Theoretical Astroparticle Physics
**Contents**

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.4.</td>
<td>Alternative models</td>
<td>494</td>
</tr>
<tr>
<td>B.4.1.</td>
<td>Shemi and Piran model</td>
<td>494</td>
</tr>
<tr>
<td>B.4.2.</td>
<td>Shemi, Piran and Narayan model</td>
<td>495</td>
</tr>
<tr>
<td>B.4.3.</td>
<td>Mészáros, Laguna and Rees model</td>
<td>496</td>
</tr>
<tr>
<td>B.4.4.</td>
<td>Approximate results</td>
<td>497</td>
</tr>
<tr>
<td>B.4.5.</td>
<td>Nakar, Piran and Sari revision</td>
<td>500</td>
</tr>
<tr>
<td>B.5.</td>
<td>Significance of the rate equation</td>
<td>502</td>
</tr>
<tr>
<td>B.6.</td>
<td>The simple model for the afterglow</td>
<td>504</td>
</tr>
<tr>
<td>B.7.</td>
<td>Blandford and McKee radiative solution</td>
<td>506</td>
</tr>
<tr>
<td>B.7.1.</td>
<td>Reaching of transparency</td>
<td>508</td>
</tr>
</tbody>
</table>

**C.** Cosmological structure formation 511

| C.1. | The Cosmological Principle | 511 |
| C.2. | Two-point Correlation Function | 512 |
| C.2.1. | Observed Galaxy Distribution | 513 |
| C.2.2. | Power Law Clustering and Fractals | 514 |
| C.3. | Gravitational instability | 516 |
| C.3.1. | Horizon scale and mass evolution | 517 |
| C.3.2. | Selfgravitating ideal fluid: linear theory | 517 |
| C.3.3. | Applications | 521 |
| C.3.4. | Initial spectrum of perturbations | 522 |
| C.3.5. | Damping of Perturbations | 523 |
| C.3.6. | Structure formation on late times | 524 |

**D.** Massive degenerate neutrinos in Cosmology 527

| D.1. | Neutrino decoupling | 527 |
| D.2. | The redshifted statistics | 528 |
| D.3. | Energy density of neutrinos | 529 |
| D.3.1. | Neutrino mass | 530 |
| D.3.2. | Chemical potential | 531 |
| D.3.3. | Neutrino oscillations | 532 |
| D.4. | The Jeans mass of neutrinos | 532 |

**E.** Fermi’s approach to the study of hadronic interactions 537

| E.1. | Introduction | 537 |
| E.2. | Fermi’s approach to the study of hadronic interactions | 537 |
| E.3. | Modern approach | 545 |
| E.4. | Conclusion | 547 |

**Bibliography** 549
1. Topics

- Electron-positron plasma
  - Thermalization of the electron-positron-photon plasma
  - Difference from Cavallo and Rees scenarios
  - Fireball vs fireshell and equations of motion

- Neutrino in cosmology
  - Massive neutrino and structure formation
  - Cellular structure of the Universe
  - Lepton asymmetry of the Universe
1. Topics
2. Participants

2.1. ICRANet participants

- Carlo Luciano Bianco
- Remo Ruffini
- Gregory Vereshchagin
- She-Sheng Xue

2.2. Past collaborators

- Marco Valerio Arbolino (DUNE s.r.l., Italy)
- Andrea Bianconi (INFN Pavia, Italy)
- Neta A. Bahcall (Princeton University, USA)
- Daniella Calzetti (University of Massachusetts, USA)
- Jaan Einasto (Tartu Observatory, Estonia)
- Roberto Fabbri (University of Firenze, Italy)
- Long-Long Feng (University of Science and Technology, China)
- Jiang Gong Gao (Xinjiang Institute of Technology, China)
- Mauro Giavalisco (University of Massachusetts, USA)
- Gabriele Ingrosso (INFN, University of Lecce, Italy)
- Yi-peng Jing (Shanghai Astronomical Observatory, China)
- Hyung-Won Lee (Inje University, South Korea)
- Marco Merafina (University of Rome “Sapienza”, Italy)
- Houjun Mo (University of Massachusetts, USA)
- Enn Saar (Tartu Observatory, Estonia)
2. Participants

- Jay D. Salmonson (Livermore Lab, USA)
- Luis Alberto Sanchez (National University Medellin, Colombia)
- Costantino Sigismondi (ICRA and University of Rome “La Sapienza”, Italy)
- Doo Jong Song (Korea Astronomy Observatory, South Korea)
- Luigi Stella (Astronomical Observatory of Rome, Italy)
- William Stoeger (Vatican Observatory, University of Arizona USA)
- Sergio Taraglio (ENEA, Italy)
- Gerda Wiedenmann (MPE Garching, Germany)
- Jim Wilson (Livermore Lab, USA)

2.3. Ongoing collaborations

- Alexey Aksenov (ITEP, Russia)
- Valeri Chechetkin (Keldysh Institute, Russia)
- Massimiliano Lattanzi (Oxford, UK)

2.4. Students

- Gustavo de Barros (IRAP PhD, Brazil)
- Ivan Siutsou (IRAP PhD, Belarus)
3. Brief description

Astroparticle physics is a new field of research emerging at the intersection of particle physics, astrophysics and cosmology. Theoretical development in these fields is mainly triggered by the growing amount of experimental data of unprecedented accuracy, coming both from the ground based laboratories and from the dedicated space missions.

3.1. Electron-positron plasma

Electron-positron plasma is of interest in many fields of astrophysics, e.g. in the early universe, gamma-ray bursts, active galactic nuclei, the center of our Galaxy, hypothetical quark stars. It is also relevant for the physics of ultraintense lasers and thermonuclear reactions. We study some properties of dense and hot electron-positron plasmas. In particular, we are interested in the issues of its creation and relaxation, its kinetic properties and hydrodynamic description, transition to transparency and radiation from such plasmas.

Two completely different states exist for electron-positron plasma: optically thin and optically thick. Optically thin pair plasma may exist in active galactic nuclei and in X-ray binaries. The theory of relativistic optically thin nonmagnetic plasma and especially its equilibrium configurations was established in the 80s by Svensson, Lightman, Gould and others. It was shown that relaxation of the plasma to some equilibrium state is determined by a dominant reaction, e.g. Compton scattering or bremsstrahlung.

Developments in the theory of gamma ray bursts from one side, and observational data from the other side, unambiguously point out on existence of optically thick pair dominated non-steady phase in the beginning of formation of GRBs. The spectrum of radiation from optically thick plasma is assumed to be thermal. However, in such a transient phenomena as gamma-ray bursts there could be not enough time for the plasma to relax into equilibrium.

3.1.1. Thermalization of electron-positron-photon plasmas

One of crucial assumptions adopted in the literature on gamma-ray bursts (Ruffini et al., 1999, 2000) is that initial state of the pair plasma, formed in the
source of the gamma-ray burst is supposed to be thermal, with equal temperature of pairs and photons. It was analyzed by Aksenov et al. (2007, 2008, in press). The electron-positron-photon plasma was assumed to be homogeneous and isotropic, in the absence of magnetic fields, with average energy per particle bracketing electron rest mass, in the range $0.1 \text{MeV} \lesssim \epsilon \lesssim 10 \text{MeV}$. Relativistic Boltzmann equations were solved numerically for pairs and photons, starting from arbitrary initial configurations described by the corresponding distribution functions. All binary and triple collisions were accounted for, by the corresponding collisional integrals.

The evolution of the plasma was followed up to reaching thermal equilibrium with the same temperature of photons and pairs, and vanishing chemical potentials of both pairs and photons, for details see Appendix A. It was shown that thermal equilibrium is reached on a short timescale $t < 10^{-12} \text{sec}$, much shorter than the dynamical timescale. The conclusion was reached that initial state of the plasma in GRB sources is indeed thermal.

### 3.1.2. Difference from Cavallo and Rees scenarios

Cavallo and Rees (1978) performed qualitatively study of cosmic fireballs and gamma ray bursts. They suggested a number of different scenarios, assuming in all of them that the pair plasma, formed in the source of a GRB cools down due to direct bremsstrahlung process until the temperature reaches 0.511 MeV and electron-positron pairs disappear prior to expansion.

Among our results we do not find support for the Cavallo-Rees scenarios, where the average energies per particle in the lepton fireball degrade until the threshold for pair production is reached, and electron-positron pairs disappear. However, we found very similar result when artificially switched off inverse triple interactions, see Appendix A. In that case thermal equilibrium is never reached.

### 3.1.3. Fireball vs fireshell and equations of motion

Gamma-ray bursts are very different from the fireball phenomenon (e.g. atomic bomb explosion) with high temperature inside, blast wave propagating into the surrounding dense atmosphere and a characteristic self-similar motion described by Sedov solution, which can be obtained easily by dimensionality considerations.

In a gamma-ray burst explosion, after appearance of dense and hot optically thick plasma composed of electron-positron pairs and photons it starts to expand adiabatically, governed by equations of relativistic hydrodynamics. The surrounding medium is either rarefied or even absent and the motion is purely relativistic expansion into the vacuum.

The work of Ruffini et al. (1999, 2000) on the pair electromagnetic pulse
created by a black hole have shown that electron-positron pairs can self-accelerate outwards from the source, and form a relativistic shell expanding with unprecedented large Lorentz factors, of the order of several hundreds. It was shown, that non-equilibrium effects should be taken into account, in particular dynamical approach of the plasma to transparency was described by the usual hydrodynamic equations together with the rate equation for electron-positron pairs.

During acceleration phase, occurring because of large radiative pressure, the plasma engulfs certain amount of baryonic matter, and continue to accelerate until the plasma becomes transparent, electron-positron pairs disappear and all photons escape.

Although initially uniform plasma remains uniform during expansion in the comoving frame, it appears as a narrow shell, which we refer to as a fireshell, in the laboratory frame. From hydrodynamic equations it is possible to show (Vereshchagin, 2007) that the thickness of the fireshell indeed remains constant under the condition $\gamma >> 1$, which is approached soon after beginning of expansion since the scaling law $\gamma \propto r$, characterizes expansion of the radiation-dominated pair plasma. This conclusion was obtained in (Ruffini et al., 1999) by the analysis of different geometries of the plasma.

In the work of Bianco et al. (2006) we have compared and contrasted existing models of hydrodynamic evolution of gamma-ray bursts in the literature. It was pointed out, that in spite of many qualitative similarities, several crucial quantitative differences exist, namely a) the appropriate model for geometry of expanding fireshell (PEM-pulse) is given by the constant width approximation; there is no broadening of the fireshell; b) there is a bound on parameter $B$ which comes from violation of constant width approximation, $B \leq 10^{-2}$; c) the rate equation for electron-positron pairs plays crucial role in description of the approach to transparency. Details are given in the Appendix B. All these differences are crucial when the theory is confronted with observations, as shown by Ruffini et al. (2007).
3. Brief description

3.2. Neutrino in cosmology

Since experiments on neutrino oscillations these particles are known to be massive, and thus contribute to the density budget of the Universe, making them candidates for dark matter. We study the role of massive neutrinos in cosmology and in the large scale structure formation, with particular attention to the possible lepton asymmetry of the Universe.

Many observational facts make it clear that luminous matter alone cannot account for the whole matter content of the Universe. Among them there is the cosmic background radiation anisotropy spectrum, that is well fitted by a cosmological model in which just a small fraction of the total density is supported by baryons.

In particular, the best fit to the observed spectrum is given by a flat $\Lambda$CDM model, namely a model in which the main contribution to the energy density of the Universe comes from vacuum energy and cold dark matter. This result is confirmed by other observational data, like the power spectrum of large scale structures.

Another strong evidence for the presence of dark matter is given by the rotation curves of galaxies. In fact, if we assume a spherical or ellipsoidal mass distribution inside the galaxy, the orbital velocity at a radius $r$ is given by Newton’s equation of motion. The peculiar velocity of stars beyond the visible edge of the galaxy should then decrease as $1/r$. What is instead observed is that the velocity stays nearly constant with $r$. This requires a halo of invisible, dark, matter to be present outside the edge. Galactic size should then be extended beyond the visible edge. From observations it follows that the halo radius is at least 10 times larger than the radius of visible part of the galaxy. Then it follows that a halo is at least 10 times more massive than all stars in a galaxy.

Neutrinos were considered as the best candidate for dark matter about twenty years ago. Indeed, it was shown that if these particles have a small mass $m_\nu \sim 30\text{ eV}$, they provide a large energy density contribution up to critical density. Tremaine and Gunn (1979) have claimed, however, that massive neutrinos cannot be considered as dark matter. Their paper was very influential and turned most of cosmologists away from neutrinos as cosmologically important particles.

Tremaine and Gunn paper was based on estimation of lower and upper bounds for neutrino mass; when contradiction with these bounds was found, the conclusion was made that neutrinos cannot supply dark matter. The upper bound was given by cosmological considerations, but compared with the energy density of clustered matter. It is possible, however, that a fraction of neutrinos lays outside galaxies.

Moreover, their lower bound was found on the basis of considerations of galactic halos and derived on the ground of the classical Maxwell-Boltzmann
3.2. Neutrino in cosmology

statistics. Gao and Ruffini (1980) established a lower limit on the neutrino mass by the assumption that galactic halos are composed by degenerate neutrinos. Subsequent development of this approach by Arbolino and Ruffini (1988) has shown that contradiction with two limits can be avoided.

At the same time, the paper by Lee and Weinberg (1977) appeared, in which authors turned their attention to massive neutrinos with $m_\nu > 2\text{ GeV}$. Such particles could also provide a large contribution into the energy density of the Universe, in spite of much smaller value of number density.

