Generalizations of the Kerr-Newman solution

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2 Brief description

One of the most important metrics in general relativity is the Kerr-Newman solution that describes the gravitational and electromagnetic fields of a rotating charged mass. For astrophysical purposes, however, it is necessary to take into account the effects due to the moment of inertia of the object. To attack this problem we have derived exact solutions of Einstein-Maxwell equations which posses an infinite set of gravitational and electromagnetic multipole moments. Several analysis have been performed that investigate the physical effects generated by a rotating deformed mass distribution in which the angular momentum and the quadrupole determine the dominant multipole moments.

We investigated the motion of test particles in the gravitational field of a static naked singularity generated by a mass distribution with quadrupole moment. We use the quadrupole-metric ($q-$metric) which is the simplest generalization of the Schwarzschild metric with a quadrupole parameter. We study the influence of the quadrupole on the motion of massive test particles and photons and show that the behavior of the geodesics can drastically depend on the values of the quadrupole parameter. In particular, we prove explicitly that the perihelion distance depends on the value of the quadrupole. Moreover, we show that an accretion disk on the equatorial plane of the quadrupole source can be either continuous or discrete, depending on the value of the quadrupole. The inner radius of the disk can be used in certain cases to determine the value of the quadrupole parameter. The case of a discrete accretion is interpreted as due to the presence of repulsive gravity generated by the naked singularity. Radial geodesics are also investigated and compared with the Schwarzschild counterparts.

We studied stationary axially symmetric solutions of the Einstein vacuum field equations that can be used to describe the gravitational field of astrophysical compact objects in the limiting case of slow rotation and slight deformation. We derive explicitly the exterior Sedrakyan-Chubaryan approximate solution, and express it in analytical form, which makes it practical in the context of astrophysical applications. In the limiting case of vanishing angular momentum, the solution reduces to the well-known Schwarzschild solution in vacuum. We demonstrate that the new solution is equivalent to the exterior Hartle-Thorne solution. We establish the mathematical equivalence between the Sedrakyan-Chubaryan, Fock-Abdildin and Hartle-Thorne formalisms.

We presented a method for generating exact interior solutions of Einstein's equations in the case of static and axially symmetric perfect-fluid spacetimes. The method is based upon a transformation that involves the metric functions as well as the density and pressure of the seed solution. In the limiting vacuum case, it reduces to the Zipoy-Voorhees transformation that can be used to generate metrics with multipole moments. All the metric functions of the new solution can be calculated explicitly from the seed solution in a simple manner. The physical properties of the resulting new solutions are shown to be completely different from those of the seed solution.

3 Introduction

It is hard to overemphasize the importance of the Kerr geometry not only for general relativity itself, but also for the very fundamentals of physics. It assumes this position as being the most physically relevant rotating generalization of the static Schwarzschild geometry. Its charged counterpart, the Kerr-Newman solution, representing the exterior gravitational and electromagnetic fields of a charged rotating object, is an exact solution of the Einstein-Maxwell equations.

Its line element in Boyer–Lindquist coordinates can be written as

$$
\begin{aligned}
ds^2 \; = \; & \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 \\
& - \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[(r^2 + a^2)d\varphi - adt]^2 \\
& - \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 \, , \qquad (3.0.1)
\end{aligned}
$$

where M is the total mass of the object, $a = J/M$ is the specific angular momentum, and Q is the electric charge. In this particular coordinate system, the metric functions do not depend on the coordinates t and ϕ, indicating the existence of two Killing vector fields $\xi^I = \partial_t$ and $\xi^{II} = \partial_\varphi$ which represent the properties of stationarity and axial symmetry, respectively.

An important characteristic of this solution is that the source of gravity is surrounded by two horizons situated at a distance

$$
r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \qquad (3.0.2)
$$

from the origin of coordinates. Inside the interior horizon, r_-, a ring singularity is present which, however, cannot be observed by any observer situated outside the exterior horizon. If the condition $M^2 < a^2 + Q^2$ is satisfied, no horizons are present and the Kerr–Newman spacetime represents the exterior field of a naked singularity.

Despite of its fundamental importance in general relativity, and its theoretical and mathematical interest, this solution has not been especially useful for describing astrophysical phenomena, first of all, because observed astrophysical objects do not possess an appreciable net electric charge. Furthermore, the limiting Kerr metric takes into account the mass and the rotation, but does not consider the moment of inertia of the object. For astrophysical applications it is, therefore, necessary to use more general solutions with higher multipole moments which are due not only to the rotation of the body but also to its shape. This means that even in the limiting case of a static spacetime, a solution is needed that takes into account possible deviations from spherically symmetry.

4 The general static vacuum solution

In general relativity, stationary axisymmetric solutions of Einstein's equations [1] play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of know exact solutions drastically increased after Ernst [2] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies lead finally to the development of solution generating techniques [1] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [4] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

4.1 Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindric coordinates (t, ρ, z, φ). Stationarity implies that t can be chosen as the time coordinate and the metric does not depend on time, i.e. $\partial g_{\mu\nu}/\partial t = 0$. Consequently, the corresponding timelike Killing vector has the components δ^μ_t. A second Killing vector field is associated to the axial symmetry with respect to the axis $\rho = 0$. Then, choosing φ as the azimuthal angle, the metric satisfies the conditions $\partial g_{\mu\nu}/\partial\varphi = 0$, and the components of the corresponding spacelike Killing vector are δ^μ_φ.

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form $g_{\mu\nu} = g_{\mu\nu}(\rho, z)$, it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [4, 5, 6]

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] , \tag{4.1.1}$$

where f, ω and γ are functions of ρ and z, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$\rho\partial_\rho\Omega = f^2\partial_z\omega , \qquad \rho\partial_z\Omega = -f^2\partial_\rho\omega , \tag{4.1.2}$$

the vacuum field equations $R_{\mu\nu} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \partial_z^2 f + \frac{1}{f}[(\partial_\rho\Omega)^2 + (\partial_z\Omega)^2 - (\partial_\rho f)^2 - (\partial_z f)^2] = 0 , \tag{4.1.3}$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Omega) + \partial_z^2\Omega - \frac{2}{f}\left(\partial_\rho f\,\partial_\rho\Omega + \partial_z f\,\partial_z\Omega\right) = 0 , \tag{4.1.4}$$

$$\partial_\rho\gamma = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho\Omega)^2 - (\partial_z f)^2 - (\partial_z\Omega)^2\right] , \tag{4.1.5}$$

$$\partial_z\gamma = \frac{\rho}{2f^2}\left(\partial_\rho f\,\partial_z f + \partial_\rho\Omega\,\partial_z\Omega\right) . \tag{4.1.6}$$

It is clear that the field equations for γ can be integrated by quadratures, once f and Ω are known. For this reason, the equations (4.1.3) and (4.1.4) for f and Ω are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only. It is interesting to mention that this set of equations can be geometrically interpreted in the context of nonlinear sigma models [7].

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation $\varphi \to -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (4.1.1) with $\omega = 0$, and the field equations can be written as

$$\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi + \partial_z^2\psi = 0\,, \quad f = \exp(2\psi)\,, \tag{4.1.7}$$

$$\partial_\rho\gamma = \rho\left[(\partial_\rho\psi)^2 - (\partial_z\psi)^2\right]\,, \quad \partial_z\gamma = 2\rho\partial_\rho\psi\,\partial_z\psi\,. \tag{4.1.8}$$

We see that the main field equation (4.1.7) corresponds to the linear Laplace equation for the metric function ψ.

4.2 Static solution

The general solution of Laplace's equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as [4, 1]

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2+z^2)^{\frac{n+1}{2}}} P_n(\cos\theta)\,, \qquad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}\,, \tag{4.2.1}$$

where a_n $(n = 0, 1, ...)$ are arbitrary constants, and $P_n(\cos\theta)$ represents the Legendre polynomials of degree n. The expression for the metric function γ can be calculated by quadratures by using the set of first order differential equations (4.1.8). Then

$$\gamma = -\sum_{n,m=0}^{\infty} \frac{a_n a_m (n+1)(m+1)}{(n+m+2)(\rho^2+z^2)^{\frac{n+m+2}{2}}} \left(P_n P_m - P_{n+1}P_{m+1}\right)\,. \tag{4.2.2}$$

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one of the most interesting special solutions which is Schwarzschild's spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants a_n in such a way that the infinite sum (4.2.1) converges to the Schwarzschild solution in cylindric coordinates. But, or course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzchild's metric.

In fact, it turns out that to investigate the properties of solutions with multipole moments it is more convenient to use prolate spheroidal coordinates (t, x, y, φ) in which the line element can be written as

$$ds^2 = f dt^2 - \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right]$$

where

$$x = \frac{r_+ + r_-}{2\sigma}\,, \quad (x^2 \geq 1), \quad y = \frac{r_+ - r_-}{2\sigma}\,, \quad (y^2 \leq 1) \tag{4.2.3}$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2\,, \quad \sigma = const\,, \tag{4.2.4}$$

and the metric functions are f, ω, and γ depend on x and y, only. In this coordinate system, the general static solution which is also asymptotically flat can be expressed as

$$f = \exp(2\psi)\,, \quad \psi = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n(y) Q_n(x)\,, \quad q_n = const$$

where $P_n(y)$ are the Legendre polynomials, and $Q_n(x)$ are the Legendre functions of second kind. In particular,

$$P_0 = 1, \quad P_1 = y, \quad P_2 = \frac{1}{2}(3y^2-1)\,, ...$$

$$Q_0 = \frac{1}{2}\ln\frac{x+1}{x-1}\,, \quad Q_1 = \frac{1}{2}x\ln\frac{x+1}{x-1} - 1\,,$$

$$Q_2 = \frac{1}{2}(3x^2-1)\ln\frac{x+1}{x-1} - \frac{3}{2}x\,, ...$$

The corresponding function γ can be calculated by quadratures and its general expression has been explicitly derived in [8]. The most important special cases contained in this general solution are the Schwarzschild metric

$$\psi = -q_0 P_0(y) Q_0(x) \,, \quad \gamma = \frac{1}{2} \ln \frac{x^2-1}{x^2-y^2} \,,$$

and the Erez-Rosen metric [9]

$$\psi = -q_0 P_0(y) Q_0(x) - q_2 P_2(y) Q_2(x) \,, \quad \gamma = \frac{1}{2} \ln \frac{x^2-1}{x^2-y^2} + ... $$

In the last case, the constant parameter q_2 turns out to determine the quadrupole moment. In general, the constants q_n represent an infinite set of parameters that determines an infinite set of mass multipole moments.

5 Stationary generalization

The solution generating techniques [12] can be applied, in particular, to any static seed solution in order to obtain the corresponding stationary generalization. One of the most powerful techniques is the inverse method (ISM) developed by Belinski and Zakharov [13]. We used a particular case of the ISM, which is known as the Hoenselaers–Kinnersley-Xanthopoulos (HKX) transformation to derive the stationary generalization of the general static solution in prolate spheroidal coordinates.

5.1 Ernst representation

In the general stationary case ($\omega \neq 0$) with line element

$$
\begin{aligned}
& ds^2 = f(dt - \omega d\varphi)^2 \\
- \quad & \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right]
\end{aligned}
$$

it is useful to introduce the the Ernst potentials

$$E = f + i\Omega\,, \quad \xi = \frac{1-E}{1+E}\,,$$

where the function Ω is now determined by the equations

$$\sigma(x^2-1)\Omega_x = f^2\omega_y\,, \quad \sigma(1-y^2)\Omega_y = -f^2\omega_x\,.$$

Then, the main field equations can be represented in a compact and symmetric form:

$$(\xi\xi^* - 1)\left\{[(x^2-1)\xi_x]_x + [(1-y^2)\xi_y]_y\right\} = 2\xi^*[(x^2-1)\xi_x^2 + (1-y^2)\xi_y^2]\,.$$

This equation is invariant with respect to the transformation $x \leftrightarrow y$. Then, since the particular solution

$$\xi = \frac{1}{x} \to \Omega = 0 \to \omega = 0 \to \gamma = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2}$$

represents the Schwarzschild spacetime, the choice $\xi^{-1} = y$ is also an exact solution. Furthermore, if we take the linear combination $\xi^{-1} = c_1 x + c_2 y$ and introduce it into the field equation, we obtain the new solution

$$\xi^{-1} = \frac{\sigma}{M}x + i\frac{a}{M}y\,,\ \sigma = \sqrt{M^2 - a^2}\,,$$

which corresponds to the Kerr metric in prolate spheroidal coordinates.

In the case of the Einstein-Maxwell theory, the main field equations can be expressed as

$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\xi\,,$$
$$(\xi\xi^* - \mathcal{F}\mathcal{F}^* - 1)\nabla^2\mathcal{F} = 2(\xi^*\nabla\xi - \mathcal{F}^*\nabla\mathcal{F})\nabla\mathcal{F}$$

where ∇ represents the gradient operator in prolate spheroidal coordinates. Moreover, the gravitational potential ξ and the electromagnetic $\mathcal{F}$ Ernst potential are defined as

$$\xi = \frac{1 - f - i\Omega}{1 + f + i\Omega}\,, \quad \mathcal{F} = 2\frac{\Phi}{1 + f + i\Omega}\,.$$

The potential Φ can be shown to be determined uniquely by the electromagnetic potentials A_t and A_φ One can show that if ξ_0 is a vacuum solution, then the new potential

$$\xi = \xi_0\sqrt{1 - e^2}$$

represents a solution of the Einstein-Maxwell equations with effective electric charge e. This transformation is known in the literature as the Harrison transformation [10]. Accordingly, the Kerr–Newman solution in this representation acquires the simple form

$$\xi = \frac{\sqrt{1-e^2}}{\frac{\sigma}{M}x + i\frac{a}{M}y}\,, \qquad e = \frac{Q}{M}\,, \qquad \sigma = \sqrt{M^2 - a^2 - Q^2}\,.$$

In this way, it is very easy to generalize any vacuum solution to include the case of electric charge. More general transformations of this type can be used in order to generate solutions with any desired set of gravitational and electromagnetic multipole moments [11].

5.2 Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds (M,γ) and (N,G) of dimension m and n, respectively. Let M be coordinatized by x^a, and N by X^μ, so that the metrics on M and N can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $G = G(X)$. A harmonic map is a smooth map $X : M \rightarrow N$, or in coordinates $X : x \longmapsto X$ so that X becomes a function of x, and the X's satisfy the motion equations following from the action [14]

$$S = \int d^m x \sqrt{|\gamma|}\, \gamma^{ab}(x)\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \tag{5.2.1}$$

which sometimes is called the "energy" of the harmonic map X. The straightforward variation of S with respect to X^μ leads to the motion equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b \left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\, \gamma^{ab}\, \partial_a X^\nu\, \partial_b X^\lambda = 0\,, \tag{5.2.2}$$

where $\Gamma^\mu_{\ \nu\lambda}$ are the Christoffel symbols associated to the metric $G_{\mu\nu}$ of the target space N. If $G_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $G_{\mu\nu} = \eta_{\mu\nu} = diag(\pm 1, ..., \pm 1)$, the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (5.2.1) is usually referred to as the Polyakov action [16].

Consider now the case in which the base space M is a stationary axisymmetric spacetime. Then, γ^{ab}, $a, b = 0, ..., 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (4.1.1), i.e.

$$\gamma_{ab} = \begin{pmatrix} f & 0 & 0 & -f\omega \\ 0 & -f^{-1}e^{2k} & 0 & 0 \\ 0 & 0 & -f^{-1}e^{2k} & 0 \\ -f\omega & 0 & 0 & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}. \tag{5.2.3}$$

Let the target space N be 2-dimensional with metric $G_{\mu\nu} = (1/2)f^{-2}\delta_{\mu\nu}$, $\mu,\nu = 1,2$, and let the coordinates on N be $X^{\mu} = (f,\Omega)$. Then, it is straightforward to show that the action (5.2.1) becomes

$$S = \int \mathcal{L}\, dt d\varphi d\rho dz\,, \qquad \mathcal{L} = \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right]\,, \tag{5.2.4}$$

and the corresponding motion equations (5.2.2) are identical to the main field equations (4.1.3) and (4.1.4).

Notice that the field equations can also be obtained from (5.2.4) by a direct variation with respect to f and Ω. This interesting result was obtained originally by Ernst [2], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \to 2)-$nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2,R)/SO(2)$ [15], and the Lagrangian (5.2.4) can be written explicitly [17] in terms of the generators of the Lie group $SL(2,R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [14].

The form of the Lagrangian (5.2.4) with two gravitational field variables, f and Ω, depending on two coordinates, ρ and z, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (5.2.4). Indeed, if we consider γ^{ab} as a 2-dimensional metric that depends on the parameters ρ and z, the diagonal form of the Lagrangian (5.2.4) implies that $\sqrt{|\gamma|}\gamma^{ab} = \delta^{ab}$. Clearly, this choice is not compatible with the factor ρ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian (5.2.4) cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to "absorb" the unpleasant factor ρ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2,R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields [1]. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

5.3 Representation as a generalized harmonic map

Consider two (pseudo-)Riemannian manifolds (M,γ) and (N,G) of dimension m and n, respectively. Let x^a and X^μ be coordinates on M and N, respectively. This coordinatization implies that in general the metrics γ and G become functions of the corresponding coordinates. Let us assume that not only γ but also G can explicitly depend on the coordinates x^a, i.e. let $\gamma = \gamma(x)$ and $G = G(X,x)$. This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map $X : M \to N$ will be called an $(m \to n)-$generalized harmonic map if it satisfies the Euler-Lagrange equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\mu\right) + \Gamma^\mu_{\ \nu\lambda}\,\gamma^{ab}\,\partial_a X^\nu \partial_b X^\lambda + G^{\mu\lambda}\gamma^{ab}\,\partial_a X^\nu\,\partial_b G_{\lambda\nu} = 0\,, \tag{5.3.1}$$

which follow from the variation of the generalized action

$$S = \int d^m x \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a\, X^\mu \partial_b X^\nu G_{\mu\nu}(X,x)\,, \tag{5.3.2}$$

with respect to the fields X^μ. Here the Christoffel symbols, determined by the metric $G_{\mu\nu}$, are calculated in the standard manner, without considering the explicit dependence on x. Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term $G_{\mu\nu}(X,x)$ in the Lagrangian density implies that we are taking into account the "interaction" between the base space M and the target space N. This interaction leads to an extra term in the motion equations, as can be seen in (5.3.1). It turns out that this interaction is the result of the effective presence of the gravitational field.

Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (5.3.1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^{\mu}_{\ \nu\lambda}\,\partial_b X^{\lambda} + G^{\mu\lambda}\,\partial_b G_{\lambda\nu})\partial_a X^{\nu} = 0\,, \tag{5.3.3}$$

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma^{\mu}_{\ \nu\lambda} = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\lambda}\partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma^{\mu}_{\ \nu\lambda} \neq 0$ and (5.3.3) cannot be satisfied. In the general case of a curved target metric, conditions (5.3.3) represent a system of m first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

As we mentioned before, the generalized action (5.3.2) includes an interaction between the base space N and the target space M, reflected on the fact that $G_{\mu\nu}$ depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To see this explicitly we calculate the covariant derivative of the generalized Lagrangian density

$$\mathcal{L} = \sqrt{|\gamma|}\,\gamma^{ab}(x)\,\partial_a X^{\mu}\partial_b X^{\nu} G_{\mu\nu}(X,x)\,, \tag{5.3.4}$$

and replace in the result the corresponding motion equations (5.3.1). Then, the final result can be written as

$$\nabla_b \widetilde{T}_a^{\ b} = -\frac{\partial \mathcal{L}}{\partial x^a} \tag{5.3.5}$$

where $\widetilde{T}_a{}^b$ represents the canonical energy-momentum tensor

$$\widetilde{T}_a{}^b = \frac{\partial \mathcal{L}}{\partial(\partial_b X^\mu)}(\partial_a X^\mu) - \delta_a^b \mathcal{L} = 2\sqrt{\gamma}G_{\mu\nu}\left(\gamma^{bc}\partial_a X^\mu\,\partial_c X^\nu - \frac{1}{2}\delta_a^b\gamma^{cd}\partial_c X^\mu\,\partial_d X^\nu\right). \tag{5.3.6}$$

The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space $\gamma_{ab} = \eta_{ab}$, the explicit dependence of the metric of the target space $G_{\mu\nu}(X, x)$ on x generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.

$$T_{ab} = \frac{\delta \mathcal{L}}{\delta \gamma^{ab}}\,. \tag{5.3.7}$$

A straightforward computation shows that for the action under consideration here we have that $\widetilde{T}_{ab} = 2T_{ab}$ so that the generalized conservation law (5.3.5) can be written as

$$\nabla_b T_a{}^b + \frac{1}{2}\frac{\partial \mathcal{L}}{\partial x^a} = 0\,. \tag{5.3.8}$$

For a given metric on the base space, this represents in general a system of m differential equations for the "fields" X^μ which must be satisfied "on-shell".

