# Quantum Gravity, Quantum Fields and Unification Theories

# Contents

# 1. Topics

**Quantum Gravity**

Canonical Quantum Gravity without the time gauge

The time gauge problem in the path integral formalism

Quantum Mixmaster

The problem of time in Quantum Gravity

Evolutionary Quantum Gravity

Loop Quantum Cosmology

Polymer Quantum Cosmology

Minisuperspace and Generalized Uncertainty Principle

Lorentz Gauge Theory

**Quantum Fields on Background**

Dirac equation on a curved spaces and classical trajectories

Quantum Fields in accelerated systems

Quantum Fields in Black Hole Space-Time

**Unification Theories**

5-Dimensional Kaluza-Klein model

Spinning particles in Kaluza-Klein

Generalized 5-Dimensional Theories

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- Marco Valerio Battisti
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# 3. Brief Description of Quantum Gravity

## 3.1. Canonical Quantum Gravity without the time gauge

In section "Canonical Quantum Gravity without the time gauge" our investigation has been focused on describing quantum gravitational degrees of freedom, without fixing the time gauge. A Quantum Gravity theory is expected to provide a discrete structure for the space-time geometry. In this respect, it is worth noting the achievement of Loop Quantum Gravity of a discrete structure for spatial geometrical operators, at least on a kinematical level. However, this formulation is based on fixing the so-called time-gauge condition, *i.e.*, the choice of the vier-bein is adapted to the space-time splitting, such that the time-like vector $e_0$ is normal to spatial hypersurfaces. This way, boosts are frozen out and the investigation on the behavior of quantum geometrical operators in different Lorentz frames is highly non-trivial [6].

The people involved in this line of research are Francesco Cianfrani and Giovanni Montani.

## 3.2. The time gauge problem in the path integral formalism

In section "The time gauge problem in the path integral formalism" we turn our attention to general relativity expressed in first-order formalism, in order to investigate [10] the physicality condition for the states of the gravitational field arising from BRST invariance of the theory, following the same procedure employed for non-Abelian gauge theories. In this procedure we will intentionally avoid to use canonical quantization methods. We are to determine a physical state condition on quantum states without thinking of classical Hamiltonian constraints in order to compare, at the end of our calculation, our physicality condition required by BRST symmetry and derived with path-integral methods with the one obtained using the Dirac quantization method employed within Ashtekar's canonical formulation.

The people involved in this line of research are Michele Castellana and

Giovanni Montani.

## 3.3. The problem of time in quantum gravity

In section "The problem of time in quantum gravity" the so called Kučhar-Brown mechanism for a perfect fluid in the Schutz velocity potential representation is analyzed [11],[18]. When treated with the canonical formalism, this model turns out to be a constrained system, with numerous secondary and tertiary constraints, which require some restrictions on the phase space variables. We find, as a result, that the Schutz model is a good matter clock when coupled with General Relativity.

The people involved in this line of research are Simone Zonetti and Giovanni Montani.

## 3.4. Evolutionary Quantum Gravity

In section "Evolutionary Quantum Gravity" we review the fundamental aspects of the so-called evolutionary quantum gravity.

An evolutionary paradigm is inferred by restricting the covariance principle within a Gaussian gauge and the corresponding implications for a generic cosmological scenario are investigated both on a classical and a quantum level [13]. A dualism between time and the reference frame fixing is then inferred.

The people involved in this line of research are M.Valerio Battisti, Francesco Cianfrani and Giovanni Montani (past collaborator: Simone Mercuri).

## 3.5. Minisuperspace and Generalized Uncertainty Principle

In section "Minisuperspace and Generalized Uncertainty Principle" we explain some results obtained in a recent approach to quantum cosmology, in which the notion of a fundamental scale naturally appears. This scheme realizes in quantizing a cosmological model by using a deformed Heisenberg algebra, which reproduces a Generalized Uncertainty Principle as arises from studies on string theory. We find that the classical cosmological singularity of the Taub model is solved by this approach in the sense that the quantum Universe can be regarded as probabilistically singularity-free [4], [12]. Moreover, the Taub GUP wave packets provide the right behavior in the establishment of a quasi-isotropic configuration for the Universe. The Bianchi I, II and IX cosmological models are also analyzed in the GUP framework and the ordinary dynamics appears to be deeply modified and, in particular, the

Mixmaster Universe can be still considered a chaotic system. Furthermore, in the context of a deformed Heisenberg minisuperspace algebra framework, a deep phenomenological relation between loop quantum cosmology, brane cosmology and the $\kappa$-Poincaré scheme is obtained.

The people involved in this line of research are Marco Valerio Battisti and Giovanni Montani.

## 3.6. Quantum isotropization mechanism

In section “Quantum isotropization mechanism” we explain some results of a research line in which a wave function of the inhomogeneous Mixmaster Universe which has a meaningful probabilistic interpretation in agreement with the Copenhagen school is obtained. Our result is that this wave function of the Universe is spread over all values of anisotropy near the cosmological singularity but, when the radius of the Universe grows, it is asymptotically peaked around the isotropic configuration. Therefore, the FRW cosmological model is naturally the privileged state when a sufficient large volume of the Universe is taken into account and a semi-classical isotropization mechanism for the Universe naturally arises.

The people involved in this line of research are Marco Valerio Battisti, Riccardo Belvedere and Giovanni Montani.

## 3.7. Polymer quantum cosmology

In section “Polymer quantum cosmology” we explain some results obtained applying the polymer quantization paradigm to the Taub Universe. The polymer approach is based on a inequivalent representation of the Weyl algebra and its physical relevance arises from consideration on the mechanical-system-limit of the loop quantum gravity theory. As a result of our analysis, the cosmological singularity is not probabilistically removed, as in the GUP approach, since the dynamics of the wave packets is not able to stop the evolution toward the classical singularity.

The people involved in this line of research are Marco Valerio Battisti, Orchidea Maria Lecian and Giovanni Montani.

## 3.8. Lorentz Gauge Theory

In section “Lorentz Gauge Theory” we implement a non-standard gauge theory of the local Lorentz group both in flat and in curved space-time, based on diffeomorphism induced Lorentz transformation and the ambiguity which emerges in the transformation laws of the usual spin connection and spinors.

We propose a model [8][22] to analyze the interaction of a 4-spinor with the new connections of the Lorentz group (addressed in flat space). This scheme exhibits strong analogies with the electro-magnetic case and the so-called Pauli equation. The analysis of this interaction is devoted to find out anomalous selection rules for a hydrogen-like model and, of course, energy-level splits. According to standard quantum mechanics new energy levels are present, but no new transitions arise.

The peoples involved in this line of research are Giovanni Montani, Nakia Carlevaro and Orchidea M. Lecian (past collaborator: Simone Mercuri).

# 4. Brief Description of Quantum Fields on Classical Background

## 4.1. Dirac equation on a curved spaces and classical trajectories

In section "Dirac equation on a curved space-time and classical trajectories", the interaction between geometry and internal spinor-like degrees of freedom has been investigated with the aim to infer the analogous of Papapetrou equations for a quantum spin. This task has been approached by an eikonal approximation, and a localization hypothesis along the integral curve of the momentum [25]. Hence, a dispersion relation has been recovered starting from the squared Dirac equation and by virtue of an integration on spatial coordinates. It is worth noting the emergence of a Papapetrou-like interaction between the Riemann tensor and a tensor, which characterizes the internal structure of spinors.

The persons involved in this research line are Giovanni Montani and Francesco Cianfrani.

# 5. Brief Description of Unification Theories

## 5.1. 5 Dimensional Kaluza-Klein Model

In section "5 Dimensional Kaluza-Klein model", we analyze 5-dimensional Kaluza - Klein schemes. Within the unification picture provided by the Kaluza Klein (KK) theory, the 5- Dimensional (5D) model is the simplest one and the starting point for the investigation of the breaking of multidimensional gravity into the usual gravity plus Yang-Mills fields. It is characterized by an abelian structure; indeed, it provides the coupling between gravity, a U(1) gauge field and an extra scalar field. If the scalar field it is assumed to be constant from the beginning, the 5D model reproduces exactly the Einstein-Maxwell theory in vacuum. The research line about this topic is focused on following points: Hamiltonian formulation [31], test-particles dynamics, coupling with matter, geodesic deviation, search of cosmological and spherical solutions for the KK model with source.

The people involved in this research line are Valentino Lacquaniti, Giovanni Montani and Francesco Vietri.

# 6. Selected Publications before 2005

## 6.1. Quantum Gravity

[1] G. Montani, Canonical Quantization of Gravity without "Frozen Formalism", *Nucl. Phys. B*, **634**, 370 (2002).

We write down a quantum gravity equation which generalizes the WheelerDeWitt one in view of including a time dependence in the wave functional. The obtained equation provides a consistent canonical quantization of the 3-geometries resulting from a "gauge-fixing" (3 + 1)-slicing of the spacetime. Our leading idea relies on a criticism to the possibility that, in a quantum spacetime, the notion of a (3+1)-slicing formalism (underlying the WheelerDeWitt approach) has yet a precise physical meaning. As solution to this problem we propose of adding to the gravity-matter action the so-called kinematical action (indeed in its reduced form, as implemented in the quantum regime), and then we impose the new quantum constraints. As consequence of this revised approach, the quantization procedure of the 3-geometries takes place in a fixed reference frame and the wave functional acquires a time evolution along a one-parameter family of spatial hypersurfaces filling the spacetime. We show how the states of the new quantum dynamics can be arranged into an Hilbert space, whose associated inner product induces a conserved probability notion for the 3-geometries. Finally, since the constraints we quantize violate the classical symmetries (i.e., the vanishing nature of the super-Hamiltonian), then a key result is to find a (non-physical) restriction on the initial wave functional phase, ensuring that general relativity outcomes when taking the appropriate classical limit. However, we propose a physical interpretation of the kinematical variables which, based on the analogy with the so-called Gaussian reference fluid, makes allowance even for such classical symmetry violation.

[2] G. Montani, Cosmological Issues for revised canonical quantum gravity, *Int. J. Mod. Phys. D*, **12**, 8, 1445 (2003)

In a recent work we presented a reformulation of the canonical quantum gravity, based on adding the so-called kinematical term to the gravity-matter action. This revised approach leads to a self-consistent canonical quantization of the 3-geometries, which referred to the external time as provided via the added term. Here, we show how the kinematical term can be interpreted in terms of a non- relativistic dust fluid which plaies the role of a "real clock" for the quantum gravity theory, and, in the WKB limit of a cosmological problem, makes account for a dark matter component which, at present time, could play a dynamical role.

[3] G. Aprea, G. Montani and R. Ruffini, Test particles behavior in the framework of a Lagrangian geometric theory with propagating torsion, *Int. J. Mod. Phys. D*, **12**, 10, 1875 (2003)

Working in the Lagrangian framework, we develop a geometric theory in vacuum with propagating torsion; the antisymmetric and trace parts of the torsion tensor, considered as derived from local potential fields, are taken and, using the minimal action principle, their field equations are calculated. Actually these will show themselves to be just equations for propagating waves giving torsion a behavior similar to that of metric which, as known, propagates through gravitational waves. Then we establish a principle of minimal substitution to derive test particles equation of motion, obtaining, as result, that they move along autoparallels. We then calculate the analogous of the geodesic deviation for these trajectories and analyze their behavior in the nonrelativistic limit, showing that the torsion trace potential $\phi$ has a phenomenology which is indistinguishable from that of the gravitational Newtonian field; in this way we also give a reason for why there have never been evidence for it.

[4] G. Imponente and G. Montani, Mixmaster Chaoticity as Semiclassical Limit of the Canonical Quantum Dynamics, *Int. J. Mod. Phys. D*, **12**(6), 977-984 (2003).

Within a cosmological framework, we provide a Hamiltonian analysis of the Mixmaster Universe dynamics on the base of a standard Arnowitt-Deser-Misner approach, showing how the chaotic behavior characterizing the evolution of the system near the cosmological singularity can be obtained as the semiclassical limit of the canonical quantization of the model in the same dynamical representation. The relation between this intrinsic chaotic behavior and the indeterministic quantum dynamics is inferred through the coincidence between the microcanonical probability distribution and the semiclassical quantum one.

[5] S. Mercuri and G. Montani, Revised Canonical Quantum Gravity via the Frame Fixing, *Int. J. Mod. Phys. D*, **13**, 165 (2004).

We present a new reformulation of the canonical quantum geometrodynamics, which allows one to overcome the fundamental problem of

the frozen formalism and, therefore, to construct an appropriate Hilbert space associate to the solution of the restated dynamics. More precisely, to remove the ambiguity contained in the Wheeler-DeWitt approach, with respect to the possibility of a (3 + 1)-splitting when space-time is in a quantum regime, we fix the reference frame (i.e. the lapse function and the shift vector) by introducing the so-called kinematical action. As a consequence the new super-Hamiltonian constraint becomes a parabolic one and we arrive to a Schrödingerlike approach for the quantum dynamics. In the semiclassical limit our theory provides General Relativity in the presence of an additional energy-momentum density contribution coming from non-zero eigenvalues of the Hamiltonian constraints. The interpretation of these new contributions comes out in natural way that soon as it is recognized that the kinematical action can be recasted in such a way that it describes a pressureless, but, in general, non-geodesic perfect fluid.

[6] S. Mercuri and G. Montani, Dualism between physical frames and time in quantum gravity, *Mod. Phys. Lett. A*, **19**, 20, 1519 (2004).

In this work we present a discussion of the existing links between the procedures of endowing the quantum gravity with a real time and of including in the theory a physical reference frame. More precisely, as a first step, we develop the canonical quantum dynamics, starting from the Einstein equations in presence of a dust fluid and arrive at a Schrödinger evolution. Then, by fixing the lapse function in the path-integral of gravity, we get a Schrödinger quantum dynamics, of which eigenvalues problem provides the appearance of a dust fluid in the classical limit. The main issue of our analysis is to claim that a theory, in which the time displacement invariance, on a quantum level, is broken, is indistinguishable from a theory for which this symmetry holds, but a real reference fluid is include.

[7] G. Montani, Minisuperspace model for revised canonical quantum gravity, *Int. J. Mod. Phys. D*, **13**, 8, 1703 (2004)

We present a reformulation of the canonical quantization of gravity, as referred to the minisuperspace; the new approach is based on fixing a Gaussian (or synchronous) reference frame and then quantizing the system via the reconstruction of a suitable constraint; then the quantum dynamics is re-stated in a generic coordinates system and it becomes dependent on the lapse function. The analysis follows a parallelism with the case of the non-relativistic particle and leads to the minisuperspace implementation of the so-called kinematical action as proposed in Ref. 1 (here almost coinciding also with the approach presented in Ref. 2). The new constraint leads to a Schrödinger equation for the system, i.e. to nonvanishing eigenvalues for the super-Hamiltonian operator; the

physical interpretation of this feature relies on the appearance of a "dust fluid" (non-positive definite) energy density, i.e. a kind of "materialization" of the reference frame. As an example of minisuperspace model, we consider a Bianchi type IX Universe, for which some dynamical implications of the revised canonical quantum gravity are discussed. We also show how, on the classical limit, the presence of the dust fluid can have relevant cosmological issues. Finally we upgrade our analysis by its extension to the generic cosmological solution, which is performed in the so-called long-wavelength approximation. In fact, near the Big-Bang, we can neglect the spatial gradients of the dynamical variables and proceed to implement, in each space point, the same minisuperspace paradigm valid for the Bianchi IX model.

[8] G.V. Vereshchagin, On stability of simplest nonsingular inflationary cosmological models within general relativity and gauge theories of gravity, *Int. J. Mod. Phys. D*, **13**, 695 (2004).

In this paper we provide approximate analytical analysis of stability of nonsingular inflationary chaotic-type cosmological models. Initial conditions for nonsingular solutions at the bounce correspond to dominance of potential part of the energy density of the scalar field over its kinetic part both within general relativity and gauge theories of gravity. Moreover, scalar field at the bounce exceeds the planckian value and on expansion stage these models correspond to chaotic inflation. Such solutions can be well approximated by explicitly solvable model with constant effective potential (cosmological term) and massless scalar field during the bounce and on stages of quasi-exponential contraction and expansion. Perturbative analysis shows that nonsingular inflationary solutions are exponentially unstable during contraction stage. This result is compared with numerical calculations.

[9] G.V. Vereshchagin, Qualitative Approach to Semi-Classical Loop Quantum Cosmology, *JCAP*, **0407**, 013 (2004).

Recently the mechanism was found which allows avoidance of the cosmological singularity within the semi-classical formulation of Loop Quantum Gravity. Numerical studies show that the presence of self-interaction potential of the scalar field allows generation of initial conditions for successful slow-roll inflation. In this paper qualitative analysis of dynamical system, corresponding to cosmological equations of Loop Quantum Gravity is performed. The conclusion on singularity avoidance in positively curved cosmological models is confirmed. Two cases are considered, the massless (with flat potential) and massive scalar field. Explanation of initial conditions generation for inflation in models with massive scalar field is given. The bounce is discussed in models with zero spatial curvature and negative potentials.

## 6.2. Quantum Field on Classical Background

[10] V. Belinski, On the existence of quantum evaporation of a black hole, *Phys. Lett. A*, **209**, 13 (1995).

A conjecture is made that the standard derivation of the black hole evaporation effect which uses infinite frequency wave modes is inadequate to describe black hole physics. The proposed resolution is that the problem is not due to the absence of the as yet unknown "correct" derivation but rather that the effect does not exist.

[11] A. Fedotov, V. Mur , N. Narozhny, V. Belinski and B.Karnakov, Quantum field aspect of the Unruh problem, *Phys. Lett. A*, **254**, 126, (1999).

It is shown using a both conventional and algebraic approach to quantum field theory that it is impossible to perform quantization on Unruh modes in Minkowski space-time. Such a quantization implies setting a boundary condition for the quantum field operator which changes the topological properties and symmetry group of space-time and leads to a field theory in two disconnected left and right Rindler space-times. It means that the "Unruh effect" does not exist.

[12] N.B. Narozhny, A.M. Fedotov, B.M. Karnakov, V.D. Mur and V.A. Belinski, Boundary conditions in the Unruh problem, *Phys. Rev. D*, **65**, 025004, (2002).

According to Unruh, a detector moving with constant proper acceleration in empty Minkowski spacetime reveals universalnot depending on the inner structure of the detectorthermal response. We have analyzed the Unruh problem using both conventional and algebraic approaches to quantum field theory. It is shown that the Unruh quantization procedure implies setting a boundary condition for the quantum field operator which changes the topological properties and symmetry group of the spacetime and leads to a field theory in two disconnected left and right Rindler spacetimes instead of Minkowski spacetime. Thus we conclude that, in spite of the work over the last 25 years, there still remain serious gaps in grounding of the Unruh effect, and as of now there is no compelling evidence for the universal behavior attributed to all uniformly accelerated detectors.

[13] A.M. Fedotov, N.B. Narozhny, B.M. V.D. Mur and V.A. Belinski, An example of a uniformly accelerated particle detector with non-Unruh response, *Phys. Lett. A*, **305**, 211 (2002).

We propose a scalar background in Minkowski spacetime imparting constant proper acceleration to a classical particle. In contrast to the case of a constant electric field the proposed scalar potential does not create

particleantiparticle pairs. Therefore, an elementary particle accelerated by such field is a more appropriate candidate for an "Unruh-detector" than a particle moving in a constant electric field.We show that the proposed detector does not reveal the universal thermal response of the Unruh type.

[14] N.B. Narozhny, A.M. Fedotov, B.M. Karnakov, V.D. Mur, and V.A. Belinskii, Reply to "Comment on 'Boundary conditions in the Unruh problem' ", *Phys. Rev. D*, **70**, 048702 (2004).

We reply to the preceding Comment by Fulling and Unruh criticizing our conclusion that principles of quantum field theory as of now do not give convincing arguments in favor of a universal thermal response of detectors uniformly accelerated in Minkowski space [Phys. Rev. D 65, 025004 2002]. We maintain our conclusion and present additional arguments to confirm it.

[15] V.A. Belinski, N.B. Narozhny, AM. Fedotov and V.D. Mur, Unruh quantization in the presence of a condensate, *Phys. Lett. A*, **331**, 349 (2004).

We have shown that the Unruh quantization scheme can be realized in Minkowski spacetime in the presence of BoseEinstein condensate containing infinite average number of particles in the zero boost mode and located basically inside the light cone. Unlike the case of an empty Minkowski spacetime the condensate provides the boundary conditions necessary for the Fulling quantization of the part of the field restricted only to the Rindler wedge of Minkowski spacetime.

[16] G. Montani, A scenario for the dimensional compactification in eleven-dimensional space-time, *Int. J. Mod. Phys. D*, **13**, 6, 1029 (2004).

We discuss the inhomogeneous multidimensional mixmaster model in view of the appearing, near the cosmological singularity, of a scenario for the dimensional compactification in correspondence to an 11-dimensional spacetime. Our analysis candidates such a collapsing picture toward the singularity to describe the actual expanding 3-dimensional Universe and an associated collapsed 7-dimensional space. To this end, a conformal factor is determined in front of the 4-dimensional metric to remove the 4-curvature divergences and the resulting Universe expands with a power-law inflation. Thus we provide an additional peculiarity of the eleven space-time dimensions in view of implementing a geometrical theory of unification.

# 7. Publications 2005-2008

## 7.1. Quantum Gravity

[1] E. Cerasti and G. Montani, Generating functional for the gravitational filed: implementation of an evolutionary quantum dynamics, *Int. J. Mod. Phys. D*, **14**, 10, 1739 (2005)

We provide a generating functional for the gravitational field that is associated with the relaxation of the primary constraints by extending to the quantum sector. This requirement of the theory relies on the assumption that a suitable time variable exists, when taking the T-products of the dynamical variables. More precisely, we start from the gravitational field equations written in the Hamiltonian formalism and expressed via Misner-like variables; hence we construct the equation to which the T-products of the dynamical variables obey and transform this paradigm in terms of the generating functional, as taken on the theory phase-space. We show how the relaxation of the primary constraints (which corresponds to the breakdown of the invariance of the quantum theory under the four-diffeomorphisms) is summarized by a free functional taken on the Lagrangian multipliers, accounting for such constraints in the classical theory. The issue of our analysis is equivalent to a Gupta-Bleuler approach on the quantum implementation of all the gravitational constraints; in fact, in the limit of small $\hbar$, the quantum dynamics is described by a Schrödinger equation as soon as the mean values of the momenta, associated to the lapse function and the shift vector, are not vanishing. Finally we show how, in the classical limit, the evolutionary quantum gravity reduces to General Relativity in the presence of an Eckart fluid, which corresponds to the classical counterpart of the physical clock, introduced in the quantum theory.

[2] M.V. Battisti and G. Montani, Evolutionary Quantum Dynamics of a Generic Universe, *Phys. Lett. B*, **637**, 203 (2006).

The implications of an evolutionary quantum gravity are addressed in view of formulating a new dark matter candidate. We consider a Schroedinger dynamics for the gravitational field associated to a generic cosmological model and then we solve the corresponding eigenvalue problem, inferring its phenomenological issue for the actual universe. The spectrum of the super-Hamiltonian is determined including a free inflaton field, the ultrarelativistic thermal bath and a perfect gas into the dynamics. We show that, when a Planckian cut-off is imposed in the theory and the classical limit of the ground state is taken, then a dark matter contribution cannot arise because its critical parameter $\Omega_{dm}$ is negligible today when the appropriate cosmological implementation of the model is provided. Thus, we show that, from a phenomenological point of view, an evolutionary quantum cosmology overlaps the Wheeler-DeWitt approach and therefore it can be inferred as appropriate to describe early stages of the universe without significant traces on the later evolution.

[3] P. Singh, K. Vandersloot and G.V. Vereshchagin, Nonsingular bouncing universes in loop quantum cosmology *Phys. Rev. D*, **74**, 043510 (2006).

Nonperturbative quantum geometric effects in loop quantum cosmology (LQC) predict a $\rho^2$ modification to the Friedmann equation at high energies. The quadratic term is negative definite and can lead to generic bounces when the matter energy density becomes equal to a critical value of the order of the Planck density. The nonsingular bounce is achieved for arbitrary matter without violation of positive energy conditions. By performing a qualitative analysis we explore the nature of the bounce for inflationary and cyclic model potentials. For the former we show that inflationary trajectories are attractors of the dynamics after the bounce implying that inflation can be harmoniously embedded in LQC. For the latter difficulties associated with singularities in cyclic models can be overcome. We show that nonsingular cyclic models can be constructed with a small variation in the original cyclic model potential by making it slightly positive in the regime where scalar field is negative.

[4] M.V. Battisti and G. Montani, The big-bang singularity in the framework of a generalized uncertainty principle, *Phys. Lett. B*, **656**, 96 (2006).

We analyze the quantum dynamics of the FriedmannRobertsonWalker Universe in the context of a Generalized Uncertainty Principle. Since the isotropic Universe dynamics resembles that of a one-dimensional particle, we quantize it with the commutation relations associated to an extended formulation of the Heisenberg algebra. The evolution of the system is described in terms of a massless scalar field taken as a relational time. We construct suitable wave packets and analyze their dynamics from a quasi-classical region to the initial singularity. The appearance of a non-singular dynamics comes out as far as the behavior of the probability density is investigated. Furthermore, reliable indications arise about the absence of a big-bounce, as predicted in recent issues of loop quantum cosmology.

[5] M.V. Battisti and G. Montani, Evolutionary Quantization of Cosmological Models, *Nuovo Cimento B*, **122**, 179-184 (2007).

