa\lambda}$

$$\Pi^{\theta}_{\kappa\lambda}(\delta^{\sigma}_{\theta}\delta^{\kappa}_{\nu}\delta^{\lambda}_{\rho} + g^{\sigma\lambda}\delta^{\kappa}_{\rho}a_{\theta\nu} + g^{\sigma\kappa}\delta^{\lambda}_{\nu}a_{\rho\theta}) = \Gamma^{\sigma}_{\nu\rho} + \Delta^{\sigma}_{\nu\rho} + \mathbf{C}^{\sigma}_{\nu\rho} - \mathbf{D}^{\sigma}_{\nu\rho}, \qquad \text{(A.4.0.1)}$$

with the standard metric connection coefficients $\Gamma^{\sigma}_{\nu\rho}$, the metric asymmetricity object $\Delta^{\sigma}_{\nu\rho} = \frac{1}{2}s^{\sigma\mu}(a_{\mu\nu,\rho} + a_{\rho\mu,\nu} - a_{\nu\rho,\mu})$, the generalized contorsion tensor $\mathbf{C}^{\sigma}_{\nu\rho} = \frac{1}{2}\left[s^{\sigma\mu}(T^{\varepsilon}_{\nu\mu}g_{\varepsilon\rho} + T^{\varepsilon}_{\rho\mu}g_{\varepsilon\nu}) + T^{\varepsilon}_{\nu\rho}g^{\cdot\sigma}_{\varepsilon}\right]$ and the non-metricity tensor $\mathbf{D}^{\sigma}_{\nu\rho} = \frac{1}{2}s^{\sigma\mu}(N_{\mu\nu\rho} + N_{\rho\mu\nu} - N_{\nu\rho\mu})$. The determinant of the "hypercubic" *structure matrix* $J^{\sigma\kappa\lambda}_{\theta\nu\rho} = \delta^{\sigma}_{\theta}\delta^{\kappa}_{\nu}\delta^{\lambda}_{\rho} + g^{\sigma\lambda}\delta^{\kappa}_{\rho}a_{\theta\nu} + g^{\sigma\kappa}\delta^{\lambda}_{\nu}a_{\rho\theta}$ is related to the existence of solutions of the system of *inhomogeneous linear algebraic* equations (A.4.0.1) for the unknowns $\Pi^{\alpha}_{\beta\gamma}$ similar to the case of usual quadratic matrixes. When the determinant is not equal to zero the system has non-trivial solutions which can be expressed through the inverse structure matrix $\widetilde{J}^{\alpha\nu\rho}_{\sigma\beta\gamma} = (J^{-1})^{\alpha\nu\rho}_{\sigma\beta\gamma}$, $\widetilde{J}^{\alpha\nu\rho}_{\sigma\beta\gamma}J^{\sigma\beta\gamma}_{\mu\varepsilon\lambda} = \delta^{\alpha}_{\mu}\delta^{\nu}_{\varepsilon}\delta^{\rho}_{\lambda}$. The espression of the Riemannian curvature tensor $M^{\alpha}{}_{\beta\rho\sigma}$ from $\widetilde{M}^{\alpha}{}_{\beta\rho\sigma}$ is given by

$$R^{\alpha}{}_{\beta\rho\sigma} = M^{\varepsilon}{}_{\nu\rho\lambda}\widetilde{J}^{\alpha\nu\lambda}_{\varepsilon\beta\sigma} + \Sigma^{\alpha}{}_{\beta\rho\sigma}(\widetilde{A}^{\alpha}{}_{\beta\sigma}, \Delta^{\alpha}{}_{\beta\sigma}, \widetilde{J}^{\alpha\nu\rho}_{\sigma\beta\gamma}), \qquad \text{(A.4.0.2)}$$