Recent experimental results from laboratory, see (Dolgov, 2002) for a review, rule out massive neutrinos with $m_\nu > 2\text{ GeV}$. However, the paper by Lee and Weinberg was among the first where very massive particles were considered as candidates for dark matter. This can be considered as the first of cold dark matter models.

Today the interest toward neutrinos as a candidate for dark matter came down, since from one side, the laboratory limit on its mass do not allow for significant contribution to the density of the Universe, and from other side, conventional neutrino dominated models have problems with formation of structure on small scales. However, in these scenarios the role of the chemical potential of neutrinos was overlooked, while it could help solving both problems.

3.2.1. Massive neutrino and structure formation

Lattanzi et al. (2003) have studied the possible role of massive neutrinos in the large scale structure formation. Although now it is clear, that massive light neutrinos cannot be the dominant part of the dark matter, their influence on the large scale structure formation should not be underestimated. In particular, large lepton asymmetry, still allowed by observations, can affect cosmological constraints on neutrino mass.

3.2.2. Cellular structure of the Universe

One of the interesting possibilities, from a conceptual point of view, is the change from the description of the physical properties by a continuous function, to a new picture by introducing a self-similar fractal structure. This approach has been relevant, since the concept of homogeneity and isotropy formerly apply to any geometrical point in space and leads to the concept of a Universe observer-homogeneous (Ruffini, 1989). Calzetti et al. (1987); Giavalisco (1992); Calzetti et al. (1988) have defined the correlation length of a fractal

$$r_0 = \left(1 - \frac{\gamma}{3}\right)^{1/\gamma} R_S,$$

(3.2.1)

where $R_S$ is the sample size, $\gamma = 3 - D$, and $D$ is the Hausdorff dimension of the fractal. Most challenging was the merging of the concepts of fractal, Jeans
mass of dark matter and the cellular structure in the Universe, advanced by Ruffini et al. (1988). The cellular structure emerging from this study is represented in Fig. 3.1. There the upper cutoff in the fractal structure $R_{\text{cutoff}} \approx 100$ Mpc, was associated to the Jeans mass of the “ino” $M_{\text{cell}} = \left(\frac{m_{\text{pl}}}{m_{\text{ino}}}\right)^2 m_{\text{pl}}$. Details see in Appendix C.

Figure 3.1.: Cellular structure of the Universe.

3.2.3. Lepton asymmetry of the Universe

Lattanzi et al. (2005, 2006) studied how the cosmological constraints on neutrino mass are affected by the presence of a lepton asymmetry. The main conclusion is that while constraints on neutrino mass do not change by the inclusion into the cosmological model the dimensional chemical potential of neutrino, as an additional parameter, the value of lepton asymmetry allowed by the present cosmological data is surprisingly large, being

$$L = \sum n_{\nu} - n_{\bar{\nu}} \lesssim 0.9,$$ (3.2.2)

Therefore, large lepton asymmetry is not ruled out by the current cosmological data. Details see in Appendix D.
4. Publications (before 2005)


The distribution function of massive Fermi and Bose particles in an expanding universe is considered as well as some associated thermodynamic quantities, pressure and energy density. These considerations are then applied to cosmological neutrinos. A new limit is derived for the degeneracy of a cosmological gas of massive neutrinos.


The cosmological limits on the abundances and masses of weakly interacting neutral particles are strongly affected by the nonzero chemical potentials of these leptons. For heavy leptons (m_x > GeV), the value of the chemical potential must be much smaller than unity in order not to give very high values of the cosmological density parameter and the mass of heavy leptons, or they will be unstable. The Jeans’ mass of weakly interacting neutral particles could give the scale of cosmological structure and the masses of astrophysical objects. For a mass of the order 10 eV, the Jeans’ mass could give the scenario of galaxy formation, the supercluster forming first and then the smaller scales, such as clusters and galaxies, could form inside the large supercluster.


Data obtained by Einasto et al. (1986) on the amplitude of the correlation function of galaxies in the direction of the Coma cluster are compared with theoretical predictions of a model derived for a self-similar observer-homogeneous structure. The observational samples can be approximated by cones of angular width alpha of about 77 deg. Eliminating sources of large observational error, and by making a specified correction, the observational data are found to agree very well with the theoretical predictions of Calzetti et al. (1987).

Within the theoretical framework of a Gamow cosmology with massive "inos", the authors show how the observed correlation functions between galaxies and between clusters of galaxies naturally lead to a "cellular" structure for the Universe. From the size of the "elementary cells" they derive constraints on the value of the masses and chemical potentials of the cosmological "inos". They outline a procedure to estimate the "effective" average mass density of the Universe. They also predict the angular size of the inhomogeneities to be expected in the cosmological black body radiation as remnants of this cellular structure. A possible relationship between the model and a fractal structure is indicated.


It is shown that the spatial two-point correlation functions for galaxies, clusters and superclusters depend explicitly on the spatial volume of the statistical sample considered. Rules for the normalization of the correlation functions are given and the traditional classification of galaxies into field galaxies, clusters and superclusters is replaced by the introduction of a single fractal structure, with a lower cut-off at galactic scales. The roles played by random and stochastic fractal components in the galaxy distribution are discussed in detail.


Observed rotation curves for galaxies with values of the visible mass ranging over three orders of magnitude together with considerations involving equilibrium configurations of massive neutrinos, impose constraints on the ratio between the masses of visible and dark halo components in spiral galaxies. Upper and lower limits are derived for the mass of the particles making up the dark matter.


Constraints on chemical potentials and masses of ‘inos’ are calculated using cosmological standard nucleosynthesis processes. It is shown that the electron neutrino chemical potential (ENCP) should not be greater than a value of the order of 1, and that the possible effective chemical potential of the other neutrino species should be about 10 times the ENCP in order not to conflict with observational data. The allowed region (consistent with the He-4 abundance observations) is insensitive to the baryon to proton ratio \( \eta \), while those imposed by other light elements strongly depend on \( \eta \).

We study the relativistically expanding electron-positron pair plasma formed by the process of vacuum polarization around an electromagnetic black hole (EMBH). Such processes can occur for EMBH’s with mass all the way up to $6 \cdot 10^5 M_\odot$. Beginning with a idealized model of a Reissner-Nordstrom EMBH with charge to mass ratio $\xi = 0.1$, numerical hydrodynamic calculations are made to model the expansion of the pair-electromagnetic pulse (PEM pulse) to the point that the system is transparent to photons. Three idealized special relativistic models have been compared and contrasted with the results of the numerically integrated general relativistic hydrodynamic equations. One of the three models has been validated: a PEM pulse of constant thickness in the laboratory frame is shown to be in excellent agreement with results of the general relativistic hydrodynamic code. It is remarkable that this precise model, starting from the fundamental parameters of the EMBH, leads uniquely to the explicit evaluation of the parameters of the PEM pulse, including the energy spectrum and the astrophysically unprecedented large Lorentz factors (up to $6 \cdot 10^3$ for a $10^3 M_\odot$ EMBH). The observed photon energy at the peak of the photon spectrum at the moment of photon decoupling is shown to range from 0.1 MeV to 4 MeV as a function of the EMBH mass. Correspondingly the total energy in photons is in the range of $10^{52}$ to $10^{54}$ ergs, consistent with observed gamma-ray bursts. In these computations we neglect the presence of baryonic matter which will be the subject of forthcoming publications.


The interaction of an expanding Pair-Electromagnetic pulse (PEM pulse) with a shell of baryonic matter surrounding a Black Hole with electromagnetic structure (EMBH) is analyzed for selected values of the baryonic mass at selected distances well outside the dyadosphere of an EMBH. The dyadosphere, the region in which a super critical field exists for the creation of $e^+e^-$ pairs, is here considered in the special case of a Reissner-Nordstrom geometry. The interaction of the PEM pulse with the baryonic matter is described using a simplified model of a slab of constant thickness in the laboratory frame (constant-thickness approximation) as well as performing the integration of the general relativistic hydrodynamical equations. Te validation of the constant-thickness approximation, already presented in a previous paper Ruffini et al. (1999) for a PEM pulse in vacuum, is here generalized to the presence of baryonic matter. It is found that for a baryonic shell of mass-energy less than 1% of the total energy of the dyadosphere, the constant-thickness approximation is in excel-
lent agreement with full general relativistic computations. The approximation breaks down for larger values of the baryonic shell mass, however such cases are of less interest for observed Gamma Ray Bursts (GRBs). On the basis of numerical computations of the slab model for PEM pulses, we describe (i) the properties of relativistic evolution of a PEM pulse colliding with a baryonic shell; (ii) the details of the expected emission energy and observed temperature of the associated GRBs for a given value of the EMBH mass; $10^3 M_\odot$, and for baryonic mass-energies in the range $10^{-8}$ to $10^{-2}$ the total energy of the dyadosphere.


In addition to the problem of galaxy formation, one of the greatest open questions of cosmology is represented by the existence of an asymmetry between matter and antimatter in the baryonic component of the Universe. We believe that a net lepton number for the three neutrino species can be used to understand this asymmetry. This also implies an asymmetry in the matter-antimatter component of the leptons. The existence of a nonnull lepton number for the neutrinos can easily explain a cosmological abundance of neutrinos consistent with the one needed to explain both the rotation curves of galaxies and the flatness of the Universe. Some propedeutic results are presented in order to attack this problem.

Refereed publications


Starting from a nonequilibrium configuration we analyze the role of the direct and the inverse binary and triple interactions in reaching thermal equilibrium in a homogeneous isotropic pair plasma. We focus on energies in the range $0.1 - 10$ MeV. We numerically integrate the relativistic Boltzmann equation with the exact QED collisional integrals taking into account all binary and triple interactions. We show that first, when a detailed balance is reached for all binary interactions on a time scale $t_k < 10^{-14}$ sec, photons and electron-positron pairs establish kinetic equilibrium. Subsequently, when triple interactions satisfy the detailed balance on a time scale $t_{eq} < 10^{-12}$ sec, the plasma reaches thermal equilibrium. It is shown that neglecting the inverse triple interactions prevents reaching thermal equilibrium. Our results obtained in the theoretical physics domain also find application in astrophysics and cosmology.


We compare and contrast the different approaches to the optically thick adiabatic phase of GRB all the way to the transparency. Special attention is given to the role of the rate equation to be self consistently solved with the relativistic hydrodynamic equations. The works of Shemi and Piran (1990), Piran, Shemi and Narayan (1993), Meszaros, Laguna and Rees (1993) and Ruffini, Salmonson, Wilson and Xue (1999,2000) are compared and contrasted. The role of the baryonic loading in these three treatments is pointed out. Constraints on initial conditions for the fireball produced by electro-magnetic black hole are obtained.

Nonperturbative quantum geometric effects in loop quantum cosmology predict a $p^2$ modification to the Friedmann equation at high energies. The quadratic term is negative definite and can lead to generic bounces when the matter energy density becomes equal to a critical value of the order of the Planck density. The nonsingular bounce is achieved for arbitrary matter without violation of positive energy conditions. By performing a qualitative analysis we explore the nature of the bounce for inflationary and cyclic model potentials. For the former we show that inflationary trajectories are attractors of the dynamics after the bounce implying that inflation can be harmoniously embedded in LQC. For the latter difficulties associated with singularities in cyclic models can be overcome. We show that nonsingular cyclic models can be constructed with a small variation in the original cyclic model potential by making it slightly positive in the regime where scalar field is negative.


We use the Wilkinson Microwave Anisotropy Probe (WMAP) data on the spectrum of cosmic microwave background anisotropies to put constraints on the present amount of lepton asymmetry $L$, parametrized by the dimensionless chemical potential (also called degeneracy parameter) $\xi$ and on the effective number of relativistic particle species. We assume a flat cosmological model with three thermally distributed neutrino species having all the same mass and chemical potential, plus an additional amount of effectively massless exotic particle species. The extra energy density associated to these species is parametrized through an effective number of additional species $\Delta N_{\text{others}}^{\text{eff}}$. We find that $0 < |\xi| < 1.1$ and correspondingly $0 < |L| < 0.9$ at 2$\sigma$, so that WMAP data alone cannot firmly rule out scenarios with a large lepton number; moreover, a small preference for this kind of scenarios is actually found. We also discuss the effect of the asymmetry on the estimation of other parameters and, in particular, of the neutrino mass. In the case of perfect lepton symmetry, we obtain the standard results. When the amount of asymmetry is left free, we find at 2sigma. Finally we study how the determination of $|L|$ is affected by the assumptions on $\Delta N_{\text{others}}^{\text{eff}}$. We find that lower values of the extra energy density allow for larger values of the lepton asymmetry, effectively ruling out, at 2sigma level, lepton symmetric models with $\Delta N_{\text{others}}^{\text{eff}} \simeq 0$.


Brief introduction into gauge theories of gravity is presented. The most general gravitational lagrangian including quadratic on curvature, torsion and non-
metricity invariants for metric-affine gravity is given. Cosmological implications of gauge gravity are considered. The problem of cosmological singularity is discussed within the framework of general relativity as well as gauge theories of gravity. We consider the role of scalar field in connection to this problem. Initial conditions for nonsingular homogeneous isotropic Universe filled by single scalar field are discussed within the framework of gauge theories of gravity. Homogeneous isotropic cosmological models including ultrarelativistic matter and scalar field with gravitational coupling are investigated. We consider different symmetry states of effective potential of the scalar field, in particular restored symmetry at high temperatures and broken symmetry. Obtained bouncing solutions can be divided in two groups, namely nonsingular inflationary and oscillating solutions. It is shown that inflationary solutions exist for quite general initial conditions like in the case of general relativity. However, the phase space of the dynamical system, corresponding to the cosmological equations is bounded. Violation of the uniqueness of solutions on the boundaries of the phase space takes place. As a result, it is impossible to define either the past or the future for a given solution. However, definitely there are singular solutions and therefore the problem of cosmological singularity cannot be solved in models with the scalar field within gauge theories of gravity.