If the base space is 2-dimensional, we can use a reparametrization of x to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, $T_a{}^a = 0$.

In Section 5.1 we described stationary, axially symmetric, gravitational fields as a $(4 \to 2)-$nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \to 2)-$generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space M, and $X^\mu = (f, \Omega)$ the coordinates on the target space N. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

$$\gamma_{ab} = \delta_{ab} \qquad \text{and} \qquad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \qquad (a, b = 1, 2;\ \mu, \nu = 1, 2). \tag{5.3.9}$$

A straightforward computation shows that the generalized Lagrangian (5.3.4) coincides with the Lagrangian (5.2.4) for stationary axisymetric fields, and that the equations of motion (5.3.1) generate the main field equations (4.1.3) and (4.1.4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$. Then

$$T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2}\left[(\partial_\rho f)^2 + (\partial_\rho \Omega)^2 - (\partial_z f)^2 - (\partial_z \Omega)^2\right], \tag{5.3.10}$$

$$T_{\rho z} = \frac{\rho}{2f^2}\left(\partial_\rho f\, \partial_z f + \partial_\rho \Omega\, \partial_z \Omega\right). \tag{5.3.11}$$

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (5.3.8) on-shell:

$$\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\rho} = 0\,, \tag{5.3.12}$$

$$\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0\,. \tag{5.3.13}$$

Incidentally, the last equation coincides with the integrability condition for the metric function k, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs.(4.1.5,4.1.6) and (5.3.10,5.3.11), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (5.3.13) becomes an identity. Although we have eliminated from the starting Lagrangian (5.2.4) the variable k by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [17] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about k at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)-$generalized harmonic map with metrics given as in (5.3.9). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space M is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on M. Then, M represents the world-sheet of a bosonic string in which τ is measures the time and z is the parameter along the string. The string is "embedded" in the target space N whose metric is conformally flat and explicitly depends on the time parameter τ. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates ρ and z are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a\to\infty} f = 1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \omega = c_1 + O\left(\frac{1}{x^a}\right) , \quad \lim_{x^a\to\infty} \Omega = O\left(\frac{1}{x^a}\right) \tag{5.3.14}$$

where c_1 is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate φ. If we choose the domain of the spatial coordinates as $\rho \in [0, \infty)$ and $z \in (-\infty, +\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space N satisfy the boundary conditions

$$\dot{X}^\mu(\rho, -\infty) = 0 = \dot{X}^\mu(\rho, \infty) , \quad X'^\mu(\rho, -\infty) = 0 = X'^\mu(\rho, \infty) \tag{5.3.15}$$

where the dot stands for a derivative with respect to ρ and the prime represents derivation with respect to z. These relationships are known in string theory [16] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume ρ as a "time" parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D-$branes situated at plus and minus infinity in the $z-$direction.

5.4 Dimensional extension

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space N, and study the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an $(m \to D)-$generalized harmonic map. As before we denote by $\{x^a\}$ the coordinates on M. Let $\{X^\mu, X^\alpha\}$ with $\mu = 1, 2$ and $\alpha = 3, 4, ..., D$ be the coordinates on N. The metric structure on M is again $\gamma = \gamma(x)$, whereas the metric on N can in general depend on all coordinates of M and N, i.e. $G = G(X^\mu, X^\alpha, x^a)$. The general structure of the corresponding field equations is as given in (5.3.1). They can be divided into one set of equations for X^μ and one set of equations for X^α. According to the results of the last section, the class of gravitational fields under consideration can be represented as a $(2 \to 2)-$generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates X^μ of the target space. Then, the gravitational sector of the target space will be contained in the components $G_{\mu\nu}$ $(\mu, \nu = 1, 2)$ of the metric, whereas the components $G_{\alpha\beta}$ $(\alpha, \beta = 3, 4, ..., D)$ represent the sector of the dimensional extension.

Clearly, the set of differential equations for X^μ also contains the variables X^α and its derivatives $\partial_a X^\alpha$. For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing X^α and its derivatives in the equations for X^μ. It is easy to show that this can be achieved by imposing the conditions

$$G_{\mu\alpha} = 0\,, \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0\,, \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0\,. \tag{5.4.1}$$

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., $G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a)$, $\gamma = 3, 4, ..., D$. Furthermore, the variables X^α must satisfy the differential equations

$$\frac{1}{\sqrt{|\gamma|}}\partial_b\left(\sqrt{|\gamma|}\gamma^{ab}\partial_a X^\alpha\right) + \Gamma^\alpha_{\ \beta\gamma}\,\gamma^{ab}\,\partial_a X^\beta \partial_b X^\gamma + G^{\alpha\beta}\gamma^{ab}\,\partial_a X^\gamma\,\partial_b G_{\beta\gamma} = 0\,. \tag{5.4.2}$$

This shows that any given $(2 \to 2)-$generalized map can be extended, without affecting the field equations, to a $(2 \to D)-$generalized harmonic map.

It is worth mentioning that the fact that the target space N becomes split in two separate parts implies that the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. $T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)$. The generalized conservation law as given in (5.3.8) is satisfied by the sum of both parts.

Consider the example of stationary axisymmetric fields given the metrics (5.3.9). Taking into account the conditions (5.4.1), after a dimensional extension the metric of the target space becomes

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\ 0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\ . & . & \cdots & \cdots & \cdots \\ 0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x) \end{pmatrix} . \tag{5.4.3}$$

Clearly, to avoid that this metric becomes degenerate we must demand that $\det(G_{\alpha\beta}) \neq 0$, a condition that can be satisfied in view of the arbitrariness of the components of the metric. With the extended metric, the Lagrangian density gets an additional term

$$\begin{aligned} \mathcal{L} = \quad & \frac{\rho}{2f^2}\left[(\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2\right] \\ & + \left(\partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta\right) G_{\alpha\beta} \, , \end{aligned} \tag{5.4.4}$$

which nevertheless does not affect the field equations for the gravitational variables f and Ω. On the other hand, the new fields must be solutions of the extra field equations

$$\left(\partial_\rho^2 + \partial_z^2\right) X^\alpha + \Gamma^\alpha{}_{\beta\gamma}\left(\partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma\right) \tag{5.4.5}$$

$$+ G^{\alpha\gamma}\left(\partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma}\right) = 0 \, . \tag{5.4.6}$$

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice $G_{\alpha\beta} = \eta_{\alpha\beta}$ with additional fields X^α given as arbitrary harmonic functions. This choice opens the possibility of introducing a "time" coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section 4.1, these fields are obtained from the general stationary fields in the limiting case $\Omega = 0$ (or equivalently, $\omega = 0$). If we consider the representation as an $SL(2,R)/SO(2)$ nonlinear sigma model or as a $(2 \to 2)-$generalized harmonic map, we see immediately that the limit $\Omega = 0$ is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case $\Omega = 0$. In the most simple case of an extension with $G_{\alpha\beta} = \delta_{\alpha\beta}$, the resulting $(2 \to 2)-$generalized map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and

$$G = \begin{pmatrix} \frac{\rho}{2f^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4.7}$$

where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable f. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string "living" in a D-dimensional target space N. The string world-sheet is parametrized by the coordinates ρ and z. The gravitational sector of the target space depends explicitly on the metric functions f and Ω and on the parameter ρ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)-$dimensional Minkowski spacetime with time parameter τ. Then, the string world-sheet is a 2-dimensional flat hypersurface which is "frozen" along the time τ.

5.5 The general solution

If we take as seed metric the general static solution, the application of two HXK transformations generates a stationary solution with an infinite number of gravitoelectric and gravitomagnetic multipole moments. The HKX method is applied at the level of the Ernst potential from which the metric functions can be calculated by using the definition of the Ernst potential E and the field equations for γ. The resulting expressions in the general case are quite cumbersome. We quote here only the special case in which only an arbitrary quadrupole parameter is present. In this case, the result can be written as

$$\begin{aligned} f &= \frac{R}{L} e^{-2qP_2Q_2} , \\ \omega &= -2a - 2\sigma \frac{\mathcal{M}}{R} e^{2qP_2Q_2} , \\ e^{2\gamma} &= \frac{1}{4}\left(1+\frac{M}{\sigma}\right)^2 \frac{R}{x^2-y^2} e^{2\hat{\gamma}} , \end{aligned} \tag{5.5.1}$$

where

$$\begin{aligned} R &= a_+a_- + b_+b_- , \qquad L = a_+^2 + b_+^2 , \\ \mathcal{M} &= \alpha x(1-y^2)(e^{2q\delta_+} + e^{2q\delta_-})a_+ + y(x^2-1)(1-\alpha^2 e^{2q(\delta_+ + \delta_-)})b_+ , \\ \hat{\gamma} &= \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q(1-P_2)Q_1 + q^2(1-P_2)\Big[(1+P_2)(Q_1^2-Q_2^2) \\ &\quad + \frac{1}{2}(x^2-1)(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2')\Big] . \end{aligned} \tag{5.5.2}$$

Here $P_l(y)$ and $Q_l(x)$ are Legendre polynomials of the first and second kind respectively. Furthermore

$$\begin{aligned} a_\pm &= x(1-\alpha^2 e^{2q(\delta_+ + \delta_-)}) \pm (1+\alpha^2 e^{2q(\delta_+ + \delta_-)}) , \\ b_\pm &= \alpha y(e^{2q\delta_+} + e^{2q\delta_-}) \mp \alpha(e^{2q\delta_+} - e^{2q\delta_-}) , \\ \delta_\pm &= \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2 \mp xy) + \frac{3}{4}[x(1-y^2) \mp y(x^2-1)]\ln\frac{x-1}{x+1} , \end{aligned}$$

the quantity α being a constant

$$\alpha = \frac{\sigma - M}{a} , \qquad \sigma = \sqrt{M^2 - a^2} . \tag{5.5.3}$$

The physical significance of the parameters entering this metric can be clarified by calculating the Geroch-Hansen [18, 19] multipole moments

$$M_{2k+1} = J_{2k} = 0 , \quad k = 0, 1, 2, \ldots \tag{5.5.4}$$

$$M_0 = M\,, \quad M_2 = -Ma^2 + \frac{2}{15}qM^3\left(1-\frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.5}$$

$$J_1 = Ma\,, \quad J_3 = -Ma^3 + \frac{4}{15}qM^3a\left(1-\frac{a^2}{M^2}\right)^{3/2}\,, ... \tag{5.5.6}$$

The vanishing of the odd gravitoelectric (M_n) and even gravitomagnetic (J_n) multipole moments is a consequence of the symmetry with respect to the equatorial plane. From the above expressions we see that M is the total mass of the body, a represents the specific angular momentum, and q is related to the deviation from spherical symmetry. All higher multipole moments can be shown to depend only on the parameters M, a, and q.

We analyzed the geometric and physical properties of the above solution. The special cases contained in the general solution suggest that it can be used to describe the exterior asymptotically flat gravitational field of rotating body with arbitrary quadrupole moment. This is confirmed by the analysis of the motion of particles on the equatorial plane. The quadrupole moment turns out to drastically change the geometric structure of spacetime as well as the motion of particles, especially near the gravitational source.

We investigated in detail the properties of the Quevedo-Mashhoon (QM) spacetime which is a generalization of Kerr spacetime, including an arbitrary quadrupole. Our results show [20] that a deviation from spherical symmetry, corresponding to a non-zero electric quadrupole, completely changes the structure of spacetime. A similar behavior has been found in the case of the Erez-Rosen spacetime. In fact, a naked singularity appears that affects the ergosphere and introduces regions where closed timelike curves are allowed. Whereas in the Kerr spacetime the ergosphere corresponds to the boundary of a simply-connected region of spacetime, in the present case the ergosphere is distorted by the presence of the quadrupole and can even become transformed into non simply-connected regions. All these changes occur near the naked singularity which is situated at $x = 1$, a value that corresponds to the radial distance $r = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates. In the limiting case $a/M > 1$, the multipole moments and the metric become complex, indicating that the physical description breaks down. Consequently, the extreme Kerr black hole represents the limit of applicability of the QM spacetime.

Since standard astrophysical objects satisfy the condition $a/M < 1$, we can conclude that the QM metric can be used to describe their exterior gravitational field. Two alternative situations are possible. If the characteristic radius of the body is greater than the critical distance $M + \sqrt{M^2 - a^2}$, i.e. $x > 1$, the exterior solution must be matched with an interior solution in order to describe the entire spacetime. If, however, the characteristic radius of the body is smaller than the critical distance $M + \sqrt{M^2 - a^2}$, the QM metric describes the field of a naked singularity.

The presence of a quadrupole and higher multipole moments leads to interesting consequences in the motion of test particles. In several works to be presented below, the For instance, repulsive effects can take place in a region very closed to the naked singularity. In that region stable circular orbits can exist. The limiting case of static particle is also allowed. Due to the complexity of the above solution, the investigation of naked singularities can be performed only numerically. To illustrate the effects of repulsive gravity analytically, we used the simplest possible case which corresponds to the Reissner-Nordströn spacetime.

6 Motion of test particles in the field of a naked singularity

6.1 Introduction

The black hole uniqueness theorems state that in Einstein's general relativity theory the most general black hole solution in empty space is described by the Kerr metric [21], which represents the exterior gravitational field of a rotating mass m with specific angular momentum $a = J/m$. In Boyer-Lindquist coordinates (t, r, θ, φ), a true curvature singularity is determined by the equation $r^2 + a^2 \cos^2\theta = 0$ that corresponds to a ring singularity situated on the equatorial plane $\theta = \pi/2$. The ring singularity is isolated from the exterior spacetime by an event horizon situated on a sphere of radius $r_h = m + \sqrt{m^2 - a^2}$.

In the case that $a^2 > m^2$, no event horizon exists and the ring singularity becomes naked. Different studies [22, 23, 24] show, however, that in realistic situations, where astrophysical objects are mostly surrounded by accretion disks, a Kerr naked singularity is an unstable configuration that rapidly decays into a Kerr black hole. Moreover, it now seems established that in generic situations a gravitational collapse cannot lead to the formation of a final configuration corresponding to a Kerr naked singularity. These results seem to indicate that rotating Kerr naked singularities cannot be very common objects in nature.

The above results seem to corroborate the validity of the cosmic censorship hypothesis [25] according to which a physically realistic gravitational collapse, which evolves from a regular initial state, can never lead to the formation of a naked singularity; that is, all singularities formed as the result of a realistic collapse should always be enclosed within an event horizon; hence, the singularities are always invisible to observers situated outside the horizon. Many attempts have been made to prove this conjecture with the same mathematical rigor used to show the inevitability of singularities in general relativity [26]. No general proof has been formulated so far. Instead, particular scenarios of gravitational collapses have been investigated some of which indeed corroborate the correctness of the conjecture.

Nevertheless, other studies [27] indicate that under certain circumstances naked singularities can appear as the result of a realistic gravitational collapse. Indeed, it turns out that in the case of the collapse of an inhomogeneous matter distribution, there exists a critical degree of inhomogeneity below which black holes form. Naked singularities appear if the degree of inhomogeneity is higher than the critical value. The speed of the collapse and the shape of the collapsing object are also factors that play an important role in the determination of the final configuration. It turns out that naked singularities form more frequently if the collapse occurs very rapidly and if the object is not spherically symmetric.

In view of this situation it seems reasonable to investigate the effects of naked singularities on the surrounding spacetime. This is the main aim of the present work. We will study a naked singularity without black hole counterpart. In fact, we investigate the quadrupole metric ($q-$metric) which is the simplest generalization of the Schwarzschild metric containing a naked singularity. We will see that, starting from the Schwarzschild metric, the Zipoy–Voorhees [28, 29] transformation can be used to generate a static axisymmetric spacetime which describes the field of a mass with a particular quadrupole moment. For any values of the quadrupole, the spacetime is characterized by the presence of naked singularities situated at a finite distance from the origin of coordinates. In this work, we are interested in analyzing the spacetime outside the outer naked singularity.

We perform an analysis of the motion of test particles (massive and nonmassive) outside the naked singularity, comparing in all the cases our results with the corresponding situation in the Schwarzschild spacetime in order to establish the exact influence of the quadrupole on the parameters of the trajectories.

6.2 The q-metric

In 1917 [1], it was shown that the most general static axisymmetric asymptotically flat solution of Einstein's vacuum equations is represented by the Weyl class. In terms of multipole moments, the simplest static solution contained in the Weyl class is the Schwarzschild metric which is the only one that possesses a mass monopole moment only. From a physical point of view, the next interesting solution should describe the exterior field of a mass with quadrupole moment. In this case, it is possible to find a large number of exact solutions with the same quadrupole (see [32] and the references cited therein) that differ only in the set of higher multipoles. A common characteristic of the solutions with quadrupole is that their explicit form is rather cumbersome, making them difficult to be handled analytically [33]. An alternative exact solution was presented by one of us in [34] by applying on the Schwarzschild metric a Zipoy-Voorhees transformation with parameter $\delta = 1 + q$, where q represents the quadrupole parameter. In spherical coordinates, the resulting metric can be written in a compact and simple form as

$$\begin{aligned} ds^2 = & \left(1 - \frac{2m}{r}\right)^{1+q} dt^2 - \left(1 - \frac{2m}{r}\right)^{-q} \\ & \times \left[\left(1 + \frac{m^2 \sin^2\theta}{r^2 - 2mr}\right)^{-q(2+q)} \left(\frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 \right) \right. \\ & \left. + r^2 \sin^2\theta d\varphi^2 \right] . \end{aligned} \tag{6.2.1}$$

In the literature, this solution is known as the $\delta-$metric or as the $\gamma-$metric for notational reasons [35]. We propose to use the term quadrupole metric ($q-$metric) to emphasize the role of the parameter q which determines the quadrupole moment, as we will see below.

The $q-$metric is an axially symmetric exact vacuum solution that reduces to the spherically symmetric Schwarzschild metric only for $q \to 0$. It is asymptotically flat for any finite values of the parameters m and q. Moreover, in the limiting case $m \to 0$ it can be shown that, independently of the value of q, there exists a coordinate transformation that transforms the resulting metric into the Minkowski solution. This last property is important from a physical point of view because it means that the parameter q is related to a genuine mass distribution.

An important quantity that characterizes the physical properties of any exact solution is the Arnowitt-Deser-Misner (ADM) mass. If one tries to use the common formula [36, 37] for calculating the ADM mass, one would obtain incorrect results because these formula are adapted to a particular coordinate system. Therefore, we perform here a detailed calculation of the ADM mass, using the original approach. All the details are given in the Appendix. In the case of the $q-$metric (6.2.1), the final result is simply

$$M_{ADM} = m(1+q) \; . \tag{6.2.2}$$

It follows that if we take the parameter m as positive, the parameter q must satisfy the condition $q > -1$ in order for the ADM mass to be positive. Nevertheless, one can choose a negative m so that for $q < -1$ the mass is still positive. For the sake of simplicity we assume from now on that m is positive.

We also calculate the multipole moments of the $q-$metric by using the invariant definition proposed by Geroch [38]. The lowest mass multipole moments M_n, $n = 0, 1, \ldots$ are given by

$$M_0 = (1+q)m \; , \quad M_2 = -\frac{m^3}{3} q(1+q)(2+q) \; , \tag{6.2.3}$$

whereas higher moments are proportional to mq and can be completely rewritten in terms of M_0 and M_2. This means that the arbitrary parameters m and q determine the mass and quadrupole which are the only independent multipole moments of the solution. In the limiting case $q = 0$ only the monopole $M_0 = m$ survives, as in the Schwarzschild spacetime. In the limit $m \to 0$, with $q \neq 0$, and $q \to -1$, with $m \neq 0$, all multipoles vanish identically, implying that no mass distribution is present and the spacetime must be flat. Furthermore, notice that all odd multipole moments are zero because the solution possesses an additional reflection symmetry with respect to the equatorial plane $\theta = \pi/2$.