where $\Sigma^{\alpha}{}_{\beta\rho\sigma}$ is a *tensor* constructed from generalised affine deformation tensor, metric asymmetricity object and the inverse structure matrix and their derivatives. The determinant of $J^{\sigma\kappa\lambda}_{\theta\nu\rho}$ has been calculated in a perturbation expansion in terms of small asymmetric metric, $\mid a_{\mu\nu} \mid \ll \mid s_{\mu\nu} \mid$. Then in linear approximation the matrix $J^{\sigma\kappa\lambda}_{\theta\nu\rho} = \delta^{\sigma}_{\theta}\delta^{\kappa}_{\nu}\delta^{\lambda}_{\rho} + s^{\sigma\lambda}\delta^{\kappa}_{\rho}a_{\theta\nu} + s^{\sigma\kappa}\delta^{\lambda}_{\nu}a_{\rho\theta}$ has its inverse as $\widetilde{J}^{\alpha\nu\rho}_{\sigma\beta\gamma} = \delta^{\alpha}_{\sigma}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma} - s^{\alpha\nu}\delta^{\rho}_{\beta}a_{\gamma\sigma} - s^{\alpha\rho}\delta^{\nu}_{\gamma}a_{\sigma\beta}$. The expressions (A.4.0.1) and (A.4.0.2) are the main relations describing the structure of the affine connection geometries with asymmetric metric.

A.5 Approximate Symmetries, Inhomogeneous Spaces and Gravitational Entropy

The problem of finding an appropriate geometrical/physical index for measuring a degree of inhomogeneity for a given space-time manifold is posed. Interrelations with the problem of understanding the gravitational/informational entropy are pointed out. An approach based on the notion of approximate symmetry is proposed (Zalaletdinov, 2000),(Montani et al., 2000). A number of related results on definitions of approximate symmetries known from literature are briefly reviewed with emphasis on their geometrical/physical content. A definition of a Killing-like symmetry is given and a classification theorem for all possible averaged space-times acquiring Killing-like symmetries upon averaging out a space-time with a homothetic Killing symmetry is proved.

The main idea of the Killing-like symmetry is to consider the most general form of deviation from the Killing equations. Let us consider the equation for a Killing-like vector $\xi^{\alpha}(x^{\mu})$

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 2\epsilon_{\alpha\beta} \tag{A.5.0.1}$$

where a symmetric tensor $\epsilon_{\alpha\beta}(x^{\mu})$ measures deviation from the Killing symmetry. The tensor can be small in order to enable a continuous limit to the case $\epsilon_{\alpha\beta} \to 0$.

The equation (A.5.0.1) covers the cases of semi-Killing, almost-Killing and almost symmetries with additional equations for the tensor $\epsilon_{\alpha\beta}(x^{\mu})$. Also covered are standard generalizations of Killing symmetry such as conformal and homothetic Killing vectors The algebraic classification of the symmetric tensor $\epsilon_{\alpha\beta}$ gives an invariant way to introduce a set of scalar indexes measuring the degree of inhomogeneity of the space-time with (A.5.0.1) compared with that with isometries, or even weaker symmetry, for example, conformal Killing's. For the most general case $A_1[111,1]$ in Segre's notation$\epsilon_{\alpha\beta}$ has the form

$$\epsilon_{\mu\nu} = \lambda g_{\mu\nu} + \rho x_{\mu} x_{\nu} + \sigma y_{\mu} y_{\nu} + \tau z_{\mu} z_{\nu} \tag{A.5.0.2}$$

where $g_{\mu\nu}$ if the space-time metric, $\lambda(x^{\mu})$, $\rho(x^{\mu})$, $\sigma(x^{\mu})$ and $\tau(x^{\mu})$ are eigenvalues of $\epsilon_{\alpha\beta}$ and $\{t^{\mu}, x^{\mu}, y^{\mu}, z^{\mu}\}$ is the eigentetrad . If all eigenvalues vanish the space-time has an isometry (A.5.0.1), if $\rho = \sigma = \tau = 0$ then there is a conformal Killing vector for $\lambda(x) \neq 0$ and a homothetic Killing vector for $\lambda = \text{const}$. For other algebraic types of Killing-like symmetry the space-time has the following sets of eigenvalues: two complex conjugated to each other and two real scalars for $A_2[11, ZZ^*]$, three real scalars for $A_3[11, 2]$ and two real scalars for $B[1, 3]$.