Gamma-Ray Bursts (GRBs) represent very likely “the” most extensive computational, theoretical and observational effort ever carried out successfully in physics and astrophysics. The extensive campaign of observation from space based X-ray and γ-ray observatory, such as the Vela, CGRO, BeppoSAX, HETE-II, INTEGRAL, Swift, R-XTE, Chandra, XMM satellites, have been matched by complementary observations in the radio wavelength (e.g. by the VLA) and in the optical band (e.g. by VLT, Keck, ROSAT). The net result is unprecedented accuracy in the received data allowing the determination of the energetics, the time variability and the spectral properties of these GRB sources. The very fortunate situation occurs that these data can be confronted with a mature theoretical development. Theoretical interpretation of the above data allows progress in three different frontiers of knowledge: a) the ultrarelativistic regimes of a macroscopic source moving at Lorentz gamma factors up to ~ 400; b) the occurrence of vacuum polarization process verifying some of the yet untested regimes of ultrarelativistic quantum field theories; and c) the first evidence for extracting, during the process of gravitational collapse leading to

the formation of a black hole, amounts of energies up to $10^{55}$ ergs of black-
hollic energy — a new form of energy in physics and astrophysics. We outline
how this progress leads to the confirmation of three interpretation paradigms
for GRBs proposed in July 2001. Thanks mainly to the observations by Swift
and the optical observations by VLT, the outcome of this analysis points to the
existence of a “canonical” GRB, originating from a variety of different initial
astrophysical scenarios. The communality of these GRBs appears to be that
they all are emitted in the process of formation of a black hole with a negligi-
ble value of its angular momentum. The following sequence of events appears
to be canonical: the vacuum polarization process in the dyadosphere with the
creation of the optically thick self accelerating electron-positron plasma; the
engulfment of baryonic mass during the plasma expansion; adiabatic expa-
sion of the optically thick “fireshell” of electron-positron-baryon plasma up
to the transparency; the interaction of the accelerated baryonic matter with
the interstellar medium (ISM). This leads to the canonical GRB composed of a
proper GRB (P-GRB), emitted at the moment of transparency, followed by an
extended afterglow. The sole parameters in this scenario are the total energy of
the dyadosphere $E_{dya}$, the fireshell baryon loading $M_B$ defined by the dimen-
sionless parameter $B = \frac{M_B c^2}{E_{dya}}$, and the ISM filamentary distribution
around the source. In the limit $B \rightarrow 0$ the total energy is radiated in the P-
GRB with a vanishing contribution in the afterglow. In this limit, the canonical
GRBs explain as well the short GRBs. In these lecture notes we systematically
outline the main results of our model comparing and contrasting them with
the ones in the current literature. In both cases, we have limited ourselves to
review already published results in refereed publications. We emphasize as
well the role of GRBs in testing yet unexplored grounds in the foundations of
general relativity and relativistic field theories.

7. M. Lattanzi, R. Ruffini and G.V. Vereshchagin, “Do WMAP data con-
straint the lepton asymmetry of the Universe to be zero?” in Albert Ein-
stein Century International Conference, edited by J.-M. Alimi, and A.
pp.912-919.

It is shown that extended flat $Λ$CDM models with massive neutrinos, a size-
able lepton asymmetry and an additional contribution to the radiation content
of the Universe, are not excluded by the Wilkinson Microwave Anisotropy
Probe (WMAP) first year data. We assume a flat cosmological model with
three thermally distributed neutrino species having all the same mass and
chemical potential, plus an additional amount of effectively massless exotic
particle species $X$. After maximizing over seven other cosmological parama-
ters, we derive from WMAP first year data the following constraints for the
lepton asymmetry $L$ of the Universe (95% CL): $0 < |L| < 0.9$, so that WMAP
data alone cannot firmly rule out scenarios with a large lepton number; more-
over, a small preference for this kind of scenarios is actually found. We also
find for the neutrino mass $m_\nu < 1.2eV$ and for the effective number of relativistic particle species $-0.45 < \Delta N_{\text{eff}} < 2.10$, both at 95% CL. The limit on $\Delta N_{\text{eff}}$ is more restrictive than others found in the literature, but we argue that this is due to our choice of priors.


The expansion of the electron-positron plasma in the GRB phenomenon is compared and contrasted in the treatments of Meszaros, Laguna and Rees, of Shemi, Piran and Narayan, and of Ruffini et al. The role of the correct numerical integration of the hydrodynamical equations, as well as of the rate equation for the electron-positron plasma loaded with a baryonic mass, are outlined and confronted for crucial differences.


Recent constraints on neutrino mass and chemical potential are discussed with application to large scale structure formation. Power spectra in cosmological model with hot and cold dark matter, baryons and cosmological term are calculated in newtonian approximation using linear perturbation theory. All components are considered to be ideal fluids. Dissipative processes are taken into account by initial spectrum of perturbations so the problem is reduced to a simple system of equations. Our results are in good agreement with those obtained before using more complicated treatments.


The recent analysis of the cosmic microwave background data carried out by the WMAP team seems to show that the sum of the neutrino mass is $<0.7$ eV. However, this result is not model-independent, depending on precise assumptions on the cosmological model. We study how this result is modified when the assumption of perfect lepton symmetry is dropped out.


In addition to the problem of galaxy formation, one of the greatest open questions of cosmology is represented by the existence of an asymmetry between matter and antimatter in the baryonic component of the Universe. We believe that a net lepton number for the three neutrino species can be used to understand this asymmetry. This also implies an asymmetry in the matter-antimatter component of the leptons. The existence of a nonnull lepton number for the neutrinos can easily explain a cosmological abundance of neutrinos consistent with the one needed to explain both the rotation curves of galaxies and the flatness of the Universe. Some propedeutic results are presented in order to attack this problem.

Invited talks at international conferences

1. “Thermalization of the pair plasma”
   (G.V. Vereshchagin, R. Ruffini, and A.G. Aksenov)

2. “Non-singular solutions in Loop Quantum Cosmology”
   (G.V. Vereshchagin)

   2nd Stueckelberg Workshop, Pescara, Italy, 3-7 September, 2007.

3. “(From) massive neutrinos and inos and the upper cutoff to the fractal structure of the Universe (to recent progress in theoretical cosmology)”
   (G.V. Vereshchagin, M. Lattanzi and R. Ruffini)


4. “Pair creation and plasma oscillations”
   (G.V. Vereshchagin, R. Ruffini, and S.-S. Xue)


5. “Thermalization of electron-positron plasma in GRB sources”
   (with R. Ruffini, and A.G. Aksenov)


6. “Kinetics and hydrodynamics of the pair plasma”
   (G.V. Vereshchagin, R. Ruffini, C.L. Bianco, A.G. Aksenov)
7. “Pair creation and plasma oscillations”
   (G.V. Vereshchagin, R. Ruffini and S.-S. Xue)
   Cesare Lattes Meeting on GRBs, Black Holes and Supernovae, Mangaratiba-Portobello, Brazil, 26 February - 3 March 2007.

8. “Cavallo-Rees classification revisited”
   (G.V. Vereshchagin, R. Ruffini and S.-S. Xue)

9. “Kinetic and thermal equilibria in the pair plasma”
   (G.V. Vereshchagin)
   The 1st Bego scientific rencontre, Nice, 5-16 February 2006.

10. “From semi-classical LQC to Friedmann Universe”
    (G.V. Vereshchagin)
    Loops ‘05, Potsdam, Golm, Max-Plank Institut für Gravitationsphysik (Albert-Einstein-Institut), 10-14 October 2005.

11. “Equations of motion, initial and boundary conditions for GRBs”
    (G.V. Vereshchagin, R. Ruffini and S.-S. Xue)

12. “On the Cavallo-Rees classification and GRBs”
    (G.V. Vereshchagin, R. Ruffini and S.-S. Xue)
    II Italian-Sino Workshop on Relativistic Astrophysics, Pescara, Italy, 10-20 June, 2005.
6. APPENDICES
A. Thermalization of electron-positron-photon plasmas

Following (Aksenov et al., 2007) consider a uniform isotropic electron-positron-photon plasma in the absence of external electromagnetic fields and describe its evolution starting from arbitrary nonequilibrium initial conditions up to reaching thermal equilibrium. We are interested in the range of final temperatures in thermal equilibrium, bracketing the electron rest mass energy

\[ 0.1 \text{ MeV} \lesssim T_{th} \lesssim 10 \text{ MeV}. \]  
(A.0.1)

These boundaries are required for the study of electron-positron pairs in absence of the production of other particles such as muons. We assume that the energy density of the plasma is constant and is, correspondingly, in the range

\[ 1.6 \times 10^{22} \text{ erg cm}^{-3} < \rho < 3.8 \times 10^{30} \text{ erg cm}^{-3}. \]  
(A.0.2)

The corresponding number densities at thermal equilibrium will be

\[ 3.1 \times 10^{28} \text{ cm}^{-3} < n_{th} < 7.9 \times 10^{34} \text{ cm}^{-3}. \]  
(A.0.3)

If one takes typical energies and sizes of GRB sources, one finds energy densities in the region (A.0.2), which means that the pair plasma may thermalize in GRB sources and have temperature in the range (A.0.1).

A.1. The Method

We adopt a kinetic description for the distribution functions of electrons, positrons and photons. In our case the plasma parameter is small, \( g = (n_{\pm} r_D^3)^{-1} \ll 1 \), where \( r_D \) is the Debye length and \( n_{\pm} \) is concentration of pairs, and therefore we use one-particle distribution functions and neglect particle correlations.

Besides, in our case electrons and positrons are non-degenerate, since the
A. Thermalization of electron-positron-photon plasmas

degeneracy temperature

\[
\theta_F = \left\{ 1 + \left[ 3\pi^2 n_\pm \left( \frac{\hbar}{m_e c} \right)^3 \right]^{2/3} \right\}^{1/2} - 1, \quad (A.1.1)
\]

where \( \theta = k_B T / (m_e c^2) \) is dimensionless temperature, \( \hbar \) is the Planck’s constant, \( m_e \) is the electron mass, \( k_B \) is Boltzmann’s constant and \( n_\pm \) is number density of electron-positron pairs, is below the temperature range (A.0.1).

We solve numerically the relativistic Boltzmann equations in spherically symmetric case (Belyaev and Budker, 1956; Mihalas and Mihalas, 1984)

\[
\frac{1}{c} \frac{\partial f_i}{\partial t} + \beta_i \left( \frac{\partial f_i}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial f_i}{\partial \mu} \right) - \nabla U \frac{\partial f_i}{\partial p} = \sum_q \left( \eta_{iq} f_i - \chi_{iq} f_i \right), \quad (A.1.2)
\]

where \( \mu = \cos \vartheta \), \( \vartheta \) is the angle between the radius vector \( \mathbf{r} \) from the origin and the particle momentum \( \mathbf{p} \), \( U \) is a potential due to some external force, \( \beta_i = v_i / c \) are particles velocities, \( f_i(\epsilon, t) \) are their distribution functions, the index \( i \) denotes the type of particle, \( \epsilon \) is their energy, and \( \eta_{iq} \) and \( \chi_{iq} \) are the emission and the absorption coefficients for the production of a particle of type “i” via the physical process labeled by \( q \).

For homogeneous and isotropic distribution functions of electrons, positrons and photons (A.1.2) reduces to

\[
\frac{1}{c} \frac{\partial f_i}{\partial t} = \sum_q \left( \eta_{iq} f_i - \chi_{iq} f_i \right). \quad (A.1.3)
\]

In order to solve equations (A.1.3) we use a finite difference method by introducing a computational grid in the phase space to represent the distribution functions and to compute collisional integrals following (Aksenov et al., 2004). Our goal is to construct the conservative (for the energy) scheme. For this reason we prefer to use, instead of distribution functions \( f_i \), spectral energy densities

\[
E_i(\epsilon_i) = \frac{4\pi \epsilon_i^3 \beta_i f_i}{c^3}, \quad (A.1.4)
\]

where \( \beta_i = \sqrt{1 - (m_i c^2 / \epsilon_i)^2} \), in the phase space \( \epsilon_i \). Then

\[
\epsilon_i f_i(\mathbf{p}, t) d\mathbf{r} d\mathbf{p} = \frac{4\pi \epsilon_i^3 \beta_i f_i}{c^3} r d\epsilon_i = E_i d\epsilon_i \quad (A.1.5)
\]

is the energy in the volume of the phase space \( d\mathbf{r} d\mathbf{p} \). The particle density is

\[
n_i = \int f_i d\mathbf{p} = \int \frac{E_i}{\epsilon_i} d\epsilon_i, \quad d n_i = f_i d\mathbf{p}. \quad (A.1.6)
\]
We can rewrite Boltzmann equations (A.1.3) in the form
\[
\frac{1}{c} \frac{\partial E_i}{\partial t} = \sum_q (\eta^q_i - \chi^q_i E_i), \tag{A.1.7}
\]
where \( \eta^q_i = (4\pi \epsilon^3 \beta_i / c^3) \tilde{\eta}^q_i \).

