Self Gravitating Systems, Galactic Structures and Galactic Dynamics

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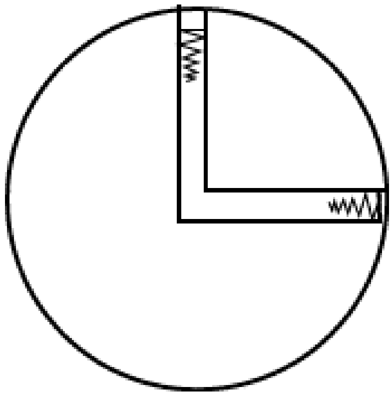

Figure 0.1: Newton's argument for an oblate earth.

0.3 Brief description

0.3.1 Self Gravitating Systems and Galactic Structures: historical Review of the Problem.

The investigation of the gravitational equilibrium of self-gravitating masses started with Newton in the third book of his Principia (1687). In this book Newton developed for the first time the idea of an oblate form of the earth due to rotation. In fact, using an argument based on the picture of a hole drilled up to the center of the earth from a point of the equator, and another one drilled from the pole, both meeting at the center and both filled with water in equilibrium in a rotating earth, Newton proposed the relationship among the ellipticity and the ratio centrifugal force/gravitational force in the form for slow rotation (Figure 1).

In this theory the earth is a spheroidal object and it is assumed that the distribution of mass is homogeneous. Newton concluded that the ellipticity is 1/230; however, the actual ellipticity is 1/294, a smaller result than Newton's predicted value; this discrepancy is interpreted in terms of the inhomogeneity of the earth. This work initiated the study of the rotation and configuration of the celestial bodies. A further step was given by Maclaurin (1742) who generalized the theory on the case when rotation cannot be considered slow but the density is homogeneous. The main result of this research is summarized in Maclaurin's formula giving the connection among angular velocity, eccentricity and density (Figure 2).

In Maclaurin's work is asserted that the direction of the composition of gravitational and centrifugal forces is perpendicular to the surface of the configuration. From Maclarin's formula the existence of two types of oblong

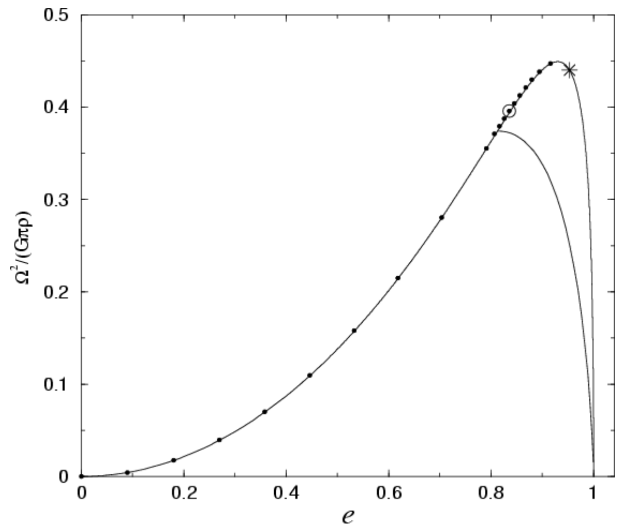

Figure 0.2: Bifurcation diagram of the Maclaurin's spheroids into Jacobi sequences

configurations can be deduced, one of them of small ellipticity ($e \to 0$, in the limit $\Omega^2 \to 0$), the other one being highly flattened ($e \to 1$, in the limit $\Omega^2 \to 0$). These figures are known as Maclaurin's spheroids. In 1834 Jacobi recognized that ellipsoids with three unequal axes, studied before by Lagrange (1811), can be indeed equilibrium configurations. In fact, the existence of such figures can be established by a direct extension of Newton's original arguments. In the triaxial case we may imagine three canals drilled along the direction of the principal axis of the ellipsoid and filled with water. From this argument it is possible to find the relationship among the size of the axes, the angular velocity and the density. As a consequence, the following inequality among the axis can be calculated $1/a_3^2 > 1/a_1^2 + 1/a_2^2$. According to Mayer (1842), the Jacobian sequence bifurcates from the Maclaurin spheroids at the point where the eccentricity is $e = 0.81267$, a result that can be deduced from the Jacobi's formula. In fact, if $a_1 = a_2$, it can be demonstrated that $e = 0.81267$ and $\Omega^2/\pi G\rho = 0.37423$. As the maximum value of $\Omega^2/\pi G\rho$ along the Maclaurin sequence is 0.4493, it follows that for $\Omega^2/\pi G\rho < 0.37423$ there are three equilibrium figures: two Maclaurin spheroids and one Jacobi ellipsoid; for $0.4493 > \Omega^2/\pi G\rho > 0.3742$ only the Maclaurin figures are possible; for $\Omega^2/\pi G\rho > 0.4493$ there are not equilibrium figures. These results were found by Liouville again (1846) using angular momentum instead of the angular velocity as the variable. Liouville demonstrated that increasing angular velocity from zero the Jacobi configurations are possible only for angular momenta up a critical value, that on the point of bifurcation. Later, on 1857, Dedekind, considering configurations with a linear profile of velocity as seen from an inertial frame, proved explicitly a theorem regarding the existence of a relation among this configuration and another one with the same form but uniformly rotating. Thus, a rotating ellipsoid without vorticity has the same dynamics as the same ellipsoid with a linear profile of velocity and without rotation. The next step was given by Riemann (1860), who showed that for linear profiles of velocity, the more general type of motion compatible with an ellipsoidal figure of equilibrium consists in a superposition of a uniform rotation and an internal motion with constant vorticity ζ in the rotating frame. Exactly, he showed that the possible motions are given by: (1) uniform rotation without vorticity; this case leads to sequences of Maclaurin and Jacobi, (2) directions of Ω and ζ coinciding with a principal axis of the ellipsoid; these configurations are known as Riemann-S ellipsoids and spheroids. This case leads as special cases to Jacobi and Dedekind sequences, and (3) the directions of Ω and ζ lie in a principal plane of the ellipsoid. This case leads to three other classes of ellipsoids, namely I, II, III. The subsequent important discovery, due to Poincare (1885), was that, along the Jacobian sequence, there is a point of bifurcation similar to that found in the Maclaurin sequence and that a new sequence of pear-shaped configurations branches off from the Jacobi sequence, corresponding to neutral modes of oscillation of the third harmonics. As a further conclusion there are neutral modes belonging to fourth, fifth and higher harmonics. The fission theory of the origin of the double stars comes from this considerations as conjectured by Poincare and Darwin (1906). In 1924 Cartan established that the Jacobi ellipsoids become unstable at the first point of bifurcation and behave in a different mode with respect to the Maclaurin sequence that is stable on both sides of the bifurcation point. When the density of the configuration is inhomogeneous it is necessary to provide an equation of state. In this case the barotropes are particularly relevant for the construction of the theory. In 1889 an important theorem was proved by Hamy: a mass ellipsoidally stratified cannot have a uniform rotation. Another classical theorem is the one given by Dive (1930): a stratified heterogeneous spheroid, rotating and without differential rotation, cannot be a barotrope. A new approach to the problem of the equilibrium configurations was started in the works by Chandrasekhar and Lebovitz (1961-69) using an integral formulation of the hydrodynamical problem. This approach, known as virial theory allows one to recover the fundamental results of Maclaurin, Jacobi, Dedekind and Riemann and sheds a new light on the problem of stability of these configurations. The tensor virial equations are integral relations, consequences of the equations of stellar hydrodynamics, and they yield necessary conditions that can furnish useful insights for the construction of ellipsoidal models. The virial method developed in Chandrasekhar (1987), shows that only in the case of homogeneous self-gravitating masses having a linear velocity field, the virial equations of second order result equivalent to the complete set of hydrodynamic equations. In the general case of heterogeneous density and non linear velocity field, this equivalence does not exist, and the n-th order virial equations represent necessarily global conditions to be satisfied by any equilibrium configuration.