We conclude that the above metric describes the exterior gravitational field of a static deformed mass. The deformation is described by the quadrupole moment M_2 which is positive for a prolate mass distribution and negative for an oblate one. Notice that the condition $q > -1$ must be satisfied in order to avoid the appearance of a negative total mass M_0. Therefore, in the interval $q \in (-1, 0)$ the $q-$metric describes a prolate mass distribution and in the interval $(0, \infty)$ an oblate one.

To investigate the structure of possible curvature singularities, we consider the Kretschmann scalar $K = R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau}$. A straightforward computation leads to

$$K = \frac{16m^2(1+q)^2}{r^{4(2+2q+q^2)}} \frac{(r^2 - 2mr + m^2 \sin^2\theta)^{2(2q+q^2)-1}}{(1-2m/r)^{2(q^2+q+1)}} L(r,\theta) \; , \tag{6.2.4}$$

with

$$L(r,\theta) = 3(r-2m-qm)^2(r^2-2mr+m^2\sin^2\theta) + q(2+q)m^2\sin^2\theta[q(2+q)m^2+3(r-m)(r-2m-qm)]\ . \quad (6.2.5)$$

In the limiting case $q = 0$, we obtain the Schwarzschild value $K = 48m^2/r^6$ with the only singularity situated at the origin of coordinates $r \to 0$. In general, we can see that the singularity at the origin, which occurs at $r^{4(2+2q+q^2)} = 0$, is present for all real values of q. Moreover, an additional singularity appears at the radius $r = 2m$ which, according to the form of the metric (6.2.1), is also a horizon in the sense that the norm of the timelike Killing vector vanishes at that radius. Outside the hypersurface $r = 2m$ no additional horizon exists, indicating that the singularities situated at the origin and at $r = 2m$ are naked. In addition, a curvature singularity occurs at the surface determined by the equation

$$r^2 - 2mr + m^2\sin^2\theta = 0 \quad (6.2.6)$$

under the condition that the value of the quadrupole parameter is within the interval $q \in (-1, -1+\sqrt{3/2}]\backslash\{0\}$. We conclude that the main consequence of the existence of a quadrupole determined by the parameter q is that the Schwarzschild horizon becomes a naked singularity. The geometric structure of the curvature singularities of the $q-$metric is illustrated in Fig. 6.1.

Several physical properties of the $q-$metric have been investigated in the literature (see, for instance, [39, 40, 41, 42, 43, 44, 35, 34, 45] and references therein). In this work, we are interested in continuing the analysis of the corresponding gravitational field as observed by exterior test particles.

6.3 Geodesic motion

Consider the trajectory $x^\alpha(\tau)$ of a test particle with 4-velocity $u^\alpha = dx^\alpha/d\tau = \dot{x}^\alpha$. The moment $p^\alpha = \mu\dot{x}^\alpha$ of the particle can be normalized so that

$$g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \epsilon, \quad (6.3.1)$$

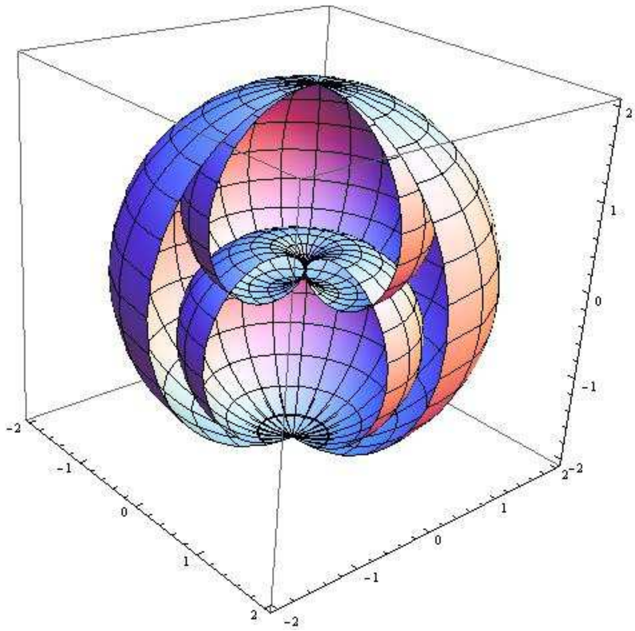


Figure 6.1: Curvature singularities of the $q-$metric. The outer singularity at $r = 2m$ and the singularity at the origin $r = 0$ exist for any value of the parameter q. The singularity which depends on the angle θ exists only for certain values of the quadrupole (see text).

where $\epsilon = 0, 1, -1$ for null, timelike, and spacelike curves, respectively [30]. For the $q-$metric we obtain from (6.3.1) that

$$\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)}\dot{r}^2 = \tilde{E}^2 - \Phi^2, \tag{6.3.2}$$

where

$$\begin{aligned}\Phi^2 = \ & \left(1-\tfrac{2m}{r}\right)^{1+q}\left[r^2\left(1-\tfrac{2m}{r}\right)^{-q}\left(1+\tfrac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)}\dot{\theta}^2\right.\\ & \left. +\tfrac{\tilde{l}^2}{r^2\sin^2\theta}\left(1-\tfrac{2m}{r}\right)^{q}+\epsilon\right],\end{aligned} \tag{6.3.3}$$

and we have used the expression for the energy $E = \mu\tilde{E}$ and the angular moment $l = \mu\tilde{l}$ of the test particle which are constants of motion

$$\begin{aligned} E &= g_{\alpha\beta}\xi^{\alpha}_{t}p^{\beta} = \left(1-\frac{2m}{r}\right)^{1+q}\mu\dot{t}, &(6.3.4)\\ l &= -g_{\alpha\beta}\xi^{\alpha}_{\varphi}p^{\beta} = \left(1-\frac{2m}{r}\right)^{-q} r^2\sin^2\theta\mu\dot{\varphi}, &(6.3.5)\end{aligned}$$

associated with the Killing vector fields $\xi_t = \partial_t$ and $\xi_\varphi = \partial_\varphi$, respectively. For the sake of simplicity we set $\mu = 1$ so that $\tilde{E} = E$ and $\tilde{l} = l$.

In addition, the equation for the acceleration along the polar angle can be expressed as

$$\begin{aligned}\ddot{\theta} = \ & -\left(\tfrac{q(2+q)m^2\sin\theta\cos\theta}{r^2-2mr+m^2\sin^2\theta}\right)\left(\tfrac{\dot{r}^2}{r^2-2mr}-\dot{\theta}^2\right)\\ & +\left(\tfrac{r^2-2mr+m^2\sin^2\theta}{r^2-2mr}\right)^{q(2+q)}\sin\theta\cos\theta\ \dot{\varphi}^2\\ & -\tfrac{r^3-(4+q)mr^2+\left[2(2+q)+(1+q)^2\sin^2\theta\right]m^2r-(1+q)(2+q)m^3\sin^2\theta}{(r^2-2mr)(r^2-2mr+m^2\sin^2\theta)}\ \dot{r}\ \dot{\theta}.\end{aligned} \tag{6.3.6}$$

From this equation it follows that the acceleration $\ddot{\theta}$ depends on the polar angle. Consider for instance the motion around the central object with initial values $\dot{r} = \dot{\theta} = 0$ and $\dot{\varphi} \neq 0$. Then, Eq.(6.3.6) reduces to

$$\left(1 + \frac{m^2 \sin^2\theta}{r^2 - 2mr}\right)^{-q(2+q)} \ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0 \, , \tag{6.3.7}$$

It then follows that the only places at which the acceleration vanishes are $\theta = 0$ and $\theta = \pi/2$, showing that the polar and equatorial planes are geodesic planes at which the investigation of the test particle motion can be considerably simplified due to the symmetry of the configuration. For all the remaining values of θ, there is always a non-zero value of the acceleration that induces a velocity in the direction of θ towards the equatorial plane or the poles.

Equation (6.3.2) can be considered as describing the motion along the radial coordinate in terms of the "effective potential" Φ^2. However, the velocity $\dot{\theta}$ enters the expression for Φ^2 explicitly so that strictly speaking it is not a potential. Nevertheless, its dependence on the coordinates can shed some light on the behavior of the geodesics for particular values of $\dot{\theta}$. In Fig. 6.2, we plot the behavior of Φ^2 for a particular initial velocity $\dot{\theta} = 1$ and different values of the q parameter. We see that at the singularity $r = 2m$ the behavior of the potential drastically depends on the value of q. Indeed, for positive values of q the potential either tends to infinity near the singularity or crosses it and becomes divergent inside the singularity. Instead, if q is negative, the potential tends to infinity as the singularity is approached from outside. In the case of vanishing quadrupole, the singularity turns into a horizon and the potential crosses it until it becomes infinity near the central singularity located at the origin of coordinates.

An interesting particular case of the geodesic motion is that of circular orbits and their stability. The geodesic equation in this case is given as in Eq.(6.3.2) with $\dot{r} = 0$, i.e. $\Phi^2 = E^2$, whose solutions strongly depend on the value of the velocity $\dot{\theta}$. In fact, the condition for circular orbits can be expressed as

$$\begin{aligned} \dot{\theta}^2 = \quad & (r^2 - 2mr)^{-1} \left(1 + \tfrac{m^2 \sin^2\theta}{r^2 - 2mr}\right)^{q(2+q)} \\ & \times \left[E^2 - \tfrac{l^2}{r^2 \sin^2\theta}\left(1 - \tfrac{2m}{r}\right)^{1+2q} - \epsilon\left(1 - \tfrac{2m}{r}\right)^{1+q}\right] . \end{aligned} \tag{6.3.8}$$

We perform a numerical investigation of this condition for a particular radius

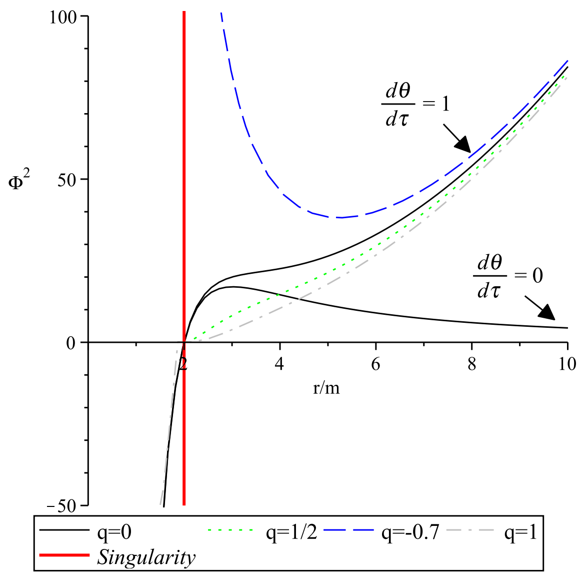


Figure 6.2: Behavior of the "effective potential" Φ^2 in terms of the radial distance r for the particular velocity $\dot{\theta} = 1$ with $\theta = \pi/4$, $l = 15$ and different values of q. The effective potential for the Schwarzschild solution $q = 0$ on a fixed plane with $\dot{\theta} = 0$ is also depicted for comparison.

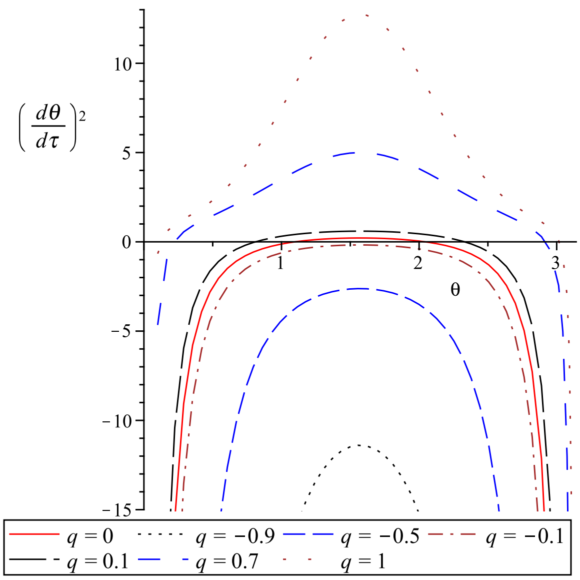


Figure 6.3: Angular velocity $\dot{\theta}$ in terms of the angle $\theta \in (0, \pi)$ for different quadrupoles. Here, we consider timelike geodesics ($\epsilon = 1$) with $m = 1$, $l = 5$. For concreteness, the radius of the orbit has been set to $r = 2.35$.

and different values of the quadrupole parameter in Fig. 6.3. If we consider, for instance, the case of a circular orbit with radius $r = 2.35m$, we can see that such a motion is not possible for arbitrary values of q and θ. In fact, for most negative values of q no circular orbit with this radius is allowed, because $\dot{\theta}^2$ is negative for all values of θ. Instead, for all positive values of q it is always possible to find an interval of θ within which it is possible to have circular orbits. For other values of the orbit radius similar results can be found.

The study of the stability of circular orbits is important to establish the possibility of having accretion disks around the central gravitational source. In particular, the radius of the last stable circular orbit determines the inner radius of the accretion disk, and corresponds to an inflection point of the potential, i.e., the point where the conditions $\partial\Phi^2/\partial r = 0$ and $\partial^2\Phi^2/\partial r^2 = 0$ are satisfied. We use these two conditions to find the explicit values of l^2 and

$\dot{\theta}^2$ for the last stable circular orbit and obtain

$$l^2_{lsco} = \frac{\epsilon m(1+q)\sin^2\theta\, r^2}{r-(3+2q)m}\left(1-\frac{2m}{r}\right)^{-q}\left[1-\frac{(r-m)\,G}{r(r-2m)\,[r-(3+2q)m]\,H}\right], \tag{6.3.9}$$

and

$$\dot{\theta}^2_{lsco} = -\frac{\epsilon m(1+q)(1-\frac{2m}{r})^{q-1}(r^2-2mr+m^2\sin^2\theta)^{(1+2q+q^2)}G}{r^4[r-(3+2q)m](r^2-2mr)^{q(2+q)}\left[r^2-2mr+m^2(1+q)^2\sin^2\theta\right]H}, \tag{6.3.10}$$

where

$$G \equiv r^2 - 2m(4+3q)r + 2m^2(2+q)(3+2q)\,, \tag{6.3.11}$$

$$H \equiv 1 + \frac{r-m}{(r^2-2mr)[r-(3+2q)m)]}\left[3r^2-6m(3+2q)r+4m^2(2+q)(3+2q) + \frac{2q(2+q)(1+q)^2m^4\sin^4\theta(r-m)[r-(3+2q)m]}{(r^2-2mr+m^2\sin^2\theta)[r^2-2mr+m^2(1+q)^2\sin^2\theta]}\right]. \tag{6.3.12}$$

In the limiting case of the Schwarzschild spacetime ($q=0$), due to the spherical symmetry we can set $\dot{\theta}_{lsco}=0$ and so we obtain the value $r^{Sch}_{lsco}=6m$, as expected. The value of l^2_{lsco} determines the angular momentum of the test particle on the last stable circular orbit and $\dot{\theta}^2_{lsco}$ the velocity of the orbit with respect to the polar angle. In Fig. 6.4, we analyze numerically the above equations for different values of the quadrupole. The condition for the existence of a last stable circular orbits is that both l^2_{lsco} and $\dot{\theta}^2_{lsco}$ be positive and finite. From Fig. 6.4 we see that for $q=0.5$ and $q=-0.5$ these conditions are not satisfied. For $q=0.1$ and $q=-0.1$, however, there exists an interval of r in which both l^2_{lsco} and $\dot{\theta}^2_{lsco}$ are positive. The first thing one can notice is that this interval is always inside the Schwarzschild radius $r^{Sch}_{lsco}=6m$. This means that the quadrupole moment diminishes the value of the radius of the last stable circular orbit. For a particular positive value of $\dot{\theta}^2_{lsco}$ one can find from the graph the corresponding radius r_{lsco}. If this value happens to correspond to a positive value of l^2_{lsco}, then there exists a stable circular orbit with that radius and that angular momentum.

The motion of test particles for arbitrary values of $\dot{\theta}$ and θ can be investigated by performing a numerical integration of the geodesic equations (6.3.2)–

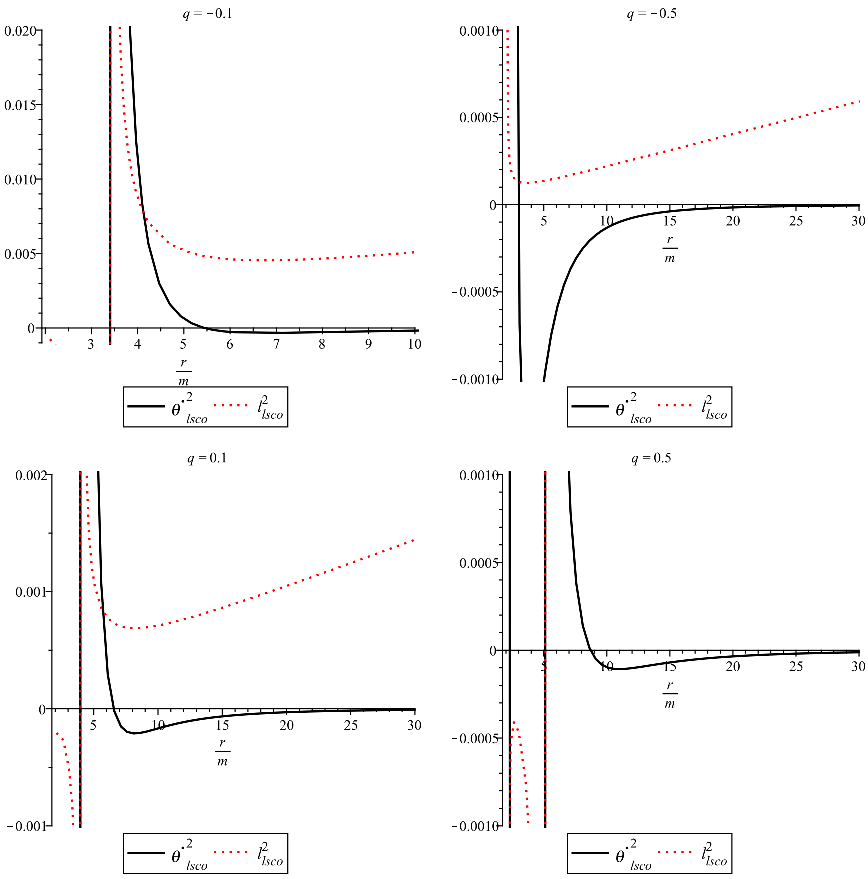


Figure 6.4: Angular momentum and polar angle velocity of the last stable circular timelike orbit as a function of the radial coordinate for different values of the quadrupole parameter q. Here we set $\theta = \pi/4$. The value of l^2_{lsco} has been rescaled in all the graphs for comparison with the graph of $\dot{\theta}^2_{lsco}$.

(6.3.6). Several cases must be considered, depending on the initial velocity $\dot{\theta}$ and the initial plane θ. This requires a detailed numerical analysis in which several aspects must be considered like the stability of the trajectories and the appearance of chaotic motion. We will present these results elsewhere. In this work, we will focus on the analytic investigation of the most important families of geodesics of the $q-$ metric.

6.4 Equatorial geodesics

As mentioned above the equatorial plane $\theta = \pi/2$ is a geodesic plane due to the symmetry of the gravitational source. In this case, the geodesic equations reduce to

$$\dot{t} = E\left(1 - \frac{2m}{r}\right)^{-(1+q)} , \tag{6.4.1}$$

$$\dot{\varphi} = \frac{l}{r^2}\left(1 - \frac{2m}{r}\right)^{q} , \tag{6.4.2}$$

$$\left(1 + \frac{m^2}{r^2 - 2mr}\right)^{-q(2+q)} \dot{r}^2 = E^2 - \left(1 - \frac{2m}{r}\right)^{q+1}\left[\frac{l^2}{r^2}\left(1 - \frac{2m}{r}\right)^{q} + \epsilon\right] . \tag{6.4.3}$$

To investigate the behavior of the geodesics it is convenient to introduce the new coordinate $u = 1/r$ and use the azimuthal angle φ instead of the affine parameter τ so that

$$\frac{du}{d\tau} = lu^2\left(1 - 2mu\right)^q \frac{du}{d\varphi} . \tag{6.4.4}$$

Then, the geodesic equation (6.4.3) can be expressed as

$$\frac{(1-2mu)^{q(4+q)}}{(1-mu)^{2q(2+q)}}\left(\frac{du}{d\varphi}\right)^2 = F(u,q) , \tag{6.4.5}$$

where

$$F(u,q) = \frac{1}{l^2}\left\{E^2 - (1-2mu)^{1+q}\left[l^2u^2(1-2mu)^q + \epsilon\right]\right\} . \tag{6.4.6}$$

The properties of the geodesic motion are then determined by the function $F(u,q)$ which must be positive for the velocity to be a real valued function.

