Integrable SUGRA

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In any space-time of dimension $n+2$ with interval

$$ds^2 = g_{\alpha\beta}\left(x^0, x^1\right) dx^\alpha dx^\beta + g_{ik}\left(x^0, x^1\right) dy^i dy^k \tag{1}$$

(where $\alpha, \beta, \gamma, ... = 0, 1$ and $i, k, l, ... = 1, 2, ...n$) the Einstein equations are integrable since the equations for the metric coefficients $g_{ik}\left(x^0, x^1\right)$ follows from the Lax pair as its self-consistency conditions and the metric components $g_{\alpha\beta}\left(x^0, x^1\right)$ can be found by quadratures in terms of the known $g_{ik}\left(x^0, x^1\right)$. Without loss of generality the 2-dimensional block $g_{\alpha\beta}\left(x^0, x^1\right)$ can be chosen in conformally flat form:

$$g_{\alpha\beta} = \lambda^2 \eta_{\alpha\beta}\ ,\ \ \eta_{\alpha\beta} = diag\left(\eta_{00}, \eta_{11}\right) = diag\left(1, -1\right)\ . \tag{2}$$

It is convenient to represent the metric coefficients g_{ik} as

$$g_{ik} = \alpha^{2/n} G_{ik} \ , \ \det(G_{ik}) = 1 \ , \ \det\left(g_{ik}\right) = \alpha^2. \tag{3}$$

Then the components $R_{\alpha i}$ of the Ricci tensor vanish identically and equations $R_{\alpha\beta} = 0$ and $R_{ik} = 0$ can be written in matrix form using the matrix $\mathbf{G}$ of dimension $n \times n$ with components G_{ik}. In the light-like variables ζ, η :

$$x^0 = \eta + \zeta \ , \ x^1 = \eta - \zeta \tag{4}$$

the equations $R_{ik} = 0$ are:

$$\alpha_{,\zeta\eta} = 0 \ , \tag{5}$$

$$(\alpha \mathbf{G}^{-1} \mathbf{G}_{,\zeta})_{,\eta} + (\alpha \mathbf{G}^{-1} \mathbf{G}_{,\eta})_{,\zeta} = 0. \tag{6}$$

Equations $R_{\alpha\beta} = 0$ are equivalent to the system:

$$\frac{f_{,\zeta}}{f} = \frac{\alpha}{4\alpha_{,\zeta}} Tr\left[\left(\mathbf{G}^{-1}\mathbf{G}_{,\zeta}\right)^2\right] \ , \tag{7}$$

$$\frac{f_{,\eta}}{f} = \frac{\alpha}{4\alpha_{,\eta}} Tr\left[\left(\mathbf{G}^{-1}\mathbf{G}_{,\eta}\right)^2\right] \ , \tag{8}$$

where

$$f = \frac{\alpha^{(n-1)/n}\lambda^2}{\alpha_{,\zeta}\alpha_{,\eta}} \ . \tag{9}$$

If equations (5) and (6) are solved then f (that is the conformal factor λ^2) can be found by quadratures from equations (7) and (8) [the integrability conditions of which are satisfied automatically if α and $\mathbf{G}$ are the solutions of the equations (5) and (6)].

From the point of view of space-time with coordinates x^0, x^1 and metric $g_{\alpha\beta}\left(x^0, x^1\right)$ the fields $\alpha\left(x^0, x^1\right)$ *and* $G\left(x^0, x^1\right)$ *are scalars and can be treated as matter living in this two-dimensional manifold.*
The spectral linear problem associated with the main equation (6) we used in (**??**) contains the differentiation also with respect to the spectral parameter but this is not the only possible construction. Our original Lax representation for the equation (6) can be transformed to the following form:

$$\hat{\mathbf{G}}^{-1}\hat{\mathbf{G}}_{,\zeta} = \frac{\alpha}{\alpha - s}\mathbf{G}^{-1}\mathbf{G}_{,\zeta}\ ,\ \hat{\mathbf{G}}^{-1}\hat{\mathbf{G}}_{,\eta} = \frac{\alpha}{\alpha + s}\mathbf{G}^{-1}\mathbf{G}_{,\eta}\ , \tag{10}$$

where $\hat{\mathbf{G}}\left(\zeta, \eta, s\right)$ depends on the complex spectral parameter s which depends on the coordinates ζ, η in accordance with the following differential equations:

$$\frac{s_{,\zeta}}{s} = \frac{2\alpha_{,\zeta}}{\alpha - s}\ ,\ \frac{s_{,\eta}}{s} = \frac{2\alpha_{,\eta}}{\alpha + s}\ . \tag{11}$$

The self-consistency requirement for the last equations is satisfied due to the condition (5). The solution of the equations (11) contains one arbitrary complex constant w then parameter $s = s(\zeta, \eta, w)$ has one arbitrary degree of freedom independent of those due to the changing of coordinates. This means that in the integrability conditions for the pair (10) all terms containing the different powers of s must vanish separately. The matrix $\mathbf{G}$ in the right hand side of (10) are function on the two coordinates ζ and η only, that is it is treated as unknown "potential" independent on the parameter s (that is on the arbitrary complex constant w). The function $\alpha(\zeta, \eta)$ which is a solution of the wave equation (5) should be considered in (10) as some given external field. The equation of interest (6) should result from the linear spectral system (10) as its self-consistency (integrability) conditions and it is easy to check that this is indeed the case.