0.3.2 The ICRANET project on galactic structures.

A series of papers of the ICRANET group (I to IX) have been devoted on the generalization of the theory of ellipsoidal figures of equilibrium, endowed with rotation Ω and vorticity ζ, obtained for the homogeneous case in the classic work of Maclaurin, Jacobi, Dedekind and Riemann, and treated in a unified manner by Chandrasekhar in the virial equation formalism in his book "Ellipsoidal figures of equilibrium" (1987). This series of papers has followed a variety of tentative approaches, consisting of subsequent generalizations of known results: looking at more general density distributions, non-linear velocity fields, selected forms of the pressure tensor, and finally analysing the constraints imposed by the n-th order virial equations. Clearly we have proceeded step by step. The first new step in the theory of ellipsoidal figures in equilibrium was the introduction by Pacheco, Pucacco and Ruffini (paper I, 1986a) of an heterogeneous density distribution and an anisotropic pressure. Using only the second order virial equations, the equilibrium and stability of heterogeneous generalized Riemann ellipsoids was analysed for the case of a linear velocity field with a corresponding uniform vorticity. The stability of second harmonic perturbations of these equilibrium solutions was also analyzed. In paper II, (1986b) by Pacheco, Pucacco and Ruffini additional special solutions of the equations introduced in paper I were considered: some generalized Maclaurin-Dedekind spheroids with anisotropic pressure and their stability properties were analyzed. It was shown how the presence of anisotropic pressure extends the region of stability towards greater values of the eccentricity, which is similar to the homogeneous case considered by Wiegand (1980). In paper III, (1988) by Busarello, Filippi and Ruffini, a second step was made to the generalization of the solutions by introducing a fully general stratified density distribution of the form $\rho = \rho(m^2)$, where ρ is an arbitrary function of the equidensity surfaces. As in the previous papers the pressure is still anisotropic and the velocity field linear. The equilibrium and stability properties of anisotropic and heterogeneous generalized Maclaurin spheroids and Jacobi and Dedekind ellipsoids were studied. The Dedekind theorem, originally proved for homogeneous and isotropic configurations is still valid for this more general case. In Pacheco, Pucacco, Ruffini and Sebastiani (paper IV, 1989) several applications of the previous treatment of the generalized Riemann sequences were studied. Special attention was given to the axial ratios of the equilibrium figures, compatible with given values of anisotropy. A stability analysis of the equilibrium was performed against odd modes of second harmonic perturbations. In Busarello, Filippi and Ruffini, (paper V, 1989), the heterogeneous and anisotropic ellipsoidal Riemann configurations of equilibrium, obtained in the previous paper and characterized by non zero angular velocity of the figure and constant vorticity were used to model a class of elliptical galaxies. Their geometrical and physical properties were discussed in terms of the anisotropy, the uniform figure rotation, and the internal streaming motion. In Busarello, Filippi and Ruffini, (paper VI, 1990), the equilibrium, stability, and some physical properties of a special case of oblate spheroidal configurations which rotate perpendicularly to the symmetry axis were analyzed, still within the framework of the second order virial equations. In Filippi, Ruffini and Sepulveda (paper VII, 1990), the authors made an additional fundamental generalization by introducing a non linear velocity field with a cylindrical structure an a density distribution originally adopted in paper I of the form $\rho = \rho_c(1 - m^2)^n$. The generalized anisotropic Riemann sequences coming from second order virial equations were studied. Some of the results obtained in that article have been modified by the consideration of the virial equations of n-th order, specially the claim regarding the validity of the Dedekind theorem, made on the basis of an unfortunate definition of certain coefficients. In Filippi, Ruffini and Sepulveda (paper VIII, 1991), following the theoretical approach of its predecessor, the nonlinear velocity field was extended to cover the most general directions of the vorticity and angular velocity. The more general form for the density $\rho = \rho(m^2)$ was adopted. Equilibrium sequences were determined, and their stability was analyzed against odd and even modes of second harmonic perturbations. In the next paper, discussed below in some detail, (Filippi, Ruffini and Sepulveda, 1996, Paper IX), the authors have considered an heterogeneous, rotating, self-gravitating fluid mass with anisotropic pressure and internal motions that are nonlinear functions of the coordinates in an inertial frame. We present here the complete results for the virial equations of n-th order, and we discuss the constraints for the equilibrium of spherical, spheroidal, and ellipsoidal configurations imposed by the higher order virial equations. In this context, the classical results of Hamy (1887) and Dive (1930) are also confirmed and generalized. In particular, (a) the Dedekind theorem is proved to be invalid in this more general case: the Dedekind figure with $\Omega = 0$ and $Z \neq 0$ cannot be obtained by transposition of the Jacobi figure, endowed $\Omega \neq 0$ and $Z = 0$; (b) the considerations contained in the previous eight papers on the series, concerning spherical or spheroidal configurations, are generalized to recover as special cases; (c) the n-th order virial equations severely constraint all heterogeneous ellipsoidal figures: as shown from tables in figures 3, 4, and 5, all the heterogeneous ellipsoidal figures cannot exist.