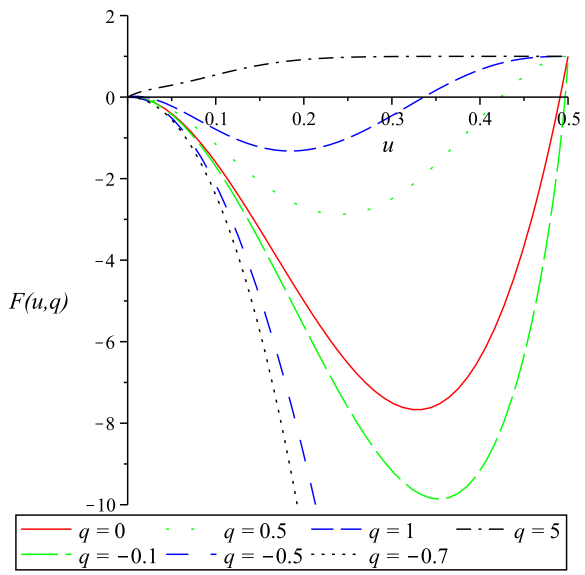


Figure 6.5: Behavior of the function $F(u,q)$ for particular values of the parameters $m = 1$, $E = 1$, and $l = 15$, and $\epsilon = 1$.

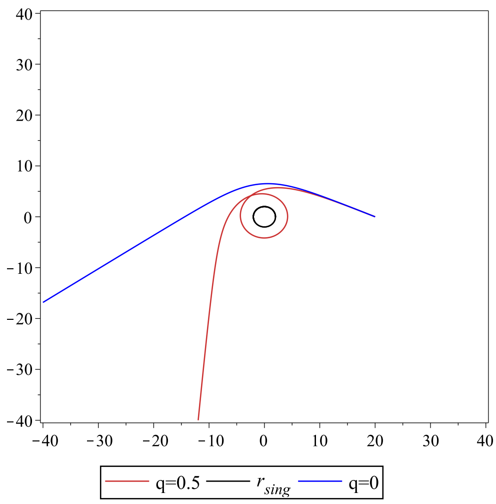
Furthermore, the zeros of $F(u,q)$ are the points where the radial velocity vanishes, i.e., the perihelion or aphelion distance in the case of bounded orbits. In turn, the behavior of $F(u,q)$ can be explored by comparison with the function $f(u)$ which is defined as

$$f(u) = F(u, q = 0) = \frac{1}{l^2}\left[E^2 - (1 - 2mu)(l^2u^2 + \epsilon)\right] . \tag{6.4.7}$$

This means that the function $f(u)$ determines the behavior of the geodesics in the case of the Schwarzschild spacetime and, consequently, a comparison between $F(u,q)$ and $f(u)$ will determine the main differences between geodesics of the $q-$metric and those of the Schwarzschild spacetime. It is easy to show that

$$F(u,q) > f(u) \quad \text{for} \quad q > 0 \quad \text{and} \quad F(u,q) < f(u) \quad \text{for} \quad q < 0 \tag{6.4.8}$$

in the interval $u \in \left(0, \frac{1}{2m}\right)$ i.e., in the spatial region located between the outer singularity $(r = 2m)$ and infinity. This behavior is illustrated in Fig. 6.5. This simple observation leads to an interesting physical result. Suppose that the



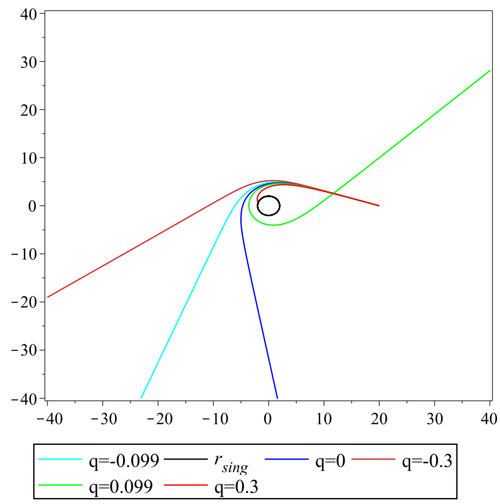


Figure 6.6: Influence of the quadrupole on unbounded Schwarzschild orbits with non-vanishing initial radial velocity ($\dot{r}(0) \neq 0$). The initial point $\varphi(0) = 0$ and $r(0) = 20$ is the same for both plots. Initial velocities: $\dot{r}(0) = -2$ and $\dot{\varphi}(0) = 0.04305$ (left plot); $\dot{r}(0) = -2.5$ and $\dot{\varphi}(0) = 0.039$ (right plot).

zeros of $f(u)$ are u_1 and u_2 with $u_1 < u_2$ so that the trajectory is bounded between $r_1 = 1/u_1$ and $r_2 = 1/u_2$; consequently, the perihelion distance is $r_2 < r_1$. Suppose that the value of q is so chosen that $F(u,q)$ has also two zeros at $u_1' < u_2'$. Then, from the behavior of $F(u,q)$ (cf. Fig. 6.5) we conclude that

$$u_1 < u_1' \,, \quad \text{i. e.} \quad r_1 > r_1' \quad \text{and} \quad u_2' < u_2 \,, \quad \text{i. e.} \quad r_2' > r_2 \quad \text{for} \quad q > 0 \,. \tag{6.4.9}$$

This means that the perihelion distance for positive q is greater that the one for vanishing quadrupole. A similar analysis shows that the perihelion distance decreases for negative values of the quadrupole. We conclude that the presence of the quadrupole in the $q-$metric leads to a change of the perihelion distance, an effect that has been predicted also for other metrics with quadrupole moment [46, 33]. Some examples of geodesics for different values of the quadrupole parameter are given in Figs. 6.6-6.8, where for simplicity the radial coordinate has been redefined as r/m for all the trajectories.

In Fig. 6.6, we consider unbounded Schwarzschild orbits with non-vanishing initial radial velocities under the influence of the quadrupole. We see that for

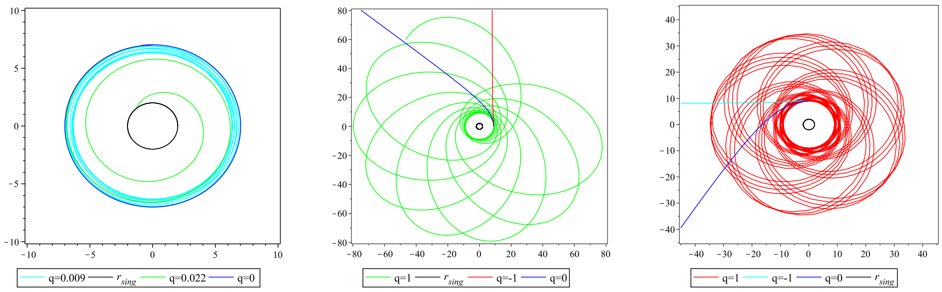


Figure 6.7: Influence of the quadrupole on unbounded Schwarzschild orbits with vanishing initial radial velocity ($\dot{r}(0) = 0$). Initial conditions: $\varphi(0) = \pi/2$, $r(0) = 7$, $\dot{\varphi}(0) = 0.07145$ (left plot); $\varphi(0) = 0$, $r(0) = 9$, $\dot{\varphi}(0) = 0.075$ (center plot); $\varphi(0) = \pi/2$, $r(0) = 9$, $\dot{\varphi}(0) = 0.07145$ (right plot).

the chosen initial radial and angular velocities all the test particles are able to escape from the gravitational field of the naked singularity. The value of the quadrupole determines only the direction along which the particle escapes towards infinity.

In Fig. 6.7, we investigate how the quadrupole acts on unbounded Schwarzschild orbits with zero initial radial velocity. First, we study the change of a stable circular Schwarzschild geodesic under the influence of a quadrupole (left panel). For a small quadrupole ($q = 0.009$), the geodesic moves slowly towards the central singularity and finally reaches a stable circular orbit with a radius which is smaller than the initial Schwarzschild radius. In the central and right plots, we consider the same values for the quadrupole moment, but different initial angular velocities $\dot{\varphi}(0)$. In both cases, the test particles move along bounded orbits with different perihelion shifts. From the behavior of these trajectories, we conclude that the quadrupole is able to transform unbounded Schwarzschild trajectories into bounded trajectories.

In Fig. 6.8, we consider a Schwarzschild bounded orbit under the influence of the quadrupole with zero initial radial velocity and the same non-zero value for the initial angular velocity. The left panel shows the Schwarzschild geodesic, whereas the central and right plots are for a particular negative and positive value of the quadrupole, respectively. In this case, we see that the quadrupole does not affect the bounded character of the geodesic. However, it can drastically modify the geometric structure of the trajectory.

In general, we see that the quadrupole of the naked singularity always affects the motion of test particles. The specific change can manifest itself in

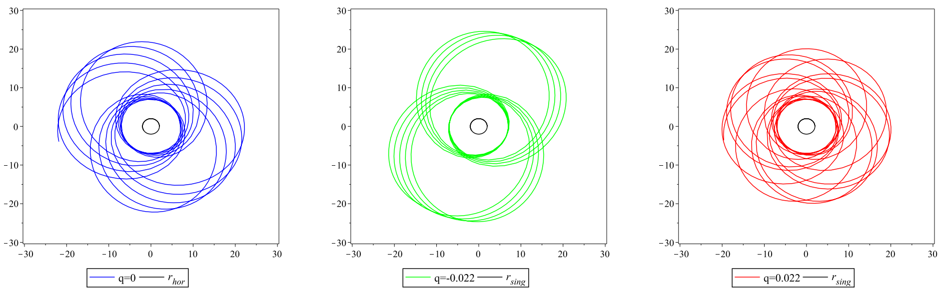


Figure 6.8: Influence of the quadrupole on Schwarzschild bounded orbits with vanishing initial radial velocity ($\dot{r}(0) = 0$). The initial conditions are $\varphi(0) = 0$, $r(0) = 7$, and $\dot{\varphi}(0) = 0.08$ for all the trajectories.

different ways, depending on the explicit value of the quadrupole.

6.4.1 Circular orbits

Circular orbits on the equatorial plane can be investigated analytically. In fact, in this case the motion under the condition $\dot{r} = 0$ is equivalent to the motion of a test particle in the effective potential [cf. Eq.(6.4.3)]

$$V_{eff}^2(r,q) = \left(1 - \frac{2m}{r}\right)^{q+1} \left[\frac{l^2}{r^2}\left(1 - \frac{2m}{r}\right)^q + \epsilon\right]. \tag{6.4.10}$$

The presence of the quadrupole parameter influences the behavior of the effective potential, as can be seen in Fig. 6.9. The effective potential of the Schwarzschild is also shown for comparison. For positive values of q the value of the effective potential at a given point outside the outer singularity is always less than the Schwarzschild value. For negative values of the quadrupole the opposite is true. This indicates that the distribution of circular orbits on the equator of the $q-$metric can be modified drastically by the quadrupole. We will now explore this point in detail.

The explicit value of the angular momentum of a test particle along a circular orbit can be derived from the condition $\partial V_{eff}^2/\partial r = 0$. A straightforward computation yields that this condition is satisfied if the angular momentum is given by

$$l^2 = \frac{m(q+1)\left(1 - \frac{2m}{r}\right)^{-q} r^2}{r - (3+2q)m}, \tag{6.4.11}$$

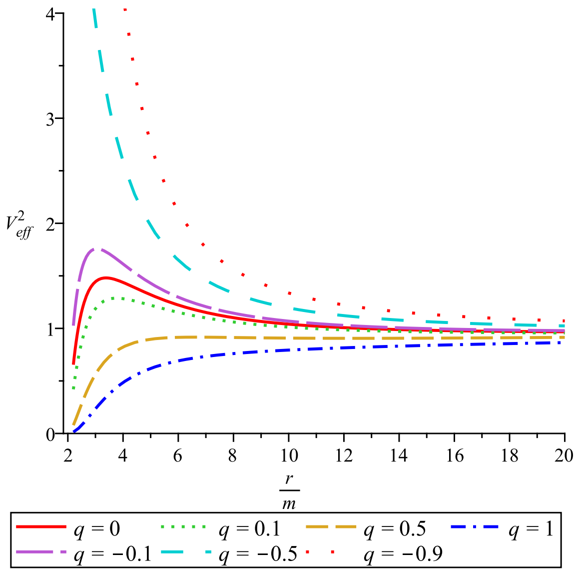


Figure 6.9: The effective potential for timelike circular geodesics on the equatorial plane as a function of the radius for different values of the quadrupole parameter. Here we set $l^2 = 30$ for concreteness.

an expression from which it follows immediately that the radius of any circular orbit must satisfy the condition

$$r_c > m(3+2q) \ . \tag{6.4.12}$$

This radius is greater than the Schwarzschild radius for a positive quadrupole parameter. However, for negative values of q the radius is smaller than the Schwarzschild value and, in principle, can be made as close as possible to the outer singularity located at $r = 2m$. Below we will show that in the limiting case $r_c = m(3+2q)$, the orbit becomes lightlike.

Furthermore, the energy of a test particle in circular motion can be expressed as

$$E^2 = \left(1 - \frac{2m}{r}\right)^{q+1} \left[\frac{m(q+1)}{r - m(3+2q)} + 1\right] \ , \tag{6.4.13}$$

which is positive only within the range of allowed radii (6.4.12). To completely characterize the parameters of circular orbits, we also calculate their angular velocity $\Omega(r)$ and period $T(r)$ and obtain

$$\Omega(r) = \dot{\varphi} = \frac{1}{r}\left[\frac{m(1+q)\left(1-\frac{2m}{r}\right)^q}{r-m(3+2q)}\right]^{1/2} \ , \tag{6.4.14}$$

$$T(r) = \int \frac{\dot{t}}{\dot{\varphi}} d\varphi = 2\pi \frac{dt}{d\varphi} = 2\pi r^{3/2}\left[\frac{r-m(2+q)}{m(1+q)(r-2m)}\right]^{1/2}\left(1-\frac{2m}{r}\right)^{-q} \ , \tag{6.4.15}$$

respectively.

6.4.2 Stability analysis

We now analyze the stability properties of the circular motion. In particular the last stable circular orbit is determined by the inflection points of the effective potential (6.4.10), i.e., by the zeros of the equation

$$\frac{\partial^2 V_{eff}^2}{\partial r^2} = 2\left(1-\frac{2m}{r}\right)^{q-1} \times \frac{m(1+q)\left(r^2-8\,mr-6\,mqr+4\,m^2q^2+14\,m^2q+12\,m^2\right)}{r^4[r-m(3+2q)]} \tag{6.4.16}$$

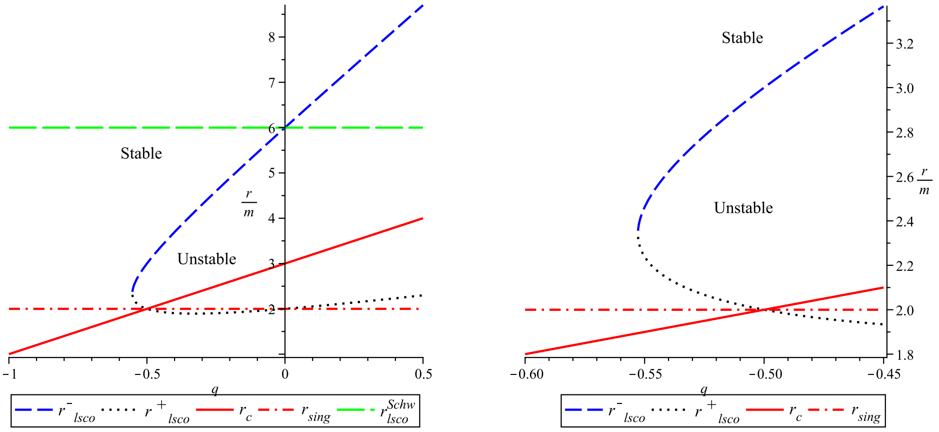


Figure 6.10: The last stable circular radius for a timelike particle as a function of the quadrupole parameter q. The critical radius $r_c = m(3+2q)$ and the outer singularity $r_{sing} = 2m$ are also plotted. The left plot is a zoom of the region of intersection.

where we have replaced the value for the angular momentum of the circular orbit (6.4.11). Then, we obtain the radii

$$r^{\pm}_{lsco} = m(4 + 3q \pm \sqrt{5q^2 + 10q + 4}) \tag{6.4.17}$$

which are positive and lie outside the outer singularity for all allowed values of q. The behavior of $r^{\pm}_{lsco}$ is plotted in Fig. 6.10 where we also plot the radius r_c that denotes the minimum value at which the angular momentum l^2 is positive. It follows from the graph that it is necessary to consider three different intervals: I for $q \in (\infty, -0.5]$, II for $q \in (-0.5, -1 + 1/\sqrt{5}]$ and III for $q \in (-1 + 1/\sqrt{5}, -1)$. The particular values of q used to determine the different intervals follow from the conditions $r_- = r_c$ $(q = -0.5)$ and $r_+ = r_-$ $(q = -1 + 1/\sqrt{5} \approx -0.5527)$.

In the interval I, the only valid radius is r^{+}_{lsco} which is greater (smaller) than the Schwarzschild radius for positive (negative) values of the quadrupole parameter q. Notice that on the boundary of this interval $(q = -0.5)$ the zero at r^{-}_{lsco} cannot be considered because it is located on the outer singularity. In this interval we have that $\partial^2 V^2_{eff}/\partial r^2 > 0$ so that all the circular orbits are stable. If we imagine a hypothetical accretion disk made of test particles around the central source so that the inner radius of the disk coincides with r^+_{lsco}, the role of the quadrupole in the interval I consists in changing the inner radius. From an observational point of view, this implies that by measuring the inner radius of the disk, one can determine the value of the quadrupole. The smallest disk inner radius corresponds to the value $r^+_{lsco}(q = -0.5) = 3m$ whereas the outer radius can in principle be extended to infinity.

In the interval II with $q \in (-0.5, -1 + 1/\sqrt{5})$, the second derivative of V^2_{eff} is negative within the two zeros located at r^+_{lsco} and r^-_{lsco}. This means that circular orbits are unstable within this interval. The radial extension L of the unstable region is

$$L = r^+_{lsco} - r^-_{lsco} = 2m\sqrt{5q^2 + 10q + 4} = \begin{cases} m & \text{for } q = -0.5 \\ 0 & \text{for } q = -1 + 1/\sqrt{5} \end{cases} \qquad (6.4.18)$$

The interesting fact in this case is that there exists an additional stable region located between the outer singularity and r^-_{lsco} (cf. Fig. 6.10) which is separated by the unstable region (r^-_{lsco}, r^+_{lsco}) from the exterior stable region with $r \geq r^+_{lsco}$. An accretion disk around such an object would consist of an external disk which can extend from r^+_{lsco} to infinity and an internal ring located inside the region $(2m, m + 3m/\sqrt{5} \approx 2.34m)$, where the outer boundary corresponds to the point where $r^+_{lsco} = r^-_{lsco}$. Since the internal stable ring is located so close to the outer singularity one could imagine that the angular momentum of the test particles should be very high in order to maintain the orbit. A detailed analysis, however, shows that this is not true. In Fig. 6.11, we plot the value of the angular momentum as a function of the radial distance for a particular quadrupole parameter which allows the existence of an internal ring. We can see that the maximum value of the angular momentum is reached on the external boundary of the ring and then it decreases as the singularity is approached. The explicit value of the angular momentum is not high and in fact it is comparable with the values for test particles located inside the external stable disk at distances of several times m. The peculiarity of the angular momentum is that its value decreases as the singularity is approached, an effect that contradicts the physical expectations for an attractive gravitational field. The only possible explanation for this unusual behavior is that there is an additional force that compensates the gravitational attraction. Similar effects has been found in the gravitational field of other naked singularities and have been interpreted as a manifestation of repulsive gravity

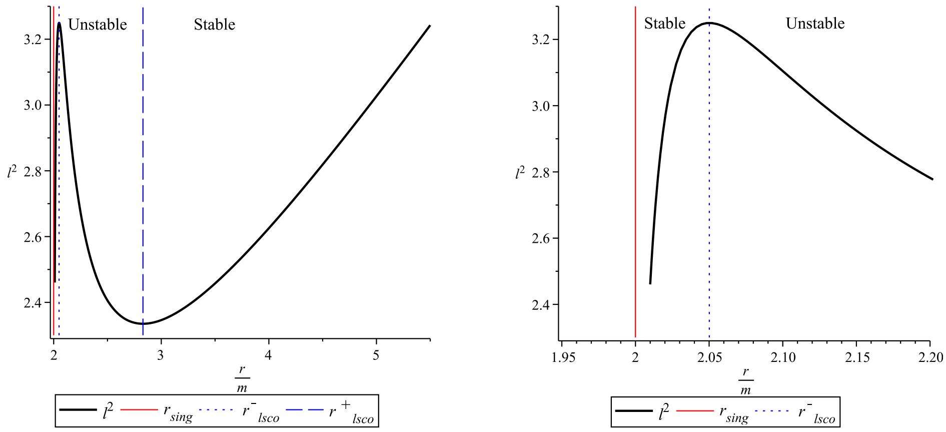


Figure 6.11: The angular momentum of test particles around a naked singularity with $q = -0.52$. The left plot is a zoom of the region where the internal stable ring is located.