We introduced the computational grid for phase space \( \{\epsilon_i, \mu, \varphi\} \), where \( \mu = \cos \theta \), \( \theta \) and \( \varphi \) are angles between radius vector \( \mathbf{r} \) and the particle momentum \( \mathbf{p} \). The zone boundaries are \( \epsilon_{i,\omega+1/2}, \mu_{k+1/2}, \varphi_{l+1/2} \) for \( 1 \leq \omega \leq \omega_{max}, 1 \leq k \leq k_{max}, 1 \leq l \leq l_{max} \). The length of the \( i \)-th interval is \( \Delta \epsilon_{i,\omega} \equiv \epsilon_{i,\omega+1/2} - \epsilon_{i,\omega-1/2} \). On the finite grid the functions (A.1.4) become
\[
E_{i,\omega} \equiv \frac{1}{\Delta \epsilon_{i,\omega}} \int_{\Delta \epsilon_{i,\omega}} d\epsilon E_i(\epsilon). \tag{A.1.8}
\]

Now we can replace the collisional integrals in (A.1.7) by the corresponding sums. After this procedure we get the set of ordinary differential equations (ODE’s), instead of the system of partial differential equations for the quantities \( E_{i,\omega} \) to be solved. There are several characteristic times for different processes in the problem, and therefore our system of differential equations is stiff. (Eigenvalues of Jacobi matrix differs significantly, and the real parts of eigenvalues are negative.) We use Gear’s method (Hall and Watt, 1976) to integrate ODE’s numerically. This high-order implicit method was developed for the solution of stiff ODE’s. The method we use is conserves for the number of particles, where appropriate.

To get more details of the method consider the Compton scattering. Without taking into account induced radiation and occupied states the emission and absorption coefficients for photons are (Berestetskii et al., 1982)
\[
\tilde{\eta}^\gamma_{Cs}(k) = \int dk' dp' dp w_{k',p',k,p} f_{\gamma}(k') f_{\epsilon}(p',t), \tag{A.1.9}
\]
\[
\chi^\gamma_{Cs} f_{\gamma}(k) = \int dk' dp' dp w_{k',p',k,p} f_{\gamma}(k) f_{\epsilon}(p,t), \tag{A.1.10}
\]
where
\[
w_{k',p',k,p} = \frac{e \delta(e_\gamma + e_\epsilon - e_\gamma' - e_\epsilon')}{(2\pi \hbar)^2} \delta(k + p - k' - p') \frac{|M_{fi}|^2}{16\epsilon_\gamma \epsilon_\epsilon' \epsilon_\gamma' \epsilon_\epsilon'} \tag{A.1.11}
\]
is the probability of the process, the matrix element \( M_{fi}(s,u) \) depends on the kinematic invariants \( s = (p + \epsilon)^2 \) and \( u = (p - \epsilon')^2 \), \( \epsilon = (e_\gamma / c)(1, e_\gamma) \) and \( p = (e_\epsilon / c)(1, \beta e_\epsilon) \) are four energy-momentum vectors of photons and electrons, respectively, and \( dp = dedo2\beta / c^3 \), and \( do = d\mu d\phi \), \( \mathbf{p} \) is the momentum of electrons, while \( \mathbf{k} \) is the momentum of photons.
A. Thermalization of electron-positron-photon plasmas

In order to calculate e.g. the absorption coefficient $\chi_{\gamma}^{Cs}$ we proceed as follows. Substituting the expression for the probability $\omega$ from equation (A.1.9) we can perform one integration over $dp'$

$$\int dp' \delta(dk + dp - dk' - dp') \rightarrow 1, \quad (A.1.12)$$

but we should take into account the fixed relation

$$p' = k + p - k'. \quad (A.1.13)$$

In the next integration over $dk' = d\epsilon'_{\gamma}d\omega'^{2}_{\gamma}$ we have

$$\int d\epsilon' \delta(\epsilon_{\gamma} + \epsilon - \epsilon'_{\gamma} - \epsilon') = \int d(\epsilon'_{\gamma} + \epsilon') \frac{1}{|\partial(\epsilon'_{\gamma} + \epsilon')/\partial\epsilon_{\gamma}|} \delta(\epsilon_{\gamma} + \epsilon - \epsilon'_{\gamma} - \epsilon') \rightarrow \frac{1}{|\partial(\epsilon'_{\gamma} + \epsilon')/\partial\epsilon_{\gamma}|} \equiv J_1.$$

Finally, we have

$$\chi_{\gamma}^{Cs} = \int d\epsilon_{\gamma} dp J_1 \frac{\epsilon \epsilon' |\tilde{M}_{fi}|^2}{16 \epsilon \epsilon' e \epsilon' e} f_{e}(p, t), \quad (A.1.14)$$

where

$$\epsilon'_{\gamma} = \frac{\epsilon \epsilon_{\gamma}(1 - \beta b_{\gamma} b'_{\gamma})}{\epsilon(1 - \beta b_{\gamma} b'_{\gamma}) + \epsilon_{\gamma}(1 - b_{\gamma} b'_{\gamma})}, \quad (A.1.15)$$

$$\epsilon' = \epsilon + \epsilon - \epsilon'_{\gamma}, \quad (A.1.16)$$

$$b_{i} = p_{i}/p, b'_{i} = p'_{i}/p', \quad (A.1.17)$$

$$b'_{\gamma} = (\beta \epsilon_{\gamma} b_{\gamma} + \epsilon_{\gamma} b_{\gamma} - \epsilon'_{\gamma} b'_{\gamma})/(\beta' \epsilon'_{\gamma}), \quad (A.1.18)$$

$$|\tilde{M}_{fi}|^2 = |M_{fi}|^2 / [c^3(2\pi\hbar)^2]. \quad (A.1.19)$$

The average absorption coefficient for the Compton scattering in the finite grid is

$$(\chi E)_{\gamma,\omega}^{Cs} = \frac{\int_{\epsilon_{\gamma} \Delta \epsilon_{\gamma}, \omega} d\epsilon_{\gamma} \chi_{\gamma}^{Cs} \epsilon_{\gamma} E_{\gamma}}{\Delta \epsilon_{\gamma,\omega}} = \frac{1}{\Delta \epsilon_{\gamma,\omega}} \int_{\epsilon_{\gamma} \Delta \epsilon_{\gamma,\omega}} dn_{\gamma} d\epsilon_{\gamma} J_1 \frac{\epsilon'_{\gamma} |\tilde{M}_{fi}|^2}{16 \epsilon \epsilon' e}. \quad (A.1.20)$$

Similar integrations can be performed for the emission coefficient, and we have

$$(\eta_{\gamma,\omega}^{Cs} = \frac{\int_{\epsilon'_{\gamma} \Delta \epsilon_{\gamma,\omega}} d\epsilon_{\gamma} d\mu_{\gamma} \eta_{\gamma}^{Cs}}{\Delta \epsilon_{\gamma,\omega} \Delta \mu_{\gamma}} = \frac{1}{\Delta \epsilon_{\gamma,\omega} \Delta \mu_{\gamma}} \int_{\epsilon'_{\gamma} \Delta \epsilon_{\gamma,\omega}} dn_{\gamma} d\epsilon_{\gamma} J_1 \frac{\epsilon''_{\gamma} |\tilde{M}_{fi}|^2}{16 \epsilon \epsilon'' e}. \quad (A.1.21)$$
A.1. The Method

For binary interactions we use exact QED matrix elements, see textbooks (Berestetskii et al., 1982; Greiner and Reinhardt, 2003). All expressions for emission and absorption coefficients in the binary reactions can be found in (Aksenov et al., 2004). For calculation of those coefficients we use the grid functions $E_{i\omega}$. Both the exact energy conservation law and the particles number conservation law are satisfied as we adopt interpolation of grid functions $E_{i\omega}$ inside the energy intervals. In order to calculate the $e^{\pm}$ scattering we introduced the minimal scattering angles following (Haug, 1988).

For such a dense plasma collisional integrals in (A.1.3) should include not only binary interactions, having order $\alpha$ in Feynmann diagrams, where $\alpha$ is the fine structure constant, but also triple ones (Berestetskii et al., 1982), having order $\alpha^2$. Calculations of emission and absorption coefficients for triple interactions we illustrate here for bremsstrahlung

$$e_1 + e_2 \leftrightarrow e'_1 + e'_2 + \gamma'.$$

(A.1.22)

The ref. (Berestetskii et al., 1982) gives the differential probability in the unit of time in the unit volume (in relativistic units $\hbar = 1, c = 1$) as

$$dw = A \prod_{\text{all particles on exit}} \frac{d\mathbf{p}'_a}{(2\pi)^3},$$

(A.1.23)

where

$$A = (2\pi)^4 \delta^{(4)}(P_f - P_i) \frac{|M_{fi}|^2}{2\epsilon_1..}.$$  

(A.1.24)

For the time derivative, for instance, of the distribution function $f_2$ in the direct and in the inverse reactions (A.1.22) one has

$$f_2 = \int d\mathbf{p}_2' d\mathbf{k}' A \left( \int \frac{d\mathbf{p}_1' f_1' d\mathbf{p}_1 f_1 f_2}{(2\pi \hbar)^6} - \int \frac{d\mathbf{p}_1 d\mathbf{p}_1' f_1 f_2}{(2\pi \hbar)^9} \right).$$

(A.1.25)

In the case of the distribution functions (A.1.27), see below, we have multipliers proportional to $\exp \frac{\mu}{\hbar k_T}$ in front of the integrals. The calculation of emission and absorption coefficients is then reduced to the well known thermal equilibrium case (Svensson, 1984). In fact, since reaction rates of triple interactions are $\alpha$ times smaller than binary reaction rates, we expect that binary reactions come to detailed balance first. Only when binary reactions are all balanced, triple interactions become important. In addition, when binary reactions come into balance, distribution functions acquire the form (A.1.27), see below. Although there is no principle difficulty in computations using exact matrix elements for triple reactions as well, our simplified scheme allows for much faster numerical computation.
A. Thermalization of electron-positron-photon plasmas

### Table A.1.: Microphysical processes in the pair plasma.

<table>
<thead>
<tr>
<th>Binary interactions</th>
<th>Radiative variants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Møller, Bhabha</td>
<td>Bremsstrahlung</td>
</tr>
<tr>
<td>$e^\pm e^\mp \leftrightarrow e^\pm' e^\mp'$</td>
<td>$e^\pm e^\mp \leftrightarrow e^\pm' e^\mp'$</td>
</tr>
<tr>
<td>Single Compton</td>
<td>Double Compton</td>
</tr>
<tr>
<td>$e^\pm \gamma \leftrightarrow e^\pm' \gamma'$</td>
<td>$e^\pm \gamma \leftrightarrow e^\pm' \gamma'$</td>
</tr>
<tr>
<td>Pair production and annihilation</td>
<td>Radiative pair production and three photon annihilation</td>
</tr>
<tr>
<td>$\gamma \gamma' \leftrightarrow e^\mp e^\mp' \gamma'$</td>
<td>$\gamma' \leftrightarrow e^\mp e^\mp' \gamma''$</td>
</tr>
</tbody>
</table>

We consider all possible binary and triple interactions between electrons, positrons and photons as summarized in Tab. A.1.

Each of the above mentioned reactions is characterized by the corresponding timescale and optical depth. For Compton scattering, for instance, we have

$$t_{cs} = \frac{1}{\sigma_{T_n \pm c}}, \quad \tau_{cs} = \sigma_{T_n \pm R_{er}}. \quad (A.1.26)$$

There are two timescales in our problem that characterize the condition of detailed balance between direct and inverse reactions, $\sim t_{cs}$ for binary and $a^{-1}t_{cs}$ for triple interactions respectively.

Starting from arbitrary distribution functions we find a common development: at the time $t_{cs}$ the distribution functions always have evolved in a functional form on the entire energy range, depending only on two parameters. We find in fact for the distribution functions the expressions

$$f_i(\varepsilon) = \exp \left( -\frac{\varepsilon - \varphi_i}{\theta_i} \right), \quad (A.1.27)$$

with chemical potential $\varphi_i \equiv \frac{\mu_i}{m_e c^2}$ and temperature $\theta_i \equiv \frac{k_B T_i}{m_e c^2}$, where $\varepsilon \equiv \frac{\varepsilon}{m_e c^2}$ is the energy of the particle. Such a configuration corresponds to a kinetic equilibrium (Pilla and Shaham, 1997; Ehlers, 1973) in which electrons, positrons and photons acquire a common temperature and nonzero chemical potential. At the same time we found that triple interactions become essential for $t > t_{cs}$, after the establishment of kinetic equilibrium. Such triple interactions, both direct and inverse, are indeed essential in achieving the thermal equilibrium.

Notice that similar method to ours was applied in (Pilla and Shaham, 1997) in order to compute spectra of particles in kinetic equilibrium. However, it was never shown how particles evolve down to thermal equilibrium.
A.2. Numerical results

In (A.1.27) analogously to the temperature, defining the average kinetic energy in the system, the chemical potential represents deviation from the thermal equilibrium through the relation

$$\varphi = \theta \ln(n/n_{th}),$$  \hspace{1cm} (A.1.28)

where $n_{th}$ are concentrations of particles in thermal equilibrium. We do not absorb the chemical potentials into the normalization factors since they depend on time and describe the approach to thermal equilibrium.

A.2. Numerical results

The results of numerical simulations are reported below. We choose two limiting initial conditions with flat spectra: a) electron-positron pairs with a $10^{-5}$ energy fraction of photons and b) the reverse case, i.e., photons with a $10^{-5}$ energy fraction of pairs. Our grid consists of 60 energy intervals and $16 \times 32$ intervals for two angles characterizing the direction of the particle momenta. In both cases the total energy density is $\rho = 10^{24}$ erg/cm$^3$. In the first case initial concentration of pairs is $3.1 \times 10^{29}$ cm$^{-3}$, in the second case the concentration of photons is $7.2 \times 10^{29}$ cm$^{-3}$.