The n-th order virial approach. Paper IX

Let us consider an ideal self-gravitating fluid of density ρ, stratified as concentric ellipsoidal shells and with an anisotropic diagonal pressure P_i. The fluid will be considered from a rotating frame with angular velocity Ω respect to an inertial frame. If u is the velocity of any point of the fluid and v is the gravitational potential, the hydrodynamical equation governing the motion, referred to the rotating frame is given by (Goldstein 1980)

$$\rho \frac{du}{dt} = -\nabla P + \rho \nabla v + \frac{1}{2} \rho \nabla |\Omega \times x|^2 + 2\rho u \times \Omega - \rho \dot{\Omega} \times x \,.$$

In this equation appear, respectively, the contributions of pressure, gravitation, centrifugal force, Coriolis force and a ”Faraday” term, corresponding to the time change in the angular velocity. As usual, the gravitational potential satisfies the Poisson equation $\nabla^2 v = -4\pi G\rho$. We want to generalize the form of the virial equations, given for the second, third and fourth order by Chandrasekhar (1987), to the n-th order and for non-linear velocities. In the rotating frame the generalization can be done in complete analogy with the treatment presented by Chandrasekhar, multiplying the hydrodynamical equation by $x_i^{a-1} x_j^b x_k^c$ and integrating over the volume V of an ellipsoid. That means that the boundary of the configuration is defined; on the surface the pressure vanishes. The index i appears a-1 times, the indices j and k appear b and c times, respectively, and $a, b, c \geq 1$. Now, the velocity field considered in Chandrasekhar (1987, pag 69) is a linear function of the coordinates, corresponding to a constant vorticity. In order to generalize the velocity profile we propose a more general form, preserving the ellipsoidal stratification and the ellipsoidal boundary. Specifically, we assume the existence of a constant unit vector n fixed in the rest frame of the ellipsoid such that the velocity field circulates in planes perpendicular to n and having the same direction as Ω. Additionally the continuity equation must be preserved so that the velocity profile can be written as $u = \hat{n} \times (\mathcal{M}\dot{r})\hat{\phi}$. The dimensionless function ϕ describes the characteristic features of the velocity field. Thus, the velocity is linear if ϕ is 1. In this equation $\mathcal{M} = \sum_{i=1}^{3} \hat{e}_i x_i$. Then, the equation of the ellipse has the form $m^2 = r \cdot \mathcal{M} \cdot r$: In the steady state regime we may rewrite the generalized virial equations by introducing generalizations of kinetic energy tensor, angular momentum and moment of inertia respectively. Note first that the virial equations with odd values of $n = a + b + c$ are identically zero if the density, , and the function contain powers of $x_i x_j$ with $i, j = 1, 2$. We now turn our attention to the virial equations with even values of n. The analysis can be performed easily by classifying the powers of the coordinates. In fact, there are only the following possibilities: (1) a=even, b,c= odd; (2) b=even, a,c=odd; (3) c=even, a,b=odd; (4) a,b,c=even. Cases (2) and (3) are equivalent because of the interchangeable positions of j and k in eq (1). Cases (1) and (2) are non-equivalent owing to the privileged position of the index i. So, the steady state virial equations can be classified in three families. In this way the classical homogeneous and linear Maclaurin spheroids, Jacobi, Dedekind and Riemann ellipsoids can be generalized to cover heterogeneous systems with non uniform vorticity and anisotropic pressure, denoted as generalized

TABLE 1

EQUILIBRIUM CONFIGURATIONS WITH $\Omega \neq 0$ AND $Z = 0$[a]

Configurations	Density	Shape	Pressure
Spheres	Homogeneous	...	$P_1 < P_3$, B
	Heterogeneous	...	$P_1 < P_3$, B
Generalized Maclaurin spheroids......	Homogeneous	Oblate	isotropic ($P_1 = P_2 = P_3$), B
			anisotropic ($P_1 = P_2 \neq P_3$), B
		Prolate	anisotropic ($P_1 < P_3$), B
	Heterogeneous	..	isotropic, BC
			anisotropic ($P_1 = P_2 \neq P_3$), BC
Generalized Jacobi ellipsoids...........	Homogeneous	Oblate	isotropic $P_1 = P_2 = P_3$), B
			anisotropic ($P_1 \neq P_2 \neq P_3$, B
		Prolate	anisotropic ($P_1 < P_3$; $P_2 < P_3$), B
	Heterogeneous	...	...

[a] Homogeneous and heterogeneous figures of equilibrium having a uniform angular velocity $\mathbf{\Omega}$, with isotropic or anisotropic pressure (the different components P_1, P_2, and P_3 can be barotropic or stratified as the density [denoted by B], or baroclinic, $P_i = P_i(\tilde{x}_1^2 + \tilde{x}_2^2, \tilde{x}_3^2)$ [denoted by BC]).

Figure 0.3: Table 1

Maclaurin spheroids, and generalized Jacobi, Dedekind and Riemann ellipsoids.

The generalized S-type Riemann ellipsoids (homogeneous and heterogeneous systems with uniform figure rotation Ω parallel to the vorticity Z, and isotropic or anisotropic pressure) encompass as special cases the generalized Dedekind ellipsoids (homogeneous and heterogeneous systems with $\Omega = 0, Z \neq 0$ and isotropic or anisotropic pressure), the generalized Maclaurin spheroids and and the generalized Jacobi ellipsoids (homogeneous and heterogeneous axysymmetric or ellipsoidal, respectively, having $\Omega \neq 0, Z = 0$, with isotropic or anisotropic pressure). Its interesting to note that Dedekind's theorem, which transforms Dedekind ellipsoids into Jacobi ellipsoids and vice versa, no longer applies in the non linear velocity regime, being limited to the linear case. The density may be inhomogeneous and the pressure may be anisotropic. For generalized S-type ellipsoids we may write $n = (0, 0, 1), \Omega = (0, 0, \Omega), Z = (0, 0, Z)$. With this choice the explicit virial equation are reduced to the infinite set. It is easy to show that in the linear and homogeneous case ($\tilde{\phi} = P_i = P_{ic}(m^2)$), this infinite set reduces just to three equations which coincide with hydrodynamical equations (Chandrasekhar 1987, pp 74-75). The analysis of the possible configurations can be performed for two independent cases in which the form of the velocity profile is decided by $\tilde{\phi} = \tilde{\phi}(m^2)$ and $\tilde{\phi} = \tilde{\phi}^*$, the last one containing combinations of m^2, r^2, x_3^2. The analysis of equilibrium configurations are summarized by the tables in figures 3, 4, and 5 for Maclaurin, Dedekind and Riemann S configurations. These tables list the existent configurations in this specific sequence. Some

TABLE 2

EQUILIBRIUM CONFIGURATIONS WITH $\boldsymbol{\Omega} = 0$ AND $\boldsymbol{Z} \neq 0$[a]