We consider a Schrödinger quantum dynamics for the gravitational field associated to a FRW spacetime and then we solve the corresponding eigenvalue problem. We show that, from a phenomenological point of view, an Evolutionary Quantum Cosmology overlaps the Wheeler-DeWitt approach. We also show how a so peculiar solution can be inferred to describe the more interesting case of a generic cosmological model.

[6] F. Cianfrani and G. Montani, Boost invariance of the gravitational field dynamics: quantization without time gauge, *Class. Quant. Grav.*, **24**, 4161 (2007).

We perform a canonical quantization of gravity in a second-order formulation, taking as configuration variables those describing a 4-bein, not adapted to the spacetime splitting. We outline how, if we either fix the Lorentz frame before quantizing or perform no gauge fixing at all, the invariance under boost transformations is affected by the quantization.

[7] R. Benini and G. Montani, Inhomogeneous Quantum Mixmaster: from Classical toward Quantum Mechanics, *Class. Quant. Grav.*, **24**, 387 (2007).

Starting from the Hamiltonian formulation for the inhomogeneous Mixmaster dynamics, we approach its quantum features through the link of the quasiclassical limit. We fix the proper operator-ordering which ensures that the WKB continuity equation overlaps the Liouville theorem as restricted to the configuration space. We describe the full quantum dynamics of the model in some detail, providing a characterization of the (discrete) spectrum with analytic expressions for the limit of high occupation number. One of the main achievements of our analysis relies on the description of the ground state morphology, showing how it is characterized by a non-vanishing zero-point energy associated with the universe anisotropy degrees of freedom.

[8] N. Carlevaro, O.M. Lecian and G. Montani, Macroscopic and microscopic paradigms for the torsion field: from the test-particles motion to a Lorentz gauge theory, *Ann. Fond. Louis de Broglie*, **32**, 281 (2007).

Torsion represents the most natural extension of General Relativity and it attracted interest over the years in view of its link with fundamental properties of particle motion. The bulk of the approaches concerning the torsion dynamics focus their attention on their geometrical nature and they are naturally lead to formulate a non-propagating theory. Here

we review two different paradigms to describe the role of the torsion field, as far as a propagating feature of the resulting dynamics is concerned. However, these two proposals deal with different pictures, i.e., a macroscopic approach, based on the construction of suitable potentials for the torsion field, and a microscopic approach, which relies on the identification of torsion with the gauge field associated with the local Lorentz symmetry. We analyze in some detail both points of view and their implications on the coupling between torsion and matter. In particular, in the macroscopic case, we analyze the test-particle motion to fix the physical trajectory, while, in the microscopic approach, a natural coupling between torsion and the spin momentum of matter fields arises

[9] F. Cianfrani and G. Montani, The role of the time gauge in the 2nd order formalism, *Int. J. Mod. Phys. A*, **23**, 8, 1214 (2008).

We perform a canonical quantization of gravity in a second-order formulation, taking as configuration variables those describing a 4-bein, not adapted to the space-time splitting. We outline how, neither if we fix the Lorentz frame before quantizing, nor if we perform no gauge fixing at all, is invariance under boost transformations affected by the quantization.

[10] M. Castellana and G. Montani, Physical state condition in Quantum General Relativity as a consequence of BRST symmetry, *Class. Quant. Grav.*, **25**, 105018 (2008).

Quantization of systems with constraints can be carried on with several methods. In the Dirac formulation the classical generators of gauge transformations are required to annihilate physical quantum states to ensure their gauge invariance. Carrying on BRST symmetry it is possible to get a condition on physical states which, differently from the Dirac method, requires them to be invariant under the BRST transformation. Employing this method for the action of general relativity expressed in terms of the spin connection and tetrad fields with path integral methods, we construct the generator of BRST transformation associated with the underlying local Lorentz symmetry of the theory and write a physical state condition consequence of BRST invariance. We observe that this condition differs form the one obtained within Ashtekar's canonical formulation, showing how we recover the latter only by a suitable choice of the gauge fixing functionals. We finally discuss how it should be possible to obtain all the requested physical state conditions associated with all the underlying gauge symmetries of the classical theory using our approach.

[11] G. Montani and S. Zonetti, Parametrizing fluids in canonical quantum gravity, *Int. J. Mod. Phys. A*, **23**, 8, 1240-1243 (2008).

The problem of time is an unsolved issue of canonical General Relativity. A possible solution is the Brown-Kuchar mechanism which couples matter to the gravitational field and recovers a physical, i.e. non vanishing, observable Hamiltonian functional by manipulating the set of constraints. Two cases are analyzed. A generalized scalar fluid model provides an evolutionary picture, but only in a singular case. The Schutz' model provides an interesting singularity free result: the entropy per baryon enters the definition of the physical Hamiltonian. Moreover in the co-moving frame one is able to identify the time variable tau with the logarithm of entropy.

[12] M.V.Battisti and G.Montani, Quantum dynamics of the Taub Universe in a generalized uncertainty principle framework, *Phys. Rev. D*, **77**, 023518 (2008).

The implications of a Generalized Uncertainty Principle on the Taub cosmological model are investigated. The model is studied in the ADM reduction of the dynamics and therefore a time variable is ruled out. Such a variable is quantized in a canonical way and the only physical degree of freedom of the system (related to the Universe anisotropy) is quantized by means of a modified Heisenberg algebra. The analysis is performed at both classical and quantum level. In particular, at quantum level, the motion of wave packets is investigated. The two main results obtained are as follows. i) The classical singularity is probabilistically suppressed. The Universe exhibits a stationary behavior and the probability amplitude is peaked in a determinate region. ii) The GUP wave packets provide the right behavior in the establishment of a quasi-isotropic configuration for the Universe.

[13] G. Montani and F. Cianfrani, General Relativity as Classical Limit of Evolutionary Quantum Gravity, *Class. Quant. Grav.*, **25**, 065007 (2008).

In this paper we analyze the dynamics of the gravitational field when the covariance is restricted to a synchronous gauge. In the spirit of the Noether theorem, we determine the conservation law associated to the Lagrangian invariance and we outline that a non-vanishing behavior of the Hamiltonian comes out. We then interpret such resulting non-zero "energy" of the gravitational field in terms of a dust fluid. This new matter contribution is co-moving to the slicing and it accounts for the "materialization" of a synchronous reference from the corresponding gauge condition. Further, we analyze the quantum dynamics of a generic inhomogeneous Universe as described by this evolutionary scheme, asymptotically to the singularity. We show how the phenomenology of such a model overlaps the corresponding Wheeler-DeWitt picture. Finally, we study the possibility of a Schrödinger dynamics of the gravitational field as a consequence of the correspondence inferred between the ensemble dynamics of stochastic systems and the WKB limit of their quantum evolution. We demonstrate that the time dependence of the ensemble distribution is associated with the first order correction in $\hbar$ to the WKB expansion of the energy spectrum.

[14] F. Cianfrani and G. Montani, Synchronous Quantum Gravity, *Int. J. Mod. Phys. A*, **23**, 8, 1105-1112 (2008).

The implications of restricting the covariance principle within a Gaussian gauge are developed both on a classical and a quantum level. Hence, we investigate the cosmological issues of the obtained Schrödinger Quantum Gravity with respect to the asymptotically early dynamics of a generic Universe. A dualism between time and the reference frame fixing is then inferred.

[15] M.V. Battisti, O.M. Lecian and G. Montani, Quantum cosmology with a minimal length, *Int. J. Mod. Phys. A*, **23**, 1257-1265 (2008).

Quantum cosmology in the presence of a fundamental minimal length is analyzed in the context of the flat isotropic and the Taub cosmological models. Such minimal scale comes out from a generalized uncertainty principle and the quantization is performed in the minisuperspace representation. Both the quantum Universes are singularity-free and (i) in the isotropic model no evidences for a Big-Bounce appear; (ii) in the Taub one a quasi-isotropic configuration for the Universe is predicted by the model.

[16] N. Carlevaro, O.M. Lecian and G. Montani, Lorentz Gauge Theory and Spinor Interaction, *Int. J. Mod. Phys. A*, **23**(8), 1282 (2008).

A gauge theory of the Lorentz group, based on the different behavior of spinors and vectors under local transformations, is formulated in a flat space-time and the role of the torsion field within the generalization to curved space-time is briefly discussed. The spinor interaction with the new gauge field is then analyzed assuming the time gauge and stationary solutions, in the non-relativistic limit, are treated to generalize the Pauli equation.

[17] M.V. Battisti, O.M. Lecian and G. Montani, Polymer Quantum Dynamics of the Taub Universe, *Phys. Rev. D*, in press.

Within the framework of non-standard (Weyl) representations of the canonical commutation relations, we investigate the polymer quantization of the Taub cosmological model. The Taub model is analyzed within the Arnowitt-Deser-Misner reduction of its dynamics, by which a time variable arises. While the energy variable and its conjugate momentum are treated as ordinary Heisenberg operators, the anisotropy variable and its conjugate momentum are represented by the polymer technique. The model is analyzed at both classical and quantum level. As a result, classical trajectories flatten with respect to the potential wall, and the cosmological singularity is not probabilistically removed. In fact, the dynamics of the wave packets is characterized by an interference phenomenon, which, however, is not able to stop the evolution towards the classical singularity.

[18] G. Montani and S. Zonetti,Definition of a time variable with Entropy of a perfect fluid in Canonical Quantum Gravity, submitted to *Phys. Rev. D*.

The Brown-Kuchar mechanism is applied in the case of General Relativity coupled with the Schutz' model for a perfect fluid. Using the canonical formalism and manipulating the set of modified constraints one is able to recover the definition of a time evolution operator, i.e. a physical Hamiltonian, expressed as a functional of gravitational variables and the entropy. Entropy then reveals to be, in the comoving frame, the time variable for the system, and a simple evolution operator is obtained.

[19] M.V. Battisti and G. Montani, The Mixmaster Universe in a generalized uncertainty principle framework, submitted to *Phys. Rev. D*.

The Bianchi IX cosmological model is analyzed in a generalized uncertainty principle framework. The Arnowitt-Deser-Misner reduction of the dynamics is performed and a time-coordinate, namely the volume of the Universe, naturally arises. Such a variable is treated in the ordinary way while the anisotropies (the physical degrees of freedom of the Universe) are described by a deformed Heisenberg algebra. The analysis of the model (passing through Bianchi I and II) is performed at classical level by studying the modifications induced on the symplectic geometry by the deformed algebra. We show that, the triangular allowed domain is asymptotically stationary with respect to the particle (Universe) and that its bounces against the walls are not interrupted by the deformed effects. Furthermore, no reflection law can be in general obtained since the Bianchi II model is no longer analytically integrable. This way, the deformed Mixmaster Universe can be still considered a chaotic system.

[20] M.V. Battisti, Loop and braneworlds cosmologies from a deformed Heisenberg algebra, submitted to *Phys. Rev. D*.

The implications of a deformed Heisenberg algebra on the Friedmann-Robertson-Walker cosmological models are investigated. In particular, we consider generalized commutation relations which leave undeformed the translation group and preserve the rotational invariance. The resulting algebra is related to the ?-Poincare one and no sign in the deformation term is selected at all. The analysis of the models is performed at classical level by studying the modifications induced on the symplectic geometry by the deformed algebra. We show that this framework leads to a modified Friedmann equation which coincide with that one found in loop quantum cosmology as well as in the Randall-Sundrum braneworlds scenario. In fact, the complementary sign of the loop and brane term, in the effective cosmological dynamics, naturally emerges from the free sign of the deformed algebra. This way, a common phenomenological description for both these theories is obtained and a relation with the low energy quantum gravity framework established.

[21] M.V. Battisti, R. Belvedere and G. Montani, Semi-classical isotropization mechanism for a generic Universe, submitted to *Phys. Rev. D*.

A semi-classical mechanism which leads to an isotropic configuration for a generic Universe is developed. In particular, we construct a wave function of the inhomogeneous Mixmaster Universe which has a meaningful probabilistic interpretation in agreement with the Copenhagen school one. It describes the evolution of the anisotropies of the Universe with respect to the isotropic scale factor, which is regarded as a semi-classical variable, i.e. plays the role of the external observer. We show that, near the cosmological singularity the solution is spread over all values of the anisotropies while, when the Universe expands enough, the closed Friedmann-Robertson-Walker model appears to be the favorite state.

[22] N. Carlevaro, O.M. Lecian and G. Montani, Fermion dynamics by internal and space-time symmetries, submitted to *Mod. Phys. Lett. A*.

This manuscript is devoted to introduce a gauge theory of the Lorentz Group based on the ambiguity emerging in dealing with isometric diffeomorphism-induced Lorentz transformations. The behaviors under local transformations of fermion fields and spin connections (assumed to be ordinary world vectors) are analyzed in flat space-time and the role of the torsion field, within the generalization to curved space-time, is briefly discussed. The fermion dynamics is then analyzed including the new gauge fields and assuming time-gauge. Stationary solutions of the problem are also studied in the non-relativistic limit, to study the spinor structure of an hydrogen-like atom.

## 7.2. Quantum Field on Classical Background

[23] V. Belinski , On the existence of black hole evaporation yet again, *Phys. Lett. A*, **354**, 249 (2006).

A new argument is presented confirming the point of view that a Schwarzshild black hole formed during a collapse process does not radiate.

[24] F. Cianfrani, G. Montani, Curvature-spin coupling from the semi-classical limit of the Dirac equation, *Int. J. Mod. Phys. A*, **23**, 8, 1274-1277 (2008).

The notion of a classical particle is inferred from Dirac quantum fields on a curved space-time, by an eikonal approximation and a localization hypothesis for amplitudes. This procedure allows to define a semiclassical version of the spin-tensor from internal quantum degrees of freedom, which has a Papapetrou-like coupling with the curvature.

[25] F. Cianfrani and G. Montani, Dirac equations in curved space-time versus Papapetrou spinning particles, *Europhys. Lett.*, in press.

We recover classical particles, starting from Dirac quantum fields on a curved space-time, by an eikonal approximation and a localization hypothesis for amplitudes. We conclude that the semi-classical dynamics of spinors is neither a geodesics one, nor resembling a Papapetrou-like spinning body. However, the spin-curvature coupling predicted by the Papapetrou theory is recovered in the weak-gravitational-field limit, but still an additional contribution to the dynamics arises

## 7.3. Unification Theories

[26] G. Montani, Geometrization of the Gauge Connection within a Kaluza-Klein Theory, *Int. J. Theor. Phys.*, **44**, 43-52 (2005).

Within the framework of a Kaluza-Klein theory, we provide the geometrization of a generic (Abelian and non-Abelian) gauge coupling, which comes out by choosing a suitable matter fields dependence on the extra-coordinates. We start by the extension of the Nother theorem to a multidimensional spacetime being the direct sum of a 4-dimensional Minkowski space and of a compact homogeneous manifold (whose isometries reflect the gauge symmetry); we show, how on such a "vacuum" configuration, the extra-dimensional components of the field momentum correspond to the gauge charges. Then we analyze the structure of a Dirac algebra as referred to a spacetime with the Kaluza-Klein restrictions and, by splitting the corresponding free-field Lagrangian, we show how the gauge coupling terms outcome.

[27] E. Alesci and G. Montani, Can gravitational waves be markers for an extra-dimension?, *Int. J. Mod. Phys. D*, **14**, 6, 923 (2005).

The main issue of the present paper is to fix specific features (which turn out being independent of extradimension size) of gravitational waves generated before a dimensional compactification process. Valuable is the possibility to detect our prediction from gravitational wave experiment without high energy laboratory investigation. In particular we show how gravitational waves can bring information on the number of Universe dimensions. Within the framework of Kaluza-Klein hypotheses, a different morphology arises between waves generated before than the compactification process settled down and ordinary 4-dimensional waves. In the former case the scalar and tensor degrees of freedom cannot be resolved. As a consequence if gravitational waves having the feature predicted here were detected (anomalous polarization amplitudes), then they would be reliable markers for the existence of an extra dimension.

[28] F. Cianfrani, A. Marrocco and G. Montani, Gauge Theories as a Geometrical Issue of a Kaluza-Klein Framework, *Int. J. Mod. Phys. D*, **14**(7), 1095 (2006).

We present a geometrical unification theory in a Kaluza-Klein approach that achieve the geometrization of a generic gauge theory bosonic component. We show how it is possible to derive gauge charge conservation from the invariance of the model under extra-dimensional translations and to geometrize gauge connections for spinors, in order to make possible to introducing matter just through free spinorial fields. Then we present the applications to (i) a pentadimensional manifold so reproducing the original Kaluza-Klein theory with some extensions related to the rule of the scalar field contained in the metric and to the introduction of matter through spinors with a phase dependance from the fifth coordinate, (ii) a seven-dimensional manifold, in which we geometrize the electroweak model by introducing two spinors for every leptonic family and quark generation and a scalar field with two components with opposite hypercharge responsible for spontaneous symmetry breaking.

[29] F. Cianfrani and G. Montani, Non Abelian gauge symmetries induced by the unobservability of extra-dimensions in a Kaluza-Klein approach, *Mod. Phys. Lett. A*, **21**(3), 265 (2006).

In this work we deal with the extension of the Kaluza-Klein approach to a non-Abelian gauge theory; we show how we need to consider the link between the n-dimensional model and a four-dimensional observer physics, in order to reproduce field equations and gauge transformations in the four-dimensional picture. More precisely, in field equations

any dependence on extra coordinates is canceled out by an integration, as consequence of the unobservability of extra dimensions. Thus, by virtue of this extra dimension unobservability, we are able to recast the multidimensional Einstein equations into the four-dimensional Einstein-Yang-Mills ones, as well as all the right gauge transformations of fields are induced. The same analysis is performed for the Dirac equation describing the dynamics of the matter fields and, again, the gauge coupling with Yang-Mills fields are inferred from the multidimensional free fields theory, together with the proper spinors transformations.

[30] O.M. Lecian and G. Montani, On the Kaluza-Klein geometrization of the Electro-Weak model within a gauge theory of the 5-dimensional Lorentz group, *Int. J. Mod. Phys. D*, **15**, 717 (2006).

The geometrization of the Electroweak Model is achieved in a five-dimensional RiemannCartan framework. Matter spinorial fields are extended to 5 dimensions by the choice of a proper dependence on the extracoordinate and of a normalization factor. weak hypercharge gauge fields are obtained from a KaluzaKlein scheme, while the tetradic projections of the extradimensional contortion fields are interpreted as weak isospin gauge fields. generators are derived by the identification of the weak isospin current to the extradimensional current term in the Lagrangian density of the local Lorentz group. The geometrized U(1) and SU(2) groups will provide the proper transformation laws for bosonic and spinorial fields. Spin connections will be found to be purely Riemannian.

[31] V. Lacquaniti and Giovanni Montani, On the ADM decomposition of the 5D Kaluza-Klein model, *Int.J. Mod. Phys. D*, **15**, 559 (2006).

Our purpose is to recast the KK model in terms of ADM variables. We examine and solve the problem of the consistency of this approach, with particular care about the role of the cylindricity hypothesis. We show in detail how the KK reduction commutes with the ADM slicing procedure and how this leads to a well-defined and unique ADM reformulation. This allows us to consider the Hamiltonian formulation of the model and moreover it can be viewed as the first step for the Ashtekar reformulation of the KK scheme. Moreover, we show how the time component of the gage vector arises naturally from the geometrical constraints of the dynamics; this is a positive check for the autoconsistency of the KK theory and for an Hamiltonian description of the dynamics which will take into account the compactification scenario; this result enforces the physical meaning of the KK model.

[32] F. Cianfrani and G. Montani, Geometrization of the electro-weak model bosonic component, *Int. J. Theor. Phys.*, **46**(3), 471 (2007).

In this work we develop a geometrical unification theory for gravity and the electro-weak model in a Kaluza-Klein approach; in particular, from the curvature dimensional reduction Einstein-Yang-Mills action is obtained. We consider two possible space-time manifolds: 1)$V^4 \otimes S^1 \otimes S^2$ where isospin doublets are identified with spinors; 2) $V^4 \otimes S^1 \otimes S^3$ in which both quarks and leptons doublets can be recast into the same spinor, such that the equal number of quark generations and leptonic families is explained. Finally a self-interacting complex scalar field is introduced to reproduce the spontaneous symmetry breaking mechanism; in this respect, at the end we get an Higgs fields whose two components have got opposite hypercharges.

[33] F. Cianfrani and G. Montani, The Electro-Weak model as low-energy sector of 8-dimensional General Relativity, *Nuovo Cimento B*, **122**, 213 (2007).

In a Kaluza-Klein background $V^4 \otimes S^3$, we provide a way to reproduce, by the dimensional reduction, a 4-spinor with a SU(2) gauge coupling. Since additional gauge violating terms cannot be avoided, we compute their order of magnitude by virtue of the application to the Electro-Weak model.

[34] F. Cianfrani, I. Milillo and G. Montani, Dixon-Souriau equations from a 5-dimensional spinning particle in a Kaluza-Klein framework, *Phys. Lett. A*, **366**, 7 (2007).

The dimensional reduction of Papapetrou equations is performed in a 5-dimensional KaluzaKlein background and DixonSouriau results for the motion of a charged spinning body are obtained. The splitting provides an electric dipole moment, and, for elementary particles, the induced parity and time-reversal violations are explained.

[35] F. Cianfrani and G. Montani, Spinning particles in General Relativity, *Nuovo Cimento B*, **122**, 173 (2007).

We analyze the behavior of a spinning particle in gravity, both from a quantum and a classical point of view. We infer that, since the interaction between the space-time curvature and a spinning test particle is expected, then the main features of such an interaction can get light on which degrees of freedom have physical meaning in a quantum gravity theory with fermions. Finally, the dimensional reduction of Papapetrou equations is performed in a 5-dimensional Kaluza-Klein background and Dixon-Souriau results for the motion of a charged spinning body are obtained.

[36] O.M. Lecian and G. Montani, Electro-weak Model within the framework of Lorentz gauge theory: Ashtekar variables?, *Nuovo Cimento B*, **122**, 207-212 (2007).

The Electroweak (EW) model is geometrized in the framework of a 5D gauge theory of the Lorentz group, after the implementation of the Kaluza-Klein (KK) paradigm. The possibility of introducing Ashtekar variables on a 5D KK manifold is considered on the ground of its geometrical structure.

[37] V. Lacquaniti and G. Montani, Hamiltonian Formulation of 5-dimensional Kaluza-Klein Theory, *Nuovo Cimento B*, **122**, 201-206 (2007).

We analyze the consistency of the ADM approach to KK model; we prove that KK reduction commute with ADM splitting. This leads to a well defined Hamiltonian; we provide the outcome. The electromagnetic constraint is derived from a geometrical one and this result enforces the physical meaning of KK model. Moreover we study the role of the extra scalar field we have in our model; classical hints from geodesic motion and cosmological solutions suggest that the scalar field can be an alternative time variable in the relational point of view.

[38] F. Cianfrani and G. Montani, Low-energy sector of 8-dimensional General Relativity: Electro-Weak model and neutrino mass, *Int. J. Mod. Phys. D*, **17**(5), 785 (2008).

In this paper we demonstrate that in a Kaluza-Klein space-time $V^4 \otimes S^3$ the dimensional reduction of spinors provides a 4-field, whose associated SU(2) gauge connections are geometrized. However, additional and gauge-violating terms arise, but they are highly suppressed by a factor $\beta$, which fixes the amount of the spinor dependence on extra-coordinates. The application of this framework to the Electro-Weak model is performed, thus giving a lower bound for $\beta$ from the request of the electric charge conservation. Moreover, we emphasize that also the Higgs sector can be reproduced, but neutrino masses are predicted and the fine-tuning on the Higgs parameters can be explained, too.

[39] F. Cianfrani and G. Montani, Elementary particle interaction from a Kaluza-Klein scheme, *Int. J. Mod. Phys. A*, **23**, 8, 1182-1189 (2008).

We discuss properties of particles and fields in a multi-dimensional space-time, where the geometrization of gauge interactions can be performed. As far as spinors are concerned, we outline how the gauge coupling can be recognized by a proper dependence on extra-coordinates and by the dimensional reduction procedure. Finally applications to the Electro-Weak model are presented.

[40] O.M. Lecian and G. Montani, Fundamental Symmetries of the extended Spacetime, *Int. J. Mod. Phys. A*, **23**, 1266-1269 (2008).

On the basis of Fourier duality and Stone-von Neumann theorem, we will examine polymer-quantization techniques and modified uncertainty relations as possible 1-extraD compactification schemes for a phenomenological truncation of the extraD tower.

[41] V. Lacquaniti and G. Montani, On matter coupling in 5D Kaluza-Klein framework, *Int. J. Mod. Phys. A* **23**, 1270-1273 (2008).

We analyze some unphysical features of the geodesic approach to matter coupling in a compactified Kaluza-Klein scenario, like the q/m puzzle and the huge massive modes. We propose a new approach, based on Papapetrou multipole expansion, that provides a new equation for the motion of a test particle. We show how this equation provides right couplings and does not generate huge massive modes.

[42] V. Lacquaniti and Giovanni Montani, Dynamics of Matter in a Compactified Kaluza-Klein Model, *Int.J. Mod. Phys. D*, in press.

A longstanding problem in Kaluza-Klein models is the description of matter dynamics. Within the 5D model, the dimensional reduction of the geodesic motion for a 5D free test particle formally restores electrodynamics, but the reduced 4D particle shows a charge-mass ratio that is upper bounded, such that it cannot fit to any kind of elementary particle. At the same time, from the quantum dynamics viewpoint, there is the problem of the huge massive modes generation. We present a criticism against the 5D geodesic approach and face the hypothesis that in Kaluza-Klein space the geodesic motion does not deal with the real dynamics of test particle. We propose a new approach: starting from the conservation equation for the 5D matter tensor, within the Papapetrou multipole expansion, we prove that the 5D dynamical equation differs from the 5D geodesic one. Our new equation provides right coupling terms without bounding and in such a scheme the tower of massive modes is removed.