In Fig. A.1 we show concentrations of photons and pairs as well as their sum for both our initial conditions. After calculations begin, concentrations and energy density of photons (pairs) increase rapidly with time, due to annihilation (creation) of pairs by the reaction $\gamma \gamma' \leftrightarrow e^\pm e^\mp$. Then, in the kinetic equilibrium phase, concentrations of each component stay almost constant, and the sum of concentrations of photons and pairs remains unchanged. Finally, both individual components and their sum reach stationary values. If one compares and contrasts both cases as reproduced in Fig. A.1 one can see that, although the initial conditions are drastically different, in both cases the same asymptotic values of the concentration are reached.

We now describe in detail the case when initially pairs dominate. To find temperature and chemical potential at any given moment of time, we use the
A. Thermalization of electron-positron-photon plasmas

Figure A.1.: Dependence on time of concentrations of pairs (black), photons (red) and both (thick) when all interactions take place (solid) with small energy density. Upper (lower) figure corresponds to the case when initially there are mainly pairs (photons). Dotted curves on the upper figure show concentrations when inverse triple interactions are neglected. In this case an enhancement of the pairs occurs with the corresponding increase in photon number and thermal equilibrium is never reached.
A.2. Numerical results

following integral equations

\[ n_k = n_\gamma + n_\pm = \]
\[ = \frac{1}{\pi^2 \lambda_c^3} \int_0^\infty z^2 dz \left[ \frac{1}{\exp(z - \frac{\varphi_k}{\theta_k}) - 1} + 2 \frac{1}{\exp(\sqrt{1 + z^2} - \frac{\varphi_k}{\theta_k}) + 1} \right], \]  
\hspace{2cm} \text{(A.2.2)}

\[ \rho_k = \rho_\gamma + \rho_\pm = \]
\[
= \frac{m_ec^2}{\pi^2 \lambda_c^3} \int_0^\infty z^3 dz \left[ \frac{1}{\exp(z - \frac{\varphi_k}{\theta_k}) - 1} + 2 \frac{\sqrt{1 + z^2}}{\exp(\sqrt{1 + z^2} - \frac{\varphi_k}{\theta_k}) + 1} \right], \]  
\hspace{2cm} \text{(A.2.4)}

where \( \lambda_c = \frac{\hbar}{m_ec} \) is the Compton length. These two equations for two variables can be solved numerically at any instant of time, so the corresponding spectra (A.2.5) with the values of temperature and chemical potential, found from (A.2.1) and (A.2.3) with distribution functions (A.1.27) can be obtained.

One can see in Fig. A.2 that the spectral density of photons and pairs (Akesson et al., 2004)

\[ \frac{dp_i}{d\epsilon} = 4\pi c^3 f(\epsilon, t) \epsilon^3 \beta_i, \]  
\hspace{2cm} \text{(A.2.5)}

where \( \beta_\pm = \sqrt{1 - (m_ec^2/\epsilon)^2} \) for pairs and \( \beta_\gamma = 1 \) for photons, can be fitted already at \( t_k \approx 20t_{cs} \approx 7 \times 10^{-15} \) sec by distribution functions (A.1.27) with definite values of temperature \( \theta_k(t_{cs}) \approx 1.2 \) and chemical potential \( \varphi_k(t_k) \approx -4.5 \), common for pairs and photons.

As expected, after \( t_k \) the distribution functions preserve their form (A.1.27) with the values of temperature and chemical potential changing in time, as shown in Fig. A.3. As one can see from Fig. A.3 the chemical potential evolves with time and reaches zero at the moment \( t_{th} \approx a^{-1}t_k \approx 7 \times 10^{-13} \) sec, corresponding to the final stationary solution with \( T_{th} = 0.26 \) MeV. For comparison we also plot the results of calculations for higher initial energy density, being \( \rho = 10^{27} \text{erg/cm}^3 \). Number densities are shown in Fig. A.4, temperatures in Fig. A.5 and chemical potentials in Fig. A.6. Unlike the previous case, we find the final temperature \( T_{th} = 1.4 \) MeV and therefore concentrations of photons and pairs in thermal equilibrium almost coincide in Fig. A.4, while in Fig. A.1 they are different because of exponential suppression of pairs in equilibrium state with temperatures below 0.511 MeV. The ratio of temperatures in two cases is in agreement with Stefan-Boltzmann law.

Also we find shorter timescales for approach to kinetic as well as thermal equilibrium with higher initial energy density. Here \( t_{th} \approx 3 \times 10^{-15} \) sec, which is two orders of magnitude larger with respect to the previous case, in agreement with timescale estimations (A.1.26). This is because the timescale linearly depends on concentrations which are two order of magnitude higher in the latter case.
Figure A.2.: Spectra of pairs (upper figure) and photons (lower figure) when initially only pairs are present. The black curve represents the results of numerical calculations obtained successively at \( t = 0 \), \( t = t_k \) and \( t = t_{th} \) (see the text). Both spectra of photons and pairs are initially taken to be flat. The yellow curves indicate the spectra obtained form (A.1.27) at \( t = t_k \). The perfect fit of the two curves is most evident in the entire energy range leading to the first determination of the temperature and chemical potential both for pairs and photons. The orange curves indicate the final spectra as thermal equilibrium is reached.
Figure A.3.: Time dependence of temperatures, measured on the left axis (solid), and chemical potentials, measured on the right axis (dotted), of electrons (black) and photons (red). The dashed lines correspond to the reaching of the kinetic and the thermal equilibria. Upper (lower) figure corresponds to the case when initially there are mainly pairs (photons).
A. Thermalization of electron-positron-photon plasmas

Figure A.4.: Dependence on time of concentrations of pairs (black), photons (red) and both (thick) when all interactions take place, for high energy density.

Figure A.5.: Time dependence of temperatures of electrons (black) and photons (red) for high energy density.
We now discuss the results. Let us consider the distribution functions (A.1.27) with different temperatures $\theta_i$ and chemical potentials $\phi_i$ for pairs and photons. The requirement of vanishing reaction rate for the Compton scattering $f_{\pm} f_{\gamma} = f_{\pm} f_{\gamma}^\prime$ leads to the equal temperature of pairs and photons, see also (Pilla and Shaham, 1997; Ehlers, 1973). Indeed, from (A.1.27) we have

$$\exp \left( -\frac{\epsilon_{\pm} - \phi_{\pm}}{\theta_{\pm}} \right) \exp \left( -\frac{\epsilon_{\gamma} - \phi_{\gamma}}{\theta_{\gamma}} \right) = \exp \left( -\frac{\epsilon_{\pm}^\prime - \phi_{\pm}}{\theta_{\pm}} \right) \exp \left( -\frac{\epsilon_{\gamma}^\prime - \phi_{\gamma}}{\theta_{\gamma}} \right).$$

(A.2.6)

Now energy conservation $\epsilon_{\pm} + \epsilon_{\gamma} = \epsilon_{\pm}^\prime + \epsilon_{\gamma}^\prime$ leads to

$$\frac{\epsilon_{\pm} - \phi_{\pm}}{\theta_{\pm}} + \frac{\epsilon_{\gamma} - \phi_{\gamma}}{\theta_{\gamma}} = \frac{\epsilon_{\pm}^\prime - \phi_{\pm}}{\theta_{\pm}} + \frac{\epsilon_{\gamma} + \epsilon_{\pm} - \epsilon_{\pm}^\prime - \phi_{\gamma}}{\theta_{\gamma}},$$

$$\frac{\epsilon_{\pm} - \epsilon_{\gamma}^\prime}{\theta_{\pm}} = \frac{\epsilon_{\pm} - \epsilon_{\gamma}^\prime}{\theta_{\gamma}},$$

(A.2.7)

$$\theta_{\pm} = \theta_{\gamma} \equiv \theta_k.$$

In this way the detailed balance between any direct and the corresponding inverse reactions shown in Tab. A.1 leads to relations between $\theta$ and $\phi$ collected in Tab. A.2. These relations are not imposed, but are verified through the numerical calculations. This is a powerful tool to verify the consistency of our approach and numerical calculations.
A. Thermalization of electron-positron-photon plasmas

### Table A.2.

<table>
<thead>
<tr>
<th>Interaction</th>
<th>Parameters of distribution functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compton scattering</td>
<td>$\theta_\gamma = \theta_\pm, \forall \phi_{\gamma'}, \phi_{\pm}$</td>
</tr>
<tr>
<td>Pair production</td>
<td>$\phi_{\gamma'} = \phi_\pm$, if $\theta_{\gamma'} = \theta_\pm$</td>
</tr>
<tr>
<td>Tripe interactions</td>
<td>$\phi_{\gamma'}, \phi_\pm = 0$, if $\theta_{\gamma'} = \theta_\pm$</td>
</tr>
</tbody>
</table>

Table A.2.: Relations between parameters of equilibrium distribution functions fulfilling detailed balance conditions for each of the reactions shown in Tab. A.1.

From Tab. A.2 one can see that the necessary condition for thermal equilibrium in the pair plasma is detailed balance between direct and inverse triple interactions. This point is usually neglected in the literature, where there are claims that the thermal equilibrium may be established with only binary interactions (Stepney, 1983). In order to demonstrate it explicitly we also show in Fig. A.1 the dependence of concentrations of pairs and photons when inverse triple interactions are artificially switched off. In this case, see dotted curves in the upper Fig. A.1, after kinetic equilibrium is reached concentrations of pairs decrease monotonically with time, and thermal equilibrium is never reached. In fact, since total energy density is constant and total number of particles increases with time, the temperature decreases with time while the chemical potential remains nonzero.

The existence of a non-null chemical potential for photons indicates the departure of the distribution function from the one corresponding to thermal equilibrium. Negative (positive) value of the chemical potential generates an increase (decrease) of the number of particles in order to approach the one corresponding to the thermal equilibrium state. Then, since the total number of particles increases (decreases), the energy is shared between more (less) particles and the temperature decreases (increases), see Fig. A.3. Clearly, as in thermal equilibrium is approached, the chemical potential of photons is zero.

In our example with the energy density $10^{24}\text{erg/cm}^3$ the thermal equilibrium is reached at $\sim 7 \times 10^{-13}\text{sec}$ with the final temperature $T_{th} = 0.26\text{ MeV}$. For a larger energy density the duration of the kinetic equilibrium phase, as well as of the thermalization timescale, is smaller.

Our results, obtained for the case of an uniform plasma, can only be adopted for a description of a physical system with dimensions $R_{\text{min}} \gg 1\text{ cm}$. Clearly, the sources of GRBs satisfy this requirement since $R_0 \gg R_{\text{min}}$.

The assumption of the constancy of the energy density is only valid if the dynamical timescale $t_{\text{dyn}} = \left(1 \frac{dR}{dt}\right)^{-1}$ of the plasma is much larger than the above timescale $t_{th}$ which is indeed true in all the cases of astrophysical interest. In other words, since $t_{th} \ll t_{\text{var}}$, i.e. because thermal equilibrium is established well before expansion starts, we have proven the assumption on
thermal equilibrium state of the pair plasma prior to expansion, used widely in the GRB literature (Goodman, 1986; Ruffini et al., 1999).

Since we get thermal equilibrium already on the timescale $t_{\text{th}} \lesssim 10^{-12} \text{sec}$, and such a state is independent of the initial distribution functions for electrons, positrons and photons, the sufficient condition to obtain an isothermal distribution on a causally disconnected spatial scale $R > ct_{\text{th}} = 10^{-2} \text{cm}$ is the request of constancy of the energy density on such a scale as well as, of course, the invariance of the physical laws.

Notice, that the validity of our treatment is a consequence of existence of essentially different timescales in the problem. In particular, there exists inequality $t_{\text{cs}} \ll t_{\text{th}}$. Since cross-sections depend on particle energies, this inequality may be violated for very large energy density. However, in that case different effects such as degeneracy, presence of other particles play major role and the simple kinetic description presented above does not apply.

### A.3. Cavallo and Rees scenarios

Among our results we do not find support for the Cavallo-Rees scenarios (Cavallo and Rees, 1978), where the average energies per particle in the lepton fireball degrade until the threshold for pair production is reached, and electron-positron pairs disappear. However, we found very similar result when artificially switched off inverse triple interactions. The result of the computations without inverse triple interactions is shown in Fig. A.1, upper panel, with dashed curves. In that case thermal equilibrium is, clearly, never reached.

The explanation of the result of Cavallo and Rees is as follows. In most cases in kinetic theory of dilute gases plasmas the most probable collisions are those between two particles, and triple collisions can be safely neglected. However, in the case under consideration, the plasma is extremely dense. One can argue that kinetic description in terms of Boltzmann equation is oversimplified, and two-particle and higher order distribution functions have to be studied. But we notice that the plasma parameter, $g = (n_{\pm} r_p^3)^{-1}$ is of the order $10^{-3}$ in our case, so one can safely consider just one-particle distribution functions. At the same time, the effect of many particle collisions becomes crucial for the correct description of thermalization process. Indeed, only binary collisions can in no way lead to thermal equilibrium. Only kinetic equilibrium may be established in the plasma, as we have demonstrated above. In that case, however, the number density of particles is different from the one in thermal equilibrium, and the temperature, although is well defined, cannot be determined from just the values of number (or energy) densities.
A. Thermalization of electron-positron-photon plasmas
B. Hydrodynamics of the pair plasma

We give systematic derivation of the main equations, present a critical review of existing models for isotropic relativistic fireballs, compare and contrast these models, following (Bianco et al., 2006). In the next section, following (Vereshchagin, 2007) we derive basic equations and describe approximations involved. Then we present the model (Ruffini et al., 1999, 2000) which differs from other models in the literature as it describes the dynamics of the fireshell taking into account the rate equations for electron-positron pairs. Then we compare and contrast above mentioned models.