Configuration	Density	Shape	Pressure	Velocity
Spheres	Homogeneous	...	anisotropic, $P_1 < P_3$, B	$\tilde{\phi}$
			anisotropic, P_1 (B, BC) $< P_3$ (B)	$\tilde{\phi}^*$
	Heterogeneous	...	$P_1 < P_3$, B	$\tilde{\phi}$
			$P_1 < P_3$, B	$\tilde{\phi}^*$
Spheroids................................	Homogeneous	Oblate	isotropic, B	$\tilde{\phi}$
			anisotropic ($P_1 \neq P_3$), B	$\tilde{\phi}$
		Prolate	$P_1 = P_2 < P_3$ (B)	$\tilde{\phi}$
		Oblate	isotropic, B	$\tilde{\phi}^*$
			anisotropic $P_1 = P_2$, (B, BC), P_3 (B)	$\tilde{\phi}^*$
		Prolate	$P_1 = P_2$, (B, BC) $< P_3$ (B)	$\tilde{\phi}^*$
	Heterogeneous		isotropic (BC)	$\tilde{\phi}$
			anisotropic ($P_1 = P_2 \neq P_3$), BC	$\tilde{\phi}$
			isotropic, BC	$\tilde{\phi}^*$
			anisotropic $P_1 = P_2$ (B, BC), P_3 (BC)	$\tilde{\phi}^*$
Generalized Dedekind ellipsoids......	Homogeneous	Oblate	isotropic and anisotropic, B	$\tilde{\phi}$
		Prolate	anisotropic ($P_1 < P_3$), B	$\tilde{\phi}$
		Oblate	isotropic and anisotropic, B	$\tilde{\phi}^*$
		Prolate	anisotropic ($P_1 < P_3$), B	$\tilde{\phi}^*$
	Heterogeneous	...	...	

[a] Homogeneous and heterogeneous figures of equilibrium, isotropic or anisotropic cases are shown, and the different components of the pressure, barotropic (B) or baroclinic (BC) are noted, with differential rotation $\boldsymbol{Z}$, and a velocity field defined by the functional form $\tilde{\phi} = \tilde{\phi}(m^2)$; $\tilde{\phi}^*$ includes $\tilde{\phi}(m^2, \tilde{x}_3^2)$, $\tilde{\phi}(m^2, \tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2)$, $\tilde{\phi}(\tilde{x}_3^2)$.

Figure 0.4: Table 2

TABLE 3

EQUILIBRIUM CONFIGURATIONS WITH $\Omega \neq \boldsymbol{Z} \neq 0$[a]

Configuration	Density	Shape	Pressure	Velocity	$\boldsymbol{\Omega}$, $\boldsymbol{Z}$
Spheres	Homogeneous	...	anisotropic $P_1 < P_3$, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
			isotropic and anisotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
	Heterogeneous	...	anisotropic $P_1 < P_3$, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
			isotropic and anisotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
Spheroids...............................	Homogeneous	Oblate	isotropic, anisotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
		Prolate	anisotropic, $P_1 < P_3$, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
	Heterogeneous	Oblate	isotropic (BC) and anisotropic	$\tilde{\phi}$	↑↑
		Prolate		...	↑↓
		Oblate	isotropic (BC) and anisotropic	$\tilde{\phi}^*$	↑↑
		Prolate	$P_1 = P_2$(B, BC), P_3(BC)	...	↑↓
Generalized Riemann ellipsoids......	Homogeneous	Prolate	anisotropic ($P_1 < P_3$, $P_2 < P_3$), B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
		Oblate	isotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
		Oblate	anisotropic, P_1(B), P_2(BC), P_3(B)	$\tilde{\phi}^*$	↑↓
	Heterogeneous	...	...	...	...

[a] Homogeneous and heterogeneous figures of equilibrium in which the direction $\boldsymbol{\Omega}$ and $\boldsymbol{Z}$ are parallel or antiparallel and lie along the rotation axis x_3 ($\boldsymbol{\Omega}$, $\boldsymbol{Z}$ parallel = ↑↑, $\boldsymbol{\Omega}$, $\boldsymbol{Z}$ antiparallel = ↑↓); isotropic and anisotropic cases are noted, and the different components of the pressure, barotropic (B) or baroclinic (BC), are shown. The various forms of the velocity field $\tilde{\phi}$, $\tilde{\phi}^*$ are considered.

Figure 0.5: Table 3

other classical results can be recovered from our formulation, as the Dive's theorem (a stratified heterogeneous and rotating spheroid, without differential rotation, cannot be a barotrope) that can be generalized to ellipsoidal, anisotropic configurations. The Hamy theorem (a mass ellipsoidally stratified cannot have a uniform rotation) is confirmed also in the anisotropic case. All the results in Chandrasekhar (1987) for homogeneous configurations are extended to the anisotropic case. Our analysis is restricted to ellipsoidal or spheroidal configurations. Many other, non ellipsoidal, self-gravitating figures can exist as announced by Dive's theorem and must be studied by doing a further generalization of the virial theory to cover integration on non ellipsoidal volumes. In this case the form of the configuration is unknown.

0.3.3 The functional approach

The n-th order virial method is an integral approach to the problem of configuration of self-gravitating systems. A different approach, the functional method, begins directly with the differential hydrodynamical equation and tries to solve an essential question: how may be expressed the functional dependence among velocity potential, the density, the pressure, the gravitational potential, the vorticity, and the form of the configuration? Using a functional method based on the introduction of a velocity potential to solve the Euler, continuity and Poisson equations, a new analytic study of the equilibrium of self-gravitating rotating systems with a polytropic equation of state allows the formulation of the conditions of integrability. For the polytropic index n, and a state equation $P = \alpha\rho^{1+1/n}$, in the incompressible case $\nabla \cdot \vec{v} = 0$, we are able to find the conditions for solving the problem of the equilibrium of polytropic self-gravitating systems that rotate and have a non uniform vorticity. In the paper X, an analysis of the hydrodynamic equation for self-gravitating systems is presented from the point of view of functional analysis. We demonstrate that the basic quantities such as the density, the geometric form of the fluid, the pressure, the velocity profile and the vorticity ζ can be expressed as functionals of the velocity potential Ψ and of a function $g(z)$ of the coordinate z on the fluid rotation axis. By writing the hydrodynamical equations in terms of the velocity potential, it is possible to establish the integrability conditions according to which the pressure and the third component of the vorticity ζ_3 have the functional form $P = P(\rho)$ and $\zeta_3 = \zeta_3(\Psi)$; these conclusions suggest the form $\Psi = \Psi(x, y)$ for the velocity potential. In this way the steady state non linear hydrodynamic equation can be written as a functional equation of Ψ and we may propose some simple arguments to construct analytic solutions. In the special case $n = 1$, an explicit analytic solution can be found. The axisymmetric, linear and non homogeneous configurations can be revisited and we may describe how the properties of the configurations obtained compare to the well-known homogeneous ellipsoids of Maclaurin, Jacobi and Riemann and the discussion can be extended to related works using analytical and numerical tools in Newtonian gravity.