# 8. Introduction

The gravitational interaction, as described in General Relativity, has a geometrical nature, which makes it very different from many other fundamental forces of Nature. This picture calls attention to search for a unification scheme, in which all fundamental fields are on the same footing, carrying physical interactions via a common mechanism.

Two different approaches can be pursued in this direction

- to find a gauge representation for the gravitational field, which fixes a clear strategy for its quantization. By other words, we can pursue the attempt to restate the space-time geometry as a natural issue for a gauge paradigm of the fundamental symmetries. This is the direction addressed by the Loop-Quantum-Gravity approach, and, in our investigation, attention is devoted to the extension of this point of view to include non-Riemannian features of the space-time, as the torsion field. In fact, we formulate a more general scheme, in which a Lorentz connection is introduced to take into account diffeomorphism-induced Lorentz rotations.

- It is possible to define a more general scheme, which is able to provide a geometrical interpretation for all fundamental interactions. Such an ambitious unification plan mainly relies on the introduction of extra-dimensions, which provide the necessary additional degrees of freedom to represent other physical fields beyond gravity. In our studies, this paradigm is intensely developed both within a well-established Kaluza-Klein framework and in extended non-Riemannian approaches to implement gauge symmetries. The Kaluza-Klein approach is then modified to make physical account for the non-observabilty of extra dimensions, thus expressing internal gauge symmetries as dimensional reduction of multi-dimensional properties, even when the coupling with matter fields is addressed.

An intermediate point of view is that of studying quantum field theory on a fixed background, regarding the back reaction as a negligible effect. This research line is mainly devoted to investigate features concerning the influence of choosing non-inertial systems on quantum states (for instance, the so-called Unruh effect), as well as considering space-time curvature features, i.e., the Hawking effect. The present work concernes a basic criticism to the mathematical formalism on which these effects are described, and therefore on their physical ground.

Finally, since the natural scenario to implement generalization of the Einsteinian picture is the very early universe dynamics, a significant discussion of cosmological applications concerning quantum and unification aspects is provided. These implementations give interesting tests on the viability of underlying theories, as well as important indications about the arising physics.

# 9. Quantum Gravity

## 9.1. Canonical Quantum Gravity without the time gauge

A Quantum Gravity theory is expected to provide a discrete structure for the space-time geometry. In this respect, it is worth noting the achievement of Loop Quantum Gravity of a discrete structure for spatial geometrical operators, at least on a kinematical level. However, this formulation is based on fixing the so-called time-gauge condition, *i.e.*, the choice of the vier-bein is adapted to the space-time splitting, such that the time-like vector $e_0$ is normal to spatial hypersurfaces. This way, boosts are frozen out and the investigation on the behavior of quantum geometrical operators in different Lorentz frames is highly non-trivial. Hence, our investigation has been focused on describing quantum gravitational degrees of freedom, without fixing the time gauge (Cianfrani and Montani, 2008d), (Cianfrani and Montani, 2007b), (Cianfrani and Montani, 2007a).

We also perform such an analysis in the first order formulation, thus establishing a clear connection with the Loop Quantum Gravity framework.

We start from the Holst modification, which reads as follows (in units $8\pi G = 1$)

$$S = \frac{1}{2}\int \sqrt{g} e^{\mu}_{A} e^{\nu}_{B} R^{CD}_{\mu\nu}(\omega^{FG}_{\mu})\,{}^{\gamma}p^{AB}{}_{CD}, \tag{9.1.1}$$

$g$ being the determinant of the metric tensor $g_{\mu\nu}$ with 4-bein vectors $e^{A}_{\mu}$ and spinor connections $\omega^{AB}_{\mu}$, while the expressions for $R^{AB}_{\mu\nu}$ and ${}^{\gamma}p^{AB}{}_{CD}$ are

$$R^{AB}_{\mu\nu} = \partial_{[\mu}\omega^{AB}_{\nu]} + \omega^{A}_{c[\mu}\omega^{CB}_{\nu]}, \qquad {}^{\gamma}p^{AB}{}_{CD} = \delta^{AB}{}_{CD} - \frac{1}{2\gamma}\epsilon^{AB}{}_{CD}. \tag{9.1.2}$$

Here $\gamma$ is the Immirzi parameter.

By a Legendre transformation, conjugate momenta ${}^{\gamma}\pi^{\mu}_{AB} = {}^{\gamma}p^{CD}{}_{AB}\pi^{\mu}_{CD}$ can be defined.

The full Hamiltonian turns out to be

$$\mathcal{H} = \int \left[\frac{1}{eg^{tt}}H - \frac{g^{ti}}{g^{tt}}H_i - \omega^{AB}_{t}\,{}^{\gamma}p^{CD}{}_{AB}G_{CD} + \lambda_{ij}C^{ij} + \eta_{ij}D^{ij} + \lambda^{AB}\pi^{t}_{AB}\right]d^3x, \tag{9.1.3}$$

where $1/eg^{tt}$, $g^{ti}/g^{tt}$, ${}^{\gamma}p^{CD}{}_{AB}\omega^{AB}_{t}$, $\lambda_{ij}$, $\eta_{ij}$ and $\lambda^{AB}$ behave as Lagrangian multipliers, while constraints are given by

$$\begin{cases} H = \pi^i_{CF}\pi^{jF}_{\ \ D}{}^{\gamma}p^{CD}_{\ \ \ AB}R^{AB}_{ij} = 0 \\ H_i = {}^{\gamma}p_{AB}^{\ \ \ CD}\pi^j_{CD}R^{AB}_{ij} = 0 \\ G_{AB} = D_i\pi^i_{AB} = \partial_i\pi^i_{AB} - \omega_{i[A}^{\ \ \ C}\pi^i_{|C|B]} = 0 \\ C^{ij} = \epsilon^{ABCD}\pi^{(i}_{AB}\pi^{j)}_{CD} = 0 \\ D^{ij} = \epsilon^{ABCD}\pi^k_{AF}\pi^{(iF}_{\ \ \ B}D_k\pi^{j)}_{CD} = 0 \end{cases} \quad . \tag{9.1.4}$$

The interpretation of such constraints is straightforward, since $H$ and $H_i$ denote the super-Hamiltonian and the super-momentum, respectively, while $G_{AB}$ is the Gauss constraint of the Lorentz symmetry. As far as $C^{ij}$ and $D^{ij}$ are concerned, they are the main difficulty of this analysis, since they make the constraint algebra second-class.

A solution for $C^{ij}$ and $D^{ij}$ for constant $\chi_a$ can be written as

$$\pi^i_{ab} = \chi_{[a}\pi^i_{b]}, \qquad \omega_a^{\ \ b}{}_i = {}^{\pi}\omega_a^{\ \ b}{}_i + \chi_a\omega^{ob}{}_i + \chi^b(\omega_a^{\ \ 0}{}_i + {}^{\pi}\omega_a^{\ \ c}{}_i\chi_c) \tag{9.1.5}$$

$\pi^i_b$ being $\pi^i_{0b}$, while ${}^{\pi}\omega_a^{\ \ b}{}_i = \frac{1}{\pi^{1/2}}\pi^{b3}_l\nabla_i(\pi^{1/2}\pi^l_a)$ with $\pi$ the determinant of $\pi^a_i$ and $T^{-1}_{ab} = \eta_{ab} + \chi_a\chi_b$.

In the reduced set of variables $\{\omega^{a0}{}_i, \pi^j_b\}$ the boost constraints is redundant, since $D_i\pi^i_a = \chi^b D_i\pi^i_{ab}$.

In order to define constraints with a closed algebra we sum up to the rotation constraints a vanishing contribution, so finding the following Gauss constraints

$$G_a = \partial_i\widetilde{\pi}^i_a + \gamma\epsilon_{abc}\widetilde{A}^b_i\widetilde{\pi}^i_c = 0, \qquad \{G_a, G_b\} = \gamma\epsilon_{ab}^{\ \ \ c}G_c, \tag{9.1.6}$$

with $\widetilde{\pi}^j_b$ "densitized" 3-bein vectors of the metric, whose expressions read as

$$\widetilde{\pi}^i_a = S^b_a\pi^i_b, \qquad S^a_b = \sqrt{1+\chi^2}\delta^a_b + \frac{1-\sqrt{1+\chi^2}}{\chi^2}\chi_a\chi_b, \tag{9.1.7}$$

while new connection $\widetilde{A}^a_i$ are

$$\widetilde{A}^a_i = (1+\chi^2)S_a^{-1b}T^c_b(\omega_{c0i} - {}^{\pi}\omega_{ic}^{\ \ \ d}\chi_d) - \frac{1}{2\gamma}S_a^{-1b}\epsilon_{bcd}{}^{\pi}\omega^{cf}{}_iT^{-1d}_{\ \ f}. \tag{9.1.8}$$

Since the symplectic structure is trivial one, $\widetilde{A}^a_i$ are the extension of Barbero-

Immirzi connections to a generic global Lorentz frame.

The above results demonstrate that it can be inferred a phase space structure similar to a $SU(2)$ one also without the time-gauge condition. Therefore, the LQG quantization procedure can be extended to a generic global Lorentz frame and no modification occurs (for instance we can probe that the area spectrum is not modified).

The action of the boost constraints can be represented by the operators $\epsilon_a{}^{bc}\chi_b G'_c$, thus by gauge transformations. Hence the boost symmetry is actually preserved on a quantum level.

As far as local Lorentz frames are concerned, Gauss constraints cannot be inferred by this procedure, thus the investigation is going on the possibility to implement the local Lorentz invariance after the quantization.

Finally, further investigations will consider the case with matter fields, which can give a better characterization to the space-time slicing.

## 9.2. The time gauge problem in the path integral formalism

The problem of quantization of constrained systems arises in many contexts of physical interest. The presence of constraints at a classical level avoids us to threat all the dynamical variables as independent ones, and entails several difficulties when we are to construct the quantum theory. In a program of canonical quantization which promotes all classical canonical variables to quantum operators one has to deal with the problem of quantum-mechanically imposing the constraints. In the procedure *à la Dirac*, the constraint operators are imposed to annihilate physical states. This procedure stems from the observation that in the classical theory, the constraint functions are generators of infinitesimal canonical transformations which don't alter the physical state of the system.
The Dirac procedure is widely used in different contexts, including quantization of general relativity. Nevertheless this procedure of quantization encounters several difficulties when we require the Dirac conditions on physical states to be consistent with each other and the physical states selected by constraint operators to posses a finite scalar product allowing a probabilistic interpretation: moreover, in some cases this procedure can lead to a physical subspace of the entire Hilbert space that is curiously empty. Other difficulties arise when one tries to implement the Dirac procedure, which are not properly to be ascribed to the Dirac theory for constrained systems, but to the canonical quantization framework this procedure is developed in. As a matter of fact, our experience on quantum field theory in special relativity showed us how canonical quantization methods, when applied to systems with infinite degrees of freedom, lead to several inconsistencies: for example,

it is a remarkable fact that the Glashow - Weinberg - Salam theory for electroweak interactions cannot be consistently formulated by canonical quantization methods, while the only way by which can be coherently written by is the Feynman path integral technique. Even if Feynman's path integral can be derived after constructing the quantum theory by means of canonical quantization methods, such inconsistencies need to postulate the path-integral approach as a founding element of the quantum theory when we deal with systems with infinite degrees of freedom. It is for these reasons that we developed all of our work (Castellana and Montani, 2008) avoiding using the Dirac procedure for constrained systems and canonical quantization methods at all, employing a method to derive conditions on physical states based on BRST symmetry and path=integral methods uniquely.

BRST symmetry was conceived at first within non-Abelian gauge theories and shown to apply to a really wide class of systems of physical interest. Anyway, in the literature, there are different formulations for the BRST formalism, with substantial differences from each other. First of all, there exists a formulation of BRST symmetry for constrained systems based on canonical quantization methods which is widely diffused, being also employed in quantization of general relativity. Another approach, the one we followed in this work, is to derive BRST symmetry, based entirely on path integral methods, and it is applicable to systems with infinite degrees of freedom, avoiding those inconsistencies proper of canonical quantization methods we discussed above.

We start with an enlightening and more or less known example, considering BRST symmetry for a non-Abelian gauge theory. In order to compare path integral methods with canonical quantization ones, one can consider the Nöether charge following from BRST symmetry of the action and, taking an appropriate choice for the gauge fixing functionals in the DeWitt - Fadeev - Popov method, show it to be the generator of quantum BRST transformation within a canonical quantization framework.
Otherwise, using solely path integral methods, we show the BRST Nöether charge

$$Q \equiv \int d^3x J^0(x) \tag{9.2.1}$$

related to the BRST current $J^\mu$ to generate quantum BRST transformation by means of Ward's identities for the ensemble of gauge fields, ghost and antighost fields and Nakanishi - Lautrup fields, designed by $\psi_i(x)$, i. e.

$$\begin{aligned} 0 \;=\; & \partial^x_\mu \left\langle \psi_{i_k}(x_k) \cdots \psi_{i_1}(x_1)\, J^\mu(x) \right\rangle_{j=0} - i \sum_{l=1}^{k} \sigma^{i_1} \cdots \sigma^{i_l} \langle \psi_{i_k}(x_k) \cdots \\ & \cdots \psi_{i_{l+1}}(x_{l+1})\, s\psi_{i_l}(x) \psi_{i_{l-1}}(x_{l-1}) \cdots \psi_{i_1}(x_1) \rangle_{j=0} \delta^{(4)}(x - x_l)\,. \end{aligned} \tag{9.2.2}$$

where $\sigma^i = \pm 1$ for $\psi_i$ bosonic or fermionic respectively. The fact that in (9.2.2) the gauge fixing functionals are completely arbitrary allows us to infer a physical-state condition on states of the gauge fields following from BRST invariance, given by the usual Gauss'

$$\mathcal{D}_a F^{0a\alpha}(x)\,|\psi\rangle = 0. \tag{9.2.3}$$

Afterward, we turn our attention to general relativity expressed in first-order formalism, in order to investigate the physicality condition for the states of the gravitational field arising from BRST invariance of the theory, following the same procedure employed for non-Abelian gauge theories. In this procedure we will intentionally avoid to use canonical quantization methods. We are to determine a physical state condition on quantum states without thinking of classical Hamiltonian constraints in order to compare, at the end of our calculation, our physicality condition required by BRST symmetry and derived with path-integral methods with the one obtained using the Dirac quantization method employed within Ashtekar's canonical formulation. Employing the same method leading us to the usual Gauss' constraint for non-Abelian gauge theories, we arrive at the following physical state condition for the densitized triad $E^a_i$

$$\mathcal{D}_a \left[E^a_j(x) + i e_{jb}(x) e_{0c}(x) \epsilon^{abc}\right] |\psi\rangle = 0. \tag{9.2.4}$$

Comparing our physicality condition with the one used in loop quantum gravity, we find they differ by an additional non-vanishing term. We think the origin of this discrepancy is in the choice of a particular gauge in the classical theory which is made within Ashtekar's approach and which was intentionally avoided in our work. Finally, we show how we recover the Dirac canonical condition in our BRST quantization only by a suitable choice of gauge fixing functionals within the DeWitt - Fadeev - Popov method.

## 9.3. The problem of time in quantum gravity

The problem of time arises in General Relativity when the canonical formalism is applied: the Hamiltonian function is constrained to vanish as a result of the 4-dim diffeomorphism invariance of the theory, which in this picture acquires a non evolutionary behaviour.
In the Dirac quantization program, where one tries to quantize the whole phase space and to impose constraints as conditions on quantum states, the famous Wheeler-DeWitt equation appears, stating that for physical states:

$$\hat{\boldsymbol{H}}|\psi\rangle = 0. \tag{9.3.1}$$

These have to be annihilated by the quantum Hamiltonian operator, which in the ordinary quantum theories is also the generator of infinitesimal time evolution.
Hence the lost of the evolutionary character: the Schrödinger equation is replaced by the WDW one, and physical state cannot depend on the time variable.
Different solutions have been proposed so far, one among them is to use some kind of matter as a physical clock. The proposal by Brown and Kučhar is to introduce a fluid coupled to the gravitational field in the standard ADM formulation: in the canonical formalism *the modified constraint equations are solved*, in order to recover a meaningful Schrödingher equation. The Hamiltonian constraint will take the local form:

$$\pi(x) - h(q, P)(x) = 0 \tag{9.3.2}$$

where $\pi$ is the momentum conjugate to one of the fluid variables, $\tau$. The whole procedure has to be performed with smeared quantities, i.e. integrating in space local quantities on suitable functions. Under the condition of a closed constraint algebra under time evolution, i.e. $\{\pi - h, \boldsymbol{H}\} = 0$ for the Hamiltonian constraint, one can write down the equation for the physical *evolutionary* quantum states:

$$-i\hbar\frac{d}{d\tau}\psi = \hat{h}\psi. \tag{9.3.3}$$

This is the usual Schrödinger equation, where the notion of time is recovered from the coupling of GR with the fluid. This is the so called Kučhar-Brown mechanism.

This procedure can be applied [Montani and Zonetti (2008)] to different fluid models: a scalar field model, with a Lagrangian density in the form:

$$\mathcal{L}_F = \sqrt{-g}(\partial_\mu\phi\partial^\mu\phi)^\gamma, \tag{9.3.4}$$

where $\gamma$ takes real values, has been analyzed. The associated equation of state depends on $\gamma$, being:

$$p = \frac{\rho}{2\gamma - 1}. \tag{9.3.5}$$

Some values seem to allow the KB procedure to be applied, and this includes some singular cases, like $\gamma \to \infty$. The result is that only for $\gamma = 1/2$ a non vanishing Hamiltonian is recovered: but this model exhibits a singular equation of state, with $p \to \infty$.
The Schutz velocity potential representation of a perfect fluid is another potential candidate.
In this model a relativistic description of the fluid is performed, using six

scalar fields, which combine in the 4-velocity as:

$$\boldsymbol{U}_\nu = \mu^{-1}(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}) = \mu^{-1}v_\nu, \tag{9.3.6}$$

and a Lagrangian is constructed following the standard classical thermodynamics:

$$\mathcal{L}_F = \sqrt{-g}p \tag{9.3.7}$$

where $p$ is the pressure, expressed as $p = \rho_0(\sqrt{v^\mu v_\mu} - TS)$, $\rho_0$ is the rest mass energy distribution and $T$ is the temperature. Only the field $S$ has a direct physical interpretation: it is the entropy per barion.
When treated with the canonical formalism this model turns out to be a constrained system, with numerous secondary and tertiary constraints, which require some restrictions on the phase space variables. Despite the number and the complexity of the constraints one can perform the KB procedure, and finally obtain a meaningful Schrödinger equation, where the *entropy* is linked to the time variable by the constraint equation:

$$Sp^S \pm \frac{\theta}{T}H^G = Sp^S + h = 0. \tag{9.3.8}$$

So that taking the logarithm of $S$ as the time variable $\tau$ one can write down the infinitesimal time evolution of observables as:

$$\frac{dO(\tau)}{d\tau} = \{H_{phys}, O(\tau)\} \tag{9.3.9}$$

where $H_{phys}$ is nothing else that the integration on the 3-dim hypersurfaces of function $h$ in 9.3.8. This ensures that the Schutz model is a good matter clock when coupled with General Relativity.

## 9.4. Evolutionary Quantum Gravity

We establish a fundamental link between the identification of a reference and the appearance of a matter term from the point of view of Lagrangian symmetries. In particular, by fixing a synchronous frame of reference, which is characterized by a metric tensor having the following fixed components $g_{00} = 1$ and $g_{0i} = 0$, general covariance is restricted to the invariance under the following set of coordinate transformations

$$t' = t + \xi(x^l)\,, \quad x^{i'} = x^i + \partial_j\xi \int h^{ij}dt + \phi^i(x^l)\,, \tag{9.4.1}$$

$\xi$ and $\phi^i$ being three generic space functions.

This feature implies replacing the super-Hamiltonian and the super-momentum constraints with the following ones,

$$H^* \equiv H - \mathcal{E}(x^l) = 0, \qquad H_i = 0, \tag{9.4.2}$$

$\mathcal{E}$ being a scalar density of weight $1/2$, hence it can be written as $\mathcal{E} \equiv -2\sqrt{h}\rho(t,\ x^i)$, with $\rho$ a scalar function.

Hence the super-momentum still vanishes, while the super-Hamiltonian acquires a non-vanishing eigen-value, which can be interpreted as the emergence of a dust fluid co-moving with the slicing.

One can think at this contribution as the physical realization of the synchronous reference. However, it is clear that we are not dealing with an external matter field since its energy density $\rho$ is not always positive and $\mathcal{E}(x^i)$ is fixed, once initial conditions are assigned on a non-singular hypersurface.

We perform quantization of the synchronous gravitational field in a canonical way and we implemented according with the Dirac prescription, so fixing an evolutionary character for wave functional, which can be described by the Schrödinger equation

$$i\hbar\partial_t\chi = \int_{\Sigma_t^3} \hat{H} d^3x\chi\,. \tag{9.4.3}$$

Therefore, the quantum features of the dust contribution outline its behavior as a clock-like matter. The next task is to find out a negative portion of the super-Hamiltonian spectrum, which allows to interpret the additional contribution as a physical matter field.

This can be done in a generic inhomogeneous cosmological setting, where the 3-metric is given by

$$h_{ij} = e^{q_a}\delta_{ad}O^a_bO^d_c\partial_i y^b\partial_j y^c, \quad a,b,c,d,\alpha,\beta = 1,2,3, \tag{9.4.4}$$

with $q^a = q^a(x^l,t)$ and $y^b = y^b(x^l,t)$ six scalar functions and $O^a_b = O^a_b(x^l)$ a $SO(3)$ matrix.

The dynamics of different points decouples near the singularity and the Schrödinger functional equation splits to the sum of $\infty^3$ independent point-like contributions as follows (we denote by the subscript $x$ any minisuperspace quantity)

$$\begin{aligned} i\hbar\partial_t\psi_x = \hat{H}_x\psi_x = \frac{c^2\hbar^2k}{3}\left[\partial_\alpha e^{-3\alpha}\partial_\alpha - e^{-3\alpha}\left(\partial_+^2 + \partial_-^2\right)\right]\psi_x - \\ -\frac{3\hbar^2}{8\pi}e^{-3\alpha}\partial_\varphi^2\psi_x - \left(\frac{1}{2k\mid J\mid^2}e^\alpha V(\beta_\pm) - \frac{\Lambda}{k}e^{3\alpha}\right)\psi_x \end{aligned} \tag{9.4.5}$$

$$\psi_x = \psi_x(t,\ \alpha,\ \beta_\pm,\ \varphi)\,, \tag{9.4.6}$$

where a cosmological constant $\Lambda$ and a scalar field $\varphi$ have been added to the dynamical description.

If an integral representation is taken for the wave function $\psi_x$

$$\psi_x = \int d\mathcal{E}_x \mathcal{B}(\mathcal{E}_x)\sigma_x(\alpha,\ \beta_\pm,\ \varphi,\ \mathcal{E}_x) exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} N_x \mathcal{E}_x dt'\right\} \tag{9.4.7}$$

$$\sigma_x = \xi_x(\alpha,\ \mathcal{E}_x)\pi_x(\alpha,\ \beta_\pm,\ \varphi)\,, \tag{9.4.8}$$

where $\mathcal{B}$ is fixed by the initial conditions at $t_0$, the dynamics is given by

$$\hat{H}\sigma_x = \mathcal{E}_x\sigma_x \tag{9.4.9}$$

$$\left(-\partial_+^2 - \partial_-^2 - \frac{9\hbar^2}{8\pi c^2 k}\partial_\varphi^2\right)\pi_x - \frac{3e^{4\alpha}}{2c^2\hbar^2 k^2 \mid J\mid^2} V(\beta_\pm)\pi_x = v^2(\alpha)\pi_x \tag{9.4.10}$$

$$\left[\frac{c^2\hbar^2 k}{3}\left(\partial_\alpha e^{-3\alpha}\partial_\alpha\xi_x + e^{-3\alpha}v^2(\alpha)\right) + \frac{\Lambda}{k}e^{3\alpha}\right]\xi_x = \mathcal{E}_x\xi_x\,. \tag{9.4.11}$$

Let us now consider wave packets which are flat over the width $\Delta\beta \sim 1/\Delta v_\beta \gg 1$ ($\Delta v_\beta$ being the standard deviation in the momenta space).

In the new variable $\tau = e^{3\alpha}$, the equation (9.4.10) reads

$$\frac{c^2\hbar^2 k}{3}\left(9\frac{d^2}{d\tau^2} + \frac{v^2}{\tau^2}\right)\xi_x + \frac{\Lambda}{k}\xi_x = \frac{\mathcal{E}_x}{\tau}\xi_x\,. \tag{9.4.12}$$

A solution to equation (9.4.12) is provided by

$$\xi_x = \tau^\delta f_x(\tau), \qquad \delta = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{9}v^2}\right) \tag{9.4.13}$$

$$f = \mathcal{C}e^{-\beta^2\tau^2 + \gamma\tau}, \qquad \gamma = 2\mid\beta\mid\sqrt{\delta + \frac{1}{2} - \frac{1}{12L_\Lambda^2 l_P^4\beta^2}}, \qquad \frac{1}{L_\mathcal{E} l_P^2} = 6\delta\gamma, \tag{9.4.14}$$

$L_\mathcal{E} = \frac{\hbar c}{\mathcal{E}}$ being the characteristic length associated to the Universe "energy", while $l_P \equiv \sqrt{\hbar c k}$ denotes the Planck scale length. However, the validity of the solution above requires the condition $\beta^2\tau \ll \gamma = 2\sqrt{\delta + \frac{1}{2} - \frac{1}{12L_\Lambda^2 l_P^4\beta^2}} \mid\beta\mid$.