[30, 31, 47, 48]. An alternative approach in which repulsive gravity is defined in terms of curvature invariants leads to similar results [49, 50].

In the interval III, with $q \in (-1 + 1/\sqrt{5}, -1)$, there are no last stable orbits. All the circular trajectories are allowed, in principle, starting at the outer singularity $r = 2m$. An analysis of the angular momentum shows that it diminishes as the naked singularity is approached. Again, the simplest explanation is to assume the presence of repulsive gravity. An accretion disk around this type of naked singularities would have a continuous structure which extends from the singularity to infinity.

6.4.3 Circular null geodesics

In the case of null geodesics ($\epsilon = 0$), the effective potential (6.4.10) reduces to

$$V_{eff}^2 = \frac{l^2}{r^2}\left(1 - \frac{2m}{r}\right)^{1+2q} \tag{6.4.19}$$

and its behavior for a given l^2 is similar to that shown in Fig. 6.9. The important quantities for the analysis of circular motion of photons are

$$\frac{\partial V_{eff}^2}{\partial r} = -\frac{2l^2}{r^4}\left(1-\frac{2m}{r}\right)^{2q}[r-m(3+2q)]\,, \tag{6.4.20}$$

and

$$\frac{\partial^2 V_{eff}^2}{\partial r^2} = \frac{2l^2}{r^6}\left(1-\frac{2m}{r}\right)^{1+2q}[3r^2-12qmr-18mr+28qm^2+8q^2m^2+24m^2]\,. \tag{6.4.21}$$

The first derivative vanishes for $r = r_\gamma = m(3+2q)$, an expression which reduces to the standard Schwarzschild value for $q = 0$, as expected. We see that the circular orbit located at $r_\gamma = m(3+2q)$ corresponds to the trajectory of a photon. This is also the limiting radius for timelike circular geodesics, as explained above.

By inserting this radius value into the second derivative of the effective potential, we obtain an expression which is negative for all values of q. This implies that the circular orbit with radius $r_\gamma = m(3+2q)$ is unstable, independently of the value of the angular momentum. In contrast to the Schwarzschild spacetime, where there is only one circular orbit with $r_\gamma = 3m$, in the case of the $q-$metric the radius $r_\gamma = m(3+2q)$ can take any value within the interval $(2m,\infty)$, depending on the value of the quadrupole parameter. Nevertheless, for fixed values of m and q only one circular orbit is allowed.

6.5 Radial geodesics

In the Schwarzschild spacetime, radial geodesics representing the free fall of test particles are straight lines that connect the initial point in spacetime with the singularity located at the origin of coordinates. Therefore, it is useful to study radial geodesics in the spacetime described by the $q-$metric in order to evaluate the influence of the quadrupole. We will consider the free fall of test particles with vanishing angular momentum ($l = 0$). Then, the motion along the radial coordinate is governed by the equation

$$\dot{r} = -\left(1 + \frac{m^2 \sin^2\theta}{r^2 - 2mr}\right)^{\frac{q}{2}(2+q)} \sqrt{E^2 - \Phi_r^2}\,, \tag{6.5.1}$$

where the minus sign indicates that the particle falls inward and

$$\Phi_r^2 = \left(1 - \frac{2m}{r}\right)^{1+q} \left[r^2\left(1 - \frac{2m}{r}\right)^{-q}\left(1 + \frac{m^2 \sin^2\theta}{r^2 - 2mr}\right)^{-q(2+q)} \dot{\theta}^2 + \epsilon\right]. \tag{6.5.2}$$

In addition, we must take into account that an arbitrary polar plane ($\theta =$const) is not necessarily a geodesic plane. This means that the radial geodesic is determined by Eq.(6.5.1) together with the equation (6.3.6) for the evolution along the angle θ. We illustrate the result of the integration of these two differential equations in Fig. 6.12. The effect of the quadrupole parameter q becomes plausible from the behavior of the geodesics. For sources with negative quadrupole parameter (prolate sources), the geodesics deviate from the initial angle $\theta = \frac{\pi}{4}$ towards the axis of symmetry. In the case of positive quadrupole q (oblate body), the deviation is towards the equatorial plane of source. This result is in accordance with our physical expectations. Indeed, an oblate source is larger on the equatorial plane and so one would expect that the mass density is larger on the equator, generating a larger gravitational interaction. In the case of prolate bodies, the same intuitive reasoning is valid for the symmetry axis. This result reinforces our interpretation of the parameter q as determining the quadrupole moment of the gravitational source.

In the case of radial geodesics, it is interesting to compute the coordinate time, t, and the proper time, τ, that passes along the trajectory of a test mass between two radii r_i and r_f. For simplicity, we consider the case of radial geodesics on the equatorial plane where the geodesic equations reduce to

$$\dot{r}^2 = \left(1 + \frac{m^2}{r^2 - 2m}\right)^{q(2+q)} \left[E^2 - \left(1 - \frac{2m}{r}\right)^{1+q}\right], \quad \dot{t} = E\left(1 - \frac{2m}{r}\right)^{-1-q}. \tag{6.5.3}$$

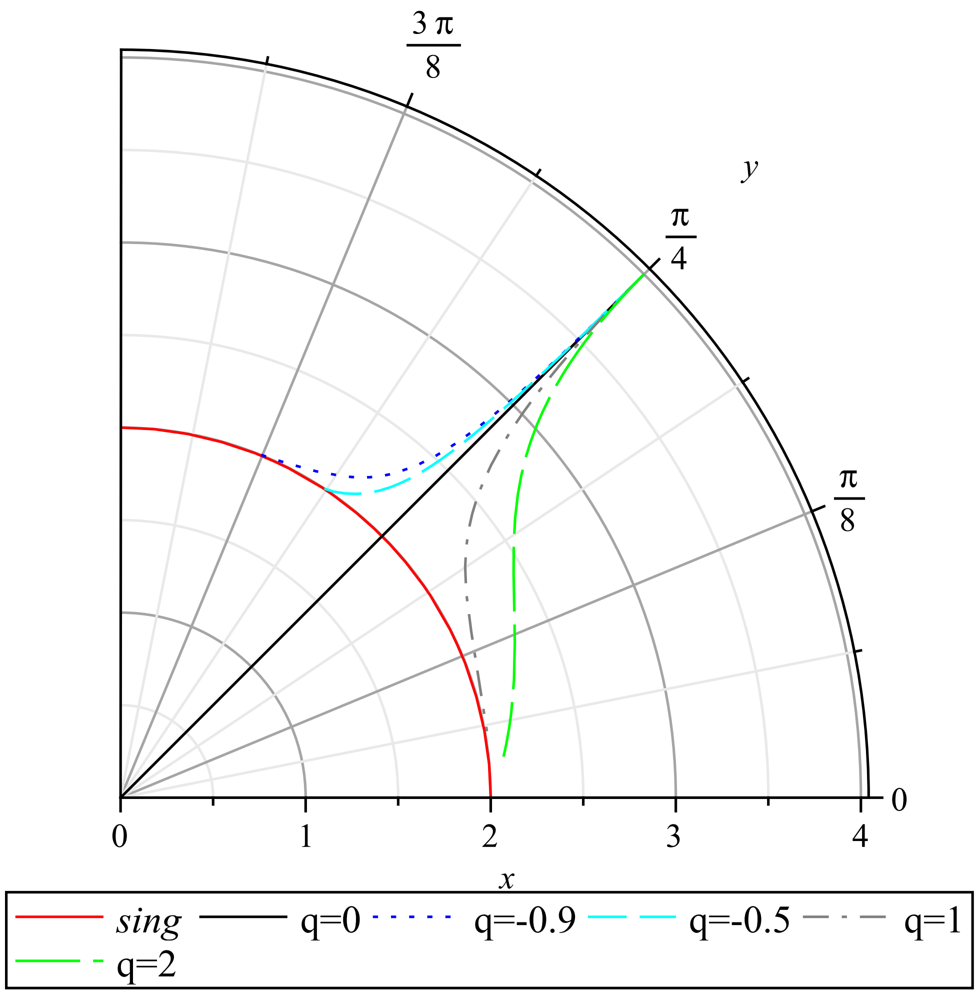


Figure 6.12: Free fall of test particles with initial point located at $r = 4$ $(m = 1)$ and $\theta = \frac{\pi}{4}$. The singularity at $r = 2m$ and the radial geodesic for the Schwarzschild spacetime are also shown for comparison.

It is then easy to obtain the relationships

$$\Delta\tau = \int_{r_i}^{r_f} \left[E^2 - \left(1 - \frac{2m}{r}\right)^{q+1}\right]^{-1/2} \left(1 + \frac{m^2}{r^2 - 2mr}\right)^{-\frac{1}{2}q(2+q)} dr \ , \quad (6.5.4)$$

and

$$\begin{aligned} \Delta t = \ & \int_{r_i}^{r_f} E \left(1 - \tfrac{2m}{r}\right)^{-(1+q)} \left[E^2 - \left(1 - \tfrac{2m}{r}\right)^{q+1}\right]^{-1/2} \\ & \times \left(1 + \tfrac{m^2}{r^2-2mr}\right)^{-\frac{1}{2}q(2+q)} dr \ , \end{aligned} \quad (6.5.5)$$

for the proper time and coordinate time, respectively. Let us consider the case of a particle located initially at $r_i \to \infty$ so that $E^2 = 1$. If the final state is at the singularity $r_f = 2m$, we obtain

$$\tau_{2m} = \int_{\infty}^{2m} \left(\frac{2m}{r}\right)^{-\frac{1}{2}(1+q)} \left(1 + \frac{m^2}{r^2 - 2mr}\right)^{-\frac{q}{2}(2+q)} dr \ . \quad (6.5.6)$$

It is not possible to find an analytical expression for this integral. Nevertheless, one can evaluate it either numerically or by means of a Taylor expansion around the value $q = 0$ and $r = 2m$. As a result we obtain that the proper time is finite for all the allowed values of q. The same result is obtained for the Schwarzschild spacetime ($q = 0$). This results indicates that the quadrupole does not affect the finiteness of the proper time that is necessary to reach the hypersurface $r = 2m$.

In a similar manner, for the coordinate time we obtain

$$\begin{aligned} t_{2m} = \ & E \int_{\infty}^{2m} \left[E^2 - \left(1 - \tfrac{2m}{r}\right)^{1+q}\right]^{-1/2} \left(1 - \tfrac{2m}{r}\right)^{-1+\frac{q^2}{2}} \\ & \times \left(1 - \tfrac{2m}{r} + \tfrac{m^2}{r^2}\right)^{-\frac{q}{2}(2+q)} \ . \end{aligned} \quad (6.5.7)$$

Again, a Taylor expansion of this quantity reveals that it is finite for $q < 0$ and infinite for $q \geq 0$. This means that a positive quadrupole does not change the main property of the coordinate time of the Schwarzschild spacetime. However, in the case of negative quadrupole the coordinate time completely changes so that an observer at infinity would observe how the test particle reaches the outer singularity located at $r = 2m$.

Let us now consider the radial motion of photons. Suppose that a photon is emitted at a radius r_{em} with frequency ν_{em} and received at a radius r_{rec} with frequency ν_{rec}. Then, the redshit z in a static spacetime is determined by the relationship [36]

$$z + 1 = \frac{\nu_{em}}{\nu_{rec}} = \left(\frac{g_{tt}|_{rec}}{g_{tt}|_{em}}\right)^{1/2} = \left(\frac{1 - 2m/r_{rec}}{1 - 2m/r_{em}}\right)^{\frac{1}{2}(1+q)} . \tag{6.5.8}$$

If the emission occurs near the outer singularity ($r_{em} \sim 2m$), the redshift diverges, independently of the value of q. This means that for an observer located outside the radius $r = 2m$, it is not possible to receive information from the singularity. In this sense, the singularity remains invisible for external observers.

6.6 Calculation of the ADM mass

Consider the space-time $(\mathcal{M}, g)$ foliated by a family $(\Sigma_t)_{t\in\Re}$ of spacelike hypersurfaces. Let

$$\mathscr{S}_t \equiv \partial\mathscr{V} \cap \Sigma_t, \tag{6.6.1}$$

where $\partial\mathscr{V}$ is the boundary of a domain $\mathscr{V}$ ($\partial\mathscr{V}$ is assumed to be a timelike hypersurface). The ADM mass of the slice Σ_t is defined by

$$M_{\mathrm{ADM}} = \frac{1}{16\pi} \lim_{\mathscr{S}_t \to \infty} \oint_{\mathscr{S}_t} [D^j h_{ij} - D_i(f^{kl} h_{kl})] S^i \sqrt{s}\, d^2y \tag{6.6.2}$$

where f is a flat background metric on Σ_t, h is the induced metric of the hypersurface Σ_t, D stands for the connection associated with the metric f, S^i stands for the components of unit normal to $\mathscr{S}_t$, $\sqrt{s}\, d^2y$ denotes the surface element induced by the spacetime metric on $\mathscr{S}_t$ with s_{ab} being the induced metric, $y^a = (y^1, y^2)$ are some coordinates on $\mathscr{S}_t$ and $s \equiv \det s_{ab}$.

Let us take for Σ_t the hypersurface of constant coordinate t. Then, in the case of the $q-$metric (6.2.1) we have for the induced metric on the hypersurface Σ_t

$$\begin{aligned} h_{ij} = \ \ & \mathrm{diag}\Bigg[\left(1-\tfrac{2m}{r}\right)^{-q-1}\left(1+\tfrac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)}, \\ & r^2\left(1-\tfrac{2m}{r}\right)^{-q}\left(1+\tfrac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)}, \\ & r^2\sin^2\theta\left(1-\tfrac{2m}{r}\right)^{-q}\Bigg], \end{aligned} \tag{6.6.3}$$

whereas for the flat metric we obtain

$$f_{ij} = \mathrm{diag}\Big[1, r^2, r^2\sin^2\theta\Big]. \tag{6.6.4}$$

For $\mathscr{S}_t$ we take the sphere $r = \text{const}$ on the hypersurface Σ_t. Then $y^a = (\theta, \varphi)$, $\sqrt{s} = r^2\sin\theta$ and $S^i = \delta^i_r$. Accordingly, Eq.(6.6.2) becomes

$$M_{\mathrm{ADM}} = \frac{1}{16\pi}\lim_{r\to\infty}\oint_{r=\mathrm{const}} [D^j h_{rj} - D_r(f^{kl}h_{kl})] r^2 \sin\theta d\theta d\varphi, \tag{6.6.5}$$

where

$$D_r(f^{kl}h_{kl}) = \frac{\partial}{\partial r}(f^{kl}h_{kl}) = \frac{\partial}{\partial r}\left(h_{rr} + r^{-2}h_{\theta\theta} + r^{-2}\sin^{-2}\theta h_{\varphi\varphi}\right). \tag{6.6.6}$$

and

$$D^j h_{rj} = f^{jk}D_k h_{rj} = \frac{\partial}{\partial r}h_{rr} + \frac{2}{r}h_{rr} - \frac{1}{r^3}h_{\theta\theta} - \frac{1}{r^3\sin^2\theta}h_{\varphi\varphi}. \tag{6.6.7}$$

Here we used the fact that $f^{kl}h_{kl}$ is a scalar field. In obtainging Eq. (6.6.7), we used the formula

$$D_r h_{rr} = \tfrac{\partial}{\partial r}h_{rr} - 2\Gamma^r_{rr}h_{rr}, \tag{6.6.8}$$

$$D_\theta h_{r\theta} = \tfrac{\partial}{\partial\theta}h_{r\theta} - \Gamma^\theta_{\theta r}h_{\theta\theta}, -\Gamma^r_{\theta\theta}h_{rr}, \tag{6.6.9}$$

$$D_\varphi h_{r\varphi} = \tfrac{\partial}{\partial\varphi}h_{r\varphi} - \Gamma^\varphi_{\varphi r}h_{\varphi\varphi} - \Gamma^r_{\varphi\varphi}h_{rr}, \tag{6.6.10}$$

with the non-zero components of the Christoffel symbols of the connection D

with respect to the coordinates x^i given by

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\varphi, \quad \Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \tag{6.6.11}$$

$$\Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta, \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta. \tag{6.6.12}$$

Accordingly, from Eq.(6.6.6) and Eq.(6.6.7), after a simple but tedious calculation, one obtains that

$$D^j h_{rj} - D_r(f^{kl}h_{kl}) = \\ \tfrac{4m(q+1)}{r^2} - q(2+q)m^2\left(1-\tfrac{2m}{r}\right)^{-q-2}\left(\tfrac{3}{r^3}+\tfrac{2mq}{r^4}\right)\sin^2\theta, \tag{6.6.13}$$

where we have used the first-order approximation

$$\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)^{-q(2+q)} \approx \left(1-\frac{q(2+q)m^2\sin^2\theta}{r^2-2mr}\right). \tag{6.6.14}$$

Thus, by substituting Eq.(6.6.13) into Eq.(6.6.5) and calculating the limit after integration we obtain $M_{\mathrm{ADM}} = m(1+q)$.

6.7 Conclusions

In this work, we investigated the motion of test particles in the gravitational field described by the $q-$metric, the simplest generalization of the Schwarzschild metric which contains a quadrupole parameter. It was shown that the $q-$ spacetime possesses two singularities at $r = 0$ and $r = 2m$ which are not covered by a horizon, i.e., they are not isolated from the exterior spacetime. For certain values of the parameter q, a third $\theta-$dependent singularity appears inside the spatial region contained within the radial interval $(0, 2m)$. If we limit ourselves to the region with $r > 2m$, the geodesic equations determine the motion of test particles around the naked singularity located at $r = 2m$. First, we studied the bounded motion on an arbitrary polar plane $\theta \neq 0, \pi/2$ and showed that in general the plane of the orbit moves towards the equator or to the axis of symmetry, depending on the value of the quadrupole. We performed a brief analysis of the conditions under which circular orbits are allowed for arbitrary values of the polar angle θ, the angular momentum of the particle and the quadrupole parameter.

We studied in detail the motion on the equatorial plane $\theta = \pi/2$. We integrated numerically the geodesics equations and found bounded and unbounded orbits for different values of the quadrupole. It was also shown explicitly that the presence of the quadrupole leads to a change of the perihelion distance of a bounded orbit. This effect could in principle be used to determine the quadrupole moment of the gravitational source.