0.3.4 Recent Studies

The theory of self-gravitating rotating bodies is well known to be quite complicated even if many simplifications are assumed. The main problem comes from the fact that the equations are in general highly nonlinear and boundary conditions refer to the surface of the configuration, which is not known at the beginning but can be located after numerical studies only. Analytical results can be obtained for constant density, incompressible bodies and linear velocity profiles only, as discussed before. Nowadays numerically two-dimensional (2D) and 3D codes allow to study complicated scenarios in the temporal domain, but there is the need to get initial values for the equations. An analysis still in progress (Cherubini, Filippi, Ruffini, Sepulveda and Zuluaga 2007) uses the notion of velocity potential to formulate the hydrodynamical problem and gives the solution for arbitrary values of the polytropic index n based on the computational method of Eruguchi-Muller (1985) that can be implemented for arbitrary profiles of the differential angular velocity. More in detail, the equations are studied with a totally general functional form which interpolates a dimensionless angular velocity profile of gaussian type due to Stoeckly (see Tassoul's monograph) and rational polynomial one (Eriguchi-Muller), i.e.

$$\tilde{\Omega}(\xi) = \frac{e^{-\alpha\xi^2}}{1+\delta^2\xi^2}\,,$$

where ξ is the non dimensional cylindrical radius. While Stoeckly's model is typically used to describe fast non-uniformly rotating configuration close to fission, Eriguchi-Muller one instead is used for j-constant law related to constant specific angular momentum near to the axes of rotation. More in detail in this case for small values of the parameter the rotation law approaches the one of a constant specific angular momentum and the rotation law tend to give rigidly rotating configuration for larger values of the parameter. The plot of our generalized choice for the non dimensional angular velocity is shown in Figure 6. Introducing a computational grid as the one shown in Figure 7, after calibration of the code with well known results of the literature (James and Williams), we have been able to get some new plots of equilibrium configurations (figure 8), with associated mass diagrams (figure 9). The stability of these configurations, a nontrivial point in the theory, is still under exam.

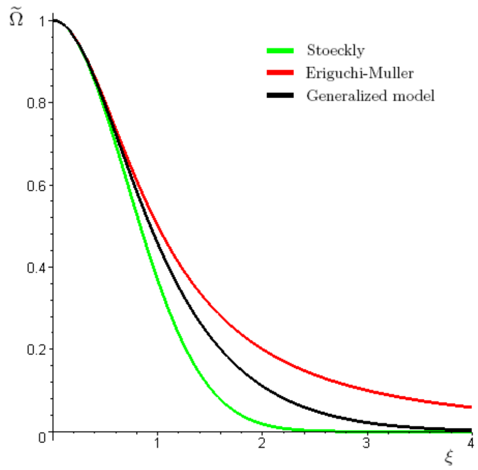

Figure 0.6: Dimensionless angular velocity versus dimensionless equatorial radius for the three models taken into exam.

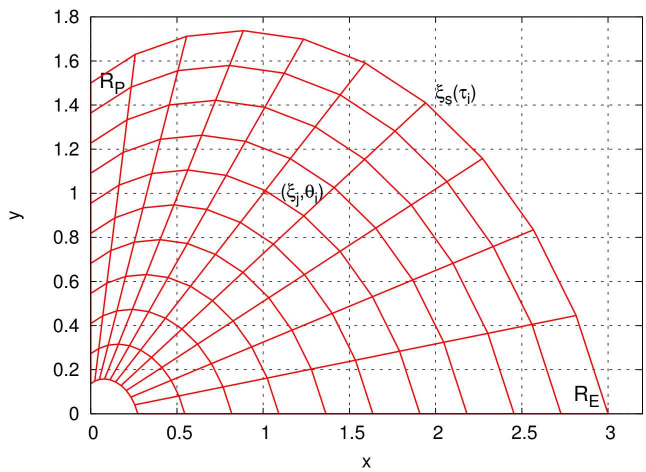

Figure 0.7: Numerical grid used in simulations.

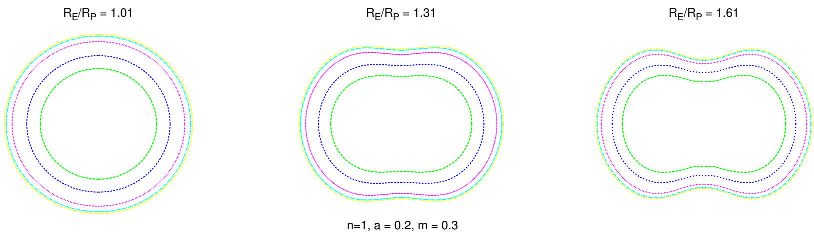

Figure 0.8: Isopicnics for $n = 1$ with $\alpha = 0.2$ and $\delta^2 = 0.3$ for increasing values of the ratio of equatorial and polar radii.

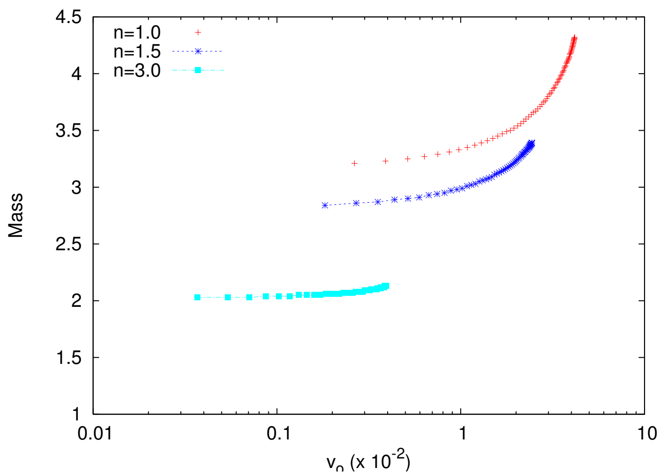

Figure 0.9: Mass Sequence for three values of n in our simulations.