A.6 Gravitational Polarization in General Relativity: Solution to Szekeres' Model of Gravitational Quadrupole

A model for the static weak-field macroscopic medium is analyzed and the equation for the macroscopic gravitational potential is derived (Montani et al., 2003). This is a biharmonic equation which is a non-trivial generalization of the Poisson equation of Newtonian gravity. In case of the strong gravitational quadrupole polarization it essentially holds inside a macroscopic matter source. Outside the source the gravitational potential fades away exponentially. The equation is equivalent to a system of the Poisson equation and the nonhomogeneous modified Helmholtz equations. The general solution to this system is obtained by using Green's function method and it does not have a limit to Newtonian gravity. In case of the insignificant gravitational quadrupole polarization the equation for macroscopic gravitational potential becomes the Poisson equation with the matter density renormalized by the factor including the value of the quadrupole gravitational polarization of the source. The general solution to this equation obtained by using Green's function method has a limit to Newtonian gravity.

Calculation of the equation for the macroscopic gravitational potential φ from the macroscopic gravity equations for the macroscopic tensor $g^{(0)}_{\mu\nu}$ brings the equation

$$\Delta\varphi = 4\pi G\mu + \frac{4\pi G\epsilon_g}{3c^2}\Delta^2\varphi \qquad \text{(A.6.0.1)}$$

where $\Delta^2\varphi \equiv \Delta(\Delta\varphi)$ is the Laplacian of the Laplacian of φ. This is a non-trivial generalization of the Poisson equation for the gravitational potential φ of Newtonian gravity. This is a biharmonic equation due the presence of the term $\Delta^2\varphi$. The equation (A.6.0.1) involves a singular perturbation, since in case of the vanishing gravitational dielectric constant, $\epsilon_g = 0$, this equation becomes the Poisson equation, but if $\epsilon_g \neq 0$, this equations change its operator structure to be of the fourth order equation in partial derivatives of φ as compared with the Poisson second order partial differential equation.

It is convenient to introduce the factor

$$\frac{1}{k^2} = \frac{4\pi G\epsilon_g}{3c^2} \tag{A.6.0.2}$$

with k having a physical dimension of inverse length, $\left[k^{-2}\right] = \text{length}^2$. Then the equation (A.6.0.1) takes the form

$$\Delta\varphi = 4\pi G\mu + \frac{1}{k^2}\Delta^2\varphi. \tag{A.6.0.3}$$

By using the definitions of the gravitational dielectric constant ϵ_g, the characteristic oscillation frequency of molecule's constituents ω_0^2, macroscopic matter density $\mu = 3m/4\pi A^3$ and the average number of molecules per unit volume $N = 4\pi D^3/3$ with D as a mean distance between molecules, the factor k^{-2} can be shown to have the following form

$$\frac{1}{k^2} = \frac{1}{4\theta}\left(\frac{A^3}{D^3}\right)A^2. \tag{A.6.0.4}$$

Here the dimensionless factor θ,

$$\theta = \frac{\omega_0^2}{4\pi G\mu/3}, \tag{A.6.0.5}$$

reflects the nature of field responsible for bounding of discrete matter constituents into molecules. If $\theta \approx 1$, the molecules of self-gravitating macroscopic medium are considered to be gravitationally bound. For instance, considering a macroscopic model of galaxy as a self-gravitating macroscopic medium consisting of gravitational molecules taken as double stars, $\theta \approx 1$ as such galactic molecules are gravitationally bound. If one takes the molecules to be of electron-proton type, like atoms, the factor $\theta \approx 10^{40}$, which makes the factor k^{-2} essentially insignificant.

The dimensionless ratio A/D reflects the structure of macroscopic medium. If $(A/D) \approx 1$, the macroscopic medium behaves itself like a liquid or solid. If $(A/D) < 1$, the macroscopic medium behaves itself like a gas. For the macroscopic galactic model for the present epoch the macroscopic medium is like a gas, since $(A/D) \approx 10^{-1} - 10^{-2}$, which makes the factor A^3/D^3 to be of order of $10^{-3} - 10^{-6}$. However, for earlier times of galaxy formulation this factor can be expected to be of much greater order of magnitude up to $1 - 10$.