The analysis of circular orbits and their stability leads to a classification of naked singularities according to which in certain cases it is possible to determine the quadrupole parameter by measuring the inner radius of the accretion disks made of test particles only. Suppose, for instance, that an accretion disk is detected with inner radius r_{inner} within the interval $(3m, \infty)$. Then, the naked singularity is of type III with a quadrupole parameter that can be determined numerically by using the formula $r_{inner} = m(4 + 3q + \sqrt{5q^2 + 10q + 4})$ and the result must be a value contained within the interval $q \in (-0.5, \infty)$. If the value of the accretion disk inner radius is inside the interval $(2.34m, 3m)$, the naked singularity belongs to the class II, and the quadrupole parameter can be determined again from the formula $r_{inner} = m(4 + 3q + \sqrt{5q^2 + 10q + 4})$. In this case, the quadrupole parameter must be contained within the interval $(-0.5527, -0.5)$, and there must exist an additional inner ring located inside the spatial region $(2m, 2.34m)$, whose exterior radius can be determined by the formula $r_{ext} = m(4 + 3q - \sqrt{5q^2 + 10q + 4})$. Finally, if the inner disk radius is inside the interval $(2m, 2.34m)$, the naked singularity is of class I with $q \in (-1, -0.5527)$. In the case of class III and II naked singularities, it is possible to find the exact value of the quadrupole parameter by measuring the inner radius of the accretion disk. This is not true in the case of class I singularities, because all the circular orbits are stable in this region. However, if we could measure the radius and the angular momentum of a particular circular orbit located inside the region $(2m, 2.34m)$, the quadrupole parameter could be determined by using the expression (6.4.11) for the angular momentum. In this manner, we see that it is always possible to determine the quadrupole parameter from the physical properties of the accretion disk. The fact that in the gravitational field of naked singularities there exist circular orbits near the outer singularity is explained by assuming the presence of repulsive gravity.

The study of radial geodesics starting at an arbitrary polar angle θ shows that the quadrupole induces a deviation from the original radial direction in such a way that the test particles tends either towards the equator or to the axis of symmetry, depending on the value of the quadrupole. We also calculated the proper time and the coordinate time along radial geodesics on the equatorial plane and established the analogies and differences with respect to radial geodesics in the Schwarzschild spacetime.

Our results show that the presence of a quadrupole can drastically affect the motion of test particles in a such a way that it is possible to determine the quadrupole by studying the properties of the test particle trajectories. If we were to compare our results with observational data from astrophysical compact objects, we would immediately notice that an important astrophysical parameter is missing in our analysis, namely, the rotation. We expect to perform such an analysis in a future work by using a stationary generalization of the $q-$metric as, for instance, the one derived in [51].

7 Equivalence of approximate solutions of Einstein field equations

7.1 Introduction

To study the gravitational field of slowly uniformly rotating and slightly deformed relativistic objects, Hartle developed in his original work [52] a method in the slow rotation approximation, extending the well-known exterior and interior Schwarzschild solutions. The method allows one to investigate the physical properties of rotating stellar objects in hydrostatic equilibrium. It was first applied to real astrophysical objects by Hartle and Thorne [53], employing the Harrison-Wheeler, Tsuruta-Cameron and the Harrison-Wakano-Wheeler equations of state. Soon after, the method has become known as the Hartle-Thorne approach and there appeared a new series of research papers extending, modifying and improving the original approach by including higher order multipole moments and corrections in the angular momentum, etc. [56, 57]. Furthermore, the Hartle formalism was tested, compared and contrasted with numerical computations in full general relativity [55, 54]. As a result, it was shown that the Hartle formalism can be safely used to study stellar objects with intermediate rotation periods. Only for higher angular velocities, close to the mass-shedding limit, it shows noticeable discrepancies from the full general relativistic simulations [56, 55, 54].

Similar approaches were developed by Bradley et. al. in [58, 59, 60, 61], where the slow rotation approximation is used in order to construct interior and exterior solution to the Einstein field equations. Unlike Hartle, Bradley et. al solved the six independent Einstein equations without involving the integral of the equation of hydrostatic equilibrium for uniformly rotating configurations. Moreover, the Darmois-Israel procedure was applied to match the interior and exterior solutions. In some particular cases, Bradley et. al.

[58, 59, 60, 61] included the electric charge by solving the Einstein-Maxwell equations.

In addition, Konno et. al [62] generalized Hartle's approach in the static case to include the deformation of relativistic stars due to the presence of magnetic fields. Afterwards, Konno and coworkers [63] calculated the ellipticity of the deformed stars due to the presence of both magnetic field and rotation, extending their previous results. This method has become popular and found its astrophysical application in the physics of all types of magnetic stars [64, 65, 66, 67, 68].

On the other hand, independently of Hartle, Sedrakyan and Chubaryan [69] formulated their own distinctive approach for calculating the exterior gravitational structure of equilibrium rigidly rotating superdense stars in the small angular velocity approximation, though it is not well-known in the scientific community. The corresponding interior solution, together with the matching procedure, was obtained in their subsequent paper [70]. The manner of solving the Einstein equations was markedly different from the Hartle's approach. Further applications of the Sedrakyan-Chubaryan solution to white dwarfs and neutron stars were considered in a number of papers e.g. [71, 72, 73]. Numerical results obtained by Arutyunyan et. al [72] were in agreement with the ones computed by Hartle and Thorne [53], implying that there was no contradiction between these two solutions.

Besides, the exterior Sedrakyan-Chubaryan solution was written in an analytic form [69, 70, 71, 72, 73] and it required the additional integration of one of the metric functions, under a careful consideration of the boundary conditions. Maybe this was one of the main causes that the Sedrakyan-Chubaryan solution is still less known in the scientific community. Indeed, this fact does not allow one to compare and contrast it with the exterior Hartle-Thorne solution straightforwardly. The main goal of the present work is to derive explicitly the exterior Sedrakyan-Chubaryan solution and to establish its relationship with the Hartle-Thorne solution. In fact, we will show that they are related by means of a coordinate transformation, whose non-trivial part includes only the radial coordinate, and a redefinition of the parameters entering the solution.

7.2 The Hartle-Thorne approximate solution

The exterior Hartle-Thorne metric describes the gravitational field of a slowly rotating slightly deformed source in vacuum. In geometric units, the metric is given by [52]

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)\left[1+2k_1P_2(\cos\theta)+2\left(1-\frac{2M}{r}\right)^{-1}\frac{J^2}{r^4}(2\cos^2\theta-1)\right]dt^2 \\ &\quad +\left(1-\frac{2M}{r}\right)^{-1}\left[1-2k_2P_2(\cos\theta)-2\left(1-\frac{2M}{r}\right)^{-1}\frac{J^2}{r^4}\right]dr^2 \\ &\quad +r^2[1-2k_3P_2(\cos\theta)](d\theta^2+\sin^2\theta d\phi^2)-\frac{4J}{r}\sin^2\theta dt d\phi \end{aligned} \tag{7.2.1}$$

where

$$\begin{aligned} k_1 &= \frac{J^2}{Mr^3}\left(1+\frac{M}{r}\right)+\frac{5}{8}\frac{Q-J^2/M}{M^3}Q_2^2\left(\frac{r}{M}-1\right), \\ k_2 &= k_1-\frac{6J^2}{r^4}, \\ k_3 &= k_1+\frac{J^2}{r^4}+\frac{5}{4}\frac{Q-J^2/M}{M^2r}\left(1-\frac{2M}{r}\right)^{-1/2}Q_2^1\left(\frac{r}{M}-1\right), \\ P_2(x) &= \frac{1}{2}(3x^2-1), \\ Q_2^1(x) &= (x^2-1)^{1/2}\left[\frac{3x}{2}\ln\frac{x+1}{x-1}-\frac{3x^2-2}{x^2-1}\right], \\ Q_2^2(x) &= (x^2-1)\left[\frac{3}{2}\ln\frac{x+1}{x-1}-\frac{3x^3-5x}{(x^2-1)^2}\right]. \end{aligned} \tag{7.2.2}$$

Here $P_2(x)$ is Legendre polynomials of the first kind, Q_l^m are the associated Legendre polynomials of the second kind and the constants M, J and Q are the total mass, angular momentum and quadrupole moment of the rotating source, respectively.

Unlike other solutions of the Einstein equations, the Hartle-Thorne solution has an internal counterpart, which makes it more practical with respect to the exact solutions. All the internal functions are interrelated with the external ones. Thus, the total mass, angular momentum and quadrupole moment of a rotating star are determined through the constants obtained by means of the numerical integration of both interior and exterior solutions, by applying the matching conditions on the surface of the star.

7.3 The Sedrakyan-Chubaryan solution

In this section, we derive the approximate Sedrakyan-Chubaryan solution [69] in detail. We will limit ourselves to the exterior solution for which we derive all the metric functions.

Following the procedure presented in [69], we consider the line element for axially symmetric rotating stars in the form

$$ds^2 = \left(\omega^2 e^{\mu} \sin^2\theta - e^{\nu}\right) dt^2 + e^{\lambda} dr^2 + e^{\mu}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + 2\,\omega\, e^{\mu} \sin^2\theta\, d\phi dt, \tag{7.3.1}$$

where $\lambda = \lambda(r,\theta),\ \mu = \mu(r,\theta),\ \omega = \omega(r,\theta)$ and $\nu = \nu(r,\theta)$ are functions of the radial r and angular θ coordinates. Note that, ω is proportional to the odd powers of the angular velocity Ω, whereas the remaining functions are proportional to the even powers of Ω. We will consider here an approximation up to the second order in Ω. We now demand that the above metric satisfies vacuum Einstein's equations in the form

$$G^{\alpha}_{\beta} = R^{\alpha}_{\beta} - \frac{1}{2} R \delta^{\alpha}_{\beta} = 0\,. \tag{7.3.2}$$

In the limiting case of a static star, the angular velocity $\Omega = 0$ and the function $\omega = 0$; then, λ, ν and μ are functions of the radial coordinate r only. Obviously, for this special case we automatically obtain the exterior Schwarzschild solution

$$e^{\nu} \;=\; e^{\nu_0} = \left(1 - \frac{2m}{r}\right), \tag{7.3.3}$$

$$e^{\lambda} \;=\; e^{\lambda_0} = \left(1 - \frac{2m}{r}\right)^{-1}, \tag{7.3.4}$$

$$e^{\mu} \;=\; e^{\mu_0} = r^2, \tag{7.3.5}$$

where m is the static mass.

We now consider the line element (7.3.1) for a slowly rotating relativistic star. In this case, we can expand the functions λ, μ, ν and ω in powers of the angular velocity of the star Ω, assuming that Ω is small. As a parameter for series expansion of the metric tensor components it is convenient to introduce the dimensionless quantity $\beta = \Omega^2/8\pi\rho_c$, where ρ_c is the central density of the configuration. Thus we define the metric functions as

$$e^{\nu(r,\theta)} = e^{\nu_0}\left[1+\beta\Phi\left(r,\theta\right)\right], \tag{7.3.6}$$

$$e^{\lambda(r,\theta)} = e^{\lambda_0}\left[1-\beta f\left(r,\theta\right)\right], \tag{7.3.7}$$

$$e^{\mu(r,\theta)} = e^{\mu_0}\left[1+\beta U\left(r,\theta\right)\right], \tag{7.3.8}$$

$$\omega\left(r,\theta\right) = \sqrt{\beta}q\left(r\right), \tag{7.3.9}$$

where the functions μ_0, ν_0 and λ_0 represent the Schwarzschild solution, and U, Φ, and f are unknown functions. To find the independent differential equations from the Einstein field equations, we make use of the following combinations

$$G_1^1 - G_0^0 = 0, \quad G_2^2 + G_3^3 = 0, \quad G_2^1 = 0, \quad G_0^3 = 0. \tag{7.3.10}$$

In order to solve each component of Eq.(7.3.10), all the metric functions are expanded in spherical harmonics. In turn, this procedure allows one to separate variables. Retaining only the terms responsible for the quadrupolar deformation, we have

$$\Phi\left(r,\theta\right) = \textstyle\sum_{l=0}^{\infty}\Phi_l\left(r\right)P_l\left(\cos\theta\right) \approx \Phi_0\left(r\right)P_0\left(\cos\theta\right) + \Phi_2\left(r\right)P_2\left(\cos\theta\right),$$
$$f\left(r,\theta\right) = \textstyle\sum_{l=0}^{\infty}f_l\left(r\right)P_l\left(\cos\theta\right) \approx f_0\left(r\right)P_0\left(\cos\theta\right) + f_2\left(r\right)P_2\left(\cos\theta\right),$$
$$U\left(r,\theta\right) = \textstyle\sum_{l=0}^{\infty}U_l\left(r\right)P_l\left(\cos\theta\right) \approx U_0\left(r\right)P_0\left(\cos\theta\right) + U_2\left(r\right)P_2\left(\cos\theta\right),$$

where $P_0(\cos\theta)$ and $P_2(\cos\theta)$ are the Legendre polynomials of the first kind

$$P_0(\cos\theta) = 1, \;\; P_2(\cos\theta) = -\frac{1}{2}\left(1 - 3\cos^2\theta\right). \tag{7.3.11}$$

Note that because the axis of symmetry is oriented along the rotation axis, the expansion in spherical harmonics contains only even values of l. Moreover, in the slow rotation approximation l accepts only two values: $l = 0$ and $l = 2$. The components of the Einstein tensor G_3^0 or G_0^3 yield a differential equation which is proportional to Ω,

$$q_{,rr} + \frac{4\, q_{,r}}{r} = 0, \tag{7.3.12}$$

where $q_{,r} = \frac{\partial q}{\partial r}$, etc. The solution to the last equation is

$$q(r) = \frac{C_q}{r^3}, \tag{7.3.13}$$

where C_q is a constant to be determined from the matching between the interior and exterior solutions.

Now we can substitute Eqs. (7.3.11), (7.3.11) and (7.3.11) in Eq.(7.3.10). The resulting equation can then be expanded up to the first order in β for the different values of l. In the case $l = 0$, we obtain the following differential equations

$$U_{0,rr} + \frac{1}{r}\left[2U_{0,r} + f_{0,r} - \Phi_{0,r}\right] = 0, \tag{7.3.14}$$

$$\Phi_{0,rr} + U_{0,rr} + \frac{1}{r\,(r-2m)}\bigg[(m+r)\,\Phi_{0,r}$$
$$+\,(r-m)\,(f_{0,r} + 2U_{0,r}) - \frac{6}{r^4}C_q^2\bigg] = 0. \tag{7.3.15}$$

In general, it is not possible to solve the above system of equations, because the number of unknown functions is greater than the number of differential equations. It is therefore necessary to impose an additional equation that closes the system of differential equations. Several possibilities are available. An analysis of of the line element (7.3.1) shows that in the lowest approximation of a spherically symmetric field, the metric components g_{tt} and g_{rr} satisfy the relationship $g_{rr} = -1/g_{tt}$. Consequently, we can assume the following condition $f_0\,(r) = \Phi_0\,(r)$ which allows one to easily solve the system of equations [69]. In addition, if at large distances we impose the conditions

$U_0\,(r \to \infty) = 0$, $\Phi_0\,(r \to \infty) = 0$ and $f_0\,(r \to \infty) = 0$, we find

$$U_0\,(r) = \frac{C_{U_0}}{r}, \tag{7.3.16}$$

$$\Phi_0\,(r) = f_0\,(r) = \frac{C_q^2 + 2C_{U_0}\,mr^2 - 2C_{f_0}r^3}{2\,r^3\,(r-2m)}, \tag{7.3.17}$$

where C_{U_0} and C_{f_0} are the integration constants of the corresponding functions.

From Eqs.(7.3.10), we can reduce the field equations with the $l = 2$ terms to:

$$U_{2,rr} + \frac{1}{r}\left[2U_{2,r} + f_{2,r} - \Phi_{2,r} + \frac{3}{r-2m}\,(\Phi_2 + f_2)\right] = 0,$$

$$\Phi_{2,rr} + U_{2,rr} + \frac{r-m}{r\,(r-2m)}$$
$$\times\left[2U_{2,r} + f_{2,r} + \frac{1}{r-m}\left((r+m)\,\Phi_{2,r} + 3\,(f_2 - \Phi_2) + \frac{6C_q^2}{r^4}\right)\right] = 0,$$

$$\Phi_{2,r} + U_{2,r} - \frac{1}{r\,(r-2m)}\left[(r-3m)\,\Phi_2 - (r-m)\,f_2\right] = 0.$$

To solve this system of equations, we isolate $U_{2,r}$, then we calculate $U_{2,rr}$ and substitute the resulting expressions in the above equation. This gives the relationship

$$f_2\,(r) = \Phi_2\,(r) - \frac{3\,C_q^2}{r^4}\,. \tag{7.3.18}$$

Subsequently, the solutions of Eqs.(7.3.18) and (7.3.18) can be expressed as

$$\Phi_2\,(r) = \frac{C_q^2}{2}\left(\frac{1}{mr^3} + \frac{1}{r^4}\right) - \frac{3C_{\Phi_2}}{4}\left(1 - \frac{2m}{r}\right)r^2\ln\left(1 - \frac{2m}{r}\right)$$
$$-\frac{\left(3r^2 - 6mr - 2m^2\right)(r-m)\,mC_{\Phi_2}}{2r\,(r-2m)}, \tag{7.3.19}$$

$$f_2\left(r\right) = \frac{C_q^2}{2}\left(\frac{1}{mr^3} - \frac{5}{r^4}\right) - \frac{3C_{\Phi_2}}{4}\left(1 - \frac{2m}{r}\right) r^2 \ln\left(1 - \frac{2m}{r}\right)$$
$$- \frac{\left(3r^2 - 6mr - 2m^2\right)\left(r - m\right) mC_{\Phi_2}}{2r\left(r - 2m\right)}, \tag{7.3.20}$$

$$U_2\left(r\right) = -\frac{C_q^2}{2}\left(\frac{1}{mr^3} + \frac{2}{r^4}\right) + \frac{3C_{\Phi_2}}{4}\left(1 - \frac{2m^2}{r^2}\right) r^2 \ln\left(1 - \frac{2m}{r}\right)$$
$$+ \frac{\left(3r^2 + 3mr - 2m^2\right) mC_{\Phi_2}}{2r}. \tag{7.3.21}$$

Note that due to the asymptotic flatness condition $U_2\left(r \to \infty\right) \to 0$ the integration constant of (7.3.21) is related to C_{Φ_2} as

$$C_{\Phi_2} = -\frac{C_{U_2}}{3m^2}. \tag{7.3.22}$$

Finally, we can rewrite the metric tensor components of the line element

(7.3.1) as:

$$
\begin{aligned}
g_{00} &= \omega^2 e^{\mu} \sin^2\theta - e^{\nu} \approx \beta \left(\frac{C_q}{r^3}\right)^2 r^2 \sin^2\theta - \left(1 - \frac{2m}{r}\right) \\
&\quad \times \left[1 + \beta \langle \Phi_0(r) + \Phi_2(r) P_2(\cos\theta) \rangle\right] \qquad (7.3.23) \\
&= -\left(1 - \frac{2m}{r}\right) \left\{1 + \beta \left\langle \frac{C_q^2 + 2C_{U_0} mr^2 - 2C_{f_0} r^3}{2\, r^3 (r - 2m)} \right.\right. \\
&+ \left[\frac{C_q^2}{2}\left(\frac{1}{mr^3} + \frac{1}{r^4}\right) - \frac{3C_{\Phi_2}}{4}\left(1 - \frac{2m}{r}\right) r^2 \ln\left(1 - \frac{2m}{r}\right)\right. \\
&- \left.\left.\left.\frac{(3r^2 - 6mr - 2m^2)(r - m)\, mC_{\Phi_2}}{2r(r - 2m)}\right] P_2(\cos\theta) - \left(1 - \frac{2m}{r}\right)^{-1} \frac{C_q^2}{r^4} \sin^2\theta \right\rangle \right\}, \\
g_{11} &= e^{\lambda} \approx \left(1 - \frac{2m}{r}\right)^{-1} \left[1 - \beta \langle f_0(r) + f_2(r) P_2(\cos\theta) \rangle\right] \qquad (7.3.24) \\
&= \left(1 - \frac{2m}{r}\right)^{-1} \left\{1 - \beta \left\langle \frac{C_q^2 + 2C_{U_0} mr^2 - 2C_{f_0} r^3}{2\, r^3 (r - 2m)} \right.\right. \\
&- \left[\frac{C_q^2}{2}\left(\frac{1}{mr^3} - \frac{5}{r^4}\right) - \frac{3C_{\Phi_2}}{4}\left(1 - \frac{2m}{r}\right) r^2 \ln\left(1 - \frac{2m}{r}\right)\right. \\
&- \left.\left.\left.\frac{(3r^2 - 6mr - 2m^2)(r - m)\, mC_{\Phi_2}}{2r(r - 2m)}\right] P_2(\cos\theta) \right\rangle \right\}, \\
g_{22} &= e^{\mu} \approx r^2 \left[1 + \beta \langle U_0(r) + U_2(r) P_2(\cos\theta) \rangle\right] \qquad (7.3.25) \\
&= r^2 \left\{1 + \beta \left\langle \frac{C_{U_0}}{r} + \left[-\frac{C_q^2}{2}\left(\frac{1}{mr^3} + \frac{2}{r^4}\right) + \frac{3C_{\Phi_2}}{4}\left(1 - \frac{2m^2}{r^2}\right) r^2 \ln\left(1 - \frac{2m}{r}\right)\right.\right.\right. \\
&+ \left.\left.\left.\frac{(3r^2 + 3mr - 2m^2)\, mC_{\Phi_2}}{2r}\right] P_2(\cos\theta) \right\rangle \right\}, \\
g_{33} &= g_{22} \sin^2\theta, \qquad (7.3.26) \\
g_{30} &= g_{03} = \omega e^{\mu} \sin^2\theta \approx \frac{C_q \sqrt{\beta}}{r} \sin^2\theta. \qquad (7.3.27)
\end{aligned}
$$

All the constants are to be determined by matching the corresponding interior solution on the surface of the star.