In recent decades an analogy between General Relativity (GR) and other branches of Physics has been noticed. The central idea is that a number of systems of non relativistic condensed matter manifest a mathematical structure similar to the dynamics of fields in a curved manifold. In biology, as an example, it has been hypothesized that stationary scroll wave filaments in cardiac tissue describe a geodesic in a curved space whose metric is represented by the inverse diffusion tensor, with a dynamics close to cosmic strings in a curved universe. In fluid dynamics, in particular, this analogy becomes more striking: given a perfect barotropic and irrotational Newtonian fluid a study for the perturbations of the velocity potential with respect to a background exact solution has been performed. The equations satisfied by the perturbed quantities can be unified in a linear second order hyperbolic equation with non constant coefficients for the velocity potential only, while other quantities like density and pressure can be obtained by differentiation. This wave equation can be rewritten as describing the dynamics of a massless scalar field on a pseudo-Euclidean four dimensional Riemannian manifold. In fact, an induced "effective gravity" in the fluid arises, in which the local speed of sound plays the role of the speed of light in GR. However, while this geometric analogy is quite appealing, finding a relation with experiments is still problematic. All the performed studies in fact, have been essentially linked with superfluid physics experiments, for which viscous contributions can be neglected and perfect fluid approximation is valid, i.e. what Feynman defined as "dry water, although the quantized nature of these systems still poses some formal problems. It is then natural to look for other systems such that the perfect fluid approximation is still valid but quantization complications may be neglected, i.e. a purely classical perfect fluid. Self-gravitating classical fluids and gaseous masses, as described by Euler's equations, appear as the best candidates to satisfy this requirement. For these systems however, one must solve a coupled problem of hydrodynamics and gravitation which is absent in acoustic analogy literature. In all the existing studies on analog models in fact, the contribution of gravitational field was assumed to be externally fixed and practically constant, with no back-reaction. On the other hand in the self-gravitating problem, one must take into account the effect of gravitational back-reaction, present both in the exact background solution as well as in its perturbations. This effect of coupled acoustic disturbances which travel at finite speed and the gravitational field which rearranges itself instantaneously has never been analyzed before using analog geometry models and has been examined by ICRA scientists, focusing in particular on the classical problem of self-gravitating polytropes. Polytropic systems, as discussed before, play an important role in galactic dynamics as well as in the theory of stellar structure and evolution. For these systems the pressure is simply related to the density, while remaining independent of the temperature. Such a choice has specific physical grounds: in the case of a degenerate electron gas, central in the theory of stars, it is well known, as an example, that the pressure and density behave as $\rho \sim p^{\frac{3}{5}}$. Assuming that such a relation exists for other states of the star one has a general relation of polytropic form $p \propto \rho^{1+\frac{1}{n}}$ with a general polytropic index n. Regarding galactic dynamics on the other hand, spherical polytropes are globally stable solutions to the collisionless Boltzmann, or Vlasov, equation of galactic dynamics. While in galactic dynamics n must be larger than $\frac{1}{2}$, in the case of the theory of stellar structure quantity n ranges in $0 \leq n < +\infty$. For selfgravitating polytropes the Lane-Emden equation is central: there exist very few analytic solutions and only for selected values of polytropic index and typically for non rotating spherically symmetric configurations or for incompressible fluids only. In more general cases, while uniformly rotating polytropes have analytic solutions in the case $n = 1$ only (assuming truncations of power series), solutions for the other non spherical configurations are obtained by numerical techniques only. The acoustic analogy has been extended here to these systems. Skipping the mathematical details we list here the equations satisfied by the velocity (ψ_1) and gravitational (Φ_1) perturbed potentials. Let us form the following symmetric 4×4 matrix

$$f^{00} = -\frac{\rho_0}{c^2}, \quad f^{0i} = -\frac{\rho_0 v_0^i}{c^2}, \quad f^{ij} = \rho_0(\delta^{ij} - \frac{v_0^i v_0^i}{c^2}) \tag{0.1}$$

where Greek indices run from 0 to 3, while Roman indices run from 1 to 3. Quantity ρ_0 is the background density function satisfying the exact nonlinear self-gravitating problem. Here the local speed of sound $c^{-2} \equiv \frac{\partial \rho}{\partial p}$, computed on the background solution, has been introduced (p is the pressure). Then, using $(1+3)$–dimensional space-time coordinates $x^\mu \equiv (t; x^i)$ we get

$$-\partial_\mu (f^{\mu\nu}\, \partial_\nu \psi_1) = \partial_t \left(\frac{\partial \rho}{\partial p}\, \rho_0 \Phi_1 \right) + \nabla \cdot \left(\frac{\partial \rho}{\partial p}\, \rho_0 \Phi_1 \vec{v}_0 \right), \tag{0.2}$$

$$\nabla^2 \Phi_1 + 4\pi G \frac{\partial \rho}{\partial p}\, \rho_0 \Phi_1 = 4\pi G \frac{\partial \rho}{\partial p}\, \rho_0 \left(\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1 \right). \tag{0.3}$$

Let us now define $\sqrt{-g}\; g^{\mu\nu} = f^{\mu\nu}$ which leads to $\sqrt{-g} = \frac{\rho_0^2}{c}$. We get the acoustic line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ with metric tensor and its inverse given by

$$g^{\mu\nu} \equiv \frac{1}{\rho_0 c} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2 \delta^{ij} - v_0^i v_0^j) \end{bmatrix} \quad , \quad g_{\mu\nu} \equiv \frac{\rho_0}{c} \begin{bmatrix} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta_{ij} \end{bmatrix}. \tag{0.4}$$

Note that ds^2 has not the dimension of a length, as it happens in GR instead. We can now rewrite the final system as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\psi_1\right) = -\frac{c}{\rho_0^2}\left[\partial_t\left(\frac{1}{c^2}\,\rho_0\Phi_1\right) + \nabla\cdot\left(\frac{1}{c^2}\,\rho_0\Phi_1\vec{v}_0\right)\right] \tag{0.5}$$

$$\left(\nabla^2 + \frac{4\pi G\rho_0}{c^2}\right)\Phi_1 = \frac{4\pi G\rho_0}{c^2}\,\left(\partial_t\psi_1 + \vec{v}_0\cdot\nabla\psi_1\right)\,. \tag{0.6}$$

The minus sign on the right hand side of first equation above comes from our choice of signature $(-,+,+,+)$.

Introducing the magnitude of a generalized Jeans' wavevector $k_J = \sqrt{\frac{4\pi G\rho_0}{c^2}}$, associated with Jeans wavelength by $\lambda_J = 2\pi/k_J$, we can finally write (∇^2 is standard Laplace operator of Euclidean space in three dimensions):

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\psi_1\right) = -\frac{c}{\rho_0^2}\left[\partial_t\left(\frac{\rho_0}{c^2}\,\Phi_1\right) + \nabla\cdot\left(\frac{\rho_0}{c^2}\,\Phi_1\vec{v}_0\right)\right] \tag{0.7}$$

$$\left[\nabla^2 + k_J^2\right]\Phi_1 = k_J^2\left(\partial_t\psi_1 + \vec{v}_0\cdot\nabla\psi_1\right)\,, \tag{0.8}$$

which mixes the hydrodynamical (with finite speed) and gravitational (instantaneous) problems through first order time and space partial derivatives of the fields. The last two equations can also be rewritten as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\psi_1\right) = -\frac{\rho_0}{\sqrt{-g}}\frac{D^{(0)}}{\partial t}\left(\frac{\Phi_1}{c^2}\right) \tag{0.9}$$

$$\left[\nabla^2 + k_J^2\right]\Phi_1 = k_J^2\frac{D^{(0)}}{\partial t}\psi_1\,, \tag{0.10}$$

where

$$\frac{D^{(0)}}{\partial t} = \partial_t + \vec{v}_0\cdot\nabla\,. \tag{0.11}$$

If Φ_1 vanishes (no gravitational back-reaction) and $G\to 0$, one recovers known results in absence of gravitational back-reaction. These equations have been studied in detail for the spherical cases solutions of Lane-Emden's equation. In some cases the PDE's have been numerically analysed using finite element techniques (Bini, Cherubini and Filippi 2008).