The following important point should be stressed here. The solution of equation (6) for the matrix $\mathbf{G}$, as follows from definitions (3), should satisfy the restriction $\det \mathbf{G} = \mathbf{1}$. However, the application of the inverse scattering integration procedure to the equation (6), if taking this restriction from the outset, technically are involved. Much more convenient approach is to solve (6) first ignoring any additional restricton for the $\det \mathbf{G}$ but at the end of calculation make the simple re-scaling of the solution in order to get the necessary condition of unity determinant**.** The trick is as follows. If we obtained a solution $\mathbf{G}_0$ of the equation (6) with $\det \mathbf{G}_0 \neq \mathbf{1}$ we can pass to the correct matrix $\mathbf{G}$ by the transformation

$$\mathbf{G} = (\det \mathbf{G}_0)^{-1/n} \, \mathbf{G}_0 \ . \tag{12}$$

It is simple task to see that matrix $\mathbf{G}$ (12) is also a solution of the same equation (6) and automatically satisfy the condition $\det \mathbf{G} = 1$.

It turnes out that the above-cited equations (6) for $\mathbf{G}$ coincide with the self-consistency conditions ($\hat{\mathbf{G}}_{,\zeta\eta} = \hat{\mathbf{G}}_{,\eta\zeta}$) of the linear spectral problem (10). If $\hat{\mathbf{G}}\,(\zeta, \eta, s)$ can be found then $\hat{\mathbf{G}}\,(\zeta, \eta, s = 0)$ gives the metric $\mathbf{G}(\zeta, \eta)$ which satisfy automatically the Einstein equations.

For any given vacuum solution $\hat{\mathbf{G}}_0\,(\zeta,\eta,s)$ of the spectral equations we can construct the exact solitonic excitations over this vacuum containing any desirable number of solitons.

Such solitonic solution has the the form $\hat{\mathbf{G}}(\zeta,\eta,s) = \hat{\mathbf{G}}_0(\zeta,\eta,s)\mathbf{K}(\zeta,\eta,s)$ where the "dressing" matrix $\mathbf{K}(\zeta,\eta,s)$ is meromorphic with respect to the spectral parameter s and the number of poles in $\mathbf{K}$ is the number of solitons.

The *miracle* is that for the s-meromorphic structure of $\mathbf{K}$ the dependence of $\mathbf{K}$ also on the coordinates ζ,η can be found exactly solely by algebraic manipulations. After this the matrix $\hat{\mathbf{G}}_0(\zeta,\eta,s)\mathbf{K}(\zeta,\eta,s)$ taken at $s=0$ gives the exact multi-solitonic solution of the Einstein equations for the metric matrix $\mathbf{G}$.

This technique works in empty spacetime and in spacetime filled by the stiff pefect fluid (Belinski, 1979) or by electromagnetic field (Alekseev, 1981). All Black Holes (Schwarzschild, Kerr, Reissner-Nordström, Kerr-Newman) have been rederived as bound states of two interecting solitons. Some equilibrium states of two interecting black holes (four-solitonic configurations) have been obtained. It was constructed exact solutions representing gravitational waves propagating on the Friedmann cosmological background and solutions describing colliding solitons, etcetera.

In 2015-2016 due to interection of ICRANet with the Albert Einstein Institute (Potsdam) appear the necessity to extend this technique for the cases when also fermionic fields are present. The first integrable supergravity have been constructed (using the different approach) by Nicolai in 1987.

However, the Nicolai's constructions had some shortcomings. First of all his Lax pairs are not complete since they contain (as their self-consistency conditions) only bosonic part of the equations of motion. The equations of motion for spinor fields do not follow from his linear spectral problems and must be added by hands. Such mixed approach to the integration can not be satisfactory in full. A manner how to apply it for the construction of the exact solutions of the whole system of equations of motion is intricate and such a way does not represents integrability in the conventional sense.

Another undesirable point which also creates some non-standard complications is appearance in these linear spectral equations the poles of the second order with respect to the spectral parameter s while the corresponding spectral problem in pure gravity has only simple poles. The final circumstance we would like to mention is characteristic for many papers dedicated to the integrable systems. The point is that the authors often became fully satisfied as soon as they showed the existence of the Lax pair and they do not pay attention to the next even more important task: how to solve these equations. However, to construct a procedure for extraction the exact solutions of the spectral problem represents the main part of the integration process.