7.4 The relation between the Hartle-Thorne and the Sedrakyan-Chubaryan metrics

In general, to establish the equivalence between two spacetimes in an invariant way it is necessary to perform a detailed analysis of the corresponding curvature tensors and their covariant derivatives [74]. The problem can be simplified significantly if it is possible to find the explicit diffeomorphism that relates the two spacetimes. In the case that the spacetimes are approximate solutions of the field equations, the problem simplifies even further because the coordinate transformation must be valid only approximately. This is the case we are analyzing in the present work.

In order to compare the Sedrakyan-Chubaryan solution with the Hartle-Thorne solution, we will find a coordinate transformation so that both solutions are written in the same coordinates. A close examination of the Sedrakyan-Chubaryan solution shows that it is indeed possible when one chooses the radial coordinate transformation of the type

$$r \to r\left(1 - \frac{\beta}{2} U_0(r)\right) , \tag{7.4.1}$$

and keeps the remaining coordinates unchanged. Notice that the practical effect of this transformation is to absorb the function $U_0(r)$. This means that, without loss of generality, we can set $U_0(r) = 0$ (or, equivalently, $C_{U_0} = 0$) in the Sedrakyan-Chubaryan solution and thus it becomes equivalent to the Hartle-Thorne solution, up to a redefinition of the constants entering the metric. Indeed, we now have only three integration constants, namely, C_{f_0}, C_q and C_{Φ_2} which are directly related to the total mass, angular momentum and quadrupole moment of the Hartle-Thorne solution. In fact, by comparing the g_{tt} and $g_{t\phi}$ components of the metric tensor, we obtain

$$M = m + \frac{\beta}{2} C_{f_0}, \tag{7.4.2}$$

$$J = -\frac{\sqrt{\beta}}{2} C_q, \tag{7.4.3}$$

$$Q = \frac{\beta}{2}\left(\frac{C_q^2}{2m} + \frac{4m^5 C_{\Phi_2}}{5}\right). \tag{7.4.4}$$

Notice that M in the Hartle-Thorne solution is actually composed of two terms, $M = m + \delta m$, where m is the "static mass" and δm is the contribution due to the rotation of the source. This means that in fact the last equations relate four constants of the Hartle-Thorne solution with four constants of the Sedrakyan-Chubaryan solution, implying that the inverse transformation is well defined. This proves the mathematical and physical equivalence of the two spacetimes up to the first order in the quadrupole moment Q and the second order in the angular momentum J.

7.5 Conclusions

In this work, we reviewed the original papers by Hartle (1967) and Hartle and Thorne (1968), and discussed their main properties, extensions and modifications. We revisited the results of Sedrakyan and Chubaryan (1968) for the metric that describes the exterior field of an axially symmetric mass distribution. Using a perturbation procedure, we derived the Sedrakyan-Chubaryan solution explicitly which includes several integration constants. Instead of using the interior Sedrakyan-Chubaryan solution in order to find the integration constants, we compare the exterior metric with the exterior Hartle-Thorne spacetime solution in the same coordinates. As a result, we obtain a set of simple algebraic expressions relating the main parameters of the Hartle-Thorne metric with the integration constants of the Sedrakyan-Chubaryan solution. In this way, we also proved the mathematical and physical equivalence of the two spacetimes.

We conclude that the Sedrakyan-Chubaryan solution can be considered as an alternative approach to describe the gravitational field of a slightly deformed stationary axially symmetric mass distribution in the slow rotation approximation. Moreover, the Sedrakyan-Chubaryan solution with its internal counterpart can be applied to various astrophysical problems together with the Hartle-Thorne solution on equal rights.

On the other hand, in a previous work [75] it was shown that the Hartle-Thorne formalism for the approximate description of rotating mass distributions is equivalent to the Fock-Abdildin approach. The last one, however, allows us to interpret the parameters of the interior solution in terms of physical quantities like the rotational kinetic energy or the mutual gravitational attraction between the particles of the source. Therefore, the results obtained in this work imply that it should be possible to find a direct relationship between the interior Sedrakyan-Chubaryan solution and the corresponding counterpart in the Fock-Abdildin approach.

It is interesting that different approaches that were developed independently in different places and under diverse circumstances turn out to be equivalent from a mathematical point of view. It would interesting to perform a more detailed analysis of all the physical characteristics of each approach in order to propose a unique formalism that would incorporate the advantages of all the known approaches.

8 Generating static perfect-fluid solutions of Einstein's equations

8.1 Introduction

General relativity is a theory of gravity and as such should be able to describe the field of all possible physical configurations in which gravity is involved. All the information about the gravitational field should be encoded in the metric tensor which must be an exact solution of Einstein's equations. In this work, we will focus on the study of the gravitational field generated by compact objects like planets or stars, which can be considered as independent of time. In this case, the problem of describing the complete field generated by the source can be split into two correlated problems, namely, the interior and the exterior rotating fields. To handle the corresponding field equations, one usually assumes axial symmetry with respect to the rotation axis. It is well known that the exterior field of an arbitrarily rotating mass can be described by the Kerr spacetime [21]. As for the interior field, the situation is more complicated. In fact, a major actual problem of classical general relativity consists in finding a physically reasonable interior solution for the exterior Kerr metric, which could be applied to describe the interior field of realistic compact objects. One usually assumes that the matter inside the object can be described by a rotating perfect fluid. Since the discovery of the Kerr metric in 1963, many attempts have been made to find the corresponding exact perfect-fluid interior solution. It seems that the rotation parameter for a perfect fluid leads to important difficulties in the context of Einstein's equations. In this work, we will concentrate on the problem of perfect-fluid solutions, without considering the rotation parameter. We expect that the understanding of the symmetry properties of perfect-fluid solutions will help to incorporate later the rotation parameter.

Although there exists quite a large number of exact solutions [1], only a few can be considered as physically meaningful. This type of solutions are usually classified in terms of their symmetry. Consider, for instance, spherically symmetric spacetimes. According to Birkhoff's theorem, the exterior field is uniquely determined by the Schwarzschild solution. For the interior field, however, a quite large number of exact solutions are known. In fact, the explicit form of the interior solutions depends on the model used to determine the energy-momentum tensor of the source. In addition, if we limit ourselves to the case of perfect-fluid sources, the search for exact solutions requires the knowledge of an additional equation which is usually taken as the equation of state of the fluid. The explicit form of the interior solution depends heavily on the properties of the equation of state. Recently, several generating techniques have been proposed that allow one to obtain all spherically symmetric perfect-fluid interior solutions with and without equations of state [76, 77, 78, 79].

In the case of stationary axisymmetric spacetimes, only a few solutions are known for the interior of a rigidly rotating perfect fluid. In [81, 82, 83] several exact solutions were found which, however, are characterized by negative pressures. In [84, 85] more realistic interior solutions were obtained with physically plausible equations of state for rigidly rotating sources. On the other hand, Herrera and Jiménez [86] proposed an alternative approach in which an extension of the Newman-Janis algorithm is applied to static spherically symmetric interior solutions to generate stationary interior spacetimes. Several new interior Kerr solutions have been obtained by using this method. For instance, in [87] an interior trial solution was obtained with a pressure that diverges at the origin of coordinates. Rotating neutral and charged solutions were obtained in [88] with non-perfect fluids as the interior source of gravity. The case of rotating spacetimes for anisotropic fluids with shear viscosity and heat flux was analyzed in [89, 90], obtaining some particular solutions whose exterior counterpart is unknown. In an attempt to generate reasonable rotating solutions, in [91] the extension of the Newman-Janis trick was applied to static physically meaningful spacetimes like the incompressible Schwarzschild interior. The same method can in principle be applied to obtain interior solutions which match a general stationary vacuum spacetime, provided the starting static metric is physically reasonable. Moreover, in [92], the field equations for anisotropic fluids were presented in an Ernst-like form which leads to a precise prescription for generating interior solutions. It turns out that, when applied to static spherically symmetric interior solutions, the extension of the Newman-Janis algorithm always destroys the perfect-fluid property [93]; nevertheless, in the case of a pure gravitomagnetic Weyl tensor, the perfect-fluid property can be preserved [94]. Although it has been argued [95] that this method would work only in Einstein's theory, it also has been applied to obtain rotating higher dimensional spacetimes [96], non-commutative black holes [97, 98], loop black holes [99], regular black holes [100, 102] and wormholes [101]. Recently, a generalization of the Newman-Janis algorithm was proposed which includes the transformation of a particular gauge field [103].

Nevertheless, the problem of describing the interior and exterior gravitational field of a rotating compact object remains open because the attempts to solve directly the resulting vacuum field equations, together with the energy conditions and the matching conditions, have not led to physically acceptable models. In view of this situation, it seems reasonable to explore other methods. In particular, we believe that investigating the symmetries of the field equations and proposing methods for generating new inner solutions from known ones could give some new insight into the problematic [1]. The present work can be considered as a first step in this direction. In this work, we focus on non-rotating perfect-fluid configurations. We propose a line element which is specially adapted to the investigation of the symmetry properties of the corresponding field equations. In fact, we will show that a particular symmetry corresponds to a transformation that can be used to generate new exact solutions. This paper is organized as follows.

8.2 Line element and field equations

We will consider a static axially symmetric spacetime with Killing vectors $\xi_t = \partial_t$ and $\xi_\varphi = \partial_\varphi$, where t is the time-coordinate and φ is the azimuthal angle. This implies that $\partial g_{\alpha\beta}/\partial t = \partial g_{\alpha\beta}/\partial\varphi = 0$. For the remaining spatial coordinates we choose θ as the polar angle and r as a radial-like coordinate. As for the line element for this type of spacetimes, several choices are possible [1]. In the case of interior solutions, a particular choice [80] has been intensively used to derive and analyze approximate numerical solutions. In principle, for a given set of symmetry conditions and coordinates, all possible line elements should be equivalent. However, the point is that the structure of the field equations depends on the explicit form of the line element and, therefore, a particular choice might be more suitable for the investigation of the field equations structure. In this work, we propose the following line element

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}\left[e^{2\gamma}\left(\frac{dr^2}{h} + d\theta^2\right) + \mu^2 d\varphi^2\right], \tag{8.2.1}$$

to study the structure of the field equations with a perfect-fluid source. Here $\psi = \psi(r,\theta)$, $\gamma = \gamma(r,\theta)$, $\mu = \mu(r,\theta)$, and $h = h(r)$. Moreover, we use geometric units with $G = c = 1$. A redefinition of the $r-$coordinate in Eq.(8.2.1) leads to a line element which was previously used in [104] to perform a detailed analysis of axisymmetric gravitational sources with anisotropic static fluids.

The Einstein equations for a perfect fluid with 4-velocity U_α, density ρ, and pressure p

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi\left[(\rho + p)U_\alpha U_\beta - pg_{\alpha\beta}\right] \tag{8.2.2}$$

for the line element (8.2.1) lead to a system of four independent differential equations. The main equations, relating the metric functions μ, ψ, and h, can be written as

$$\frac{\mu_{,rr}}{\mu} + \frac{\mu_{,\theta\theta}}{h\mu} + \frac{h_{,r}\mu_{,r}}{2h\mu} = \frac{16\pi}{h}\bar{p}\,, \tag{8.2.3}$$

$$\psi_{,rr} + \frac{\psi_{,\theta\theta}}{h} + \left(\frac{h_{,r}}{2h} + \frac{\mu_{,r}}{\mu}\right)\psi_{,r} + \frac{\mu_{,\theta}\psi_{,\theta}}{h\mu} = \frac{4\pi}{h}(3\bar{p} + \bar{\rho})\,, \tag{8.2.4}$$

where the comma represents partial differentiation with respect to the corresponding coordinates, and we have introduced the "normalized" density $\bar{\rho}$ and pressure $\bar{p}$ by means of

$$\bar{\rho} = \rho e^{2\gamma - 2\psi}\,, \quad \bar{p} = pe^{2\gamma - 2\psi}\,. \tag{8.2.5}$$

Notice that there is no independent equation for the metric function $h(r)$. The reason is that in the line element (8.2.1) it is possible to introduce a new radial-like coordinate, say $dR = dr/\sqrt{h}$, which absorbs the function h, implying that it does not represent a genuine degree of freedom. Nevertheless, we keep h as an auxiliary function which can be chosen arbitrarily, in principle. Accordingly, if we attempt to solve the above system it is necessary to specify h, by using some additional criteria. This will turn to be useful for the purpose of the present work, as we will show below. As for the metric function

γ, it is determined by two first order differential equations

$$\gamma_{,r} = \frac{1}{h\mu_{,r}^2 + \mu_{,\theta}^2}\left\{\mu\left[\mu_{,r}\left(h\psi_{,r}^2 - \psi_{,\theta}^2\right) + 2\mu_{,\theta}\psi_{,\theta}\psi_{,r} + 8\pi\mu_{,r}\bar{p}\right] + \mu_{,\theta}\mu_{,r\theta} - \mu_{,r}\mu_{,\theta\theta}\right\}, \tag{8.2.6}$$

$$\gamma_{,\theta} = \frac{1}{h\mu_{r}^2 + \mu_{,\theta}^2}\left\{\mu\left[\mu_{,\theta}\left(\psi_{,\theta}^2 - h\psi_{,r}^2\right) + 2h\mu_{,r}\psi_{,\theta}\psi_{,r} - 8\pi\mu_{,\theta}\bar{p}\right] + h\mu_{,r}\mu_{r\theta} + \mu_{,\theta}\mu_{,\theta\theta}\right\}, \tag{8.2.7}$$

which can be integrated by quadratures once the main field equations (8.2.3) and (8.2.4) are solved, and the normalized pressure $\bar{p}$ is known. It is important to notice that if we introduce the differential equations (8.2.3)-(8.2.7) into the original Einstein equations (8.2.2), a second order differential equation for γ emerges

$$\gamma_{,rr} + \frac{\gamma_{,\theta\theta}}{h} + \psi_{,r}^2 + \frac{\psi_{,\theta}^2}{h} + \frac{h_{,r}\gamma_{,r}}{2h} = \frac{8\pi}{h}\bar{p}\,, \tag{8.2.8}$$

which must also be satisfied. Nevertheless, it can be shown that this equation is identically satisfied if the two first order differential equations (8.2.6) and (8.2.7) for γ and the conservation equation for the non-normalized parameters of the perfect fluid

$$p_{,r} = -(\rho + p)\psi_{,r}\,, \quad p_{,\theta} = -(\rho + p)\psi_{,\theta} \tag{8.2.9}$$

are satisfied.

Notice that the structure of the differential equations as presented above resembles the structure of the vacuum field equations. In fact, for vanishing density and pressure with $h = 1$, the equation for μ is trivially satisfied, if μ is used as a radial-like coordinate ($\mu \sim r$), and the only remaining field equation is (8.2.4) for the function ψ whose solution determines uniquely the remaining function γ. Since the conservation law is identically satisfied in this case, the compatibility condition following from Eqs.(8.2.6)-(8.2.8) coincides with the main field equation (8.2.4).

Notice also that the present choice of the line element leads to a particularly simple generalization of the Tolman-Oppenheimer-Volkov (TOV) equation for spherically symmetric perfect-fluid sources. This has been also obtained previously with different choices of line elements [80]. In fact, the TOV equation coincides with the first equation given in (8.2.9), with r being the radial spherical coordinate. The generalization to include the axisymmetric case implies a second TOV equation for the angle coordinate θ. This simple form of the TOV equations allows us to easily calculate the pressure, if ρ and ψ are given.

8.3 The transformation

The search for physically reasonable exact solutions of the above system of differential equations is in general a very complicated task. In principle, we have a system of four independent equations (8.2.3)-(8.2.7) for the six unknowns μ, ψ, γ, h, p, and ρ. To close the system it is necessary to specify $h = h(r)$ and, for instance, an equation of state of the form $p = p(\rho)$. We investigated the simple case of a polynomial function $h(r)$ and a barotropic equation of state $p = \omega\rho$, where ω is the barotropic constant factor, and found several approximate solutions which, however, turned out to be characterized by either negative values of the pressure or singularities at the origin. Although this type of solutions could have some mathematical interest, their physical interpretation is difficult due to the presence of singularities, negative pressures or negative energy densities.

However, we investigated the structure of the field equations and found certain symmetries that allow us to generate new solutions from a given seed solution. This generating technique is based upon a transformation of the functions and coordinates entering the field equations. The result can be formulated as follows:

Theorem: Let an exact interior solution of Einstein's equations (8.2.3)-(8.2.7) for the static axisymmetric line element (8.2.1) be given explicitly by means of the functions

$$h_0 = h_0(r),\ \mu_0 = \mu_0(r,\theta),\ \psi_0 = \psi_0(r,\theta),\ \gamma_0 = \gamma_0(r,\theta), \tag{8.3.1}$$

$$\bar{p}_0 = \bar{p}_0(r,\theta),\ \bar{\rho}_0 = \bar{\rho}_0(r,\theta)\ . \tag{8.3.2}$$

Then, for any arbitrary real value of the constant parameter δ the new functions

$$h = h_0(r),\ \mu = \mu_0(r,\tilde{\theta}),\ \psi = \delta\psi_0(r,\tilde{\theta}),\ \bar{p} = \delta\bar{p}_0(r,\tilde{\theta}),\ \bar{\rho} = \delta\bar{\rho}_0(r,\tilde{\theta}),\ \tilde{\theta} = \frac{\theta}{\sqrt{\delta}}\ , \tag{8.3.3}$$

$$\gamma(r,\tilde\theta) = \qquad \delta^2\gamma_0(r,\tilde\theta) + (\delta^2-1)\int \frac{\nu_{\tilde\theta}}{h_0+\nu^2}dr$$
$$+8\pi\delta(1-\delta)\int \frac{\frac{\mu_0}{\mu_{0,r}}\bar p_0}{h_0+\nu^2}dr + \kappa, \quad \nu = \frac{\mu_{0,\tilde\theta}}{\mu_{0,r}}\ , \ \kappa = const \tag{8.3.4}$$

represent a new solution of the field equations (8.2.3)-(8.2.7).

Proof: Since the set of functions (8.3.2) represents a solution, the field equations (8.2.3) and (8.2.4)

$$\frac{\mu_{0,rr}}{\mu_0} + \frac{\mu_{0,\theta\theta}}{h_0\mu_0} + \frac{h_{0,r}\mu_{0,r}}{2h_0\mu_0} = \frac{16\pi}{h_0}\bar p_0\ , \tag{8.3.5}$$

$$\psi_{0,rr} + \frac{\psi_{0,\theta\theta}}{h_0} + \left(\frac{h_{0,r}}{2h_0} + \frac{\mu_{0,r}}{\mu_0}\right)\psi_{0,r} + \frac{\mu_{0,\theta}\psi_{0,\theta}}{h_0\mu_0} = \frac{4\pi}{h_0}(3\bar p_0 + \bar\rho_0)\ , \tag{8.3.6}$$

are identically satisfied. Notice that the normalized quantities $\bar p_0$ and $\bar\rho_0$, as written in the above identities, are just explicitly known functions of their arguments so that we can treat them as independent functions. In other words, at the level of the identities, which follow from the field equations for the seed functions, we can treat $\bar p_0$, $\bar\rho_0$, ψ_0 and γ_0 as algebraically independent functions.