0.4 Hamiltonian Dynamical Systems and Galactic Dynamics

0.4.1 Near-integrable dynamics and galactic structures.

The study of self-gravitating stellar systems has provided in several occasions important hints to develop powerful tools of analytical mechanics. We may cite the ideas of Jeans (1929) about the relevance of conserved quantities in describing the phase-space structure of large N-body systems and his introduction of the concept of *isolating integral*. Later important contribution are those of Hénon & Heiles, (1964), where a paradigmatic example of non-integrable system derived from a simple galactic model was introduced and of Hori (1966), where the theory of Lie transform was introduced in the field of canonical perturbation theory and Hamiltonian normal forms. These and other cues contributed to set up a body of methods and techniques to analyze the near integrable and chaotic regimes of the dynamics of generic non-integrable systems. For a general overview see Boccaletti & Pucacco (Theory of Orbits, Vol. I, 1996; Vol. II, 1999).

On the other side, the payback from analytical mechanics to galactic dynamics has not been as systematic and productive as it could be. The main line of research has been that pursued by Contopoulos (2002) who applied a direct approach to compute approximate forms of effective integrals of motion. The method of Hori (1966), subsequently developed by several other people (Deprit, 1969; Dragt & Finn, 1976; Efthymiopoulos et al. 2004; Finn, 1984; Giorgilli, 2002), has several technical advantages and has gradually become a standard tool in the perturbation theory of Hamiltonian dynamical systems (Boccaletti & Pucacco, 1999).

In this respect we have applied the Lie transform method to construct Hamiltonian normal forms of perturbed oscillators and investigate the orbit structure of potentials of interest in galactic dynamics (Belmonte, Boccaletti and Pucacco, 2006, 2007a, 2007b). The approach allows us to gather several informations concerning the near integrable dynamics below the stochasticity threshold (if any) of the system. Being a completely analytical approach, it has the fundamental value of a complete generality which provides simple recipes to explore the structure of the backbone of phase-space. Exploiting asymptotic properties of the series constructed via the normal form, one can also get quantitative predictions extending the validity of the approach well beyond the radius of convergence of the initial series expansion of the perturbed potential. We have shown how to exploit resonant normal forms to extract information on several aspects of the dynamics of the the logarithmic and the Schwarzschild potential. In particular, using energy and ellipticity as parameters, we have computed the instability thresholds of axial orbits, bifurcation values of low-order boxlets and phase-space fractions pertaining to the families around them. We have also shown how to infer something about the singular limit of the potential.

As in any analytical approach, this method has the virtue of embodying in (more or less) compact formulas simple rules to compute specific properties, giving a general overview of the behavior of the system. In the case in which a non-integrable system has a regular behavior in a large portion of its phase space, a very conservative strategy like the one adopted in our work provides sufficient qualitative and quantitative agreement with other more accurate but less general approaches. In our view, the most relevant limitation of this approach, common to all perturbation methods, comes from the intrinsic structure of the single-resonance normal form. The usual feeling about the problems posed by non-integrable dynamics is in general grounded on trying to cope with the interaction of (several) resonances. Each normal form is instead able to correctly describe only one resonance at the time. However, we remark that the regular dynamics of a non-integrable system can be imagined as a superposition of very weakly interacting resonances. If we are not interested in the thin stochastic layers in the regular regime, each portion of phase space associated with a given resonance has a fairly good alias in the corresponding normal form. An important subject of investigation would therefore be that of including weak interactions in a sort of higher order perturbation theory. For the time being, there are two natural lines of developement of this work: 1. to extend the analysis to cuspy potentials and/or central 'black holes'; 2. to apply this normalization algorithm to three degrees of freedom systems.

0.4.2 Geometric approach to the integrability of Hamiltonian systems

Integrable systems are still very useful benchmark to understand the properties of general non-integrable systems, not only for their relevance as starting points for perturbation theory. The topology of invariant surfaces in the phase-space of integrable systems can be highly non-trivial and give rise to complex phenomena (high-order resonances, monodromy, etc.) still not completely understood.

We have started our work by investigating quadratic integrals at fixed and arbitrary energy with a unified geometric approach (Rosquist & Pucacco, 1995; Boccaletti & Pucacco, 1997) solving the Killing tensor equations for 2nd-rank Killing tensors on 2-dim. conformally-Euclidean spaces. In Pucacco & Rosquist (2003) these systems have been shown to be endowed of a bi-Hamiltonian structure and in Pucacco & Rosquist (2004) a class of systems separable at fixed energy has been shown to be non-integrable in Poincarè sense. The case of cubic and quartic integrals of motion, respectively associated to 3rd and 4th-rank Killing tensors, has been investigated in Karlovini &

Rosquist (2000) and in Karlovini, Pucacco, Rosquist and Samuelsson (2002). In Pucacco (2004) and Pucacco & Rosquist (2005a) we have obtained new classes of integrable Hamiltonian system with vector potentials and in Pucacco & Rosquist (2005b) we have provided a general treatment of weakly integrable systems. Recently, Pucacco & Rosquist (2007), we have presented the theory of separability over 2-dimensional pseudo-Riemannian manifolds (1+1-separable metrics).

We plan to investigate the existence of higher-order polynomial integrals on general compact surfaces with the topology of the sphere and the torus and to apply the results about pseudo-Riemannian systems to treat integrable time-dependent Hamiltonian systems.

0.4.3 Stochasticity in galactic dynamics

The issue of stochasticity in galactic dynamics received in the past few years a certain degree of attention (Contopoulos, 2002). A possible approach can be that of exploiting methods of the theory of dynamical systems (Pucacco, 1992) to study instability and relaxation. In fact, in a self-gravitating N-body system, the compactness of phase space can be introduced by force. Moreover, the violently varying sign and amount of the curvatures could provide effects close to ergodicity. However, in addition to the fact that there is no strict proof neither of mixing nor of ergodicity, there is a fundamental difficulty connected with numerical simulations indicating an instability time much smaller than that predicted by the Gurzadyan & Savvidy (1996) approach (Goodman, Heggie and Hut, 1993; Hut & Heggie, 2002).