B.1. Local, global and average conservation laws

B.1.1. Particle number

The first relevant equation represents continuity of relativistic flux and reads

\[ (nU^\mu)_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} nU^\mu \right) = 0, \]

(B.1.1)

where \( n \) is the number density of relativistic fluid, \( U^\mu \) is its velocity field. Defining particle number as

\[ N = \int_V \sqrt{-g} nU^0 dV. \]

(B.1.2)

we see that

\[ \frac{dN}{dt} = - \int_V \sqrt{-g} nU^i dV = - \oint_\Sigma \sqrt{-g} nU^i dS_i, \]

(B.1.3)

where we have used the Ostrogradsky-Gauss theorem. Thus, if particles do not cross the surface \( \Sigma \) bounding considered volume \( V \), the total number of particles is constant during system evolution.

---

1Greek indices denote four-dimensional components and run from 0 to 3 while Latin indices run from 1 to 3. The general relativistic effects are neglected, which is a good approximation, but we left the general definition of the energy-momentum conservation to take into account the most general coordinate system.
Now assume spherical symmetry\(^2\), which is usually done for fireballs description. With spherical spatial coordinates \(x^i = \{r, \vartheta, \varphi\}\) the interval is
\[
\text{ds}^2 = -dt^2 + dr^2 + r^2d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2.
\]
(B.1.4)

Assuming absence of fluxes through the boundary \(\Sigma\) we rewrite (B.1.1)
\[
\frac{\partial (n\gamma)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 n \sqrt{\gamma^2 - 1} \right) = 0,
\]
(B.1.5)

Integrating this equation over the volume from certain \(r_i(t)\) to \(r_e(t)\) which we assume to be comoving with the fluid
\[
\frac{dr_i(t)}{dt} = \beta(r_i, t), \quad \frac{dr_e(t)}{dt} = \beta(r_e, t),
\]
(B.1.6)
and ignoring a factor \(4\pi\) we have
\[
\int_{r_i}^{r_e} \frac{\partial (n\gamma)}{\partial t} r^2 dr + \int_{r_i}^{r_e} \frac{\partial}{\partial r} \left( r^2 n \sqrt{\gamma^2 - 1} \right) dr = 0,
\]
(B.1.7)
\[
\frac{\partial}{\partial t} \int_{r_i}^{r_e} (n\gamma) r^2 dr - \frac{dr_e}{dt} n(r_e, t) \gamma(r_e, t) r_e^2 + \frac{dr_i}{dt} n(r_i, t) \gamma(r_i, t) r_i^2 +
\]
\[
+ r_e^2 n(r_e, t) \sqrt{\gamma^2(r_e, t) - 1} - r_i^2 n(r_i, t) \sqrt{\gamma^2(r_i, t) - 1} =
\]
\[
= \frac{d}{dt} \int_{r_i}^{r_e} (n\gamma) r^2 dr = 0,
\]

Since we deal with arbitrary comoving boundaries, this means that the number of particles in each shell between the boundaries conserves, as well as the total number of particles integrated over all shells, in other words,
\[
N = 4\pi \int_{0}^{R(t)} n \gamma r^2 dr = \text{const},
\]
(B.1.8)

where \(R(t)\) is the external radius of the fireshell.

Following (Piran et al., 1993) one can transform (B.1.5) from the variables

\(^2\)The only nonvanishing components of the energy-momentum tensor are \(T^{00}, T^{01}, T^{10}, T^{11}, T^{22}, T^{33}\). The factor \(\sqrt{-g} = r^2 \sin \vartheta\) in all expressions above becomes simply a volume measure and the differentials are \(dV = drd\vartheta d\varphi\), \(dS = d\vartheta d\varphi\), so the differential laboratory volume can be written as \(d\tilde{V} = \sqrt{-\tilde{g}}dV = r^2 \sin \vartheta drd\vartheta d\varphi\).
(t, r) to the new variables (s = t - r, r) and then show that

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 n \sqrt{\gamma^2 - 1} \right) = -\frac{\partial}{\partial s} \left( \frac{n}{\gamma + \sqrt{\gamma^2 - 1}} \right). \tag{B.1.9}
\]

Now assume the expansion velocity is ultrarelativistic,

\[
\gamma \gg 1. \tag{B.1.10}
\]

In this approximation, therefore,

\[
dN = 4\pi n \gamma r^2 dr \approx \text{const}. \tag{B.1.11}
\]

Relations (B.1.11) and (B.1.8) then imply

\[
4\pi \int_{r_i}^{r_e} \left( n \gamma r^2 \right) dr = 4\pi \left[ n(r, t) \gamma(r, t) r^2 \right] \int_{r_i}^{r_e} dr = 4\pi \left( n \gamma r^2 \right) \Delta \approx \text{const}, \tag{B.1.12}
\]

where the first argument of functions \( n(r, t) \) and \( \gamma(r, t) \) is restricted to the interval \( r_i < r < r_e \) and

\[
\Delta \equiv r_e - r_i \approx \text{const}. \tag{B.1.13}
\]

This means, that the fluid shell does not broaden, but rather has a constant thickness. This fact proves the constant thickness approximation, adopted in (Ruffini et al., 1999, 2000).

The volume element measured by the observer outside the fireshell (to be referred to as the lab frame in what follows), for which it appears moving with velocity \( \beta \) is just

\[
dV = 4\pi r^2 dr, \tag{B.1.14}
\]

while the volume element comoving with the fireshell, for which the fluid is at rest, is

\[
dV = 4\pi \gamma r^2 dr, \tag{B.1.15}
\]

with the conversion of the volumes

\[
dV = \gamma dV. \tag{B.1.16}
\]

Then the average value of the Lorentz factor is defined as follows

\[
\langle \gamma \rangle \equiv \frac{4\pi \int \gamma r^2 dr}{4\pi \int r^2 dr} = \frac{V}{\bar{V}}. \tag{B.1.17}
\]

Now we can formulate the conservation law for the average value of the
number density in the lab frame

\[ \langle n \rangle_{\text{lab}} \equiv \frac{N}{V} = \frac{4\pi}{\int_{r_i}^{r_e} r^2 dr} \int_{r_i}^{r_e} n(r) \gamma^2(r,t) r dr. \] (B.1.18)

Assuming \( r \gg \Delta \) we then obtain

\[ \langle n \rangle_{\text{lab}} \approx \frac{4\pi n \gamma^2 \Delta}{4\pi r^2 \Delta} = n(r,t) \gamma(r,t) \propto r^{-2}. \] (B.1.19)

Therefore, the average number density in the lab frame scales as \( r^{-2} \).

At the same time, recalling the expression for the divergence of the four-velocity

\[ U^\mu;\mu = \frac{1}{V} \frac{dV}{dt}, \] (B.1.20)

we get

\[ (nU^\mu);\mu = U^\mu n;\mu + nU^\mu;\mu = \frac{dn}{d\tau} + \frac{n}{V} \frac{dV}{d\tau} = 0, \]
\[ d\ln n + \frac{d}{dt} \ln V = 0. \] (B.1.21)

This means, that the number of particles is conserved along the flow lines of the fluid. The solution of this equation provides the definition for the comoving average number density

\[ \langle n \rangle_{\text{com}} \equiv \frac{N}{V} = \frac{4\pi}{\int_{r_i}^{r_e} \gamma^2 dr} \int_{r_i}^{r_e} n(r) \gamma(r,t) r dr. \] (B.1.22)

Clearly, the condition (B.1.13) gives a link between the description of the fireshell evolution in terms of local functions, entering (B.1.5) on the one side, and global quantities (B.1.17) and (B.1.19), on the other side. The presence of the global conservation (B.1.8) in both these cases ensures equivalence of the local (B.1.5) and the average (B.1.18) descriptions for the fireshell, unless its detailed structure is considered.
B.1.2. Energy-momentum conservation

The basis of description for relativistic fireshell is the energy-momentum principle. It allows to obtain relativistic hydrodynamic equations, or equations of motion for the fireshell, energy and momentum conservation equations which are used extensively to describe interaction of relativistic baryons of the fireshell with the interstellar matter, and boundary conditions which are used to understand shock waves propagation in the decelerating baryons and in the outer medium. Consider energy-momentum conservation in the most general form:

\[ (T_{\mu}^{\nu})_{;\nu} = \frac{\partial (\sqrt{-g} T_{\mu}^{\nu})}{\partial x^{\nu}} + \sqrt{-g} \Gamma^{\nu}_{\nu\lambda} T^{\nu\lambda} = 0, \]  

where \( \Gamma^{\nu}_{\nu\lambda} \) are Cristoffel symbols and \( g \) is determinant of the metric tensor.

Integrating over the whole three-dimensional volume we obtain

\[ \int_{V} T_{\mu}^{\nu} ;_{\nu} dV = 0. \]  

Integrating over the whole four-dimensional volume and applying divergence theorem we get (Taub, 1948)

\[ \int_{t} \int_{V} T_{\mu}^{\nu} ;_{\nu} dV dt = \oint_{V} T_{\mu}^{\nu} \lambda^{\nu} dV = 0, \]  

where \( \lambda^{\alpha} \) are covariant components of the outward drawn normal to the three-dimensional hypersurface (volume \( V \)).

Now suppose that there is a discontinuity on the fluid flow. Taking the volume to be a spherical shell and choosing the coordinate system where the discontinuity is at rest so that in (B.1.25) for normal vectors to the discontinuity hypersurface \( \lambda^{\alpha} \), we have

\[ \lambda^{\alpha} \lambda^{\alpha} = 1, \quad \lambda^{0} = 0. \]

Let the radius of the shell \( R_s \) be very large and shell thickness \( \Delta \) be very small. With \( R_s \rightarrow \infty \) and \( \Delta \rightarrow 0 \) from (B.1.25) we arrive to

\[ [T^{\alpha\beta}] = 0, \]  

where the brackets mean that the quantity inside is the same on both sides of the discontinuity surface. This equation together with continuity condition for particle density flux \( [nU^{i}] = 0 \) was used by Taub (1948) to obtain relativistic Rankine-Hugoniot equations. These equations govern shock waves dynamics which are supposed to appear during collision of the baryonic material left from the fireshell with the interstellar medium.
and McKee, 1976). The origin of the afterglow could be connected (Rees and Meszaros, 1992; Narayan et al., 1992; Katz, 1994) to the conversion of kinetic energy into radiative energy in these shocks.

Consider now the energy-momentum tensor of the perfect fluid in the lab frame (where the fluid was initially at rest)

\[ T_{\mu\nu} = p g_{\mu\nu} + \omega U^\mu U^\nu, \]

where \( \omega = \rho + p \) is proper enthalpy, \( p \) is proper pressure and \( \rho \) is proper internal energy density.

Rewrite (B.1.23) in spherically symmetric case

\[ \frac{\partial T_{00}}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 T_{11} \right) = 0, \]

\[ \frac{\partial T_{10}}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 T_{11} \right) - \frac{1}{r} \left( T_{22} + T_{33} \right) = 0, \]

arriving to equations of motion of relativistic fireshell (Blandford and McKee, 1976; Piran et al., 1993; Ruffini et al., 1999)

\[ \frac{\partial (\gamma^2 \omega)}{\partial t} - \frac{\partial p}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \gamma^2 \beta \omega \right) = 0, \]

\[ \frac{\partial (\gamma^2 \beta \omega)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 (\gamma^2 - 1) \omega \right] + \frac{\partial p}{\partial r} = 0, \]

where the four-velocity and the relativistic Lorentz factor are defined as follows

\[ U^\mu = (\gamma, \gamma \beta, 0, 0), \quad \gamma \equiv (1 - \beta^2)^{-1/2}, \]

the radial velocity \( \beta \).