If we now replace the functions $h = h_0/\delta$ and $\mu = \mu_0(r,\tilde\theta)$ in Eq.(8.2.3), and perform the coordinate reparametrization $\theta \to \tilde\theta = \theta/\sqrt{\delta}$, we obtain the equation

$$\frac{\mu_{0,rr}}{\mu_0} + \frac{\mu_{0,\tilde\theta\tilde\theta}}{h_0\mu_0} + \frac{h_{0,r}\mu_{0,r}}{2h_0\mu_0} = \frac{16\pi}{h_0}\delta\bar p\ . \tag{8.3.7}$$

This equation coincides with the identity (8.3.5) only if $\tilde\theta \to \theta$ and $\delta\bar p \to \bar p_0$. So, in general, it is not identically satisfied. However, one can interpret it as the definition of a new pressure $\bar p_{new}(r,\tilde\theta) = \delta\bar p(r,\tilde\theta)$. To establish the connection with the seed solution we assume that $\bar p_{new}(r,\tilde\theta) = \delta\bar p_0(r,\tilde\theta)$. It then follows that the functions $h = h_0$ and $\mu = \mu_0(r,\tilde\theta)$ satisfy the field equation (8.2.3) for a new pressure $\delta\bar p_0(r,\tilde\theta)$. The next step is to show that these functions can be made to satisfy the remaining field equations.

Consider the field equation (8.2.4) for $\psi(r,\tilde\theta)$ with $\mu = \mu_0(r,\tilde\theta)$, $h = h_0(r)$ and the new pressure $\bar p(r,\tilde\theta) = \delta\bar p_0(r,\tilde\theta)$. Then, we obtain

$$\psi_{,rr} + \frac{\psi_{\tilde\theta\tilde\theta}}{h_0} + \left(\frac{h_{0,r}}{2h_0} + \frac{\mu_{0,r}}{\mu_0}\right)\psi_{,r} + \frac{\mu_{0,\tilde\theta}\psi_{\tilde\theta}}{h_0\mu_0} = \frac{4\pi}{h_0}(3\delta\bar p_0 + \bar\rho)\ . \tag{8.3.8}$$

A simple inspection of this equation shows that it reduces to the identity (8.3.6) for the choice $\psi = \delta\psi_0(r,\tilde{\theta})$ and $\bar{\rho} = \delta\bar{\rho}_0(r,\tilde{\theta})$. This proves that the functions (8.3.3) determine a new solution of the field equations (8.2.3) and (8.2.4).

We now calculate the explicit form of the function γ. First of all, we notice that since $\gamma_0(r,\theta)$ is a solution so is also $\gamma_0(r,\tilde{\theta})$ when we change θ by $\tilde{\theta}$ everywhere in the equations. Then, the corresponding field equations become identities, i. e.,

$$\gamma_{0,r} = \tfrac{1}{H(r,\tilde{\theta})}\Big\{\mu_0\left[\mu_{0,r}\left(h_0\psi_{0,r}^2 - \psi_{0,\tilde{\theta}}^2\right) + 2\mu_{0,\tilde{\theta}}\psi_{0,\tilde{\theta}}\psi_{0,r} + 8\pi\mu_{0,r}\bar{p}_0\right] + \mu_{0,\tilde{\theta}}\mu_{0,r\tilde{\theta}} - \mu_{0,r}\mu_{0,\tilde{\theta}\tilde{\theta}}\Big\}, \tag{8.3.9}$$

$$\gamma_{0,\tilde{\theta}} = \tfrac{1}{H(r,\tilde{\theta})}\Big\{\mu_0\left[\mu_{0,\tilde{\theta}}\left(\psi_{0,\tilde{\theta}}^2 - h_0\psi_{0,r}^2\right) + 2h_0\mu_{0,r}\psi_{0,\tilde{\theta}}\psi_{0,r} - 8\pi\mu_{0,\tilde{\theta}}\bar{p}_0\right] + h_0\mu_{0,r}\mu_{0,r\tilde{\theta}} + \mu_{0,\tilde{\theta}}\mu_{0,\tilde{\theta}\tilde{\theta}}\Big\}, \tag{8.3.10}$$

with

$$H(r,\tilde{\theta}) = h_0\mu_{0,r}^2 + \mu_{0,\tilde{\theta}}^2\,. \tag{8.3.11}$$

We now introduce the new solution (8.3.3) into the field equation (8.2.6), and obtain

$$\gamma_{,r} = \frac{1}{H(r,\tilde{\theta})}\Big\{\ \delta^2\mu_0\left[\mu_{0,r}\left(h_0\psi_{0,r}^2 - \psi_{0,\tilde{\theta}}^2\right) + 2\mu_{0,\tilde{\theta}}\psi_{0,\tilde{\theta}}\psi_{0,r}\right] + 8\pi\delta\mu_0\mu_{0,r}\bar{p}_0 + \mu_{0,\tilde{\theta}}\mu_{0,r\tilde{\theta}} - \mu_{0,r}\mu_{0,\tilde{\theta}\tilde{\theta}}\Big\}\,. \tag{8.3.12}$$

Using Eq.(8.3.9) to replace the term in squared brackets of the last equation, after some algebraic rearrangements, we get

$$\gamma_{,r} = \delta^2\gamma_{0,r} + \frac{\delta^2-1}{H(r,\tilde{\theta})}\left(\frac{\mu_{0,\tilde{\theta}}}{\mu_{0,r}}\right)_{\tilde{\theta}}\mu_{0,r}^2 + 8\pi\delta(1-\delta)\frac{\mu_0\mu_{0,r}\bar{p}_0}{H(r,\tilde{\theta})}\,. \tag{8.3.13}$$

Consider now the second equation (8.2.7) for the function γ. Introducing the values of the new solution (8.3.3) into Eq.(8.2.7), we obtain

$$\gamma_{,\tilde{\theta}} = \frac{1}{H(r,\tilde{\theta})}\Big\{ \quad \delta^2\mu_0\left[\mu_{0,\tilde{\theta}}\left(\psi_{0,\tilde{\theta}}^2 - h_0\psi_{0,r}^2\right) + 2h_0\mu_{0,r}\psi_{0,\tilde{\theta}}\psi_{0,r}\right]$$
$$-8\pi\delta\mu_0\mu_{0,\tilde{\theta}}\bar{p}_0 + \left(h_0\mu_{0,r}\mu_{0,r\tilde{\theta}} + \mu_{0,\tilde{\theta}}\mu_{0,\tilde{\theta}\tilde{\theta}}\right)\Big\} . \quad (8.3.14)$$

Furthermore, we use the identity (8.3.10) to replace the expression contained in the squared brackets. It is then straightforward to get

$$\gamma_{,\tilde{\theta}} = \delta^2\gamma_{0,\tilde{\theta}} + \frac{1}{2}(1-\delta^2)\frac{\partial}{\partial\tilde{\theta}}\ln H(r,\tilde{\theta}) + 8\pi\delta(\delta-1)\frac{\mu_0\mu_{0,\tilde{\theta}}\bar{p}_0}{H(r,\tilde{\theta})} . \quad (8.3.15)$$

The above equation can immediately be integrated, and the solution is determined up to an arbitrary function of r

$$\gamma(r,\tilde{\theta}) = \delta^2\gamma_0 + \frac{1}{2}(1-\delta^2)\ln H + 8\pi\delta(\delta-1)\int\frac{\mu_0\mu_{0,\tilde{\theta}}\bar{p}_0}{H(r,\tilde{\theta})}d\tilde{\theta} + F(r) \quad (8.3.16)$$

which can be fixed by considering the second equation (8.3.13) for the function $\gamma(r,\tilde{\theta})$. We obtain

$$F(r) = \quad (1-\delta^2)\left[\int\frac{\mu_{0,\tilde{\theta}}\mu_{0,r\tilde{\theta}} - \mu_{0,r}\mu_{0,\tilde{\theta}\tilde{\theta}}}{h_0\mu_{0,r}^2 + \mu_{0,\theta}^2}dr - \frac{1}{2}\ln H(r,\tilde{\theta})\right]$$
$$+ \quad 8\pi\delta(1-\delta)\left[\int\frac{\mu_0\mu_{0,\tilde{\theta}}\bar{p}_0}{H(r,\tilde{\theta})}d\tilde{\theta} + \int\frac{\mu_0\mu_{0,r}\bar{p}_0}{H(r,\tilde{\theta})}dr\right] + \kappa , \quad (8.3.17)$$

where κ is an integration constant. Inserting this expression into Eq.(8.3.16), after some algebraic rearrangements we finally obtain

$$\gamma(r,\tilde{\theta}) = \delta^2\gamma_0(r,\tilde{\theta}) + (\delta^2-1)\int\frac{\nu_{\tilde{\theta}}}{h_0+\nu^2}dr + 8\pi\delta(1-\delta)\int\frac{\mu_0\mu_{0,r}\bar{p}_0}{H(r,\tilde{\theta})}dr + \kappa , \quad (8.3.18)$$

which is equivalent to Eq.(8.3.4). This ends the proof of the theorem.

8.4 Examples

The theorem proved in the previous section states that given a static axisymmetric solution it is possible to generate a new solution which contains an additional new parameter δ. In this section, we will apply the transformation to show that it can be used to generate new solutions that are physically different from the seed solution.

8.4.1 The vacuum q-metric

Let us consider the vacuum limiting case of the above transformation. The particular choice

$$p = 0,\ \rho = 0,\ \mu = r,\ h = 1,\ \theta = z \tag{8.4.1}$$

in Eq.(7.3.1) leads to the static axisymmetric Lewis-Papapetrou line element for vacuum spacetimes [1]

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}\left[e^{2\gamma}(dr^2 + dz^2) + r^2 d\varphi^2\right] \tag{8.4.2}$$

in cylindrical coordinates (t, r, z, φ) for which the field equations (8.2.3)-(8.2.7) reduce to

$$\psi_{,rr} + \frac{1}{r}\psi_{,r} + \psi_{,zz} = 0\ ,\ \gamma_{,r} = r(\psi_{,r}^2 - \psi_z^2),\ \gamma_{,z} = 2r\psi_{,r}\psi_z\ , \tag{8.4.3}$$

for the functions ψ and γ only. Then, according to the Theorem proved in the previous section, if $\psi_0 = \psi_0(r,z)$ and $\gamma_0 = \gamma_0(r,z)$ represent a particular solution of Eqs.(8.4.3), a new solution ψ, γ is given by

$$\psi(r,\tilde{z}) = \delta\psi_0(r,\tilde{z}),\ \gamma(r,\tilde{z}) = \delta^2\gamma_0(r,\tilde{z})\ . \tag{8.4.4}$$

Notice that for this new solution the line element corresponds to (8.4.2) with $z \to \tilde{z}$. Since the new functions depend also on $\tilde{z}$, one can use the coordinate z instead, without loss of generality. The transformation (8.4.4) was originally proposed by Zipoy [28] and Voorhees [29] and has been investigated in several works (see, for instance, [39, 40, 41, 42, 43, 44, 35, 34] and references therein). For this reason, the transformation defined in Eqs.(8.3.3) and (8.3.4) can be considered as a generalization of the Zipoy-Voorhees transformation that includes the case of perfect-fluid spacetimes.

A particular example of the Zipoy-Voorhees transformation arises if we take the Schwarzschild metric as seed solution as follows. First, we introduce spherical coordinates $(t, \tilde{r}, \vartheta, \varphi)$ into the line element (8.4.2) by means of the transformations

$$\tilde{r} = m + \frac{1}{2}(r_+ + r_-)\,, \quad \cos\vartheta = \frac{1}{2m}(r_+ - r_-)\,, \quad r_\pm = \sqrt{r^2 + (z \pm m)^2}\,, \tag{8.4.5}$$

and obtain

$$\begin{aligned} ds^2 = \;& e^{2\psi} dt^2 - e^{-2\psi} \left[e^{2\gamma} \left(1 - \frac{2m}{\tilde{r}} + \frac{m^2 \sin^2\vartheta}{\tilde{r}^2} \right) \left(\frac{d\tilde{r}^2}{1 - \frac{2m}{\tilde{r}}} + \tilde{r}^2 d\vartheta^2 \right) \right. \\ + \;& \left. \tilde{r}^2 \left(1 - \frac{2m}{\tilde{r}} \right) \sin^2\vartheta d\varphi^2 \right]. \end{aligned} \tag{8.4.6}$$

Then, the Schwarzschild metric corresponds to the particular solution

$$\psi_0 = \frac{1}{2} \ln \frac{\tilde{r} - 2m}{\tilde{r}}\,, \quad \gamma_0 = \frac{1}{2} \ln \frac{\tilde{r}^2 - 2m\tilde{r}}{\tilde{r}^2 - 2m\tilde{r} + m^2 \sin^2\vartheta}\,. \tag{8.4.7}$$

Applying the transformation (8.4.4) with $\delta = 1 + q$, i.e.,

$$\psi = \frac{1}{2}(1+q) \ln \frac{\tilde{r} - 2m}{\tilde{r}}\,, \quad \gamma = \frac{1}{2}(1+q)^2 \ln \frac{\tilde{r}^2 - 2m\tilde{r}}{\tilde{r}^2 - 2m\tilde{r} + m^2 \sin^2\vartheta}\,, \tag{8.4.8}$$

the resulting line element can be expressed as

$$\begin{aligned} ds^2 = \;& \left(1 - \frac{2m}{\tilde{r}} \right)^{1+q} dt^2 - \left(1 - \frac{2m}{\tilde{r}} \right)^{-q} \\ \times \;& \left[\left(1 + \frac{m^2 \sin^2\vartheta}{\tilde{r}^2 - 2m\tilde{r}} \right)^{-q(2+q)} \left(\frac{d\tilde{r}^2}{1 - \frac{2m}{\tilde{r}}} + \tilde{r}^2 d\vartheta^2 \right) + \tilde{r}^2 \sin^2\vartheta d\varphi^2 \right] \end{aligned} \tag{8.4.9}$$

which, as expected, reduces to the Schwarzschild metric in the limiting case $q \to 0$.

A detailed analysis of this metric shows that m and q are constant parameters that determine the total mass and the quadrupole moment of the gravitational source [34]. The metric (8.4.9) has been interpreted as the simplest generalization of the Schwarzschild metric with a quadrupole. In the literature, this metric is known as the $\delta-$metric, referring to the parameter δ, or as the $\gamma-$metric for notational reasons. We use the term $q-$metric to emphasize the physical interpretation of the new solution in terms of the quadrupole moment.

Whereas the seed metric is the spherically symmetric Schwarzschild solution, which describes the gravitational field of a black hole, the generated $q-$metric is axially symmetric and describes the exterior field of a naked singularity [34]. This shows that the transformation presented in the previous section generates non-trivial solutions even in the case of vacuum solutions.

8.4.2 Generalization of the interior Schwarzschild solution

One of the most important interior solutions of Einstein's equations is the spherically symmetric Schwarzschild solution which describes the interior field of a perfect-fluid sphere of radius R and total mass m. The corresponding line element can be written as

$$ds^2 = \left[\frac{3}{2}f(R) - \frac{1}{2}f(r)\right]^2 dt^2 - \frac{dr^2}{f^2(r)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \, , \qquad (8.4.10)$$

with

$$f(r) = \sqrt{1 - \frac{2mr^2}{R^3}} \, . \qquad (8.4.11)$$

The parameters of the perfect fluid are the constant density ρ_0 and the pressure p_0 which is a function of the radial coordinate r only

$$p_0 = \rho_0 \frac{f(r) - f(R)}{3f(R) - f(r)} \qquad (8.4.12)$$

We will use the above metric as the seed solution (8.3.2) for the general transformation (8.3.3). Then, a straightforward comparison with the general line element (8.2.1) yields

$$e^{\psi_0} = \frac{3}{2}f(R) - \frac{1}{2}f(r) \, , \; h_0 = r^2 f^2(r) \, , \; \mu_0 = r \sin\theta e^{\psi_0} \, , \; e^{\gamma_0} = r e^{\psi_0} \, . \qquad (8.4.13)$$

According to the Theorem presented in Sec.8.3, the new solution can be obtained from Eq.(8.4.13) by multiplying the corresponding metric functions with the new parameter δ. Then, the new line element can be represented as

$$ds^2 = e^{2\delta\psi_0}dt^2 - e^{-2\delta\psi_0}\left[e^{2\gamma}\left(\frac{dr^2}{r^2 f^2(r)} + d\tilde{\theta}^2\right) + r^2 e^{2\psi_0}\sin^2\tilde{\theta}d\varphi^2\right] , \quad (8.4.14)$$

where the new function γ is given by

$$\gamma = \delta^2\gamma_0 + \int \frac{(1-\delta^2) + 8\pi\delta(1-\delta)\sin^2\tilde{\theta}r^2 p_0}{rf^2(r)(1+r\psi_{0\,r})\sin^2\tilde{\theta} + \frac{r}{1+r\psi_{0\,r}}\cos^2\tilde{\theta}}dr + \kappa\,, \quad (8.4.15)$$

where

$$\psi_{0\,r} = \frac{2mr}{R^3 f(r)[3f(R) - f(r)]}\,. \quad (8.4.16)$$

Moreover, the parameters of the perfect fluid source are

$$\rho = \delta\rho_0 e^{2\gamma_0 - 2\gamma + 2(\delta-1)\psi_0} \ , \quad p = \delta p_0 e^{2\gamma_0 - 2\gamma + 2(\delta-1)\psi_0}\,. \quad (8.4.17)$$

Notice that the new function γ as given in Eq.(8.4.15) depends explicitly on the coordinate $\tilde{\theta}$, in contrast to the seed metric function γ_0 which depends on the coordinate r only. This proves that the new solution is not spherically symmetric, indicating that it is physically different from the interior Schwarzschild solution. This conclusion is corroborated also by the fact that the density and pressure of the new solution are functions of the angular coordinate too.

8.5 Conclusions

In this work, we presented a new method for generating perfect-fluid solutions of the Einstein equations, starting from a given seed solution. The method is based upon the introduction of a new parameter at the level of the metric functions of the seed solution in such a way that the generated new solution is characterized by physical properties which are different from those of the seed solutions. This means that the solution generating technique proposed in this work can be used to generate non-trivial new solutions.

To show the validity of the method we propose a line element which is especially adapted to handle the problem. In fact, the perfect-fluid field equations turn out to be split into a set of two partial differential equations that once solved can be used to integrate by quadratures the remaining equations. We also introduce in the line element an auxiliary function that is not fixed by the field equations and, therefore, can be used to simplify the analysis of the main field equations.

The method has a well-defined vacuum limit. In fact, for a particular choice of the metric functions and for vanishing density and pressure, we obtain the Lewis-Papapetrou line element for static axisymmetric vacuum fields. In this limit, the generating technique proposed here coincides with the Zipov-Voorhees transformation which can be used to generate vacuum solutions from vacuum solutions. As a particular example we derive from the Schwarzschild metric a solution with quadrupole moment, which we call the $q-$metric, and can be interpreted as describing the simplest generalization of the Schwarzschild spacetime with a quadrupole parameter. A stationary generalization of the $q-$metric, satisfying the main physical conditions of exterior spacetimes, has been obtained in [51].

In the case of perfect-fluid solutions, we use the interior spherically symmetric Schwarzschild solution with constant density to generate a new interior solution which turns out to be axially symmetric.

The generation of further new solutions does not represent any particular difficulty for the method developed in this work. The important problem of matching the so generated interior solutions with the corresponding exterior solutions is beyond the scope of the present work, and will be the subject of future investigations. In this regard, several facts must be considered. It now seems to be well established that static perfect-fluid gravitational sources with isotropic pressure are spherically symmetric [105]. A recent detailed study of axisymmetric static sources with anisotropic pressure has corroborated this result in the case of an incompressible spheroid whose metric cannot be matched smoothly to any exterior Weyl solution [104]. This implies that it is necessary to consider non-smooth matching conditions in order to match exterior spacetimes with interior solutions with isotropic pressure. Another possibility would be to generalize the generating method presented here to include the case of anisotropic pressure, as considered in [104]. Indeed, a direct computation shows that the generating method can be generalized to include this case in a straightforward manner because, in the appropriate parametrization, the anisotropic pressure appears only in the first order differential equations for the metric function γ which can be integrated by quadratures. A more detailed analysis of the anisotropic case is necessary in order to determine the physical properties of the solutions generated by applying this method.

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