If effects related to the curvature could be envisaged, they are more probably associated to 'violent relaxation' on a time scale of the order of the dynamical time (Boccaletti, Pucacco and Ruffini, 1991; Cipriani and Pucacco, 1993, 1994). We remark that there are still many open issues in this area: the qualitative picture of the exponential instability given by Hut and collaborators is itself quite uncertain and its relation to the standard Chandrasekhar relaxation still obscure. A thorough general picture is still lacking and it cannot be excluded that the geometric approach based on geodesic flows could help. However, the results obtained up to now are very far from the target.

Aim of this project is to provide the evidence of effects produced by *stochasticity* in elliptical galaxies. The fact that irregular orbits in non-integrable triaxial potentials may play an important role in determining the structure and evolution of elliptical galaxies was already realized by Goodman and Schwarzschild (1981) and studied then in many other papers (Binney, 1982; Schwarzschild, 1993). However, in the past years, the prevailing line of thinking maintained the view that effective regularity is more or less the rule: Gerhard (1985), for example, in his study of perturbed Stäckel potentials, was led to deny a relevance to stochasticity because, although it was produced by every non-integrable perturbation, its amount was always very small in the case of perturbations consistent with observations. Hence, the emergence of appreciable stochasticity only in the presence of unreasonable perturbations, was again used as an argument in favour of regularity. But, if the nature of perturbed potentials with near-homogeneous cores is to be "reluctant" to produce much chaos, as soon as one examines singular or simply higher concentrated systems, the fraction of stochastic orbits becomes suddenly non-negligible (Miralda-Escudé and Schwarzschild, 1989). Merritt and Valluri (1996) have performed high-accuracy numerical integrations of orbits in the potential Φ_{m_0} and have detected a two-peak distribution in the Liapounov numbers, with an increasing amount of positive values at smaller m_0. The origin of the breaking of integrability is the central singularity which induces a chaotic scattering on the box orbits, which pass arbitrarily close to the center, with a characteristic Liapounov time-scale:

$$\tau_{\mathrm{L}} \sim 3 \div 5\, \tau_{\mathrm{d}}. \tag{12}$$

It is superfluos to remark on the importance of this result: box orbits constitute the *bulk* of a triaxial galaxy and, due to the short time-scale on which stochasticity is expected to develop, we must necessarily reconsider the picture of elliptical galaxy modelling based on integrable potentials. It is cleat that many new aspects of the theory emerge and need a deep analysis in the light of these findings. One important consequence is just on the issue of the rotational support of elliptical galaxies: in fact, in objects more flattened than indicated by the amount of rotation, anisotropic stresses contribute to the support. Since they are originated by non-classical integrals, their permanence over the life-time of the galaxy implies that the system must settle from its origin in a state of integrable mean potential. Otherwise, in the stochastic regions of the orbit space of a generic potential, even if some effective integral is approximately conserved along the orbits, the short instability time rapidly remove the corresponding anisotropy in velocity space.

Consider for simplicity the axially symmetric case (Boccaletti & Pucacco,1998). The effective potential is

$$\Phi_{\mathrm{eff}}(r,z) = \Phi(r,z) + \frac{L_z^2}{2r^2}, \quad r = \sqrt{x^2+y^2} \tag{13}$$

and if Φ is separable in elliptical coordinates in the meridional plane, for example the potential coming from the spheroidal specialization of the density distribution (8), ("Kuzmin's spheroid")

$$\rho(r,z) = \rho_0 \left(1 + \frac{r^2}{a^2} + \frac{z^2}{c^2}\right)^{-2}, \tag{14}$$

also Φ_{eff} is separable. If Φ is a generic non-integrable potential, stochasticity instead appears in many fashions in so far the relative fraction of irregular orbits depends on several geometric and dynamical parameters. Recall that the celebrated paper by Hénon and Heiles (1964) explores just the nature of the motion of a generic $\Phi_{\rm eff}$ around its minimum; in fact, if $r = r_{\rm min}$, $z = 0$ is the location of this minimum, we can write the expansion up to the third order

$$\Phi_{\rm eff} \simeq \Phi_{\rm min} + \frac{1}{2}A(r - r_m)^2 + \frac{1}{2}Bz^2 - C(r - r_m)z^2 + \frac{1}{3}D(r - r_m)^3, \qquad (15)$$

when $\Phi_{\rm min} = \Phi_{\rm eff}(r_m, 0)$ and A, B, C, D are positive constants. After suitable rescaling, we recognize in expression (15) that of the Hénon-Heiles potential. We have therefore that there is a low-energy near integrable regime in which every system behaves in a way which is practically indistinguishable from the exact integrability. Above a certain threshold, which depends in a complex way on the parameters but that is generically well below the average stellar energy, stochasticity dominates.

Coming back to the issue of rotational support, it was already proposed by the author (Pucacco, 1992) that secular evolution in dwarf and normal elliptical galaxies, accompanied by a progressive reduction of anisotropy, accounts for the distribution of data-points in the $[(v/<\sigma>)^*$ versus $M]$-plane. In the light of what said above, the secular evolution is linked to a gradual disappearance of orbits effectively constrained by quasi-conserved phase-space functions. Using the data collected by accurate observavions with HST, we can perform a more reliable test of the scenario proposed above.

Merritt and Valluri (1996), besides to compute short-time estimates of Lyapunov numbers, have also attempted the assesment of the "mixing-time" of their systems, that is the time-scale required to reach a coarse-grained steady-state of the ensemble of the irregular orbits. Their conclusion is that in a time of the order $\tau_{mix} \sim 10 - 100\tau_L$ the system relaxes to an invariant distribution. Despite its large uncertainty, this result is in line with many other works in pointing out that the Lyapunov time is only the instability time-scale of single orbits. The time required so that instability manifests itself in the structure of the whole system is much longer. This fact helps to clarify laso some obscure point in the theory of the exponential instability of N-body systems. Goodman, Heggie and Hut (1993) and others find that the instability time-scale is of the order of τ_D, but their explanation of its relation with the classical relaxation time is not satisfactory: in particular, it is not clear why, at the end of the linear regime of the growth of perturbations, classical relaxation would ensue. In the light of the above discussion, we propose instead that the N-body system, much in the same way of chaotic ensembles in a non-integrable potential, attains a "mixed" (equilibrium) state after a $\tau_{mix} \sim 10^k\tau_L$, where the order of magnitude k is possibly dependent on the number of stars.

0.5 Publications (2008)

Refereed journals

- D. Bini, Cherubini and S. Filippi,
  "Effective geometries in self-gravitating polytropes",
  Phys. Rev. D **78**, 064024 (2008).
  Abstract: Perturbations of a perfect barotropic and irrotational Newtonian self-gravitating fluid are studied using a generalization of the so-called effective geometry formalism. The case of polytropic spherical stars, as described by the Lane-Emden equation, is studied in detail in the known cases of existing explicit solutions. The present formulation gives a natural scenario in which the acoustic analogy has relevance for both stellar and galactic dynamics.

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