Hence, the quantum dynamics in a fixed space point (*i.e.* over a causal portion of the Universe) is described, in the considered approximation ($\tau \ll 1$), by a free wave-packet for the variables $\beta_\pm$ and $\varphi$ and by a profile in $\tau$ which has a maximum in $\tau = (\gamma + \sqrt{\gamma^2 + 8\delta\beta^2})/4\beta^2$.

If a lattice structure for the space-time is assumed on the Planckian scale, to preserve the reality of $\mathcal{E}$ we have to impose some inequalities, leading to

$$| \mathcal{E}_x | \ll \frac{c^2 k \hbar^2}{l_{Pl}^3} \sim \mathcal{O}(M_{Pl} c^2) \rightarrow L_{\mathcal{E}} \gg l_P \,, \tag{9.4.15}$$

$M_{Pl} \equiv \hbar/(l_{Pl} c)$ being the Planck mass.

Therefore, the existence of a cut-off implies that a ground state exists for the evolutionary approach. Hence it is a natural request to assume the Universe to approach this state during its evolution.

The associated critical parameter turns out to be

$$\Omega_{\mathcal{E}} \equiv \frac{\rho_{\mathcal{E}}}{\rho_c} \ll \mathcal{O}\left(\frac{10^{-2} G M_{Pl}}{c^2 R_0}\right) \sim \mathcal{O}\left(\frac{10^{-2} l_{Pl}}{R_0}\right) \sim \mathcal{O}\left(10^{-60}\right) \,. \tag{9.4.16}$$

Therefore, the dust contribution cannot play the role of dark matter.

Within this scheme a proper quantum to classical transition for the Universe volume can also be described.

## 9.5. Minisuperspace and Generalized Uncertainty Principle

This section is devoted to explain some results obtained in a recent approach to quantum cosmology, in which the notion of a minimal length naturally appears. In particular, this scheme realizes in quantizing a cosmological model by using a modified Heisenberg algebra, which reproduces a Generalized Uncertainty Principle (GUP)

$$\Delta q \Delta p \geq \frac{1}{2}\left(1 + \beta(\Delta p)^2 + \beta\langle \mathbf{p}\rangle^2\right), \tag{9.5.1}$$

where $\beta$ is a "deformation" parameter. The above uncertainty principle (9.5.1) can be obtained by considering an algebra generated by $\mathbf{q}$ and $\mathbf{p}$ obeying the commutation relation

$$[\mathbf{q}, \mathbf{p}] = i(1 + \beta \mathbf{p}^2). \tag{9.5.2}$$

Such a deformed Heisenberg uncertainty principle was appeared in studies on string theory and leads to a fundamental minimal scale. More precisely, from the string theory point of view, a minimal observable length it is a consequence of the fact that strings can not probe distance below the string scale. However, we have to stress that the minimal scale predicted by the GUP is, by its nature, different from the minimal length predicted by other approaches. In fact, the equation (9.5.1) implies a finite minimal uncertainty in position $\Delta q_{min} = \sqrt{\beta}$. This way, we will introduce a minimal scale in the quantum dynamics of a cosmological model.

Of course the appearance of a nonzero uncertainty in position pose some difficulty in the construction of an Hilbert space. In fact, as well-known, no physical state which is a position eigenstate can be constructed. An eigenstate of an observable necessarily has to have vanishing uncertainty on it. Although it is possible to construct position eigenvectors, they are only formal eigenvectors but not physical states. In order to recover information on position, we have to study the so-called *quasiposition wave functions*

$$\psi(\zeta) \sim \int_{-\infty}^{+\infty} \frac{dp}{(1+\beta p^2)^{3/2}} \exp\left( i \frac{\zeta}{\sqrt{\beta}} \tan^{-1}(\sqrt{\beta} p) \right) \psi(p), \tag{9.5.3}$$

where $\zeta$ is the *quasiposition* defined by the main value of the position $\mathbf{q}$ on certain functions, *i.e.*, $\langle \mathbf{q} \rangle = \mathbf{1}$. The quasiposition wave function (9.5.3) represent the probability amplitude to find a particle being maximally localized around the position $\zeta$ (*i.e.*, with standard deviation $\Delta q_{min}$).

It is notable to stress how, the GUP approach relies on a modification of the canonical prescription for quantization, and therefore it can be reliable applied to any dynamical system. Moreover, the application of such a formalism in quantizing a cosmological model allows us to analyze some peculiar features of string theory in the minisuperspace dynamics.

Let us now extend the above framework to the Taub general cosmological model, discussing its quantization in the GUP scheme. The Taub model is a particular case of the Bianchi IX model which line element (in the Misner parametrization) reads

$$ds^2 = N^2 dt^2 - e^{2\alpha} \left( e^{2\gamma} \right)_{ij} \omega^i \otimes \omega^j, \tag{9.5.4}$$

where $N = N(t)$ is the lapse function, the variable $\alpha = \alpha(t)$ describes the isotropic expansion of the Universe and $\gamma_{ij} = \gamma_{ij}(t)$ is a traceless symmetric matrix which determines the shape change (the anisotropy) *via* $\gamma_\pm$. Since the determinant of the 3-metric is given by $h = \det e^{\alpha+\gamma_{ij}} = e^{3\alpha}$, it is easy to recognize that the classical singularity appears for $\alpha \to -\infty$.The Taub model is the Bianchi IX model in the $\gamma_- = 0$ case and thus its dynamics is equivalent to the motion of a particle in a one-dimensional closed domain. Its ADM Hamiltonian in the Poincaré-plane framework is

$$H^T_{ADM} = p_x \equiv p, \qquad x \in [x_0 \equiv \ln(1/2), \infty), \tag{9.5.5}$$

where $x = \ln v$ and the classical singularity now appears for $\tau \to \infty$.

The canonical quantization of this model is not able to solve the classical singularity problem. In fact, the incoming Universe ($\tau < 0$) bounces at the potential wall at $x = x_0$ and then falls toward the classical singularity ($\tau \to \infty$). Such situation is drastically changed in the GUP scheme and two main conclusions can be inferred: (i) The probability amplitude to find the Universe is peaked near the potential wall. In other words, the GUP Taub Universe exhibits a singularity-free behavior. (ii) The large anisotropy states, i.e. those for $|\gamma_+| \gg 1$, are probabilistically suppressed. In fact the Universe wave function appears to be peaked at values of anisotropy $|\gamma_+| \simeq \mathcal{O}(10^{-1})$. In this respect, the GUP wave packets *predict the establishment of a quantum isotropic Universe* differently from what happens in the WDW theory.

When this approach is applied to the Bianchi IX cosmological model we show that three important features. i) The velocity of the anasotropy-particle (Universe) inside the allowed domain of the Mixmaster model grows with respect to the undeformed case. Furthermore, although the dynamics is still Kasner-like, two negative Kasner indices are now allowed. Therefore, during each Kasner era, the volume of the Universe can contracts in one direction while expands in the other two. ii) The velocity $\dot{\gamma}_w$ of the potential walls, bounding the triangular domain of Bianchi IX, is increased by the deformation terms. However, it no rises so much to avoid the bounces of the $\gamma$-particle against the walls, i.e. the particle bounces are not stopped by the GUP effects. As matter of fact, when the ultra-deformed regime is reached the dynamics is that of a particle which bounces against stationary walls (no maximum incidence angle appears). iii) No BKL map (reflection law $\theta_f = \theta_f(\theta_i)$) can be in general analytically computed. In fact, such a map arises from the analysis of the Bianchi II model which is no longer analytically integrable in the deformed scheme. Thus, a non-vanishing minimal uncertainty in the anisotropies complicates so much the Mixmaster dynamics in such a way that each its wall-side is no longer an integrable system. This way, we can conclude that the chaoticity of the Bianchi IX model is not tamed by the GUP effects on the Universe anisotropies.

A relation between the effective dynamics of loop quantum cosmology and the Randall-Sundrum braneworlds scenario can by obtained quantizing the FRW models with the use of the following deformed algebra

$$[\mathbf{q}, \mathbf{p}] = i\sqrt{1 \pm \alpha \mathbf{p}^2}, \tag{9.5.6}$$

where $\alpha > 0$ is a deformation parameter such that for $\alpha = 0$ the ordinary Heisenberg algebra is recovered. In particular, such an algebra is related to the $\kappa$-Poincaré one which is the mathematical structure which describes the so-called doubly special relativity, where an other invariant, observer independent, scale (the Planck scale) is included ab initio in the theory. From this approach the deformed Friedmann equation

$$H^2_{k=0} = \frac{8\pi G}{3}\rho\left(1 \pm \frac{\rho}{\rho_P}\right), \tag{9.5.7}$$

for the flat case is obtained. The most interesting point to be stressed is the equivalence, at phenomenological level, between the $(-)$-deformed Friedmann equation (9.5.7) and the one obtained considering the effective dynamics in loop quantum cosmology. On the other hand, the string inspired Randall-Sundrum braneworlds scenario leads to a modified Friedmenn equation as in (9.5.7) with the positive sign. The opposite sign of the $\rho^2$-term in such an equation, is the well-known key difference between the effective loop quantum cosmology and the Randall-Sundrum framework. In fact, the former approach leads to a non-singular bouncing cosmology while in the latter, because of the positive sign, $\dot{a}$ can not vanish and there is not place for a big-bounce.

## 9.6. Quantum isotropization mechanism

In this section we show how a semi-classical mechanism, which leads to an isotropic configuration for an inhomogeneous quasi-isotropic Universe, can be developed. In particular, we obtain a wave function of the Universe which has a clear probabilistic interpretation when the isotropic scale factor $a$ of the Universe is regarded as a semi-classical variable. It describes the evolution of the anisotropies of the inhomogeneous Mixmaster Universe and its dynamics is traced with respect to $a$, which can be regarded as a semi-classical variable as soon as the Universe expands enough. Explicitly it satisfies the Schrödinger equation in $\tau \propto a^{-3}$

$$i\partial_\tau \chi = \hat{H}_q \chi = \frac{1}{2}\left(-\Delta_\beta + \omega^2(\tau)(\beta_+^2 + \beta_-^2)\right)\chi, \tag{9.6.1}$$

where $\omega^2(\tau) = C/\tau^{4/3}$ is a time-dependent frequency and $C$ being a constant. The exact solution can by obtained the use of the invariants method and on some time-dependent transformations. The wave function of the Universe is spread over all values of anisotropy near the cosmological singularity but, when the radius of the Universe grows, it is asymptotically peaked around the isotropic configuration. In other words, the closed FRW model is naturally the privileged state when a sufficient large volume of the Universe is taken into account. This way, a semi-classical isotropization mechanism for the Universe is obtained.

## 9.7. Polymer Quantum Cosmology

The polymer representation of quantum mechanics is based on a non-standard representation of the canonical commutation relations. In particular, in a two-dimensional phase space, it is possible to choose a discretized operator, whose conjugate variable cannot be promoted as an operator directly.

From a physical point of view, this scheme can be interpreted as the quantum-mechanical framework for the introduction of a cutoff. Its continuum limit, which corresponds to the removal of the cutoff, has to be understood as the equivalence of microscopically-modified theories at different scales. This approach is relevant in treating the quantum-mechanical properties of a background-independent canonical quantization of gravity. In fact, the holonomy-flux algebra used in Loop Quantum Gravity reduces to a polymer-likealgebra, when a system with a finite number of degrees of freedom is taken into account. From a quantum-field theoretical point of view, this is substantially equivalent to introducing a lattice structure on the space. Loop Quantum Cosmology can be regarded as the implementation of this quantization technique in the minisuperspace dynamics.
The Taub model is approached in the scheme of an Arnowitt-Deser-Misner (ADM) reduction of the dynamics in the Poincare plane. As a result, a time variable naturally emerges, and the Universe is described by an anisotropy-like variable. The anisotropy variable and its conjugate momentum are quantized within the framework of the polymer representation. More precisely, the former appears as discretized, while the latter cannot be implemented as an operator in an appropriate Hilbert space directly, but only its exponentiated version exists. The analysis is performed at both classical and quantum levels. The modifications induced by the cutoff scale on ordinary trajectories are analyzed from a classical point of view. On the other hand, the quantum regime is explored in detail by the investigation of the evolution of the wave packets of the universe (Battisti et al., 2008).
From a classical point of view, in the ordinary case, the model can be interpreted as a photon in the Lorentzian minisuperspace, and the classical trajectory is its light-cone. More precisely, the incoming particle bounces on the wall and falls into the classical cosmological singularity. Contrastingly, in the discretized case, the one-parameter family of trajectories flattens, i.e. the angle between the incoming trajectory and the outgoing one is greater than $\pi/2$.
From a quantum point of view, the modified Schroedinger equation is solved. As a result, a modified dispersion relation is found, and wave functions depend on this modified dispersion relation.
The analysis of the corresponding wavepackets shows the implications of the polymer representation of quantum mechanics mostly when a spread weighting function is taken into account. In fact, in this case, as a result, a strong interference phenomenon appears between the incoming (outgoing) wave and the wall. However, as a matter of fact, such an interference phenomenon is not able to localize the wave packet in a determined region of the configuration space, so that the probability density to find the Universe far away the singularity is not peaked, i.e. the cosmological singularity of this model is not tamed by the polymer representation from a probabilistic point of view. Consequently, the incoming particle (Universe) is initially lo-

calized around the classical polymer trajectory. It then bounces against the wall, where the wave packet spreads in the 'outer' region, regains the classical polymer trajectory and eventually falls into the cosmological singularity. This way, we claim that the classical singularity is not solved by this quantization of the model.
The result can be also discussed as compared with the application of the polymer representation of quantum mechanics to other cosmological models, as well as with the implementation of a generalized uncertainty principle to the Taub model itself. In these cases, the peculiarity of this scheme are clarified.

## 9.8. Lorentz Gauge Theory

General Relativity admits two different symmetries, namely the diffeomorphism invariance, defined in the real space-time, and the local Lorentz invariance, associated to the tangent fiber. Such two symmetries reflect the different behavior of tensors and spinors, respectively, when global Lorentz transformations become local, *i.e.*, while tensors do not experience the difference between the two transformations, spinors do. In our proposal, the diffeomorphism invariance concerns the metric structure of the space-time and it finds in the vier-bein fields the natural gauge counterpart, though the gauge picture holds on a qualitative framework. On the other hand, the real gauge symmetry corresponds to the local rotations in the tangent fiber and admits a geometrical gauge field induced by the space-time torsion and its properties.

This picture has led us to infer the existence of (metric-independent) gauge fields of the Lorentz group, identified with ${A_\mu}^{ab}$, which interacts with spinors. The Ricci spin connection ${\omega_\mu}^{ab}$ could not be identified with the suitable gauge field, for it is not a primitive object (it depends on bein vectors) and defines local Lorentz transformations on the tangent bundle.

**Perspectives on observability** We propose here a model to analyze the interaction of a *4*-spinor $\psi$ with the gauge field $A_\mu$ of the Lorentz group (addressed in flat space) (Carlevaro et al., 2008). Using the tetrad formalism, the implementation of the local Lorentz symmetry leads to the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}\,, \qquad \mathcal{L}_0 = \tfrac{i}{2}\,\bar{\psi}\gamma^a e^\mu_a \partial_\mu \psi - \tfrac{i}{2}\,e^\mu_a \partial_\mu \bar{\psi}\gamma^a \psi \,-\, m\,\bar{\psi}\psi\,, \tag{9.8.1}$$

$$\mathcal{L}_{int} = \tfrac{1}{8}\,e^\mu_c\,\bar{\psi}\{\gamma^c, \tau_{ab}\}\,A^{ab}_\mu\,\psi = \tfrac{1}{8}\,e^\mu_c\,\bar{\psi}\;\;2\epsilon^c_{abd}\,\gamma_5\,\gamma^d\,A^{ab}_\mu\,\psi\,. \tag{9.8.2}$$

To study the interaction terms, we perform a *3+1* splitting of the gauge field and impose the time-gauge condition associated to this picture (*i.e.*, $A^{ij}_0 = 0$). Using variational principles, we are able to write down the motion equations for the spinor field. In this scheme, it is convenient to express the Lorentz gauge field trough the fields $C_0 = \frac{1}{4}\,\epsilon^k_{ij0}A^i_k$, and $C_i = \frac{1}{4}\,\epsilon^k_{0ji}A^{0j}_k$, describing rotations and Lorentz boosts respectively.

Our purpose is the analysis of corrections, due to the implementation of the Lorentz gauge theory, and to a one-electron-atom model. In this respect, we look for stationary solutions of the Dirac equation and we express the 4-component spinor $\psi(t, \mathbf{x})$ in terms of two stationary 2-spinors $\chi(\mathbf{x})$ and $\phi(\mathbf{x})$, assuming standard-representation Dirac matrices. To investigate the low-energy limit, we can write the spinor-field total energy in the form $\mathcal{E} = E + m$, obtaining the expression

$$\phi \;=\; \tfrac{1}{2m}\,(\sigma^i p_i + C_0)\,\chi\;. \tag{9.8.3}$$

It is immediate to see that $\phi$ is smaller than $\chi$ by a factor of order $\frac{p}{m}$ (*i.e.*, $\frac{v}{c}$ where $v$ is the magnitude of the velocity): the 2-component spinors $\phi$ and $\chi$ form the so-called *small* and *large components*, respectively.

Using standard Pauli relations, we finally get the following equation for the large components

$$E\,\chi \;=\; \tfrac{1}{2m}\Big[p^2 \,+\, C_0^2 \,+\, 2C_0\,\sigma^i p_i \,+\, \sigma^i C_i\Big]\,\chi\;. \tag{9.8.4}$$

This equation exhibits strong analogies with the electro-magnetic case and the so-called Pauli equation

$$E\,\chi(\mathbf{x}) \;=\; \tfrac{1}{2m}\Big[(\mathbf{p}+\mathbf{A})^2 \,+\, \mu_B\,\boldsymbol{\sigma}\cdot\mathbf{B} \,+\, \Phi\Big]\,\chi(\mathbf{x})\;, \tag{9.8.5}$$

where $\mu_B = e/2m$ is the Bohr magneton and $\mathbf{A}$ denotes the vector potential ($\mathbf{B}$ and $\Phi$ are the external magnetic and electric field respectively). These eqs can be used in the analysis of the energy levels as in the Zeeman effect.

Let us now neglect the term $C_0^2$ in eq. (9.8.4) and implement the symmetry

$$\partial_\mu \to \partial_\mu + A_\mu^{U(1)} + A_\mu{}^{ab}\Sigma_{ab}\;, \tag{9.8.6}$$

with a vanishing electromagnetic potential $\mathbf{A} = 0$. This way, we can introduce a Coulomb central potential $V(r)$ ($E \to E - V(r)$), obtaining the expressions

$$H_0 = \frac{p^2}{2m} - \frac{Ze^2}{(4\pi\epsilon_0)r}\;, \tag{9.8.7}$$

$$H' = \tfrac{1}{2m}\Big[\;2C_0\,(\sigma_i\cdot\,p^i) \,+\, \sigma\cdot\,C^i\;\Big]\;, \tag{9.8.8}$$

which characterize the electron dynamics in a hydrogen-like atom in presence of a gauge field of the LG. It is worth noting the presence of a term related to the helicity of the 2-spinor: this coupling is controlled by the rotation-like component associated to $C_0$. A Zeeman-like coupling associated to the boost-like component $C_i$ is also present.

# 10. Quantum Fields on Classical Background

## 10.1. Dirac equation on a curved spaces and classical trajectories

The interaction between geometry and internal spinor-like degrees of freedom has been investigated with the aim to infer the analogous of Papapetrou equations for a quantum spin (Cianfrani and Montani, 2008a). This task has been approached by an eikonal approximation, *i.e.*, $\psi = e^{iS}u$, and a localization hypothesis for $u$ along the integral curve of the momentum $K_\mu$. Hence, a dispersion relation has been recovered starting from the squared Dirac equation and by virtue of an integration on spatial coordinates. This way, the following relation has been obtained

$$(K_\mu K^\mu - K^\mu S_\mu)(1 + O(\lambda^2)) + \mu^2 = 0, \tag{10.1.1}$$

$\lambda$ and $\mu$ being the Compton length of the particle and the mass, respectively, while the quantity $S_\mu$ reads as

$$S_\mu = 2i\frac{\bar{u}_0\gamma^{\bar{0}}D_\mu u_0 - D_\mu\bar{u}_0\gamma^{\bar{0}}u_0}{\bar{u}_0\gamma^{\bar{0}}u_0}. \tag{10.1.2}$$

Hence the dynamics of $K_\mu$ is obtained by acting on the relation (10.1.1) with the derivative operator $\nabla_\nu$ and we have

$$\left\{\begin{array}{c} U^\mu\nabla_\mu P_\nu - \frac{\hbar}{2}R_{\rho\sigma\mu\nu}U^\mu S^{\rho\sigma} - \hbar\nabla_\nu U^\mu S_\mu - \\ -2i\hbar U^\mu D_{[\nu}\bar{u}_0\gamma^{\bar{0}}D_{\mu]}u_0 + O(\lambda^2) = 0 \\ P_\nu = K_\nu - S_\nu \end{array}\right. . \tag{10.1.3}$$

Here the quantity $S^{\mu\nu}$ is given by the expression

$$S^{\mu\nu} = \frac{\int d^3x\sqrt{h}\bar{u}\{\gamma^{\bar{0}}, \Sigma^{\mu\nu}\}u}{2\int d^3x\sqrt{h}\bar{u}\gamma^{\bar{0}}u} = \frac{\bar{u}_0\{\gamma^{\bar{0}}, \Sigma^{\mu\nu}\}u_0}{2\bar{u}_0\gamma^{\bar{0}}u_0} + O(\lambda^2), \tag{10.1.4}$$

for which we have

$$S^{\nu\mu}U_\nu = 0, \tag{10.1.5}$$

Since we are performing a multi-pole expansion, it is possible to assume that

$$D_\mu u_0 = iU_\mu v, \tag{10.1.6}$$

$v$ being an arbitrary spinor. It can be shown that this hypotheses is well-grounded by an analysis on the dynamics of the wave-function.

This way, the following equations are obtained

$$U^\mu \nabla_\mu U_\nu - \frac{\hbar}{2} R_{\rho\sigma\mu\nu} U^\mu S^{\rho\sigma} = 0 \tag{10.1.7}$$

Therefore, Dirac particles follow the trajectory of classical spinning ones (according with the Mathisson-Papapetrou formulation), whose spin tensor is given by $S_{\mu\nu}$ (10.1.4).

# 11. Unification Theories

Kaluza-Klein theories aim at providing a geometrical interpretation for gauge degrees of freedom, by arranging bosons into the metric tensor. This scheme implies to deal with a space-time having more than four-dimensions, being the additional space compactified to distances not yet accessible to experiments. In these models, the metric tensor takes the following form

$$j_{AB} = \left( \begin{array}{c|c} g_{\mu\nu}(x^\rho) + \gamma_{mn}(x^\rho; y^r)\xi^m_{\bar{M}}(y^r)\xi^n_{\bar{N}}(y^r)A^{\bar{M}}_\mu(x^\rho)A^{\bar{N}}_\nu(x^\rho) & \gamma_{mn}(x^\rho; y^r)\xi^m_{\bar{M}}(y^r)A^{\bar{M}}_\mu(x^\rho) \\ \hline \gamma_{mn}(x^\rho; y^r)\xi^n_{\bar{N}}(y^r)A^{\bar{N}}_\nu(x^\rho) & \gamma_{mn}(x^\rho; y^r) \end{array} \right),$$

$x^\mu$ and $g_{\mu\nu}$ being the four-dimensional coordinates and metric, respectively, while $y^m$ and $\gamma_{mn}$ are the analogous ones on the extra-dimensional space, endowed with Killing vectors $\xi^m_{\bar{M}}$.

The possibility to geometrize a gauge theory is encoded in the existence of a homogeneous manifold, whose isometries reproduce the algebra of the gauge group in the following way

$$\xi^n_{\bar{N}} \frac{\partial \xi^m_{\bar{M}}}{\partial y^n} - \xi^n_{\bar{M}} \frac{\partial \xi^m_{\bar{N}}}{\partial y^n} = C^{\bar{P}}_{\bar{N}\bar{M}} \xi^m_{\bar{P}} , \tag{11.0.1}$$

where $C^{\bar{P}}_{\bar{N}\bar{M}}$ indicate the structure constants of the Lie group. It is possible to interpret $A^{\bar{M}}_\mu$ as gauge bosons since, by the dimensional reduction of the Einstein-Hilbert action, the Yang-Mills Lagrangian density comes out.

## 11.1. 5-Dimensional Kaluza-Klein model

Within the unification picture provided by Kaluza Klein (KK) theory, the 5-Dimensional (5D) model is the simplest one and the starting point for the investigation of the breaking of multidimensional gravity into the usual gravity plus Yang-Mills fields. It is characterized by an abelian structure; indeed, it provides the coupling between gravity, a U(1) gauge field and an extra scalar field. If the scalar field it is assumed to be constant from the beginning, the 5D model reproduces exactly the Einstein-Maxwell theory in vacuum.

However, an interesting perspectives deals with the dynamics of the scalar field and its conjugate momentum; starting from the Hamiltonian of the model it is possible to see via the Kuchar-Brown approach that the momentum $\pi_\phi$ de-parametrizes from hamiltonian constraints and we are therefore able to write down a Schroedinger-like equation. This is a hint toward the interpretation of $\phi$ as a time variable in the relational point of view. People involved are Valentino Lacquaniti, Giovanni Montani and Simone Zonetti (Lacquaniti et al., 2008b).