The total momentum of spherically symmetric expanding shell vanishes. However, from local conservation equations (B.1.31) one can find out that the radial component of the four momentum vector does not vanish. In analogy with the continuity equation (B.1.5) we integrate the first equation in (B.1.31) over volume starting from some internal radius \( r_i(t) \) up to some external ra-

---

3Throughout this chapter we put the speed of light equal to 1.
B.1. Local, global and average conservation laws

dius \( r_e(t) \) and ignoring a factor \( 4\pi \) we obtain

\[
\int_{r_i}^{r_e} \frac{\partial (\gamma^2 \omega)}{\partial t} r^2 dr - \int_{r_i}^{r_e} \frac{\partial p}{\partial t} r^2 dr + \int_{r_i}^{r_e} \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \gamma^2 \beta \omega) r^2 dr =
\]

\[
\frac{\partial}{\partial t} \int_{r_i}^{r_e} \gamma^2 \omega^2 dr + \gamma_1^2 \gamma_2^2 (r_i) \omega (r_i) \beta (r_i) - \gamma_1^2 \gamma_2^2 (r_e) \omega (r_e) \beta (r_e) - \]

\[r_e^2 \gamma^2 (r_i) \beta (r_i) = 0. \tag{B.1.34}\]

If the boundaries \( r_i(t) \) and \( r_e(t) \) are comoving with the fluid we have

\[
\frac{d}{dt} \int_{r_i}^{r_e} \left( \gamma^2 \omega - p \right) r^2 dr = r_e^2 p (r_e) - r_i^2 p (r_i). \tag{B.1.35}\]

Further, if one assumes (B.1.10), one gets the following result

\[
E = 4\pi \int_0^{R(t)} \gamma^2 \omega^2 dr = \text{const.} \tag{B.1.37}\]

The differential conservation law follows from the same arguments which lead to (B.1.11), so we also have

\[
dE = 4\pi \gamma^2 \omega^2 dr \approx \text{const}. \tag{B.1.38}\]

Analogously to (B.1.18) we introduce the average energy density in the lab frame

\[
\langle \rho \rangle_{\text{lab}} \equiv \frac{E}{V} = \frac{4\pi \int_{r_i}^{r_e} (\gamma^2 \omega) r^2 dr}{4\pi \int_{r_i}^{r_e} r^2 dr}, \tag{B.1.39}\]

Taking the polytropic equation of state with the thermal index

\[
\Gamma \equiv 1 + \frac{P}{\rho}, \tag{B.1.40}\]

and requiring also \( r \gg \Delta \) and (B.1.10) we find from (B.1.39)

\[
\langle \rho \rangle_{\text{lab}} \approx \rho (r) \gamma^2 (r) \propto r^{-2}. \tag{B.1.41}\]

The radial momentum equation follows from (B.1.32) as
B. Hydrodynamics of the pair plasma

\[
\int_{r_i}^{r_e} \frac{\partial (\gamma^2 \beta \omega)}{\partial t} r^2 dr \ + \ \int_{r_i}^{r_e} \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 (\gamma^2 - 1) \omega \right] r^2 dr \ + \ \int_{r_i}^{r_e} \frac{\partial p}{\partial r} r^2 dr = \frac{\partial}{\partial t} \int_{r_i}^{r_e} \gamma^2 \beta \omega (r) r^2 dr
\]

\[
\ + \ r_i^2 \left( \gamma^2 (r_i) - 1 \right) \omega (r_i) - r_i^2 \left( \gamma^2 (r_i) - 1 \right) \omega (r_i) + \int_{r_i}^{r_e} \frac{\partial p}{\partial r} r^2 dr = 0. \tag{B.1.42}
\]

This leads to

\[
\frac{d}{dt} \int_{r_i}^{r_e} \gamma^2 \beta \omega (r) r^2 dr = 2 \int_{r_i}^{r_e} pr dr + r_i^2 p (r_i) - r_e^2 p (r_e). \tag{B.1.43}
\]

For the radial momentum we have

\[
\frac{d \mathbf{P}_{\text{tot}}}{dt} = \frac{d}{dt} \int_0^{R(t)} 4\pi \left( \gamma^2 \beta \omega \right) r^2 dr = 8\pi \int_0^{R(t)} pr dr. \tag{B.1.44}
\]

The left hand side of this equation is the time derivative of the radial momentum, i.e. the radial “force”. The right hand side is the integral of pressure over all shells, so it is clear that unless the pressure in the fireshell is zero, it experiences self-acceleration due to internal pressure.

B.1.3. Entropy conservation

Yet another relevant equation is entropy conservation which may be obtained from (B.1.23) by projection on the flow line

\[
U^\mu \left( T^\nu_\mu \right)_\nu = (U^\mu T^\nu_\mu)_\nu - T^\nu_\mu (U^\mu)_\nu = - (\rho U^\mu)_{;\mu} - \omega U^\nu (U^\mu U^\nu)_{;\mu} - p U^\mu_{;\mu} = 0. \tag{B.1.45}
\]

The second term on the last line vanishes since \( U^\mu U^\mu = -1 \), so we have another conservation equation

\[
- U^\mu \left( T^\nu_\mu \right)_\nu = (\rho U^\mu)_{;\mu} + p U^\mu_{;\mu} = 0. \tag{B.1.46}
\]

This conservation law corresponds to another conserved quantity, the entropy. In fact, equation (B.1.46) can be rewritten as

\[
(\rho U^\mu)_{;\mu} + p U^\mu_{;\mu} = (\omega U^\mu)_{;\mu} - U^\mu p_{;\mu} = 0. \tag{B.1.47}
\]

Now using continuity equation (B.1.1) and the identity \( \omega U^\mu = n U^\mu \left( \frac{\omega}{n} \right) \) we
find
\[(\omega U^\mu)_{,\mu} - U^\mu p_{,\mu} = n U^\mu \left[ \left( \frac{\omega}{n} \right)_{,\mu} - \frac{1}{n} p_{,\mu} \right] = 0. \quad (B.1.48)\]

But inside the brackets there are scalar functions, and therefore covariant derivatives can be replaced by usual derivatives. Then we recall the second law of thermodynamics (Landau and Lifshits, 1987)
\[d \left( \frac{\omega}{n} \right) = T d \left( \frac{\sigma}{n} \right) + \frac{1}{n} d p, \quad (B.1.49)\]
and finally obtain
\[n TU^\mu \left( \frac{\sigma}{n} \right)_{,\mu} = 0, \quad (B.1.50)\]
which can be rewritten using (B.1.1) as
\[\langle \sigma U^\mu \rangle_{,\mu} = 0. \quad (B.1.51)\]
This is continuity equation for the entropy. Since it has exactly the same form as (B.1.1), all conservation equations such as (B.1.11) and (B.1.8) hold for the entropy as well,
\[d \sigma = 4\pi (\sigma \gamma) r^2 dr \approx \text{const}, \quad (B.1.52)\]
\[S = \int_{0}^{R(t)} d \sigma = \text{const.} \quad (B.1.53)\]
Assuming (B.1.40) we find from (B.1.46) and (B.1.20) the following result
\[U^\mu \rho_{,\mu} + \Gamma \rho U^\mu \left( \frac{\sigma}{n} \right)_{,\mu} = d \ln \rho + \Gamma d \ln V = 0, \quad (B.1.54)\]
\[\langle \rho \rangle_{\text{com}} V^{\Gamma} = \text{const.} \]
Finally, due to similarity of equations (B.1.11) and (B.1.52), the average entropy can be defined in the same manner as (B.1.18).

### B.1.4. Analogy with a Friedmann Universe

It is easy to show that conservation equations (B.1.11),(B.1.38) and (B.1.52) imply an analogy between the fireball and the Friedmann Universe, noticed first by Shemi and Piran (Shemi and Piran, 1990; Piran et al., 1993). In fact, this analogy is valid for polytropic equation of state (B.1.40) and ultrarelativistic expansion condition (B.1.10). First, using (B.1.40) and the integral form of
(B.1.49) we obtain
\[ \sigma = \frac{\omega}{T}, \]
which leads to
\[ \rho \propto \sigma^\Gamma, \quad (B.1.55) \]
we rewrite the above mentioned conservation equations using (B.1.40)
\[ n \gamma r^2 = \text{const}, \]
\[ \rho \Gamma \gamma r^2 = \text{const}, \quad (B.1.56) \]
\[ \rho \gamma^2 r^2 = \text{const}. \]

From these equations we then easily find
\[ \gamma \propto r^{2(\Gamma-1)/(\Gamma+1)}, \]
\[ n \propto r^{-2/\Gamma}, \quad (B.1.57) \]
\[ \rho \propto r^{-2\Gamma/\Gamma+1}. \]

Taking ultrarelativistic equation of state with \( \Gamma = 4/3 \) we immediately obtain
\[ \gamma \propto r, \]
\[ n \propto r^{-3}, \quad (B.1.58) \]
\[ \rho \propto r^{-4}, \]
as opposed to the nonrelativistic equation of state with \( \Gamma = 1 \) with different scalings
\[ \gamma = \text{const}, \]
\[ n \propto r^{-2}, \quad (B.1.59) \]
\[ \rho \propto r^{-2}. \]

Actually, scaling laws (B.1.58) take place for the homogeneous isotropic radiation-dominated Universe (Shemi and Piran, 1990; Piran et al., 1993). This fact allowed the authors of (Piran et al., 1993) to speak about the frozen-pulse profile for \( \gamma \gg 1 \) where number, energy and entropy density is conserved within each differential shell with thickness \( dr \), although radial distribution of matter and energy can be inhomogeneous.

Although for observer inside the radiation-dominated fireshell it looks indistinguishable from a portion of radiation-dominated Universe, for the observer outside it looks drastically different. In fact, validity of differential conservation laws (B.1.11),(B.1.38) and (B.1.52) together with integral ones (B.1.8),(B.1.37) and (B.1.53) implies constant thickness approximation assumed.
in (Ruffini et al., 1999, 2000).

Clearly, if the condition (B.1.10) is satisfied, also

$$\Delta \ll R(t)$$

(B.1.60)

is valid. Given the scalings (B.1.58) we then find

$$V = 4\pi \int_{R(t) - \Delta}^{R(t)} r^2 dr \simeq 4\pi \int_{R(t) - \Delta}^{R(t)} \delta r^3 dr \simeq 4\pi R^3,$$

(B.1.61)

where we put $$\gamma = \delta r$$, $$\delta$$ is a constant. At the same time,

$$V = 4\pi \int_0^{R(t)} r^2 dr = \frac{4\pi}{3} R^3.$$

(B.1.62)

Equality of (B.1.61) and (B.1.62) up to a numerical factor suggests that the initially homogeneous energy and particle number distribution looks highly compressed in the lab frame expanding with ultrarelativistic velocity, with the compression factor $$\gamma$$.

### B.2. Self acceleration of the fireshell

For a fireshell which is initially optically thick the total energy conserves. Assume, that the fireshell consists of relativistic electrons, positrons and photons, and also some admixture of a plasma in the form of photons and electrons is present, such that the total charge is zero. While electrons are relativistic, protons are not. Equation of state for pairs and electrons in such a case is given by that of ultrarelativistic fluid with a good approximation:

$$p_{e^\pm,\gamma} = \rho_{e^\pm,\gamma} / 3.$$  

At the same time for protons we have $$p_p \simeq 0$$. Therefore positrons and electrons together with photons can be considered as one fluid with $$p_r = \rho_r / 3$$ as they are strongly coupled since the medium is optically thick. Instead protons have small pressure and internal energy comparing to their rest mass energy.

According to (B.1.37) we find

$$\int_0^{R(t)} (\gamma^2 \omega - p) r^2 dr = \int_0^{R(t)} \gamma^2 \rho_p r^2 dr + \frac{4}{3} \int_0^{R(t)} \gamma^2 \rho_r r^2 dr.$$  

(B.2.1)

These two terms are the rest mass energy of protons $$M_B$$ and energy of the ultrarelativistic fluid $$E$$ correspondingly, so we arrive to a simple result, expressing the total energy of expanding relativistic shell in the lab frame:

$$\gamma (E + M) = \text{const},$$

(B.2.2)
which reads simply as $E + M = \text{const}$ in the comoving frame taking into account the conversion of volumes (B.1.16).

For homogeneous distributions of matter, energy density and pressure the integrals (B.1.7), (B.1.36) and (B.1.43) reduce to

$$n\gamma V = \text{const},$$

$$\left[\gamma^2 (\rho + p)\right] V = \text{const},$$

which means energy and number of particles do not change.

From the above we have

$$nU^0_{\text{com}} V = nV = \text{const} = nU^0_{\text{lab}} V = n\gamma V,$$

$$T^0_{0\text{com}} V = \rho V = \text{const} = T^0_{0\text{lab}} V = \left[\gamma^2 (\rho + p)\right] V,$$

remembering that all quantities $n, \rho, p$ are always defined as comoving ones.

Energy conservation (B.2.1) for (B.1.10) implies

$$\gamma = \gamma_0 \sqrt{\frac{\rho_0^0 + \Gamma \rho_0 V_0}{\rho_p + \Gamma \rho V}}.$$

Clearly all the equations given above can be written for the average values of the number and energy densities.

### B.3. Quasi-analytic model of GRBs

The first detailed models for expansion of a relativistic fireball were suggested in the beginning of nineties (Shemi and Piran, 1990; Piran et al., 1993; Mészáros et al., 1993). Independent calculations were performed in (Ruffini et al., 1999) and (Ruffini et al., 2000). The main difference of these last two works from the models in the literature is that initially not photons but pairs are created by overcritical electric field, and these pairs produce photons later. This plasma, referred to as pair-electro-magnetic (PEM) pulse expands initially into vacuum surrounding the black hole reaching very soon relativistic velocities. Then collision with the baryonic remnant of the collapsed
B.3. Quasi-analytic model of GRBs

star takes place and the PEM pulse becomes pair-electro-magnetic-baryonic (PEMB) pulse, see (Ruffini et al., 2003) for details. This difference is not large, since it was shown that the final gamma factor does not depend on the distance to the baryonic remnant and parameters of the black hole. The only crucial parameters are again the initial energy $E_0$, and baryonic admixture $\tilde{B}$.

The model is based on numerical integration of relativistic energy-momentum conservation equations (B.1.31,B.1.32) together with the baryonic number conservation equation (B.1.1). However, the most important distinct point from all previous models is that the rate equation for electron-positron pairs is added to the model and integrated simultaneously in order to give more detailed description of the transparency. This latter fact leads to quantitative differences in predictions of the model with respect to the simplified models in the literature.