A.7 Averaging Problem in Cosmology and Macroscopic Gravity

The Averaging problem in general relativity and cosmology is discussed. The approach of macroscopic gravity to resolve the problem is presented. The averaged Einstein equations of macroscopic gravity are modified on cosmological scales by the gravitational correlation tensor terms as compared with the Einstein equations of general relativity. This correlation tensor satisfies an additional set of structure and field equations. Exact cosmological solutions to the equations of macroscopic gravity for spatially homogeneous and isotropic macroscopic space-times are presented. In particular, it has been found that for a flat geometry the gravitational correlation tensor terms in the averaged Einstein equations have the form of a spatial curvature term which can be either negative or positive. Thus macroscopic gravity provides a cosmological model for a flat spatially homogeneous and isotropic Universe which obeys the dynamical law for either open or closed Universe geometry.

For a flat spatially homogeneous, isotropic macroscopic space-time

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2) \quad \text{(A.7.0.1)}$$

the averaged Einstein equations for the case of a constant macroscopic gravitational correlation tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma} = \text{const}$ read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa\rho}{3} + \frac{\varepsilon}{3a^2}, \quad \text{(A.7.0.2)}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\kappa p + \frac{\varepsilon}{3a^2}, \quad \text{(A.7.0.3)}$$

or in terms of ρ_{grav} and p_{grav}

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\left(\rho + \rho_{grav}\right), \quad \text{(A.7.0.4)}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\kappa\left(p + p_{grav}\right). \quad \text{(A.7.0.5)}$$

with the equations of state $p = p(\rho)$ and $p_{grav} = -\frac{1}{3}\rho_{grav}$. They look similar to Einstein's equations of General Relativity for either a closed or an open spatially homogeneous, isotropic FLRW space-time, but they do have different mathematical and physical, and therefore, cosmological content since

$$\frac{\varepsilon}{3} = \frac{\kappa \rho_{grav} a^2}{3} \neq -k \tag{A.7.0.6}$$

in general.

The macroscopic (averaged) Einstein's equations for a flat spatially homogeneous, isotropic macroscopic space-time have macroscopic gravitational correlation terms of the form of a spatial curvature term

$$\frac{\varepsilon}{3a^2} = \frac{\kappa \rho_{grav}}{3}. \tag{A.7.0.7}$$

Thus, the theory of Macroscopic Gravity predicts that constant macroscopic gravitational correlation tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma} = \text{const}$ for a flat spatially homogeneous, isotropic macroscopic space-time takes the form of a dark spatial curvature term it interacts only gravitationally with the macroscopic gravitational field it does not interact directly with the energy-momentum tensor of matter it exhibits a negative pressure $p_{grav} = -\frac{1}{3}\rho_{grav}$ which tends to accelerate the Universe when $\rho_{grav} > 0$.

Only if one requires $12Z^{3}{}_{23}{}^{3}{}_{32} = -\varepsilon$ to be $\varepsilon = -3k$ the macroscopic (averaged) Einstein's equations become exactly Einstein's equations of General Relativity for either a closed or an open spatially homogeneous, isotropic space-time for the macroscopic geometry of a flat spatially homogeneous, isotropic space-time.

This exact solution of the Macroscopic Gravity equations exhibits a very non-trivial phenomenon from the point of view of the general-relativistic cosmology: the macroscopic (averaged) cosmological evolution in a flat Universe is governed by the dynamical evolution equations for either a closed or an open Universe depending on the sign of the macroscopic energy density ρ_{grav} with a dark spatial curvature term $\kappa\rho_{grav}/3$.

From the observational point of view such a cosmological model gives a new paradigm to reconsider the standard cosmological interpretation and treatment of the observational data.