**(i) Matter Coupling** The problem of the matter coupling is a longstanding puzzle that affects KK models from the foundation. Indeed, while KK models are successful in vacuum, they show unsatisfactory features when the presence of matter is considered. The standard approach to the dynamics of test particles is to generalize to five dimensions the "geodesic" Action usually adopted in 4D, namely $S = m\int ds$. Therefore, starting from $S_5 = \hat{m}\int ds_5$, where $\hat{m}$ is the 5D mass parameter, it is shown via dimensional reduction, that the motion of a free 5D test particle is reduced into the motion of a 4D test particle interacting with the electromagnetic field, plus the extra scalar field. In such a scheme, the $q/m$ ratio is defined in term of the fifth component of the 5D-velocity $w_5$, which is a constant of the motion. Even if electrodynamics is formally restored, setting $\phi = 1$, the $q/m$ ratio results to be upper bounded in such a way that this bound cannot be satisfied by every known elementary particles. In the simple case $\phi = 1$, indeed we have:

$$\left\{ \begin{array}{l} \frac{d}{ds}w_5 = 0 \\ \frac{D}{Ds}u^\mu = \sqrt{4G}F^{\mu\nu}u_\nu\left(\frac{w_5}{\sqrt{1+w_5^2}}\right) \end{array} \right. ,$$

where

$$q/m\sqrt{4G} = \frac{w_5}{\sqrt{1+w_5^2}} < 1 .$$

The problem of the geodesic approach relies in a bad definition of the rest mass of the particle. By studying the Hamiltonian formulation of the dynamics we can get the dispersion relation for the $4D$ reduced particle: such a relation is consistent with an interacting particle whose charge $q$ and mass $m$ arise defined as follows:

$$q = \sqrt{4G}P_5 \qquad m^2 = \hat{m}^2 + P_5^2/\phi^2$$

Given that $P_5 = \hat{m}w_5$, in the case $\phi = 1$, we recover the previous bound. These relations show that the physical mass $m$ of the particle does not coincide with the mass parameter $\hat{m}$ we put in the Action; moreover, if we consider the compactification of the extra dimension, we get a quantized charge, as well as a tower of massive modes; but, fixing the length of the extra dimension using the value of the elementary charge, we get massive modes beyond Planck scale ( which is indeed the order of magnitude of mass requested by the $q/m$ bound ). Hence, the 5D geodesic approach is not able to take into account the definition of the rest mass for a test particle. This relies in the fact that in the 5D KK model the Equivalence Principle is violated, thus the 5D test particle does not follow a geodesic motion. We are studying a new scheme based on the multipole expansion of Papapetrou. This approach deals directly with a generic 5D matter tensor and is able, via a Taylor expansion, to takes into account the case of a test particle, *i.e.*, a localized particle, as it happens in the 4D theory, where the test particle is recognized as the single-pole order of the Papapetrou expansion. When applied in the 5D framework, the Papapetrou approach reproduces the electrodynamics motion of a 4D particle, plus the interaction with an extra scalar field. This new equation is formally the same as the old one, but coupling factors are now different and no bound appears for $q/m$. The reason is that coupling factors are now defined by means of degrees of freedom provided by the 5D matter tensor, while in the geodesic approach they are defined only via the kinematical constant of motion. Mass is defined via the component $T^{00}$, a conserved current is defined in terms of components $T_5^{\mu}$, and an additional coupling factor $A$, between matter and the KK extra scalar field, is found to be related to $T_{55}$. Moreover, this model reproduces electrodynamics and correct couplings not only at the test particle level, but also for a generic kind of 4D matter, described by a 4D tensor $T^{\mu\nu}$. If the extra scalar field is assumed to be constant, this model reproduces exactly the classical electrodynamics; if the extra scalar field is allowed to vary, the most important difference is that the model shows as a new feature a varying mass, depending on the scalar field and on the coupling factor $A$. The new equations reads

$$m\frac{Du^{\mu}}{Ds} = A(u^{\rho}u^{\mu} - g^{\mu\rho})\frac{\partial_{\rho}\phi}{\phi} + qF^{\mu\rho}u_{\rho}\ ,$$

with

$$\frac{dm}{ds} = -\frac{A}{\phi}\frac{d\phi}{ds}\ .$$

This equation for the motion can also be obtained by the Action $S_5 = -\int m\, ds + q(A_{\mu}dx^{\mu} + \frac{dx^5}{\sqrt{4G}})$, where $m$ is now a variable function whose derivatives are known. Charge is still identified with $P_5$ as in the geodesic approach, but now the parameter $m$ we put in the Action represents the physical rest mass of the particle, without carrying extra factor, as it is confirmed by the analysis of Hamiltonian and conjugate momenta. This means that the additional factor $P_5^2/\phi^2$ now does not affects the definition of mass; indeed the novelty of this scheme is the removal of the huge massive modes we usually have in KK models. People involved are Valentino Lacquaniti and Giovanni Montani (Lacquaniti and Montani, 2008a).

Most promising perspectives of this line of research are the following:

- KK model with generic matter fields: following the analysis performed on the particles dynamics is possible to consider a KK model with a generic matter source; extending the results obtained for particle we identify 4D sources starting from a conserved 5D matter tensor; therefore we can write down equations for the full model with fields plus matter:

$$G^{\mu\nu} = \frac{1}{\phi}\nabla^{\mu}\partial^{\nu}\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi + 8\pi G\phi^2 T^{\mu\nu}_{em} + 8\pi G\frac{T^{\mu\nu}_{matt}}{\phi}$$
$$\nabla_{\nu}\left(\phi^3 F^{\nu\mu}\right) = 4\pi j^{\mu}$$
$$\Box\phi = -\frac{1}{4}\phi^3(ek)^2 F^{\mu\nu}F_{\mu\nu} + \frac{8}{3}\pi G\left(T_{matter} + 2\frac{T_{55}}{\phi}\right)$$

Given this model, next step to be pursued are: i) search for the general Lagrangian formulation and definitions of appropriate conserved currents related to gauge symmetries ii) analysis of the role of the extra source term $T_55$, which is linked to the coupling factor A for particles, in the dynamics. It is possible to find scenarios where we recover the free falling universality of particles, without necessarily setting $\phi = 1$, and this topic deserves some effort to be pursued. People involved are Valentino Lacquaniti and Giovanni Montani (Lacquaniti and Montani, 2008b). Also interesting is to search for a generic cosmological solution of the equations, especially near the singularity; people involved are Riccardo Benini, Valentino Lacquaniti, Giovanni Montani.

- Dark matter models with particles of running mass depending on a scalar field: this topic involves Valentino Lacquaniti and Giovanni Montani, with an ongoing collaboration with Massimiliano Lattanzi ( University of Oxford ) as far as model with mass varying neutrinos are considered, and with an ongoing collaboration with Luca Amendola (OAR/INAF ) and Cinzia Di Porto (University of Roma "RomaTre" ) where models with a non conserved 4D matter tensor are taken into account.

**(ii) Geodesic Deviation** In a work of Kerner et al. (2000) the problem of geodesic deviation in 5D KK is faced. The 4D space-time projection of the obtained equation is identical with the equations obtained by direct variation of the usual geodesic equation in the presence of the Lorenz force, provided that the fifth component of the deviation vector satisfies an extra constraint there derived . The analysis was performed taking $\phi = 1$ and it was developed within the scheme of the geodesic approach. Therefore, our research focused on the extension of this work to the model where the presence of the scalar field is considered. Our results coincide with those of Kerner et al. when the minimal case $\phi = 1$ is considered, while it shows some departures in the general case. The novelty due to the presence of $\phi$ is that the variation of the $q/m$ between the two geodesic line is not conserved during the motion; an exact law for such a behavior has been derived. In principle such a results is interesting in order to check if it is possible to find a mark of the extra dimension via tidal effects due to the scalar field. Other perspective is to deal with this topic addressing the Papapetrou approach : the aim is to find the geodesic deviation via the localization hypothesis on the matter tensor around two world lines separated by an infinitesimal displacement, and compare the result with respect to the pure geodesic approach. People involved are Valentino Lacquaniti, Giovanni Montani, Francesco Vietri (Lacquaniti et al., 2008a).

**(iii) Spherical Solutions** This research line is in collaboration with P. Chardonnet. It consists in searching an extra-dimensional scenario for the formation of so-called "Small-Mass Black Holes" (SMBH). These mini black holes are supposed to have mass $m \sim 10^{17} g$ and a Schwarzschild radius between the size of the proton and of the atom. The SMBH model is strictly linked to an experimental evidence of 511 $keV$ annihilation line in the Galactic Center. These objects, with the mechanism of their accretion disk, could reproduce this emission. But, more important, they can be a good candidate for dark matter. Their mass is too small to be detected by micro-lensing and too big to evaporate by Hawking scenario. The problem is that the usual fluctuation scenario seems not able to explain these objects; it provides a continuous spectrum, depending on the mass, while it would be preferable to have a peak on the supposed mass that fits with the observed emission. Therefore, the scheme of this research is to apply the multidimensional KK model, that provides extra parameters (*i.e.*, scalar fields). First step is the analysis of the corresponding Schwarzschild solution : unfortunately in the 5D KK model we do not have a Birkhoff-like theorem, therefore spherical solution are not unique. Some solutions with singularity, of the form $f(r,t) = g(r)h(t)$ are known, but they do not represent black holes; in the usual KK model without matter then they represent solitons, while in our model with matter we can see them as the exterior solution for fields given by a spherical matter distribution. The black-hole like solution are to be looked for in a generic solution $f(r,t)$. People involved are Pascal Chardonnet, Valentino Lacquaniti, Giovanni Montani.

# 12. Activities

This group lives within the Relativistic Astrophysics Center at the Physics Department of "Sapienza" University of Rome (**Prof. Remo Ruffini** - $2^{nd}$ Chair in Theoretical Physics). It deals with three main research lines, each of them aimed to specific topics, according to the following scheme:

- ***Early Cosmology***:

  Chaotic Universes, Dissipative cosmologies

- ***Quantum Gravity***:

  Quantum cosmology, The problem of time

- ***Multidimensional Physics***:

  Particle and Field dynamics in Kaluza-Klein theories,
  Geometrization of the gauge connection (the electroweak model)

The group is directed by **Dr. Giovanni Montani** and it is composed of about ten members, undergraduate students, PhD students and post-docs. The main goal of this investigation paradigm is to find, through different aspects of the gravitational field, markers for a unification picture of the fundamental interactions. In this respect, the Cosmological framework is the natural arena of this expected scenario.

## 12.1. Seminars and Workshops

### 12.1.1. I Stueckelberg Workshop on Relativistic Field Theories

Period: 26-30 June 2006
***Main Lecturer: A. Ashtekar***
Introduction to Loop Quantum Gravity trough Cosmology

**Talks:**

- G. Montani: *Evolutionary Quantum Gravity*.
- L. Titarchuk

- S. Mercuri: *Nieh-Yan invariant and Fermions in Ashtekar-Barbero-Immirzi formalism*
- G. Vereshchagin: *Non-singular solutions in Loop Quantum Cosmology*
- V.A. Belinski: *New developments in Einstein-Maxwell Theory: non-perturbative approach*
- A. Geralico: *New developments in Einstein-Maxwell Theory: perturbative approach*
- D. Bini: *Relative strains in General Relativity*
- F. Cianfrani, O.M. Lecian: *Stuckelberg: a forerunner of modern physics*
- R. Benini: *Multi-Time approach to the Generic Quantum Cosmology*
- M.V. Battisti: *Generic Evolutionary Quantum Universe*
- S.S. Xue: *Gravitational instantons and the cosmological term*
- V. Laquaniti: *Hamiltonian formulation to 5-dimensional Kaluza-Klein Theory*
- F. Cianfrani: *Spinning particle in the gravitational field*
- O.M. Lecian: *Electroweak Model within the framework of Lorentz Gauge Theory: Ashtekar variables?*
- F. Cianfrani: *The Electroweak Model within Kaluza-Klein Framework*

### 12.1.2. II Stueckelberg Workshop on Relativistic Field Theories

Period: 3-7 September 2007
***Main Lecturers: T. Thiemann,***
Loop Quantum Gravity and Recent Developements
***T. Damour***
Coalescing Binary Black Holes and Chaos in String Cosmology

**Talks:**

- G. Montani: *Sincrhonous Quantum Gravity: Early Universe dynamics*
- F. Cianfrani, O.M. Lecian: *Stuckelberg: a forerunner of modern physics II*
- F. Cianfrani: *The role of the time gauge in the 2nd order formalism*
- G.V. Vereshchagin: *Semi-classical Loop Quantum Cosmology*
- E. Magliaro, C. Perini: *Comparing loop quantum gravity with the linearized theory*
- E. Alesci: *The full graviton propagator from loop quantum gravity*

- G. Vereshchagin: *Thermalization of the pair plasma*
- N. Carlevaro: *Lorentz Gauge Theory and spinor interaction*
- K. Giesel: *Dirac observables*
- O.M. Lecian: *Exponential Lagrangian for the gravitational field and the problem of vacuum energy*
- R. Zalaletdinov: *Macroscopic Gravity and Averaging Problem in Cosmology*
- S.S. Xue: *The gravitational origin of fermion masses*
- C. Sigismondi: *Meteotsunami detection in Adriatic and Tyrrenian sea*
- G. Amelino-Camelia: *Quantum Gravity Phenomenology and a generalization of the Noether theorem for quantum spacetime*
- M.V. Battisti: *Minisuperspace dynamics in the GUP framework*
- A. Geralico: *Perturbations of a Reissner-Nordström black hole by a charged massive particle at rest*
- K. Giesel: *Geometrical Operators*
- S. Xue: *Electron-positron productions in inhomogeneous electric fields*
- F. Cianfrani: *Curvature-spin coupling from the semi-classical limit of the Dirac equation*
- N. Carlevaro: *On the role of viscosity in Early Cosmology*
- S. Zonetti: *The parametrizing fluids in canonical Quantum Gravity*
- F. Cianfrani: *Elementary particle interaction from a Kaluza-Klein scheme*
- V. Lacquaniti: *On the problem of the matter coupling in a 5d Kaluza-Klein theory"*
- O.M. Lecian: *Extended fundamental space-time symmetries*
- C. Perini: *Noncommutative geometries: an overview*
- M. Pizzi: *Electric force lines of the double Reissner-Nordstrom solution*
- R. Benini: *Mixmaster dynamics in the Wheeler-DeWitt framework*
- F. Zonca: *The Physics of Burning Plasmas in Toroidal Magnetic Field Devices*
- I. Milillo: *On the coupling between spinning particles and cosmological gravitational waves*
- M.V. Battisti: *Cosmological implication of Evolutionary Quantum Gravity*

### 12.1.3. III Stueckelberg Workshop on Relativistic Field Theories - III Session

Period: 15-18 July 2008
***Main Lecturers: Prof. G. 't Hooft.***

**Talks:**

- A. Zhuk: *Early Inflation in Non-Linear Multidimensional Cosmological Models*
- G. Montani: *Perspectives in Cosmology, Gravitation and Multidimensions*
- Prof G. 't Hooft: *Quantum Mechanics, Discretization and Local Determinism*
- R. Benini, O.M. Lecian: *E.C.G. Stueckelberg: a Forerunner of Modern Physics III*
- F. Cianfrani: *The Role of Time-Gauge in Quantizing Gravity*
- M.V. Battisti: *Time Evolution of a Generic Quantum Universe*
- S. Zonetti: *Fluid Entropy as Time-Variable in Canonical Quantum Gravity*
- N. Carlevaro: *New Issues in Lorentz Guage Theories*
- E. Alesci: *Graviton Propagator in LQG: a Tool to Test Spinfoam Models*
- N. Carlevaro: *Gravitational Instability in Presence of Dissipative Effects*
- M.V. Battisti: *Quantum Cosmology in the GUP Approach*
- R. Belvedere: *Quantum Isotropization Mechanism for the Mixmaster Model*
- F. Cianfrani: *Review on Extended Approaches in the Kaluza-Klein Model*
- O.M. Lecian: *Recent Approaches to Modified-Gravity Theories*
- M.V. Battisti: *Extended Approach to the Canonincal Quantization in the Minisuperspace*
- O.M. Lecian: *The Taub Universe viewed in a Polymer Quantization Approach*
- L. Lusanna: *Towards Relativistic Atomic Physics and Relativistic Entanglement*
- R. Benini: *Review on the Generic Cosmological Solution Near the Singularity*
- T.P. Shestakova: *The Extended Phase Space Approach to Quantum Geometrodynamics*
- S. Mercuri: *From the Einstein-Cartan to the Ashketar-Barbero formulation of Gravity and a possible interpretation of the Immirzi parameter*
- G. Fodor: *Almost Periodic Localized Systems: Oscillons and Oscillatons*
- V. Lacquaniti: *Recent Developement in Particle and Field Motion within the Kaluza-Klein Picture*
- D. Pugliese: *Deformations of Spacetime Metrics*
- F. Vietri: *Geodesic Deviation on a Kaluza-Klein Background*

- A. Cardinali: *Asymptotic Techniques and Hamiltonian Formalism in the Solution of the Electromagnetic Wave Equation in Ionized Gases*
- Prof. G. 't Hooft: *A Locally Finite Model for Gravity*
- G. Barbiellini: *Fulmini e Saette*
- S. Bernuzzi: *Gravitational Waves from Neutron Star Oscillation*
- A. Nagar: *Gravitational Waves from Binary Black Hole Coalescence*
- M. Lattanzi: *Constraining Dark Matter Models Through 21cm Observations*
- S. Capozziello: *Dark Energy Models toward Observational Tests and Data*
- G. Amelino-Camelia: *Falsifiable Planck-Scale Theories of NOT Everything*
- M.F. De Laurentis: *Stochastic background of gravitational waves "tuned" by f(R) gravity*
- L. Izzo: *Detection of the CSB of GW in f(R) gravity with FASTICA*
- C. Corda: *Signals and Interferometric Response Functions in the Framework of GW Arising from Ext.Th.Grav.*

### 12.1.4. ICRA Seminars on Quantum Gravity

**Preface:** Aim of this seminars was to fix some relevant links between different approach to the cut-off physics. Particular attention has been devoted to the mathematical framework underlying Loop Quantum Gravity in view of clarifying how the notion of a minimal length lives within the local Lorentz gauge symmetry. In this respect, some implications of Loop Quantum Cosmology are discussed and the features of a Big-Bounce, replacing a Big-Bang, are outlined in some detail. As alternative approach to the cut-off physics, we dealt with the non-commutative structure of the space-time, especially on the base of a Generalized Uncertainty Principle (GUP) formalism. The issues of a GUP quantum dynamics were described in the case of a non-relativistic particles, while the proposal for a real quantum field theory was critically revised. Finally interesting aspects of a path integral theory for the gravitational field were analyzed within a metric formulation and attention to the gauge fixing procedure was devoted.

**Polymer representation and GNS construction: fundamentals and applications** - *Speakers: O.M. Lecian, M.V. Battisti*
Abstract: We briefly review the main steps of the GNS construction. We then discuss the polymer representation of quantum mechanics within this framework, and eventually outline the features of its continuum limit for the toy-model of a one-dimensional particle. Furthermore, we implement this paradigm also to the case of a scalar field, and the differences between this picture and the Schroedinger representation are pointed out. As a result, we

the possibility to apply such a framework to the quantization of cosmological models is envisaged, especially in view of the removal of the cosmological singularity.

**On the notion on Classical limit in relativistic and non-relativistic limit** - *Speaker: G. Montani*

**An introduction Wilson Loop** - *Speaker: F. Cianfrani*
Abstract: The emergence of Wilson loops in the strong coupling limit of lattice gauge theories is outlined, stressing how this formulation easily accomplished the issue of gauge invariance. Then, the appealing features of a quantization of gravity in terms of loops are sketched, in particular with respect to the appearance of non canonical commutation relations and of a numerable basis in the Hilbert space.

**Mixmaster chaos from the Loop Quantum Cosmology point of view** - *Speaker: M.V. Battisti*
Abstract: The chaotic behavior of the Bianchi IX model, in the dynamics toward the classical singularity, is investigated in the Loop Quantum Cosmology framework. Starting from an isotropic settings, we review the key points that brings to a non-chaotic dynamics for such a model, as soon as the quantum effects become important. We also point out the problems and ambiguities of such a framework, in particular in the formulation of the Hamiltonian constraint.

**Quantization of theories with constraints I** - *Speaker: S. Zonetti*
Abstract: Gauge theories, when treated with an hamiltonian formalism, behave as constrained theories, where conditions between the canonical variables hold. At first two kinds of constraints can be recognized: primary and secondary constraints. The former appearing with Lagrange multipliers, the latter arising from the consistency conditions, i.e. time indipendence, of the others. This distinction, however, is not essential, and the more usefull classification based on the Poisson algebra is adopted. This way it is possible to discard the Poisson bracktes and adopt the Dirac ones, constructed with 2nd class constraint algebra matrix, and embed these constraints in the inner structure of the theory. Now one gets a theory with at most 1st class constraints, that generate the gauge transformations.

**Quantization of theories with constraints II** - *Speaker: F. Cianfrani*
Abstract: The geometrical interpretation of first- and second-class constraints in the phase space is outlined, with the aim to demonstrate the different reduction of degrees of freedom they produce. Furthermore, the quantization

of such constrained system is analyzed, which provide us with a demonstration of how the Fadeev-Popov determinant arises in the path-integral formulation.

**BRST symmetries** - *Speaker: M. Castellana*
Abstract: After the experimental observation of neutral-currents processes in 1973, the requirement for a proof of renormalizability of non-abelian gauge theories predicting the existence of such processes became an essential point in quantum field theory. The discovery of BRST symmetry for the Yang - Mills action made the electroweak model predicting these processes a consistent theoretical framework. As a matter of fact, this underlying symmetry of the gauge fixed action allowed to show it to posses all the required term to make it renormalizable. BRST symmetry showed to apply to a really wide class of systems of physical interest, and can be easily generalized to a generic system which possessing some basic gauge symmetry. To this end, we recall that in the literature there exist different formulations for the BRST formalism, with substantial differences from each other. On one side there exists a formulation of BRST symmetry for constrained systems based on canonical quantization methods which is widely diffused and on the other hand there is another approach to derive BRST symmetry based entirely on path integral methods and is applicable to systems with infinite degrees of freedom avoiding those inconsistencies proper of canonical quantization methods we discussed above. In this paper we will follow the latter derivation.

**An introduction to spin foams** - *Speaker: O.M. Lecian*
Abstract: Spin foams will be introduced from a geometrical and field-theoretical point of view. Starting from the definition of spin networks as a generalization of Wilson loops and drawing the analogy with the concept of "plaquettes" will allow one to outline the possibility of recognizing spin-network states as basis for the functionals of the connection. Spin foams, defined as branched surfaces, will accomplish the dual transformation that leads to a physically equivalent description of lattice gauge theory. Particular attention will be paid to the description of physical observables in terms of the path-integral formulation within this formalism, and to the mathematical meaning of these operations. A spin-foam model for Yang-Mills theories will follow, and a background independent spin-foam model for quantum gravity will be obtained by slightly modifying the duality map. The relation between spin-network states and the geometry of spacetime will be further investigated. In particular, covariant quantum gravity will be approached considering spin-network states as states of the gravitational field. Equivalence classes for spin foams will be established, and the interpretation of spin foams as quantum histories will be proposed.

**Generalized uncertainty principle and noncommutative spacetime** - *Speaker: M.V. Battisti*
Abstract: The existence of a minimal observable length has long been suggested when we try to unify Enistein's theory of classical gravity with quantum mechanics principles. The first attempt to describe a Lorentz invariant spacetime with a minimal length was made by Snyder in 1947. This approach is compared with that one in which the authors study in full detail the quantum mechanical structure which underlies a generalized uncertainty relation, which implement the appearance of a nonzero minimal uncertainty in position. At the end, introducing the Moyal star-product, we discuss about the criteria for preserving Poncarè invariance in noncommutative gauge theories.

**Framework of Loop Quantum Cosmology** - *Speaker: M.V. Battisti*
Abstract: Is showed how the classical singularity, of k=0 FRW cosmological model, is removed by quantum geometry. We begin by singling out "elementary functions" on the classical phase space which are to have unambiguous quantum analogs: the almost periodic function and the momenta p. No operator corresponding to connections is defined. Then,we will express the physically interesting operators in terms of these variables. We will show that the WDW equation is transformed in a difference equation, because the area operators has a lowest eigenvalue and, thus, is physically inappropriate to try to localize the curvature on arbitrary small surfaces. References:

**Wheeler-DeWitt equation** - *Speaker: R. Benini*
Abstract: We briefly discuss some features of the Wheeler-DeWitt [WDW] equation: starting from the Arnowitt-Deser-Misner approach to the Einstein action, we discuss the meaning, the problems and some solutions of the WDW. We derive in particular the solutions in the mini super-space of the flat FRW model, with particular attention to the role that the initial conditions have in a cosmological model. Then, the problem of classical constraints while quantizing a theory is discussed from the Multi-time point of view; in this framework some features of the Bianchi type I and IX homogeneous models are presented. The last remark is about the semiclassical states in the quantum Mixmaster. References

**Phenomenology of Lorentz violations** - *Speaker: F. Cianfrani*
Abstract: The fate of Lorentz invariance in Quantum Gravity is an open issue; however, if it will induce Lorentz violating terms in the Lagrangian density, because of radiative corrections, they will be suppressed by the size of Standard Model couplings, unless some sort of fine-tuning on the parameters of such terms. Therefore, the quantum gravity phenomenology of possible Lorentz-violations has already been ruled out by experimental data.