To avoid these drawbacks the Lax representation proposed by Nicolai can be extended to the complete one (covering also fermionic equation of motion) and simultaneously can be liberated from the second order poles with respect to the spectral parameter s. Also it can be showed how one can get the exact super-solitonic solutions using this extention. The interesting point is that such development can be reached simply by the extension of the original BZ technique to the more general case which include the fields taking their values in graded algebra (including Grassmann variables).

The idea is to extend the old BZ approach to the superfields adding to $\hat{\mathbf{G}}(\zeta, \eta, s)$ some fermionic matrices depending on the variables ζ, η, s and to find an appropriate spectral representation for the such composite set. The simplest way to do this is to use the multi-dimensional superspace parametrized by the coordinates $(\zeta, \eta, \theta^1, \theta^2, ...)$ with some number M of the odd elements θ^i $(i = 1, 2, ..., M)$ and consider the old bosonic matrix $\hat{\mathbf{G}}\,(\zeta, \eta, s)$ (with even entries) and additional fermionic matrices $\Phi_i\,(\zeta, \eta, s)$ (with odd entries) as components of the single spectral supermatrix $\hat{\Psi}(\zeta, \eta, s, \theta^i)$.

Graded algebra machinery suggests introducing the super-differential operators

$$D_\zeta = A^i \frac{\partial}{\partial\theta^i} - B_i\theta^i \frac{\partial}{\partial\zeta} \ , \ \ D_\eta = -E^i \frac{\partial}{\partial\theta^i} + F_i\theta^i \frac{\partial}{\partial\eta} \ ,$$

where A^i, B_i, E^i, F_i are even constants restricted by the conditions

$$A^i F_i = 0 \ , \ \ E^i B_i = 0 \ .$$

Under these two restrictions the operators D_ζ and D_η anticommute with each other:

$$D_\zeta D_\eta + D_\eta D_\zeta = 0 \ .$$

Now consider the following superspace Lax representation for the new supermatrix $\hat{}(\zeta, \eta, s, \theta^i)$:

$$\hat{\Psi}^{-1} D_\zeta \hat{\Psi} \; = \frac{\alpha}{\alpha - s} \Psi^{-1} D_\zeta \Psi \; , \quad \hat{\Psi}^{-1} D_\eta \hat{\Psi} \; = \frac{\alpha}{\alpha + s} \Psi^{-1} D_\eta \Psi \; .$$

where $\Psi(\zeta, \eta, \theta^i) = \hat{\Psi}(\zeta, \eta, \theta^i, s = 0)$ is the superspace generalization of the original matrix $\mathbf{G}$:

$$\Psi(\zeta, \eta, \theta^i) = \mathbf{G}(\zeta, \eta) \left[\mathbf{I} + \theta^i \Omega_i(\zeta, \eta) + \theta^i \theta^k \mathbf{H}_{ik} \left(\zeta, \eta\right) + \theta^i \theta^k \theta^l \mathbf{H}_{ikl} \left(\zeta, \eta\right) + ..\right.$$

where $\Omega_i \zeta, \eta) = \Phi_i \left(\zeta, \eta, s = 0\right)$ and $\mathbf{H}$-matrices represent the new additional fields.

By the direct calculations it can be shown that the only condition of self-consistency (that is of the requirement $D_\zeta D_\eta \hat{\Psi} + D_\eta D_\zeta \hat{\Psi} = 0$) for this superspace spectral problem is:

$$D_\zeta \left[\alpha \Psi^{-1} D_\eta \Psi\right] - D_\eta \left[\alpha \Psi^{-1} D_\zeta \Psi\right] = 0$$

Now we should insert into this equation the matrix Ψ and equate to zero coefficients in front of $\theta^i, \theta^i \theta^k, ...$ and also the term independent on the odd coordinates. This gives the system of equations of motion for the fields $\mathbf{G}, \Omega_i, \mathbf{H}_{ik}, ...$ This system is formidabble for the big number of the odd coordinates θ^i but nevertheless exactly integrable!