Indeed, this macroscopic cosmological model has the Riemannian geometry of a flat homogeneous, isotropic space-time. Therefore, all measurements and data are to be considered and designed for this geometry. The dynamical interpretation of the obtained data should be considered and treated for the cosmological evolution of either a closed or an open spatially homogeneous, isotropic Riemannian space-time.

A.8 Astrophysical Topics

- M. V. Barkov, V. A. Belinskii and G.S. Bisnovatyi-Kogan Model of ejection of matter from non-stationary dense stellar clasters and chaotic motion of gravitating shells, **Mon. Not. R.A.S.**, 334, 338, (2002) (astro-ph/0107051). A model of ballistic ejection effect of matter from spherically symmetric stellar clusters it is investigated. The problem is solved in newtonian gravity but with cutoff fixing the minimal radius of selfgravitating matter shell by its relativistic gravitational radus. It is shown that during the motion of two initially gravitationally bound spherical shells, consisting of point particles moving along ballistic trajectories, one of the shell may be expelled to infinity at subrelativistic expelling velocity of the order of 0,25c. Also it is shown that the motion of two intersecting shells in the case when they do not runaway reveal a chaotic behaviour.

- M. V. Barkov, V.A. Belinskii and G.S. Bisnovatyi-Kogan An exact General Relativity solution for the Motion and Intersections of Self-Gravitating Shells in the Field of a Massive Black Hole, **JETP** 95, 371, (2002) (astro-ph/0210296). It is found the complete exact solution in the General Relativity for the intersection process of two massive selfgravitating spherically symmetric shells (in general with tangential pressure). It is shown how one can calculate all shell's parameters after intersection in terms of the parameters before the intersection. The result is quite new, the solution of this kind was known only for the massless shells (Dray and t'Hooft, 1985). The solution was applied to the analysis of matter ejection effect from relativistic stellar clusters. It is shown that in relativistic case the matter ejection effect is stronger than in newtonian gravity.

- G.S. Bisnovatyi-Kogan, R.V.E. Lovelace and V.A. Belinskii A cosmic battery reconsidered, **ApJ** 580, 380, (2002) (astro-ph/0207476).
  The problem of magnetic field generation in accretion flows onto black holes owing to the excess radiation force on electrons is revisited. This excess force may arise from the Poynting-Robertson effect. Instead of a recent claim of the generation of dynamically important magnetic fields, we show only small magnetic fields are generated. A model of the Poynting-Robertson magnetic field generation close to the horizon of a Schwarzschild black hole is solved exactly using General Relativity, and the field is found to be dynamically insignificant. These weak magnetic fields may however be important as seed fields for dynamos.

- M.V.Barkov, V.A.Belinskii, G.S.Bisnovatyi-Kogan and A.I.Neishtadt Model of Ejection of Matter from Dense Stellar Cluster and Chaotic Motion of Gravitating Shells, in **Galaxies and Chaos**, page 357, Eds. G.Contopoulos and N.Voglis, Lecture Notes in Physics, Springer (2003).
  It is shown that during the motion of two initially gravitationally bound spherical shells, consisting of point particles moving along ballistic trajectories, one of the shells may be expelled to infinity at subrelativistic speed of order 0.25c. The problem is solved in Newtonian gravity. Motion of two intersecting shells in the case when they do not runaway shows a chaotic behaviour. We hope that this simple toy model can give nevertheless a qualitative idea on the nature of the mechanism of matter outbursts from the dense stellar clusters.

- M. V. Barkov, G. S. Bisnovatyi-Kogan, A. I. Neishtadt and V.A. Belinski On chaotic behavior of gravitating stellar shells, **Chaos**, 15, 013104 (2005).
  Motion of two gravitating spherical stellar shells around a massive central body is considered. Each shell consists of point particles with the same specific angular momenta and energies. In the case when one can neglect the influence of gravitation of one ("light") shell onto another ("heavy") shell ("restricted problem") the structure of the phase space is described. The scaling laws for the measure of the domain of chaotic motion and for the minimal energy of the light shell sufficient for its escape to infinity are obtained.

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