**Covariant formulation of Loop Quantum Gravity** - *Speaker: F. Cianfrani*
Abstract: It is possible to derive the formulation of General Relativity in terms of Barbero-Immirzi connections starting from a variational principle. In particular, the ADM splitting, in the time gauge, of the Holsts action gives the set of constraints predicted by Barbero. In particular, the Super-Hamiltonian and the Super-Momentum ones reproduce the algebra of time-diffeomorphisms and three-diffeomorphisms, respectively. Then, the machinery of the quantization by virtue of holonomies can start. However, holonomies themselves, despite the case of Ashtekar variables, depend on the splitting, therefore connections have no well-defined behavior under time reparametrization. In order to solve this aesthetical issue, Alexandrov introduce a covariant theory, where the splitting is not performed in the time gauge. At the end, the ambiguity due to the Immirzi parameter disappears, but arise second class constraints, which in general cannot be solved and predict modified commutation relations between fields and their conjugates momenta.

**Resolution of the cosmological singularity in Loop Quantum Gravity** - *Speaker: F. Cianfrani*
Abstract: The quantization of the k=0 Freedman-Robertson-Walker spacetime, in a minisuperspace approach, can be performed in the Wheeler DeWitt or in the Loop Quantum Gravity framework. The main distinction between the two procedures deals with the choice of fundamental variables (in the former case they are holonomies and smeared densitized triads) and the definition of the kinemathical Hilbert space; in the dynamical sector, we end up with differential and difference equations, respectively. The origin of difference equations can be traced to the impossibility to take the limit of vanishing area for loops, when the hamiltonian constraint is rewritten in terms of holonomies, since no operator corresponding to connections is defined. Hence, by the group averaging techniques a scalar product can be introduced, so amplitudes, Dirac observables and semiclassical states can be determined. By introducing a massless scalar field as the internal time, the evolution of semiclassical states is studied: while for the Wheeler DeWitt equation, once evolved backward in time, they remain semi-classical till they reach the classical singularity, this is replaced by a bounce in Loop Quantum Cosmology. Moreover, before the bounce a semi-classical contracting phase is predicted. This way, Loop Quantum Cosmology solves the Big Bang singularity.

**A derivation of ADM splitting via the Space-Ambient embedding of a manifold** - *Speaker: V. Lacquaniti*
Abstract: The embedding of a 4D manifold in a 5D external Minkowskyan space (our so-called Space-Ambient) allows us to recover the whole features of General Relativity in a simple vectorial picture. Within this picture its easy to stress the geometrical meaning of the metrics and the covariant derivative.

A similar technique allows us to derive in a fast way the rules of the ADM splitting. It is showed hot to get the rules for the synchronous splitting, for the generic ADM splitting of the metrics and for the projection of a generic tensor; also, it is examined the splitting of the covariant derivative, the definition of the extrinsic curvature and its geometrical meaning. Finally, the Gauss-Codacci formula is provided, with a discussion of the dynamical properties of the Einstein-Hilbert action that arises from this picture.

**Generalized uncertainty principle, noncommutative spacetime and the scalar field** - *Speaker: O.M. Lecian*

Abstract: The gravity-induced breakdown of canonical quantum mechanics in the description of the spacetime at the Plank scale and the emergence of a cut off is described by different mathematical structures. In the framework of a generalized uncertainty principle, the quantization of fields is discussed, and two approaches are presented. The standard QFT established by A.Kempf, based on nonzero minimal lengths and momenta and achieved in the framework of a generalized Bargman Fock representation, is compared to the proposal by T.Matsuo et al., where nonzero minimal momenta only are taken into account, and canonical and path integral quantization are extended to higher dimensions by means of the introduction of new parameters. k-Minkowski and canonical noncommutative spacetimes are presented, and the Moyal product is introduced. In particular, in a canonical noncommutative spacetime scenario, the quantization of the scalar field is studied, and the problem of microcausality (and its possible violation) is investigated between vacuum states and between non-vacuum states: for particular choices of the commutation relations only the microcausality of the scalar field is satisfied.

**Functional approach to quantum gravity** - *Speaker: O.M. Lecian*

Abstract: Different formalisms in quantum gravity are aimed to describe the dynamics of physical processes by means of integration over all the possible states (geometries), once the initial and the final states are given. In this context, Minkowski vacuum has been studied. Minkowski vacuum, the zero-particle state, is expressed in terms of a functional that depends on the field boundary value and on the geometry of the surface that encloses the region where experiments are performed. This functional is preliminary expressed in terms of a functional Schroedinger representation; then, the concept of Hilbert space is extended to include the initial and final states of the measure operation: here a covariant vacuum is defined, which maps the initial state into the final one, and a relation connecting the two vacuum states is found. Neither spacial nor time infinities are needed: since macroscopic scales are much larger than the mass gap that ensures convergence, local particles can be treated like global particles for these purposes. Analogous for-

mulas are worked out for background independent quantum gravity, where the Minkowski vacuum can be expressed by a Euclidean gravitational functional integral. Faddeev Popovmethod in the temporal gauge is applied to the propagation Kernel: after fixing the notation in the YM case, General relativity is taken into account. Changing all the fields into a synchronous gauge and transforming the action accordingly solve the problem of fixing such a gauge without removing all four-metrics. In the compact case, the functional integral is independent of the proper time, which can be determined from the 00 component of the Einstein equations. In the asymptotically flat case, on the other hand, the functional integral depends also on the asymptotic proper time, because of the boundary conditions.

**Loop Quantum space** - *Speaker: E. Magliaro*
Abstract: In this talk I show how to quantize (in LQG) the canonical formulation of General Relativity. Starting from the Hamiltonian system defined by three constraints equations, a quantization of the theory can be obtained in terms of complex valued Schrdinger-like wave functionals; the Gaussian constraint and the vectorial constraint simply force to be invariant under SU(2) gauge transformations and 3d diffeomorphisms, the Hamiltonian constraint gives the Wheeler-De Witt equation. In order to construct the kinematical Hilbert space is necessary to find suitable functionals of the connection (cylindrical functions) and then to require internal gauge invariance (spin network states) and diffeomorphism invariance (linear functionals of spin network states) of the states and of their scalar product. The next step is to find well defined operators in our Hilbert space that are invariant and self-adjoint. We obtain two operators of this kind which are diagonal on the spin networks with discrete spectrum and have a precise physical interpretation: they are the physical area of the surface intersected by spin networks' links, and the physical volume that get contribution only from the nodes of the spin network states; we notice that in the context of Loop Quantum Gravity this discreteness is a direct consequence of a (conceptually) straightforward quantization of General Relativity. Finally I present relational interpretation of Quantum Mechanics and I observe that there is a connection between relationalism of QM and of GR due to the connection between contiguity and interaction.

**Lorentz invariance and space(-time) discretization** - *Speaker: S. Mercuri*
Abstract: One of the most interesting aspects of non-perturbative quantum gravity theories is the discretization of the space(-time). In particular in Loop Quantum Gravity the area and volume operators are not only hermitian and regularizable, but have also discrete eigenvalues in the base of spin-network. The question is: Can the Lorentz symmetry be reconcilable with such a discretization? Or, as suggested by the non-commutative geometry theories, we

might expect a modification in the Lorentz-Fitzgerald transformation at the Planck scale? We present arguments in favor of the former and latter hypothesis accordingly to the up to now results existing in literature.

**The Ashtekar-Barbero-Immirzi connections** *- Speaker: S. Mercuri*
Abstract: The canonical formulation of General Relativity leads to a consistent formulation of a Quantum Gravity theory. Even though many aspects of Quantum Gravity can be studied in the framework of ADM phase space, it is in general useful to operate a canonical transformation, passing to the Ashtekar formulation. This transformation reduces the phase space of General Relativity to that of a Yang-Mills gauge theory of the SU(2,C) group, allowing the implementation of many well known technics developed in gauge theories to the gravitational theory. We give a detailed description of the construction of Ashtekar phase space, dwelling upon the problem of Immirzi parameter and to the definition of Barbero connections, which actually complicate the constraints, but being real do not need any reality condition.

## 12.2. Review Work

### 12.2.1. Fundamentals and recent developments in non-perturbative canonical Quantum Gravity

- ***Authors***: F. Cianfrani, O.M. Lecian and G. Montani

In this work fundamental and recent aspects of canonical quantum gravity are reviewed. The aim of the presentation is to provide a pedagogical approach to the problem of quantizing the gravitational field which provides the tools for a proper understanding of recent issues in this research line.

After a detailed discussion of some relevant features concerning the classical and quantum field dynamics, the Wheeler-DeWitt formulation of canonical quantum gravity is presented with a careful discussion of its main shortcomings. Then a detailed analysis of the Loop Quantum Gravity approach is given starting from the basic mathematical notions at the ground of this modern formulation. Finally the full paradigm is developed giving emphasis on the successes and the open questions concerning the loop representation of space-time.

**Contents:**

1.3 Difference operators Vs differential operators
1.4 Time
1.5 Quantization of Hamiltonian constraints
1.6 Weyl quantization
1.7 GNS construction

- **2. Hamiltonian formulation of the geometrodynamics**
  2.1 The action for the gravitational field
  2.2 The space-time slicing
  2.3 The Hamiltonian structure
  2.4 The Hamilton-Jacobi equation
  2.5 Reduction to the canonical form
- **3. Gravity as a gauge theory**
  3.1 Gauge theories
  3.2 First-order formulation for the gravitational field
  3.3 Gravity as a gauge theory of the Lorentz group?
  3.4 Poincaré gauge theory
  3.5 The Holst formulation
  3.6 The Kodama state
- **4. Quantization of the gravitational field**
  4.1 The WDW equation
  4.2 The problem of time
  4.3 Interpretation of the wave function
  4.4 The idea of Third Quantization
- **5. Loop Quantum Gravity**
  5.1 Holonomies and Fluxes
    5.1.1 Why a reformulation in terms of Wilson loops?
    5.1.2 Lattice gauge theories
    5.1.3 Holonomies and fluxes in Quantum Gravity
  5.2 Spectrum of space-time operators
  5.3 Quantum dynamics in LQG
  5.4 Open issues in Loop Quantum Gravity
    5.4.1 Master Constraint and Algebraic QG
  5.5 Loop Quantum Cosmology

5.6 On the physical meaning of the Immirzi parameter

5.7 Time gauge and boost invariance

5.8 The picture of the space-time

# A. Brief description of Quantum Gravity

## A.1. Quantum Mixmaster

In section "Quantum Mixmaster" we propose a semiclassical treatment and a Schrödinger quantization scheme applied to the Mixmaster dynamics; the associated eigenvalue problem is solved. This approach gives a set of eigenfunctions (here we assume an ordering for the position and momentum operators such that $v^2 p_v^2 \rightarrow \hat{v}\hat{p_v}^2\hat{v}$, which is the only one able to reproduce the proper statistical dynamics). As soon as (approximated) Dirichlet boundary condition are taken into account, the energy spectrum is obtained. This spectrum is a discrete one, and it admits a minimum value given by $E_0^2 = 19.831\hbar^2$. In the figures in the section the wave function of the ground state and its probability distribution are plotted [7]. The persons working on this topic are Riccardo Benini and Giovanni Montani.

## A.2. Loop Quantum Cosmology

In section "Loop Quantum Cosmology" we perform a general analysis of the equations governing the evolution of the Universe within semi-classical Loop Quantum Cosmology by using qualitative methods of the theory of dynamical systems. Specifically, two cases are considered with different type of corrections to the Friedmann equations [8], [3]. Quadratic terms on the energy density correction to the Friedmann equation, coming from effective Hamiltonian of Loop Quantum Cosmology, and corrections due to the inverse scale factor operator (both in the gravitational and the matter part of the effective Hamiltonian) were analyzed, respectively.

Our general conclusion, considering both types of corrections, is the absence of cosmic singularity, so in all solutions the usual expansion stage follows after the generic bounce. Moreover, we have shown that in both cases there exist successful mechanisms for generation of initial conditions suitable for inflation.

This work is relevant for the development of the theory of Early Universe. In particular, better understanding of background solutions and their properties should be reached to study of the cosmological perturbations.The dynamics of these perturbations, in turn, is crucial in view of verifications of predictions of the theory, confronted with observational data.

The peoples involved in this line of research are Giovanni Montani and Gregory V. Vereshchagin.

## A.3. Lorentz gauge connection

The Yang-Mills picture of the local Lorentz transformations is approached in a second-order formalism. For the Lagrangian approach to reproduce the second Cartan structure equation, as soon as the Lorentz gauge connections are identified with the contortion tensor, an interaction term between the Lorentz gauge fields and the spin connections $\omega$ has to be postulated. This interaction term induces a Riemannian source to the Yang-Mills equations; thus, the real vacuum dynamics of the Lorentz gauge connection takes place on a Minkowski space only, when the Riemannian curvature and the spin currents provide negligible effects. In fact, it is the geometrical interpretation of the torsion field as a gauge field that generates the non-vanishing part of the Lorentz connection on flat space-time. The full picture involving gravity, torsion and spinors is described by a coupled set of field equations, which allows one to interpret both gravitational spin connections and matter spin density as the source term for the Yang-Mills equations. The contortion tensor acquires a propagating character, because of its non-Abelian feature, and the pure contact interaction is restored in the limit of vanishing Lorentz connections [8].

# B. Brief description of Quantum Fields on Classical Background

## B.1. Quantum Fields in accelerated systems

In section "Quantum Fields in accelerated systems" we concentrate on a rigorous analysis of Quantum Field Theory (QFT) from the point of view of an accelerated observer moving in the flat space and so called Unruh effect. We showed that the quantization procedure proposed by Unruh implies setting a boundary condition for the quantum field operator and this changes drastically the topological properties and symmetry group of the spacetime which lead to the field theory in two disconnected left and right Rindler spacetimes instead of Minkowski spacetime. Thus in spite of the work over last 30 years, there still remain serious gaps in grounding of the Unruh effect, and as of now there is no compelling evidence for the universal behaviour attributed to all uniformly accelerated detectors [11], [12], [14], [13], [15].

The people involved in this line of research are Vladimir A. Belinski, Nikolay B. Narozhny and Alexander M. Fedotov.

## B.2. Quantum Fields in Black Hole Space-Time

In section "Quantum Fields in Black Hole Space-Time" a careful investigation of the conventional derivation of the quantum effect of evaporation of black holes have been done. We show that there are serious doubts about the existence of such phenomenon. The main reason is due to the absence of the quasiclassical tunneling process corresponding to the particle creation by the Schwarzschild black hole created by the collapse [10], [23].

The person involved in this line of research is Vladimir A. Belinski.

# C. Brief description of Unification Theories

## C.1. Classical and Quantum spinning particles in Kaluza-Klein space-times

In the section "Classical and Quantum spinning particles in Kaluza-Klein space-times", we analyze the introduction of spinor fields in a KK model. The dynamics of a classical spinning particle, in a KK space-time, is inferred from the extension of Papapetrou equations to the 5-dimensional case, with Pirani conditions. This way, the system reproduces exactly equations of motion of a spinning particle, endowed with a charge and an electro-magnetic moment. This result demonstrates that the geometrization of electro-dynamics does not modify the dynamics of spinning objects [34].

The introduction of spinor fields in a KK model is the main open point of such an approach. The standard way to deal with them is to extend the Dirac equation to the multi-dimensional case and to try to identify extra-dimensional quantum numbers with internal ones. However this procedure fails, because of the emergence of mass terms of the compactification scale order and because quantum numbers of Standard model particles cannot be inferred. In this respect, our investigation has been focused on a more phenomenological approach, based on recovering 4-dimensional properties by an averaging procedure on the extra-dimensional manifold. This average is motivated by the undetectability of the extra-space and the need for it is not restricted to the case spinors are present. In fact, we showed that it is required in order to reproduce non-Abelian gauge transformations from extra-dimensional isometries and to get the equations of motion, proper of the 4-dimensional picture, starting from multi-dimensional ones. As far as spinors are concerned, the average produces a non-trivial effect on extra-dimensional symmetries, such that some of the above mentioned issues can be solved [28], [29], [38].

The people involved in this research line are Francesco Cianfrani, Irene Milillo, Andrea Marrocco and Giovanni Montani.

## C.2. Generalized 5-Dimensional Theories

In section "Generalized 5-Dimensional Theories", we analyze possible generalizations of the 5D Kaluza-Klein model. The introduction of torsion has been shown to produce interesting structures after dimensional reduction. In a 5D scenario, the geometrization of the Electro-weak model has been worked out on the ground of the broken 5D Lorentz group and the properties of torsion [30], and proposal for the introduction of Ashtekar variables within this scheme has been evaluated.On the other hand, the truncation of the infinite tower that characterizes KK theories has been evaluated within the framework of polymer representation and generalized uncertainty principle: in the first case, compactification is illustrated to occur because of the truncation, while, in the second case, compactification is illustrated to be compatible with the main hypotheses of the scheme.

The people involved in this research line are Orchidea M. Lecian and Giovanni Montani.

# D. Quantum Gravity

## D.1. Boost invariance in a second order formulation

Given an hyperbolic space-time manifold $V$, endowed with a metric $g_{\mu\nu}$, a $3+1$ splitting consists in a map $V \rightarrow \Sigma \otimes R$, $\Sigma$ being spatial 3-hypersurfaces (in the following $x^i \ \ (i = 1, 2, 3)$ indicate spatial coordinates, while $t$ is the coordinate on the real time-like axis). The crucial choice consists in introducing an arbitrary vier-bein, *i.e.*,

$$e^0 = N dt + \chi_a E^a_i dx^i \ , \qquad e^a = E^a_i N^i dt + E^a_i dx^i \ , \qquad (a = 1, 2, 3) \ , \quad \text{(D.1.1)}$$

where the time-gauge is obtained for $\chi_a = 0$.
From a physical point of view, $\chi_a$ gives the velocity components of the $e^A$ frame with respect to one at rest, *i.e.*, adapted to the spatial splitting.
The standard variables of the ADM formulation (the lapse function $\tilde{N}$, the shift vector $\tilde{N}^i$ and the 3-geometry $h_{ij}$) read as follows in terms of $e^A_\mu$ components

$$\tilde{N} = \frac{1}{\sqrt{1-\chi^2}}(N - N^i E^a_i \chi_a) \ , \qquad \tilde{N}^i = N^i + \frac{E^c_l \chi_c N^l - N}{1-\chi^2} E^i_a \chi^a \ , \quad \chi^a = \chi_b \delta^{ab} \ ,$$

$$h_{ij} = E^a_i E^b_j (\delta_{ab} - \chi_a \chi_b) \ . \qquad \text{(D.1.2)}$$

Once the $3+1$ splitting of the Einstein-Hilbert action has been performed, by taking as configuration variables $\tilde{N}$, $\tilde{N}^i$, $E^a_i$ and $\chi_a$, the full Hamiltonian density turns out to be

$$\mathcal{H} = \tilde{N}' H + \tilde{N}^i H_i + \lambda^{\tilde{N}} \pi_{\tilde{N}} + \lambda^i \pi_i + \lambda^{ab} \Phi_{ab} + \lambda_a \Phi^a \ , \qquad \text{(D.1.3)}$$

$\tilde{N}'$ being $\sqrt{h}\tilde{N}$, while the super-Hamiltonian and the super-momentum, $H$ and $H_i$, respectively, take the following forms

$$H = \pi^i_a \pi^j_b \left( \frac{1}{2} E^a_i E^b_j - E^b_i E^a_j \right) + h^3 R \ , \qquad \text{(D.1.4)}$$

$$H_i = D_j (\pi^j_a E^a_i) \ , \qquad \text{(D.1.5)}$$

$D_i$ being the covariant derivative built up from $h_{ij}$.
Lagrangian multipliers $\lambda^{\tilde{N}}$, $\lambda^i$, $\lambda_a$ and $\lambda^{ab} = -\lambda^{ba}$ ensure the standard first-class constraints

$$\pi_{\tilde{N}} = 0\,, \qquad \pi_i = 0\,, \tag{D.1.6}$$

and new conditions, coming out as a consequence of variables adopted,

$$\Phi^a = \pi^a - \pi^b \chi_b \chi^a + \delta^{ab} \pi^i_b \chi_c E^c_i = 0\,, \tag{D.1.7}$$

$$\Phi_{ab} = \pi^c \delta_{c[a} \chi_{b]} - \delta_{c[a} \pi^i_{b]} E^c_i = 0\,. \tag{D.1.8}$$

The investigation on these new constraints is performed by analyzing their action on the phase space, once a canonical symplectic structure is introduced. It outlines that $\Phi_{ab}$ and $\Phi^a$ generate rotations and boosts, modulo a time re-parametrization, respectively, on the phase space. Therefore, they arise because General Relativity is a Lorentz-invariant theory.
We also probe that the algebra of constraints is first-class.
Before performing the quantization, a formal fixing of the boost symmetry is performed, such that transformations between $\chi$-sectors can be studied. In this respect, we set $\chi_a = \bar{\chi}_a(t;x)$, $\bar{\chi}_a(t;x)$ being arbitrary functions of space-time coordinates. The boost constraint can be solved classically, so finding

$$\pi^a = -\left(\delta^{ab} + \frac{\chi^a \chi^b}{1-\chi^2}\right) \pi^i_b \chi_c E^c_i\,. \tag{D.1.9}$$

Hence the action becomes

$$S = -\tfrac{1}{16\pi G} \int [\pi^i_a \partial_t E^a_i + \pi_{\tilde{N}'} \partial_t \tilde{N}' + \pi_i \partial_t \tilde{N}^i - \tilde{N}' H^{\bar{\chi}} - \tilde{N}^i H^{\bar{\chi}}_i - \lambda^{ab} \Phi'_{ab} + \\ - \lambda^{\tilde{N}} \pi_{\tilde{N}} - \lambda^i \pi_i] dt d^3x\,, \tag{D.1.10}$$

where the new constraint for rotations is

$$\Phi'_{ab} = \bar{\chi}_{[a} \pi^i_{b]} E^d_i \bar{\chi}_d - \delta_{c[a} \pi^i_{b]} E^c_i\,, \tag{D.1.11}$$

while, in $H^{\bar{\chi}}$ and in $H^{\bar{\chi}}_i$, $\chi$ are replaced by functions $\bar{\chi}$.
In this picture, we have completely fixed the gauge associated with the boost symmetry, because $\bar{\chi}_a$ are three functions to be assigned explicitly together with the Cauchy data.

The canonical quantization consists in promoting to operators $\tilde{N}$, $\tilde{N}^i$, $E^a_i$ and the corresponding conjugated momenta, then Poisson brackets are replaced by commutators in a canonical way. Once an Hilbert space has been defined to which wave functionals $\psi = \psi_{\bar{\chi}}(\tilde{N}, \tilde{N}^i, E^a_i)$ belong, according with the Dirac prescription for constrained systems, physical states are defined as states annihilated by quantum constraints.

In order to investigate if the transformation between different $\bar{\chi}$-sectors can be implemented in a quantum setting, an operator connecting Hilbert spaces with different forms of $\bar{\chi}$ must be defined.
Let us now consider a wave functional $\psi_0$ in the time gauge: it is a solution of the following system of constraints (we do not consider primary constraints (D.1.6), since they are not affected by transformations changing $\bar{\chi}_a$)

$$H^0\psi_0 = 0\,, \quad H_i^0\psi_0 = 0\,, \quad -\delta_{c[a}\pi^i_{b]}E^c_i\psi_0 = 0\,, \tag{D.1.12}$$

$H^0$ and $H_i^0$ being the super-Hamiltonian and super-momentum built up from the metric tensor $h_{ij} = \delta_{ab}E^a_iE^b_j$, *i.e.*, in the case $\bar{\chi} \equiv 0$, respectively.
The action of the boost constraint $\Phi^a$, restricted to the hypersurface $\chi_a = 0$, is reproduced by the unitary operator $U_\epsilon$

$$U_\epsilon = I - \frac{i}{4}\int \epsilon^a\epsilon_b(E^b_i\pi^i_a + \pi^i_aE^b_i)d^3x + O(\epsilon^4)\,, \tag{D.1.13}$$

which maps the metric $h_{ij}$ from $\bar{\chi} = 0$ to $\bar{\chi}_a = \epsilon_a \ll 1$. The new state $\psi' = U_\epsilon\psi$ satisfies, at the $\epsilon^2$ order,

$$U_\epsilon H^0U_\epsilon^{-1}\psi' = H^\epsilon = \psi'0\,, \qquad U_\epsilon H^0_iU_\epsilon^{-1}\psi' = H^\epsilon_i\psi' = 0\,, \tag{D.1.14}$$

$$U_\epsilon(-\delta_{c[a}\pi^i_{b]}E^c_i)U_\epsilon^{-1}\psi' = -\left[\delta_{c[a}\pi^i_{b]}E^c_i + \frac{1}{2}\delta_{c[a}\epsilon_{b]}\epsilon^dE^c_i\pi^i_d - \frac{1}{2}\epsilon_d\epsilon_{[a}\pi^i_{b]}E^d_i\chi_d\right]\psi' = 0\,. \tag{D.1.15}$$

While the first two relations reproduce the vanishing of the super-Hamiltonian and of the super-momentum in the $\epsilon$-sector, the last condition can be shown to be equivalent to $\Phi'_{ab}\psi' = 0$ for $\bar{\chi}_a = \epsilon_a$.
Therefore, since the unitary operator $U_\epsilon$ maps physical states corresponding to $\bar{\chi} = 0$ and $\bar{\chi} = \epsilon$, the transformation between a frame at rest and one moving with respect to $\Sigma$ can be implemented as a symmetry on a quantum level. This provides us with an explanation for the use of the time-gauge condition, because any other choice for the Lorentz frame gives the same expectation values for observables.