In case when we have only one θ the theory is equivalent to the original BZ approach. For the case of two odd coordinates the equations of motion became:

$$\left(\alpha \mathbf{G}^{-1}\mathbf{G}_{,\zeta}+\alpha\Omega_2^2\right)_{,\eta}+\left(\alpha \mathbf{G}^{-1}\mathbf{G}_{,\eta}+\alpha\Omega_1^2\right)_{,\zeta}=0\ , \tag{13}$$

$$\begin{gathered}2\Omega_{1,\zeta}+\frac{\alpha_{,\zeta}}{\alpha}\Omega_1+\mathbf{G}^{-1}\mathbf{G}_{,\zeta}\Omega_1-\Omega_1\mathbf{G}^{-1}\mathbf{G}_{,\zeta}\\ +\frac{1}{2}\left(\Omega_2^2\Omega_1-\Omega_1\Omega_2^2\right)=0\ ,\end{gathered} \tag{14}$$

$$\begin{gathered}2\Omega_{2,\eta}+\frac{\alpha_{,\eta}}{\alpha}\Omega_2+\mathbf{G}^{-1}\mathbf{G}_{,\eta}\Omega_2-\Omega_2\mathbf{G}^{-1}\mathbf{G}_{,\eta}\\ +\frac{1}{2}\left(\Omega_1^2\Omega_2-\Omega_2\Omega_1^2\right)=0\ ,\end{gathered} \tag{15}$$

$$\mathbf{H}=\frac{1}{2}\left(\Omega_2\Omega_1-\Omega_1\Omega_2\right)\ . \tag{16}$$

We can demonstrate that spinors of interest are encoded in the odd entries of the fermionic matrices Ω_1 and Ω_2 and equations (b),(c) for these matrices are the non-linear Dirac-like equations for these spinors, namely that equations which have been missed in the Nicolai's spectral representation.

Plus to this we see that the Lax representation for the spectral supermatrix $\hat{\Psi}(\zeta, \eta, \theta_1, \theta_2, s)$ contains only the first order poles with respect to s. This is important from the technical point of view since it permits to use the BZ procedure of construction the supersolitonic solution almost literaly (the only new point is that now we are in the graded algebra and more care should be paid for disposition of the different multipliers in course of the algebraic manipulations).

The gravitational sector of the 2-D theory presented by Nicolai consists of two-dimensional metric tensor $g_{\alpha\beta}(x^0, x^1)$ and 16 gravitino $\psi^I_\mu(x^0, x^1)$. These fields have no propagating degrees of freedom of their own . The matter living in this two-dimensional space-time are represented by 128 space-time scalars $\varphi^A(x^0, x^1)$ and 128 space-time spinors $\chi^{\dot{A}}(x^0, x^1)$ (each $\chi^{\dot{A}}$ represents one physical degree of freedom although it has two space-time spinorial components). These are "normal" physical fields which can propagate. The number 128 arises because in this model matter fields are represented by an *internal* spinor of $SO(16)$ having 256 components half of which are 128 "right" Weyl components (these are φ^A which are scalars relative to the 2-dimensional space-time) and 128 "left" Weyl components (these are $\chi^{\dot{A}}$ which are spinors from the point of view of the 2-dimensional space-time).

In general the fermions can be introduced by the different means but the supersymmetric way is unic. For an *extended* supergravity such a way need for the metric $\mathbf{G}$ and the frame $\mathbf{V}$ to have some special symmetry structure.

The extension of this scheme to the case when fermions are present forced us to consider side by side with metric $\mathbf{G}$ also the matrix of the orthonormal frame $\mathbf{V}(\zeta, \eta)$ which can be introduced by the relation:

$$\mathbf{G} = \mathbf{V}\tilde{\mathbf{V}} \ , \tag{17}$$

(here and in the sequel tilde means transposition). The frame current $\mathbf{V}^{-1}\mathbf{V}_{,\alpha}$ can be decomposed into antisymmetric and symmetric parts:

$$\mathbf{V}^{-1}\mathbf{V}_{,\alpha} = \mathbf{Q}_{\alpha} + \mathbf{P}_{\alpha} \ , \ \tilde{\mathbf{Q}}_{\alpha} = -\mathbf{Q}_{\alpha} \ , \ \tilde{\mathbf{P}}_{\alpha} = \mathbf{P}_{\alpha} \ . \tag{18}$$

From the last two formulas one can get the following expression for the metric current $\mathbf{G}^{-1}\mathbf{G}_{,\alpha}$:

$$\mathbf{G}^{-1}\mathbf{G}_{,\alpha} = 2\tilde{\mathbf{V}}^{-1}\mathbf{P}_{\alpha}\tilde{\mathbf{V}} \tag{19}$$

The basic equation of motion (6) can be formulated also in terms of the frame matrix $\mathbf{V}$. Substituting $\mathbf{G}^{-1}\mathbf{G}_{,\alpha}$ from (**??**) into equation (6) (which in the original coordinates x^{α} is $\eta^{\alpha\beta}(\alpha\mathbf{G}^{-1}\mathbf{G}_{,\alpha})_{,\beta} = ...$) and taking into account the definitions (**??**) we get:

$$\eta^{\alpha\beta}\left[(\alpha\mathbf{P}_{\alpha})_{,\beta} + \alpha\mathbf{Q}_{\beta}\mathbf{P}_{\alpha} - \alpha\mathbf{P}_{\alpha}\mathbf{Q}_{\beta}\right] = ... \tag{20}$$