## D.2. Quantum Mixmaster

The quantization of the Bianchi IX geometry is investigated in the approximation of a squared potential well, after an ADM reduction of the dynamics with respect to the super-momentum constraint only. A functional representation of the quantum dynamics, equivalent to the Misner-like one, was extended point by point, since the Hilbert space factorizes into $\infty^3$ independent components, due to the parametric role that the three-coordinates assume in the asymptotic potential term. Finally, we obtain the conditions for a semi-classical behavior of the dynamics, equivalent to mean occupation numbers $n = \mathcal{O}(10^2)$ [Imponente and Montani (2006)].

A physical link between the chaoticity characterizing the system at a classical level and the quantum indeterminism appearing in the Planckian era was constructed through the canonical quantization of the model via a Schrödinger approach (equivalent to the Wheeler-DeWitt scheme) and then developed the WKB semiclassical limit to be compared with the classical dynamics [Imponente and Montani (2003a)], [Imponente and Montani (2003b)]. We found a correspondence between the continuity equation of the microcanonical distribution function and that one describing the dynamics of the first-order corrections in the wave function for $\hbar \to 0$ [Imponente and Montani (2002)].

The dynamics of the homogeneous model of the type IX of the Bianchi classification (the Mixmaster model) exhibits an oscillatory like behavior while approaching the Big Bang; furthermore, Belinskii et al. showed in the 70's how this model can be used to construct a generic cosmological solution in the neighborhood of a time-like singular point, in the sense of the correct number of physically-arbitrary functions.

However, this classical description is in conflict with the requirement of a quantum behavior of the Universe through the Planck era; there are reliable indications that the Mixmaster dynamics overlaps the quantum Universe evolution, requiring an appropriate analysis of the transition between these two different regimes. Indeed, the dynamics of the very early Universe corresponds to a very peculiar situation, with respect to the link existing between the classical and quantum regimes. The expansion of the Universe is the crucial phenomenon which maps into each other these two stages of the evolution. The appearance of a classical background takes place essentially at the end of the Mixmaster phase, when the anisotropy degrees of freedom can be treated as small perturbations; this result indicates that the oscillatory regime takes place almost during the Planck era and therefore it is a problem of quantum dynamics. However the end of the Mixmaster (and in principle the *quantum to classical* transition phase) is fixed by the initial conditions on the system and, in particular, it takes place when the cosmological horizon reaches the inhomogeneity scale of the model; therefore the question of an appropriate treatment for the semiclassical behavior arises when the inhomogeneity scale is so larger than the Planck scale, so that the horizon can approach it only in the classical limit.

In the Arnowitt-Deser-Misner (ADM) formalism, the classical dynamics of the Mixmaster can be reduced to the physical degrees of freedom: the evolution resembles that one of a billiard ball on a constant negative curved 2-dimensional surface, described in the Poincaré half-plane by the following

$$
\begin{aligned}
&Q_1(u,v) = -u/\delta \geq 0 \\
&Q_2(u,v) = (1+u)/\delta \geq 0 \\
&Q_3(u,v) = (u^2+u+v^2)/\delta \geq 0 \\
&\delta = u^2+u+1+v^2
\end{aligned}
\tag{D.2.3}
$$

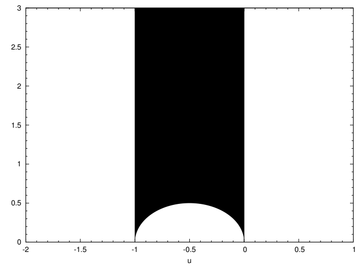


**Figure D.1.:** The billiard where the Mixmaster Universe moves.

action principle:

$$I = \int_{\Gamma_Q} (p_u \partial_t u + p_v \partial_t v - H_{ADM})\, dt\,, \tag{D.2.1}$$

$$H_{ADM} = \epsilon = v\sqrt{p_u^2 + p_v^2}\,, \tag{D.2.2}$$

where $\Gamma_Q$ is a portion of the full Poincarè plane described by the inequalities above.

A Schrödinger quantization scheme can be applied to the squared Hamiltonian operator, and the associated eigenvalue problem is solved. This approach gives a set of eigenfunctions (here we assume an ordering for the position and momentum operators such that $v^2 p_v^2 \rightarrow \hat{v}\hat{p_v}^2\hat{v}$, which is the only one able to reproduce the proper statistical dynamics). As soon as (approximated) Dirichlet boundary condition are taken into account, the energy spectrum results to be given by

$$(E/\hbar)^2 = t^2 + 1/4\,. \tag{D.2.4}$$

where the values of the parameter $t$ have to be evaluated solving $K_{it}(2n) = 0$ for a generic integer $n$. This spectrum is a discrete one, and it admits a minimum value given by $E_0^2 = 19.831\hbar^2$. In the figures below the wave function of the ground state and its probability distribution are plotted.

## D.3. Dualism between time evolution and matter fields

In this section we review the fundamental aspects of the so-called evolutionary quantum gravity as presented in (Montani, 2002), (Mercuri and Montani, 2004). First we analyze the implication of a Schrödinger formulation of the quantum dynamics for the gravitational field and then we establish a dualism between time evolution and matter fields. Finally, we stress how an evolu-

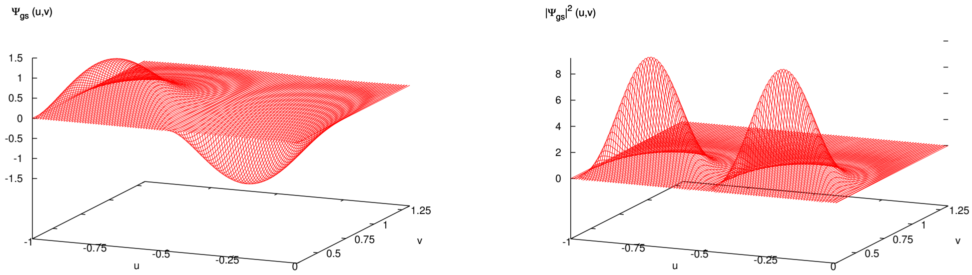


**Figure D.2.:** The ground state wave function and the probability distribution.

tionary paradigm can be fixed by restricting the admissible set of coordinate transformations to synchronous ones (Montani and Cianfrani, 2008).

Let us assume that the quantum evolution of the gravitational field is governed by the smeared Schrödinger equation

$$i\partial_t \Psi = \hat{\mathcal{H}}\Psi \equiv \int_\Sigma d^3x \left(N\hat{H}\right)\Psi \,, \tag{D.3.1}$$

being $\hat{H}$ the super-Hamiltonian operator, $N$ the lapse function and the wave functional $\Psi$ is defined on the Wheeler superspace, *i.e.*, it is annihilated by the super-momentum operator $\hat{H}_\alpha$. Let us now take the following expansion for the wave functional

$$\Psi = \int D\epsilon \chi(\epsilon, \{h_{\alpha\beta}\}) \exp\left\{-i\int_{t_0}^{t} dt' \int_\Sigma d^3x (N\epsilon)\right\}, \tag{D.3.2}$$

$D\epsilon$ being the Lebesgue measure in the space of the functions $\epsilon(x^\rho)$. Such an expansion reduces the Schrödinger dynamics to an eigenvalues problem of the form

$$\hat{H}\chi = \epsilon\chi, \qquad \hat{H}_\alpha\chi = 0, \tag{D.3.3}$$

which outlines the appearance of a non zero super-Hamiltonian eigenvalue.

In order to reconstruct the classical limit of the above dynamical constraints, we address the limit $\hbar \to 0$ and replace the wave functional $\chi$ by its corresponding zero-order WKB approximation $\chi \sim e^{iS/\hbar}$. Under these restrictions, the eigenvalues problem (D.3.3) reduces to the following classical counterpart

$$\hat{HJ}S = \epsilon \equiv -2\sqrt{h}T_{00}, \qquad \hat{HJ}_\alpha S = 0\,, \tag{D.3.4}$$

where $\hat{HJ}$ and $\hat{HJ}_\alpha$ denote operators which, acting on the phase $S$, reproduce the super-Hamiltonian and super-momentum Hamilton-Jacobi equations respectively. We see that the classical limit of the adopted Schrödinger quantum dynamics is characterized by the appearance of a new matter contribution (associated with the non zero eigenvalue $\epsilon$) whose energy density reads

$$\rho \equiv T_{00} = -\frac{\epsilon(x^\rho)}{2\sqrt{h}}, \tag{D.3.5}$$

where by $T_{ij}$ we refer to the new matter energy-momentum tensor.

Since the spectrum of the super-Hamiltonian has, in general, a negative component, we can then infer that, when the gravitational field is in the ground state, this matter out-coming in the classical limit has a positive energy density. The explicit form of (D.3.5) is that of a dust fluid co-moving with the slicing 3-hypersurfaces, *i.e.*, the field $n^i$ begin the 4-velocity normal to the 3-hypersurfaces (in other words, we deal with an energy-momentum tensor $T_{ij} = \rho n_i n_j$).

We stress that in this approach, it is possible to turn the solution space into Hilbert one and therefore a notion of probability density naturally arises, from the squared modulus of the wave-functional.

Let us now consider the opposite sector, *i.e.*, a gravitational system in the presence of a macroscopic matter source. In particular, we choice a perfect fluid having a generic equation of state $p = (\xi - 1)\rho$ ($p$ being the pressure and $\xi$ the polytropic index). The energy-momentum tensor, associated to this system reads

$$T_{ij} = \xi\rho u_i u_j - (\xi - 1)\rho g_{ij}\,. \tag{D.3.6}$$

To fix the constraints when matter is included in the dynamics, let us make use of the relations

$$G_{ij}n^i n^j = -\kappa\frac{H}{2\sqrt{h}}\,, \tag{D.3.7}$$

$$G_{ij}n^i\partial_\alpha y^j = \kappa\frac{H_i}{2\sqrt{h}}, \tag{D.3.8}$$

where $\partial_\alpha y^i$ are the tangent vectors to the 3-hypersurfaces, *i.e.*, $n_i\partial_\alpha y^i = 0$. Equations (D.3.7) and (D.3.8), by (D.3.6) and identifying $u_i$ with $n_i$ (*i.e.*, the physical space is filled by the fluid), rewrite

$$\rho = -\frac{H}{2\sqrt{h}}, \qquad H_i = 0\,; \tag{D.3.9}$$

furthermore, we get the equations

$$G_{ij}\partial_\alpha y^i\partial_\beta y^j \equiv G_{\alpha\beta} = \kappa(\xi - 1)\rho h_{\alpha\beta}\,. \tag{D.3.10}$$

We now observe that the conservation law $\nabla_j T_i^j = 0$ implies the following two conditions

$$\xi \nabla_i \left(\rho u^i\right) = (\xi - 1) u^i \partial_i \rho \; , \tag{D.3.11}$$

$$u^j \nabla_j u_i = \left(1 - \frac{1}{\xi}\right) \left(\partial_i \ln \rho - u_i u^j \partial_j \ln \rho\right) \; . \tag{D.3.12}$$

If we now adapt the spacetime slicing, looking the dynamics into the fluid frame (*i.e.*, $n^i = \delta^i_0$), then, by the relation $n^i = (1/N, \; -N^\alpha/N)$, we see that the co-moving constraint implies the synchronous nature of the reference frame. As it is well-known that a synchronous reference is also a geodesic one, the right-hand-side of equation (D.3.12) must vanish identically and, for a generic inhomogeneous case, this means to require $\xi \equiv 1$. Hence, equations (D.3.11) yields $\rho = -\bar{\epsilon}(x^\rho)/2\sqrt{h}$; substituting the last expression into (D.3.9), we get the same Hamiltonian constraints associated to the Evolutionary Quantum Gravity at the point i), as soon as the function $\bar{\epsilon}$ is turned into the eigenvalue $\epsilon$. In this respect, we stress that, while $\bar{\epsilon}$ is positive by definition, the corresponding eigenvalue can also take negative values because of the $H$-structure.

Thus, we conclude that a dust fluid is a good choice to realize a clock in Quantum Gravity, because it induces a non-zero super-Hamiltonian eigenvalue into the dynamics; furthermore, for vanishing pressure ($\xi = 1$), the equations (D.3.10) reduces to the right vacuum evolution for $h_{\alpha\beta}$. Moreover, we stress how the above two points outline, in quantum gravity, a real dualism between time evolution and the presence of a dust fluid.

The approach above was applied to a generic cosmological model in (Battisti and Montani, 2006b) where is shown how, from a phenomenological point of view, an evolutionary quantum cosmology overlaps the Wheeler-DeWitt framework.

In particular, for such a model, the eigenvalues problem (D.3.3) rewrite as

$$\left\{\kappa \left[\partial_R \frac{1}{R} \partial_R - \frac{1}{R^3}\left(\partial^2_+ + \partial^2_-\right)\right] - \frac{3}{8\pi R^3}\partial^2_\phi - \frac{R^3}{4\kappa l^2_{in}} V(\beta_\pm) + R^3(\rho_{ur} + \rho_{pg})\right\} \chi = \epsilon \chi . \tag{D.3.13}$$

where $\kappa = 8\pi l_P^2$ and we have added to the dynamics of the system an ultra-relativistic energy density ($\rho_{ur} = \mu^2/R^4$), a perfect gas contribution ($\rho_{pg} = \sigma^2/R^5$) and a scalar field $\phi$ (a free inflaton field). Such a problem can be analytically solved and the spectrum of the super-Hamiltonian reads as

$$\epsilon_{n,\gamma} = \frac{\sigma^2}{l_P^2(n + \gamma - 1/2)}. \tag{D.3.14}$$

Therefore the ground state $n = 0$ eigenvalue, for $\gamma < 1/2$, is negative and so it is associated via (D.3.5) to a positive dust energy density.

In order to analyze the cosmological implication of the new matter contribution, we have to impose a cut-off length in our model, requiring that the Planck length $l_P$ is the minimal physical length accessible by an observer ($l \geq l_P$). This way, we get $\sigma^2 \leq \mathcal{O}(l_P)$ and so $|\epsilon_0| \leq (1/l_P)$: *the spectrum is limited by below*. Moreover the contribution of such a dust fluid to the actual critical parameter is

$$\Omega_{dust} \sim \frac{\rho_{dust}}{\rho_{Today}} \sim \mathcal{O}\left(10^{-60}\right). \tag{D.3.15}$$

As matter of fact, such a parameter is much less then unity and so no phenomenology can came out (today) from our dust fluid. In this sense we claim that an evolutionary quantum cosmology overlaps the Wheeler-DeWitt approach and therefore it can be inferred as appropriate to describe early stages of the Universe without significant traces on the later evolution.

## D.4. Loop Quantum Cosmology

Standard cosmological model raises several fundamental issues such as initial singularity and the problem of horizon. We analyze these well known problems within the framework of cosmological models based on Loop Quantum Gravity.

One of the fundamental issues of the theory of Early Universe is cosmic singularity. Many researchers, such as J.A. Wheeler, believed that appearance of initial singularity in Friedmann Equations marks a breakdown of General Relativity theory and searched for a possible solution in quantization of gravity. The well known Wheeler-de Witt equation is one example of such an approach, although unsuccessful. At the same time, it is clear that attempts to construct viable nonsingular cosmologies within classical theories of gravitation did not succeed, as discussed by (Vereshchagin, 2004a, 2005).

Loop Quantum Gravity is at present the main background independent and nonperturbative candidate for a quantum theory of gravity; Loop Quantum Cosmology is the application of Loop Quantum Gravity to a homogeneous minisuperspace environment. The underlying geometry in LQG is discrete and the continuum spacetime is obtained from quantum geometry in a large eigenvalue limit. Numerical calculations performed within Loop Quantum Gravity theory established the possibility of resolution of singularities in various situations.
The underlying dynamics in LQC is governed by a discrete quantum difference equation in quantum geometry. However, using semiclassical states one can construct an effective Hamiltonian description on a continuum spacetime which has been shown to very well approximate the quantum dynamics.

We have performed a general analysis of equations governing evolution of the Universe within semiclassical Loop Quantum Cosmology by using qual-

itative methods of the theory of dynamical systems. Specifically, two cases were considered with different type of corrections to the Friedmann equations.
In the work by (Singh et al., 2006) quadratic on the energy density correction to the Friedmann equation, coming from effective Hamiltonian of Loop Quantum Cosmology was studied. The modified Friedmann equation takes the form

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho\left(1 - \frac{\rho}{\rho_{\rm crit}}\right), \tag{D.4.1}$$

where $a$ is the scale factor, $k$ denotes spatial curvature, $c$ and $G$ are the speed of light and the gravitational constant respectively. The energy density of the scalar field is

$$\rho = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V. \tag{D.4.2}$$

The critical energy density is

$$\rho_{\rm crit} = \frac{\sqrt{3}}{16\pi^2\gamma^3}\rho_{\rm pl}, \tag{D.4.3}$$

where $\gamma$ is Barbero-Immirzi parameter, $\rho_{\rm pl}$ is the Planckian density. The usual continuity equation for the real scalar field $\phi$ with effective potential $V(\phi)$ takes the form

$$\frac{d^2\phi}{dt^2} + 3\frac{1}{a}\frac{da}{dt}\frac{d\phi}{dt} + \frac{\partial V}{\partial\phi} = 0. \tag{D.4.4}$$

Equations (D.4.1) and (D.4.4) can be analysed by means of qualitative theory of dynamical systems. First of all, the derivative of the scale factor can be expressed from (D.4.1) and substituted into (D.4.4) thus reducing the phase space to two dimensions. The corresponding phase space variables are the scalar field $\phi$ and its time derivative $\dot{\phi} \equiv \frac{d\phi}{dt}$. Expample of the phase portrait is represented in Fig. D.3 The boundary of the phase space, defined as

$$\rho = \rho_{\rm crit}, \tag{D.4.5}$$

prevents appearance of singularities for positive energy density, unlike the case of General Relativity, where the boundary is absent. Details see in (Singh et al., 2006).

In the work of (Vereshchagin, 2004b) corrections due to the inverse scale factor operator both in the gravitational and the matter part of the effective Hamiltonian were analyzed. These corrections appear both in the energy

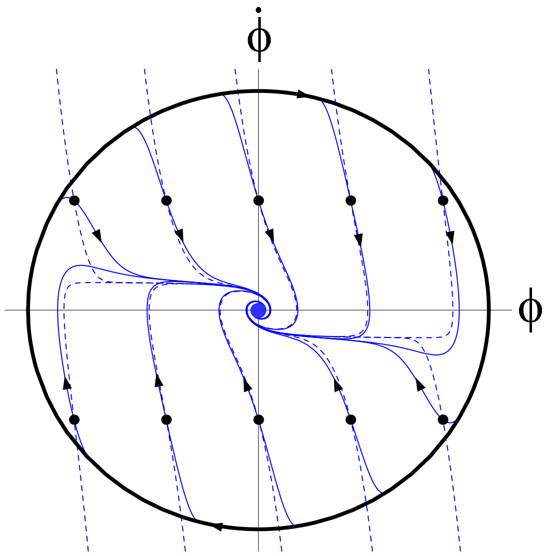


**Figure D.3.:** Phase portrait for massive scalar field $V = m^2\phi^2/2$ potential. Dashed curves represent GR case and solid curves shown LQC case.

density (D.4.2) and in the continuity equation as

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\left[\frac{1}{2D}\left(\frac{d\phi}{dt}\right)^2 + V\right], \tag{D.4.6}$$

$$\frac{d^2\phi}{dt^2} + 3\frac{1}{a}\frac{da}{dt}\frac{d\phi}{dt} - \frac{1}{D}\frac{dD}{dt}\frac{d\phi}{dt} + D\frac{\partial V}{\partial\varphi} = 0, \tag{D.4.7}$$

where the function $D$ is defined as

$$D(q) = \left(\frac{8}{77}\right)^6 q^{3/2} 7\left[(q+1)^{11/4} - |q-1|^{11/4}\right] - \\ - 11q\left[(q+1)^{7/4} - \mathrm{sign}(q-1)|q-1|^{7/4}\right]^6, \tag{D.4.8}$$

with $q = (a/a_*)^2$ and $a_*^2 = \frac{j\ln 2}{3\sqrt{3}\pi} l_P^2$ being the scale where quantum corrections become essential. The latter can be larger than the planckian length $l_P$, since the quantization parameter $j$, which must take half integer values, but is arbitrary. Equation (D.4.7) can be substituted into (D.4.6) and role of dynamical variables is then played by the scale factor and its derivative $H \equiv \frac{1}{a}\frac{da}{dt}$. Examples of phase portraits are shown in Fig. D.4. The left figure represents the case of General Relativity, while central and right figures correspond to Loop Quantum Cosmology. Due to different structure of the phase space, again singular solutions do not appear.

Our general conclusion, considering both types of corrections, is the absence of cosmic singularity, so in all solutions the usual expansion stage follows after the generic bounce. Moreover, we have shown that in both cases there exist successful mechanisms for generation of initial conditions suitable for inflation.

This work is relevant for the development of the theory of Early Universe. In particular, better understanding of background solutions and their properties should be reached prior to study of the cosmological perturbations. Dynamics of these perturbations, in turn, is crucial in view of verification of predictions of the theory, confronted with observational data.

## D.5. FRW cosmological model in the GUP framework

Let us now investigate the consequences of the an Heisenberg deformed algebra (9.5.2) of the quantum dynamics of the flat ($k = 0$) FRW model in the presence of a massless scalar field $\phi$. In particular, we will interested about the fate of the classical singularity in this framework. In which follows we summarize the discussion and results reported in [1]. The Hamiltonian constraint for this model has the form

$$H_{grav} + H_{\phi} \equiv -9\kappa p_x^2 x + \frac{3}{8\pi}\frac{p_{\phi}^2}{x} \approx 0\,, \qquad x \equiv a^3, \tag{D.5.1}$$

where $a$ is the scale factor. In the classical theory, the phase space is 4- dimensional, with coordinates $(x, p_x; \phi, p_{\phi})$ and at $x = 0$ the physical volume of the Universe goes to zero and the singularity appears. Moreover, it is not difficult to see that each classical trajectory can be specified in the $(x, \phi)$-plane, *i.e.*, $\phi$ can be considered as a relational time for the dynamics. In particular, the dynamical trajectories read as

$$\phi = \pm\frac{1}{\sqrt{24\pi\kappa}}\ln\left|\frac{x}{x_0}\right| + \phi_0\,, \tag{D.5.2}$$

where $x_0$ and $\phi_0$ are integration constants. In this equation, the plus sign describes an expanding Universe from the Big-Bang, while the minus sign a contracting one into the Big-Crunch. We now stress that the classical cosmological singularity is reached at $\phi = \pm\infty$ and every classical solution, in this model, reaches the singularity.

As well-known the canonical approach (the WDW theory) to this problem does not solve the singularity problem. More precisely, it is possible to construct a state localized at some initial time. Then, in the backward evolution, its peak will moves along the classical trajectory (D.5.2) and thus it falls into the classical singularity. This way, the classical singularity is not tamed by quantum effects.

This picture is radically changed in the GUP framework and the modifications can be realized in two different steps. At first, it is possible to show how the probability density $|\Psi(\zeta,t)|^2$ to find the Universe around $\zeta \simeq 0$ (around the Planckian region) can be expanded as

$$|\Psi(\zeta,t)|^2 \simeq |A(t)|^2 + \zeta^2 |B(t)|^2. \tag{D.5.3}$$

Here $t$ is a dimensionless time $t = \sqrt{24\pi\kappa}\phi$ and the wave packets

$$\Psi(\zeta,t) = \int_0^\infty d\epsilon g(\epsilon)\Psi_\epsilon(\zeta)e^{i\epsilon t}\ , \tag{D.5.4}$$

are such that the state is initially packed at late time, *i.e.*, the weight function $g(\epsilon)$ is a Gaussian distribution peaked at some $\epsilon^* \ll 1$ (at energy much less then the Plank energy $1/l_P$). Of course, $\Psi_\epsilon(\zeta)$ rapresent the *quasiposition eigenfunctions* (9.5.3) of this problem.

Therefore, near the Planckian region, the probability density to find the Universe is $|A(t)|^2$, which is very well approximated by a Lorentzian function packed in $t = 0$. This value corresponds to the classical time for which $x(t) = x_0$. Thus, for $x_0 \sim \mathcal{O}(l_P^3)$, the probability density to find the Universe in a Planckian volume is peaked around the corresponding classical time. As a matter of fact this probability density vanishes for $t \to -\infty$, where the classical singularity appears. This is the meaning when we claim that the classical cosmological singularity is solved by this model.

Of course the more interesting differences between the WDW and the GUP approaches can be recognized in the wave packets dynamics. In particular, we consider a wave packet initially peaked at late times and let it evolve numerically "backward in time". The result of the integration is that the probability density, at different fixed values of $\zeta$, is very well approximated by a Lorentzian function yet. Moreover, the width of this function remains, actually, the same as the states evolves from large $\zeta$ ($10^3$) to $\zeta = 0$. The peaks of Lorentzian functions, at different $\zeta$ values, move along the classically expanding trajectory (D.5.2) for values of $\zeta$ larger then $\sim 4$. Near the Planckian region, *i.e.*, when $\zeta \in [0,4]$, we observe a modification of the trajectory of the peaks. In fact they follow a power-law up to $\zeta = 0$, reached in a finite time interval and "escape" from the classical trajectory toward the classical singularity. The peaks of the Lorentzian at fixed time $t$, evolves very slowly remaining close to the Planckian region. Such behavior outlines that the Universe has a stationary approach to the cutoff volume.

An important fact has now to be stressed. The peculiar behavior of our quantum Universe is different from other approaches to the same problem. In fact, recently, it was shown how the classical Big-Bang is replaced by a Big-Bounce in the framework of Loop Quantum Cosmology (LQC). Intuitively, one can expect that the bounce and so the consequently repulsive features of the gravitational field in the Planck regime are consequences of a Planckian cut-off length. But this is not the case. As matter of fact that there is not a bounce for our quantum Universe. The main differences between the two approaches resides in the quantum modification of the classical trajectory. In fact, in the LQC framework we observe a "quantum bridge" between the expanding and contracting Universes; in our approach, contrarily, the probability density of finding the Universe reaches the Planckian region in a stationary way.

## D.6. Gauge potential of a Lorentz gauge theory

A gauge theory of the local Lorentz group has been implemented both in flat and in curved space-time, and the resulting dynamics is analyzed in view of the geometrical interpretation of the gauge potential. The Yang-Mills picture of the local Lorentz transformations in curved space-time is first approached in a first-order formalism. For the Lagrangian approach to reproduce the II Cartan Structure Equation as soon as the Lorentz gauge connections are identified with the contortion tensor, an interaction term between the new Lorentz gauge fields $A_{\mu}{}^{ab}$ and the spin connections $\omega_{\mu}{}^{ab}$,has to be postulated, *i.e.*,

$$S_{int} = 2\int \det(e)\, d^4x \;\; e^{\mu}{}_{a} e^{\nu}{}_{b}\, \omega_{\mu c}^{[a}\, A_{\nu}{}^{bc]}\ . \tag{D.6.1}$$

This interaction term induces a Riemannian source to the Yang-Mills equations; thus, the real vacuum dynamics of the Lorentz gauge connections takes place on a Minkowski space only, when the Riemannian curvature and the spin currents provide negligible effects. In fact, it is the geometrical interpretation of the torsion field as a gauge field that generates the non-vanishing part of the Lorentz connection on flat space-time. The full picture involving gravity, torsion and spinors is described by a coupled set of field equations, which allows one to interpret both gravitational spin connections and matter spin density as the source term for the Yang-Mills equations. The contortion tensor acquires a propagating character, because of its non-Abelian feature, and the pure contact interaction is restored in the limit of vanishing Lorentz connections (Carlevaro et al., 2007).

To better understand the physical implications of first- and second-order approaches, a comparison between field equations has been accomplished in the linearized regime, by considering the case of small perturbations $h_{\mu\nu}$ of a flat Minkowskian metric $\eta_{\mu\nu}$. Because of the interaction term (D.6.1) postulated in the first-order approach, it is possible to solve the structure equation and to express the connection as a sum of the pure gravitational (Ricci) connection plus other contributions, both in absence and in presence of spinor matter. The Ricci connection $\omega_{\mu}{}^{ab} = e^{b\nu}\nabla_{\mu}e_{\nu}{}^{a}$ rewrites, because of the linearization,

$$\omega_\mu{}^{ab} = \delta^{b\nu}\left(\partial_\nu \zeta^a_\nu - \tilde{\Gamma}(\zeta)^\rho_{\mu\nu}\delta^b_\rho\right) , \tag{D.6.2}$$

where $\tilde{\Gamma}(\zeta)^\rho_{\mu\nu}$ are the linearized Christoffel symbols. Since it acquires the physical meaning of a source for torsion, it can be interpreted as a spin-current density. Nevertheless, it is linear in $\zeta$, since the interaction term (D.6.1) is linear itself; as suggested by the comparison with gauge theories, and with the current

$$M^{\tau\ \beta}_\alpha = \frac{\partial L}{\partial h_{\mu\nu,\tau}}\Sigma^{\rho\beta\sigma}_{\alpha\mu\nu}h_{\rho\sigma} = \left(\delta^{c\mu}\zeta^{\nu,\tau}_c + \delta^{c\nu}\zeta^{\mu,\tau}_c\right)\Sigma^{\rho\beta\sigma}_{\alpha\mu\nu}\left(\delta_{f\rho}\zeta^f_\sigma + \delta_{f\sigma}\zeta^f_\rho\right) , \tag{D.6.3}$$

(where $\Sigma^{\rho\alpha\beta\sigma}_{\mu\nu} = \eta^{\gamma[\alpha}(\delta^\rho_\gamma\delta^{\beta]}_\mu\delta^\sigma_\nu + \delta^\rho_\mu\delta^\sigma_\gamma\delta^{\beta]}_\nu))$, the interaction term is quadratic. In this case, however, it would be very difficult to split up the solution of the structure equation as the sum of the pure gravitational connection plus other contributions.

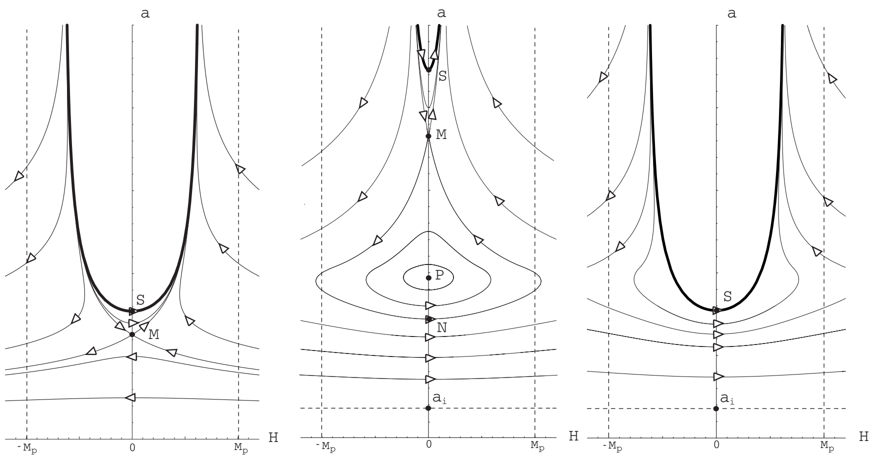


**Figure D.4.:** Phase portraits of cosmological models with the scalar field with flat effective potential. Left figure corresponds to the case of GR, $V_0 = 0.05\rho_{\text{pl}}$. Central figure represents the phase space with focus and saddle for $V_0 = 0.006\rho_{\text{pl}}$. Right figure represents the phase space with $V_0 = 0.05\rho_{\text{pl}}$. Thick curves surround regions where the derivative of the scalar field $\dot{\phi}$ is complex and there are no solutions. For central and right figure $j = 100$. Dashed lines again surround the region where semi-classical approach is valid.

# E. Quantum Fields on Classical Background

## E.1. Quantum Fields in accelerated systems

The research on this topic was dedicated to the rigorous analysis of Quantum Field Theory (QFT) from the point of view of an accelerated observer moving in the flat space and so called Unruh effect.

The field-theoretical analysis of the Unruh effect both from the point of view of canonical and algebraic approach to the quantum field theory was performed and the main results was published in the papers (Belinski et al., 1997), (Fedotov et al., 1999), (Narozhny et al., 2000), (Narozhny et al., 2002a), (Narozhny et al., 2004). We showed that the quantization procedure proposed by Unruh implies setting a boundary condition for the quantum field operator and this changes drastically the topological properties and symmetry group of the spacetime which lead to the field theory in two disconnected left and right Rindler spacetimes instead of Minkowski spacetime. The double Rindler wedge is composed of two causally disjoint regions ($R$-and $L$-wedges of Minkowski spacetime) and the Unruh construction implies existence of zero boundary condition for the quantum field operator at the common edge of $R$- and $L$-sectors of Minkowski spacetime. Such boundary condition gives rise to a superselection rule prohibiting any correlations between right and left Unruh particles. Thus the part of the field from the $L$-wedge in no way can influence a Rindler observer living in the $R$-wedge and elimination of the invisible left degrees of freedom will take no effect on him. Hence averaging over states of the field in one wedge can not lead to thermalization of the state in the other. This result is proved both in the standard and algebraic formulations of quantum field theory and conclusion is that principles of quantum field theory does not give any grounds for existence of the Unruh effect. Thus in spite of the work over last 30 years, there still remain serious gaps in grounding of the Unruh effect, and as of now there is no compelling evidence for the universal behaviour attributed to all uniformly accelerated detectors.

In parallel to the field-theoretical analysis of the Unruh effect we analyzed also the detector aspect of the Unruh problem (Narozhny et al., 2002b) in order to resolve the natural question what will be a reaction of a detector moving in the space filled by the Minkowski vacuum. The answer is that in general a detector by no means should exhibit the Unruh type behavior and such problem is sharply individual depending on the concrete detector structure and the nature of the accelerating force. There is no reasons to expect some universal answer based only on the general principles of the quantum field theory. To avoid misunderstandings it is worth to emphasize that in our work we have not asserted that one can not prepare a special type of detector which will show the Unruh reaction. Rather, we have questioned whether such reaction is universal. Certainly, the response of a particular detector can be predicted in principle by performing calculations using standard quantum mechanical technique in an inertial reference frame without any reference to Unruh modes, or the notion of Rindler particles. This could be done if one knows the nature of the accelerating force and the structure of the detector. However, there is no hope to prove universality of the Unruh effect sorting out an arbitrary number of detector models. On the contrary, non-universality of thermal response could be proved within such approach just by demonstration of at least a single example of a uniformly accelerated detector which does not reveal the Unruh behavior. The crucial point is that such examples really exist. One of the such detector construction we demonstrated in our paper (Narozhny et al., 2002b) where we showed that the proposed detector does not reveal the universal thermal response of the Unruh type. This example demonstrates non-universality of the Unruh response and constitutes a convincing argument in favor of impossibility to prove the existence of universal Unruh effect in the framework of quantum field theory.

Finally it is reasonable to ask whether one can find some physically sensible model in which the Unruh quantization procedure can be applied. In our work (Belinski et al., 2004), it is shown that such quantization procedure can be realized in Minkowski spacetime in the presence of Bose-Einstein condensate containing infinite number of particles in the zero boost mode. Unlike the case of an empty Minkowski spacetime the condensate provides the boundary conditions necessary for the Unruh quantization of the part of the field restricted only to the Rindler wedge of Minkowski spacetime. However, this treatment corresponds to the case when an accelerated observer is moving not in the Minkowski vacuum but in some medium filled by the real particles. Then there is no surprise that a detector can detect these quanta.

## E.2. Quantum Fields in Black Hole Space-Time

A careful investigation of the conventional derivation of the quantum effect of evaporation of black holes have been done and the results have been reported in papers (Belinski, 1995), (Belinski, 2006), (Belinski, 2007). These results show that there are serious doubts about the existence of such phenomenon. The main reason is due to the absence of the quasiclassical tunneling process corresponding to the particle creation by the Schwarzschild black hole created by the collapse. The manifestation of this fact can be seen also as an insolvency in the conventional second-quantization procedure analogous to the insolvency in the standard treatment of the Unruh effect.

# F. Unification theories

## F.1. Hamiltonian Formulation of the 5-dimensional Kaluza-Klein model

A first line of research is the analysis of the ADM splitting of the 5D KK model, to achieve the Hamiltonian formulation of the dynamics and get insights onto the gauge-symmetry generation. The ADM slicing of KK model, and its physical meaning, is not obvious, due to the existence of two possible procedures; we refer to these as KK-ADM and ADM-KK procedures. In KK-ADM we firstly perform the usual KK reduction of the metrics, and then a *3+1* ADM splitting of the gravitational tensor and the abelian gauge vector. The 5D metric $j_{AB}$ splits as follows

$$j_{AB} \Rightarrow \left\{ \begin{array}{l} g_{\mu\nu} \rightarrow \vartheta_{ij}, S_i, N \\ A_\mu \rightarrow A_i, A_0 \\ \phi \rightarrow \phi \end{array} \right. \rightarrow \left( \begin{array}{ccc} N^2 - S_i S^i - \phi^2 A_0^2 & -S_i - \phi^2 A_0 A_i & -\phi^2 A_0 \\ -S_i^2 \phi^2 A_0 A_i & -\vartheta_{ij} - \phi^2 A_i A_j & -\phi^2 A_i \\ -\phi^2 A_0 & -\phi^2 A_i & -\phi^2 \end{array} \right) .$$

Here $N,\ S_i,\ \vartheta_{ij}$ are the *lapse* function, the 3D *shift* vector and the 3D induced metrics ($A, B = 0,1,2,3,5;\ \mu, \nu = 0,1,2,3;\ i, j = 1,2,3$). This way, we have a non-complete space-time slicing, due to the fact that we are doing a *3+1* splitting of a 5-D background, so that the extra-dimension is not included. In the ADM-KK procedure we firstly deal with a *4+1* splitting that includes the extra-dimension and then we consider the KK reduction related to the pure spatial manifold:

$$j_{AB} \Rightarrow \left\{ \begin{array}{ll} h_{\hat{I},\hat{J}} & \rightarrow A_i, \vartheta_{ij}, \phi \\ N_{\hat{I}} & \rightarrow N_i, N_5 \\ N & \rightarrow N \end{array} \right. \Rightarrow \left( \begin{array}{ccc} N^2 - h_{\hat{I}\hat{J}} N^{\hat{I}} N^{\hat{J}} & -N_i & -N_5 \\ -N_i & -\vartheta_{ij} -^2 \phi^2 A_i A_j & -\phi^2 A_i \\ -N_5 & -\phi^2 A_i & -\phi^2 \end{array} \right) .$$

Here $N_{\hat{I}}$ and $h_{\hat{I}\hat{J}}$ are the 4D *shift* vector and the 4D spatial induced metric ($\hat{I}, \hat{J} = 1,2,3,5$). Now we have a complete slicing but, in this set of variables, the component $A_0$ is missing. Hence, both procedures are unsatisfactory and it must be checked if they commute. Despite the outcoming metric seem to be different, we are dealing with objects that must show well defined properties under pure spatial KK diffeomorphisms. This allow us to look for "conversion formulas" between this two metrics. Indeed, we can implement the KK reduction on $N_{\hat{I}}$; it is possible to recognize that $N_{\hat{I}}$ is not a pure 4D spatial vector neither simple gauge vector but a mixture of them. A detailed study of the 5-bein structure yields the following formulas for $N_{\hat{I}}$

$$\begin{cases} N_i = S_i + \phi^2 A_0 A_i \\ N_5 = \phi^2 A_0 \end{cases}, \qquad \begin{cases} N^i = S^i \\ N^5 = N^2 A^0 \end{cases}.$$

As soon as the Lagrangians resulting from these two procedures are recasted in the same set of variables, it is possible to recognize that they differ only for surface terms. Then, we conclude that we are dealing with equivalent dynamics and with a unique well defined Hamiltonian. Hence, ADM splitting is provided to commute with KK reduction and this allows us to compute the Hamiltonian. Moreover, the Hamiltonian formulation, together with conversion formulas, clearly shows how the time component of the electromagnetic field is given by a combination of the geometrical Lagrangian multipliers coming out in a 5D scheme.
People involved in this topic are Valentino Lacquaniti and Giovanni Montani (Lacquaniti and Montani, 2006a).

## F.2. Classical and Quantum spinning particles in Kaluza-Klein space-times

The dynamics of a classical spinning particle, in a KK space-time, is inferred from the extension of Papapetrou equations to the 5-dimensional case, with Pirani conditions, *i.e.*,

$$\begin{cases} \frac{D}{{}^{(5)}Ds}{}^{(5)}P^A = \frac{1}{2}{}^{(5)}R_{BCD}{}^{A}\Sigma^{BC\,(5)}u^D \\ \frac{D}{{}^{(5)}Ds}\Sigma^{AB} = {}^{(5)}P^{A\,(5)}u^B - {}^{(5)}P^{B\,(5)}u^A \\ {}^{(5)}P^A = {}^{(5)}m^{(5)}u^A - \frac{D\Sigma^{AB}}{{}^{(5)}Ds}{}^{(5)}u_B \\ \Sigma^{AB\,(5)}u_A = 0 \end{cases}. \tag{F.2.1}$$

The main new feature is the 4 additional components of the spin-tensor $\Sigma_{AB}$, whose physical meaning is going to be clarified by our analysis. At first, under coordinate transformations, proper of a KK model, $\Sigma^{\mu\nu}$ and $\Sigma_{5\mu}$ behave like 4-dimensional quantities, in particular a tensor $S^{\mu\nu}$ and a vector $S_\mu$, respectively.
By rewriting the full system above in terms of 4-dimensional quantities, $S^{\mu\nu}$,

$S_\mu$, $A_\mu$ and $g_{\mu\nu}$, one finds, after some manipulations,

$$\left\{\begin{array}{c} \frac{D}{Ds}\hat{P}^\mu = \frac{1}{2}R_{\alpha\beta\gamma}{}^{\mu}S^{\alpha\beta}u^\gamma + qF^\mu{}_\nu u^\nu + \frac{1}{2}\nabla^\mu F^{\nu\rho}M_{\nu\rho} \\ \frac{Dq}{Ds} = \frac{D}{Ds}(\alpha^2\widetilde{P}_5 + \frac{1}{4}ekF_{\alpha\beta}S^{\alpha\beta}) \\ \frac{DS^{\mu\nu}}{Ds} = \hat{P}^\mu u^\nu - \hat{P}^\nu u^\mu + F^\mu{}_\rho M^{\rho\nu} - F^\nu{}_\rho M^{\rho\mu} \\ \hat{P}^\mu = \alpha^2 P^\mu + u_5\frac{DS^\mu}{Ds} - ekF_{\rho\nu}u^\rho S^{\nu\mu}u_5 + \frac{1}{2}ekF^\mu{}_\rho S^\rho \\ S^{\nu\mu}u_\nu + S^\mu u_5 = 0 \end{array}\right. . \tag{F.2.2}$$

The quantity $M^{\mu\nu}$ has the following expression

$$M^{\mu\nu} = \frac{1}{2}ek(S^{\mu\nu}u_5 + u^\mu S^\nu - u^\nu S^\mu)\,, \tag{F.2.3}$$

and becouse of its coupling into equations of motion it has to be identified with the electro-magnetic moment.
This way, it is worth nothing that the system (F.2.2) reproduces exactly equations of motion of a spinning particle, endowed with a charge $q$ and an electromagnetic moment $M_{\mu\nu}$. This result demonstrates that the geometrization of the electro-dynamics does not modify the dynamics of spinning objects.
In this scenario, from the expression (F.2.3), the quantity $S_\mu$ is recognized as describing an electric dipole moment. The emergence of an electric dipole moment term seems to be a proper feature of a KK approach, since it arises also for spinors, in the Riemannian case.

The introduction of spinor fields in a KK model is the main open point of such an approach. The standard way to deal with them is to extend the Dirac equation to the multi-dimensional case and to try to identify extra-dimensional quantum numbers with internal ones. However this procedure fails, because of the emergence of mass terms of the compactification scale order and because quantum numbers of Standard model particles cannot be inferred.
In this respect, our investigation has been focused on a more phenomenological approach, based on recovering 4-dimensional properties by an averaging procedure on the extra-dimensional manifold. This average is motivated by the undetectability of the extra-space and the need for it is not restricted to the case spinors are present. In fact, we showed that it is required in order to reproduce non-Abelian gauge transformations from extra-dimensional isometries and to get the equations of motion, proper of the 4-dimensional picture, starting from multi-dimensional ones. As far as spinors are concerned, the average produces a non-trivial effect on extra-dimensional symmetries, such that some of the above mentioned issues can be solved.
For instance, we considered the case of a 3-sphere in view of performing the geometrization of an $SU(2)$ gauge theory. We look for a solution of the Dirac equation integrated over the sphere. Even though we do not find an exact

solution, an approximated one, with corrections controlled by an order parameter, is inferred. This is given by

$$\chi_r = \frac{1}{\sqrt{V}} e^{-\frac{i}{2}\sigma_{(p)rs}\lambda^{(p)}_{(q)}\Theta^{(q)}(y^m)} , \tag{F.2.4}$$

$V$ being the volume of $S^3$, $\sigma_{(p)}$ $SU(2)$ generators, while the constant matrix $\lambda$ satisfies

$$(\lambda^{-1})^{(p)}_{(q)} = \frac{1}{V} \int_{S^3} \sqrt{-\gamma} e^m_{(q)} \partial_m \Theta^{(p)} d^3y . \tag{F.2.5}$$

$\Theta$ functions are fixed as having the following form

$$\Theta^{(p)} = \frac{1}{\beta} c^{(p)} e^{-\beta\eta} , \qquad \eta > 0 , \tag{F.2.6}$$

with $c^{(p)}$ and $\eta$ some arbitrary functions, while $\beta$ is the order parameter, such that corrections to the Dirac equation are of the $\beta^{-1}$ order.
This form for the spinor is able to geometrize the $SU(2)$ gauge connection at the leading order in $\beta^{-1}$, while, at next orders, gauge-violating terms come out. Hence, this procedure can be used to geometrize the electro-weak model and infer a lower bound for $\beta$ from current limits on gauge-violating processes. Moreover, the introduction of the Higgs field in such a scenario succeeds in stabilizing its mass and in reproducing mass terms for neutrinos, too.

## F.3. Generalized 5-Dimensional Theories

5D KK models provide an interesting toy-model for the analysis of compactification schemes, and the features of generalized 5D models has been investigated, and the symmetries arising after dimensional reduction have been considered. In particular, alternative mechanisms that can imply compactification have been proposed, and broken 5D symmetries have been explored.
On the one hand, the presence of torsion in a 5D model has been shown to produce interesting structures after dimensional reduction. In a 5D scenario, the geometrization of the Electro-weak model has been worked out on the ground of the broken 5D Lorentz group and the properties of torsion [Lecian and Montani (2006)], and proposal for the introduction of Ashtekar variables within this scheme has been evaluated [Lecian and Montani (2007)]. Starting from the 5D Gauss-Codacci formula, and making sure that the residual symmetry of the metric components does not violate the Frobenius-Geroch requirements, evolutionary variables have been proposed.
On the other hand, a truncation of the KK towers has proposed from both

theoretical and phenomenological points of views. In particular, the simplest toy model of a scalar field in 5 dimensions has been analyzed: the truncation of such a tower has been considered as the hint of a modification of the extraD geometrical structures and related symmetries and compactification scenarios.
In the simplest toy model, i.e., a scalar field in a 5-dimensional (5D) space-time, described as the Dirac product of a 4D manifold plus a ring, $M^5 = M^4 \otimes S^1$, the Kaluza-Klein (KK) tower is defined as

$$\Psi^5(x^\rho, x^5) = \sum_{-\infty}^{+\infty} \psi_n(x^\rho) e^{ix^5 m/L}, \quad L \equiv 2\pi R, \tag{F.3.1}$$

that is the infinite sum of the Fourier harmonics, labeled by $m$. In this compactification scheme, because of the periodic (boundary) condition on the modes of the tower,

$$\psi(x^5) = \psi(x^5 + L), \quad L = 2\pi R,$$

i.e., of the identification of the points $0 \leftrightarrow 2\pi R$, $\psi(x_5)$ is defined on $S_1/\mathbb{Z} \sim \mathbb{R}$.
The scalar-field wavefunction obeys the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + {}^5M_m^2)\Psi = 0 \Rightarrow {}^5M_m^2 \equiv {}^4M^2 + (m/L)^2, \tag{F.3.2}$$

and it expression in the momentum representation reads

$$\tilde{\psi}^m(P_5) = \delta(P_5 - m/L).$$

From F.3.1, it is easy to understand the the structure of the extraD geometry can be described by means of the extraD projection of physical objects.
The analysis of truncated Kaluza-Klein (KK) tower can be performed on the ground of several considerations.
In fact, as it can be easily seen in (F.3.2), the label of the mode is deeply connected both with mass and extraD momentum, which can also be identified with the quantum number of a geometrized interaction, thus allowing for supposing a strict connection between the extraD and the internal structure. From theoretical point of view, the truncation of the tower would correspond to the introduction of a cutoff in the extra D, based on the fact that it would make little sense to specify the localization of a particle below its Schwartzschield radius. The exact localization of a particle in the extraD geometry would yield interpretative difficulties, such that a more general description of the internal structure, which does not automatically allow for an exact notion of point, should be looked for. Furthermore, an infinite spectrum of particles brings field-theoretical as well as algebraic difficulties. From

a phenomenological point of view, possible indications of the existence of an extraD would be provided both by geodesic deviation and scattering amplitudes. In the second case, the truncation would simplify the calculation of scattering amplitudes and would anyhow account for the impossibility of reaching an infinite energy in experiments.
As a result, the symmetries that characterize KK theories can be compared in the cases of infinite and truncated series. Since, in this toy model, the extraD expansion of the wavefunction of the scalar field is the only feature that accounts for the extraD, the truncation of the series would correspond to some modifications of the extraD geometry, as remarked for F.3.1. This way, it will be possible to analyze KK symmetries in both cases and possible compactification scenarios.
It has already been proposed to gain insight into the geometrical interpretation of truncated harmonics expansion on a circle by considering it as worked out from a higher-dimensional structure, thus obtaining a "fuzzy circle", in a "matrix-manifold" scheme. As a result, the ultraviolet cutoff of the model implies a minimal wavelength. As a first attempt, we propose a finite set of approximating wave functions, whose finite sum should reproduce the periodicity on the extraD coordinate, with the aim of pointing out the main difficulties of the problem.This preliminary speculation will be aimed at pointing out the main difficulties of the problem.
As a second strategy, we have considered the truncated wavefunction as a quasi-periodic function, projected on a finite set of Fourier modes. For this purpose, we have analyzed different representations of the standard operator algebra, given by the canonical commutation relations of the extraD operators $\hat{x}_5$ and $\hat{P}_5$, within the framework of the polymer representation. In this case, compactification has been illustrated to occur because of the truncation. We have then established generalized commutation relations; this way, the occurrence of compactification has been investigated through the fundamental wavelength of the theory.
The investigation of the role of the operators $\hat{x}_5$ and $\hat{P}_5$ in the extra-D symmetry and the different compactification mechanisms that arise from these scheme have motivated the comparison between the different approaches from a mathematical point of view [Lecian and Montani (2008)].

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