Here we concentrate on the simple quasi-analytical treatment presented in (Ruffini et al., 1999, 2000), see also (Ruffini et al., 2003). The PEMB pulse is supposed to contain finite number of shells each with flat density profile. The dynamics is governed by the set of equations, obtained in the previous subsections. Put (B.1.22),(B.1.54) and (B.2.8) together (omitting brackets for brevity)

\[
\frac{n^0_B}{n_B} = \frac{V}{V_0} = \frac{V}{V_0} \gamma \gamma_0, \quad (B.3.1)
\]

\[
\frac{\rho_0}{\rho} = \left( \frac{V}{V_0} \right)^\Gamma = \left( \frac{V}{V_0} \right)^\Gamma \left( \frac{\gamma}{\gamma_0} \right)^\Gamma, \quad (B.3.2)
\]

\[
\frac{\gamma}{\gamma_0} = \sqrt{\frac{\rho^0_p + \Gamma \rho_0 V_0}{\rho_p + \Gamma \rho V}}, \quad (B.3.3)
\]

where subscript "0" denotes initial values, and all quantities are assumed being averaged over finite distribution of shells with constant width and density profiles. All components such as photons, electrons, positrons and plasma ions give contribution to energy density and pressure. This set of equations is equivalent to (B.4.5) and (B.4.7) (see below). The next step is to take into account the rate equation for positrons and electrons, accounting for non-instant transparency:

\[
(n_e \pm U^\mu)_{\mu} = \sigma v (n^2_{e^\pm}(T) - n^2_{e^\pm}), \quad (B.3.4)
\]

or, being integrated over volume,

\[
\frac{\partial}{\partial t} N_{e^\pm} = -N_{e^\pm} \frac{1}{V} \frac{\partial V}{\partial t} + \sigma v \frac{1}{\gamma^2} (N^2_{e^\pm}(T) - N^2_{e^\pm}), \quad (B.3.5)
\]

where $\sigma$ is the mean pair annihilation-creation cross section, $v$ is the thermal...
velocity of $e^\pm$-pairs. The coordinate number density of $e^\pm$-pairs in equilibrium is $N_{e^\pm}(T) = \gamma n_{e^\pm}(T)$ and the coordinate number density of $e^\pm$-pairs is $N_{e^\pm} = \gamma n_{e^\pm}$ for $T > m_e c^2$. For $T > m_e c^2$ we have $n_{e^\pm}(T) \simeq n_{\gamma}(T)$, i.e. the number densities of pairs and photons are nearly equal. The pair number densities are given by appropriate Fermi integrals with zero chemical potential, at the equilibrium temperature $T$.

For an infinitesimal expansion of the coordinate volume from $V_0$ to $V$ in the coordinate time interval $t - t_0$ one can discretize the last differential equation for numerical computations.

The most important outcomes from analysis performed in (Ruffini et al., 2000) are the following:

- the appropriate model for geometry of expanding fireshell (PEM-pulse) is given by the constant width approximation (this conclusion is achieved by comparing results obtained using (B.1.31),(B.1.32) and simplified treatment described above),

- there is a bound on parameter $B$ which comes from violation of constant width approximation, $B \leq 10^{-2}$ ($\eta \geq 10^2$).

In previous subsection we have proved the applicability of the constant thickness approximation of the fireshell. The second conclusion appears to be crucial, since it shows that there is a critical loading of baryons, when their presence produce a turbulence in the outflow from the fireshell, its motion becomes very complicated and the fireshell evolution does not lead in general to the GRB.

Exactly because of this reason, the optically thick fireshell never reaches such large radius as $r_b = r_0 \eta^2$ which is discussed by Mészáros et al. (1993), see section (B.4.3), since to do this the baryonic fraction should overcome the critical value $B_c = 10^{-2}$. For larger values of $B_c$ the theory reviewed here does not apply. This means in particular, that all conclusions in (Mészáros et al., 1993) obtained for $r > r_b$ are invalid. In fact, for $B < B_c$ the gamma factor even does not reach saturation.

Notice, that another way to obtain the constraint $B < B_c$ is to require optical depth of the emitting region to be smaller than one, leading to the requirement that the Lorentz factor be greater than $\gamma \geq 10^2$, see Introduction. At the same time, there is simple relation between the Lorentz factor and the baryonic loading parameter $B = \gamma^{-1}$ in the region $10^{-2} < B < 10^{-4}$, see Fig. B.2, which leads to $B \leq 10^{-2}$.

The fundamental result coming from this model are the diagrams presented at Fig. B.1 and Fig. B.2. The first one shows basically which portion of initial energy is emitted in the form of gamma rays $E_\gamma$ when the fireshell reaches transparency condition $\tau \simeq 1$ and how much energy gets converted into the kinetic form of the baryons $E_k$ left after pairs annihilation and photons escape.
B.3. Quasi-analytic model of GRBs

![Graph](image)

**Figure B.1.** Relative energy release in the form of photons emitted at transparency point (solid line) and kinetic energy of the plasma (dashed line) of the baryons in terms of initial energy of the fireball depending on parameter $B$ obtained on the basis of quasi-analytic model. Thick line denotes the total energy of the system in terms of initial energy.

![Graph](image)

**Figure B.2.** Relativistic gamma factor of the fireball when it reaches transparency depending on the value of parameter $B$. Dashed line gives asymptotic value.
B. Hydrodynamics of the pair plasma

The second one gives the value of gamma factor at the moment when the systems reaches transparency.

The energy conservation holds, namely

\[ E_0 = E_\gamma + E_k, \]  

(B.3.6)

Clearly when the baryons abundance is low most energy is emitted when the fireshell gets transparent. It is remarkable that almost all initial energy is converted into kinetic energy of baryons already in the region of validity of constant thickness approximation \( B < 10^{-2} \), so the region \( 10^{-8} < B < 10^{-2} \) is the most interesting from this point of view.

B.4. Alternative models

B.4.1. Shemi and Piran model

In this section we discuss the model, proposed by Shemi and Piran (1990). This quantitative model gives rather good general picture of relativistic fireballs.

Shemi and Piran found that the temperature at which the fireball becomes optically thin is determined as

\[ T_{\text{esc}} = \min(T_g, T_p), \]  

(B.4.1)

where \( T_g \) and \( T_p \) is the temperature when it reaches transparency with respect to gas (plasma) or pairs:

\[ T_g^2 \approx \frac{45}{8\pi^3} \frac{m_p}{m_e} \frac{1}{\alpha^2} \frac{1}{g_0^3} \frac{1}{T_0^2} R_0^3 \eta, \]  

(B.4.2)

\[ T_p \approx 0.032, \]  

(B.4.3)

where \( m_p \) is the proton mass, \( g_0 = \frac{11}{4}, \alpha = \frac{1}{137} \), dimensionless temperature \( T \) and radius \( R \) of the fireball are measured in units of \( \frac{m_e c}{k_B} \) and \( \lambda_c \equiv \frac{\hbar}{m_e c} \) correspondingly, and the subscript “0” denotes initial values. The temperature at transparency point in the case when plasma admixture is unimportant is nearly a constant for a range of parameters of interest and it nearly equals

\[ T_p = 15 \text{ keV}. \]  

(B.4.4)

Adiabatic expansion of the fireball implies

\[ \frac{E}{E_0} = \frac{T}{T_0} = \frac{R_0}{R}, \]  

(B.4.5)
where $E = \frac{E}{mc^2}$ is a radiative energy. From the energy conservation (B.1.23), supposing the fluid to be pressureless and its energy density profile is constant we have in the coordinate frame

$$\int T_0^0 dV = \gamma E_{tot} = \text{const.} \quad (B.4.6)$$

Supposing at initial moment $\gamma_0 = 1$ and remembering that $E_{tot} = E + Mc^2$ we arrive to the following fundamental expression of relativistic gamma factor $\gamma$ at transparency point:

$$\gamma = \frac{E_0 + Mc^2}{E + Mc^2} = \frac{\eta + 1}{\left(\frac{T_{esc}}{T_0}\right)\eta + 1}, \quad (B.4.7)$$

where $M = \frac{M}{m_c}$.

One can use this relation to get such important characteristics of the GRB as observed temperature and observed energy. In fact, they can be expressed as follows:

$$T_{obs} = \gamma T_{esc}, \quad (B.4.8)$$

$$E_{obs} = E_0 \frac{T_{obs}}{T_{esc}}, \quad (B.4.9)$$

These results are presented at Fig. B.3. In the limit of small $\eta$ we have $\gamma = (1 + \eta)$, while, for very large $\eta$ the value of gamma factor at transparency point is $\gamma = T_0/T_{esc}$, and it has a maximum at intermediate values of $\eta$. We denote by dashed thick line the limiting value of $\eta$ parameter $\eta_c = \frac{B^{-1}}{c}$. For $\eta < \eta_c$ the approximations used to construct the model do not hold. It is clear that because of the presence of bound $\eta_c$ the value $\gamma = \eta$ can be reached only as asymptotic one. In effect, the value $\eta_c$ cuts the region where saturation of the gamma factor happens before the moment when the fireball becomes transparent.

It was found that for relatively large $\eta \geq 10^5$ the photons emitted when the fireball becomes transparent carry most of the initial energy. However, since the observed temperature in GRBs is smaller than initial temperature of the fireball, one may suppose that a large part of initial energy is converted to kinetic energy of the plasma.

**B.4.2. Shemi, Piran and Narayan model**

Piran et al. (1993) present generalization of this model to arbitrary initial density profile of the fireball. These authors performed numerical integrations of energy-momentum relativistic conservation equations (B.1.31),(B.1.32) and baryon number conservation equation (B.1.3). They were mainly interested
B. Hydrodynamics of the pair plasma

in the evolution of the observed temperature, gamma factor and other quantities with the radius. Their study results in the number of important conclusions, namely:

- the expanding fireball has two basic phases: a radiation dominated phase and a matter-dominated phase. In the former, the gamma factor grows linearly as the radius of the fireball: \( \gamma \propto r \), while in the latter the gamma factor reaches asymptotic value \( \gamma \approx \eta + 1 \).

- the numerical solutions are reproduced with a good accuracy by frozen-pulse approximation, when the pulse width is given by initial radius of the fireball.

The last conclusion is important, since the fireball becomes a fire shell, the volume \( V \) can be calculated as

\[
V = 4\pi R^2 \Delta,
\]

where \( \Delta \approx R_0 \) is the width of the leading shell with constant energy density profile, \( R \) is the radius of the fireball.

They also present the following scaling solution:

\[
R = R_0 \left( \frac{\gamma_0}{\gamma} \right)^{1/2},
\]

\[
\frac{1}{D} = \frac{\gamma_0}{\gamma} + \frac{3\gamma_0}{4\gamma\eta} - \frac{3}{4\eta},
\]

where subscript "0" denotes some initial time when \( \gamma \gtrsim \text{few} \), which can be inverted to give \( \gamma(R) \).

B.4.3. Mészáros, Laguna and Rees model

The next step in developing this model was made by Mészáros et al. (1993). In order to reconcile the model with observations, these authors proposed a generalization to anisotropic (jet) case. Nevertheless, their analytic results apply to the case of homogeneous isotropic fireballs and we will follow their analytical isotropic model in this section.

Starting from the same point as Shemi and Piran, consider (B.4.5) and (B.4.7). Analytic part of the paper describes the geometry of the fireball, gamma factor behavior and the final energy balance between radiation and kinetic energy. Magnetic field effects are also considered, but we are not interested in this part here.

Three basic regimes are found in (Mészáros et al., 1993) for evolution of the fireball. In two first regimes there is a correspondence between the analysis in
B.4. Alternative models

the paper and results of Piran et al. (1993), so the constant thickness approximation holds. It is claimed in (Mészáros et al., 1993), that when the radius of the fireball reaches very large values such as \( R_b = R_0 \eta^2 \) the noticeable departure from constant width of the fireball occurs. However, it is important to note, that the fireball becomes transparent much earlier and this effect never becomes important (see section B.3).

The crucial quantity, presented in the paper is \( \Gamma_m \) – the maximum possible bulk Lorentz factor achievable for a given initial radiation energy \( E_0 \) deposited within a given initial radius \( R_0 \):

\[
\Gamma_m \equiv \eta_m = (\tau_0 \eta)^{1/3} = (\Sigma_0 \kappa \eta)^{1/3}, \tag{B.4.13}
\]

\[
\Sigma_0 = \frac{M}{4\pi R_0^2}, \quad \kappa = \frac{\sigma_T}{m_p}, \tag{B.4.14}
\]

where \( \Sigma_0 \) is initial baryon (plasma) mass surface density.

All subsequent calculations in the paper (Mészáros et al., 1993) involves this quantity. It is evident from (B.4.13) that the linear dependence between the gamma factor \( \Gamma \) and parameter \( \eta \) is assumed. However, this is certainly not true as can be seen from Fig. B.2. We will come back to this point in the following section.

Another important quantity is given in this paper, namely

\[
\Gamma_p = \frac{T_0}{T_p}. \tag{B.4.15}
\]

This is just the asymptotic behavior of the gamma factor at Fig. B.3 for very large \( \eta \). Using it, the authors calculate the value of \( \eta \) parameter above which the pairs dominated regime occurs:

\[
\eta_p = \frac{\Gamma_m^3}{\Gamma_p^2}. \tag{B.4.16}
\]

This means, above \( \eta_p \) the presence of baryons in the fireball is insufficient to keep the fireball opaque after pairs are annihilated and almost all initial energy deposited in the fireball is emitted immediately.

The estimate of the final radiation to kinetic energy ratio (Mészáros et al., 1993) is incorrect, because kinetic and radiation energies do not sum up to initial energy of the fireball thus violating energy conservation. This is illustrated in Fig. B.4. The correct analytic diagram is presented in Fig. B.1 instead.

B.4.4. Approximate results

All models for isotropic fireballs are based on the following points:
B. Hydrodynamics of the pair plasma

Figure B.3.: The relativistic gamma factor (upper dashed line), the observed temperature (solid line), and the ratio of observed energy to the initial energy of the fireball (lower dashed line) as a function of baryonic loading parameter, see (Shemi and Piran, 1990). The values of parameters are the same as in the cited paper. Thick dashed line denotes the limiting value of the baryonic loading. Its values when gamma factor reaches maximum and gets constant are also shown.

Figure B.4.: The ratios of radiation and kinetic energy to the initial energy of the fireball predicted by Mészáros, Laguna and Rees model. Thick line denotes the total energy of the system in terms of initial energy. Energy conservation does not hold.
1. Flat space-time,
2. Relativistic energy-momentum principle,
3. Baryonic number conservation.

Although the model (Ruffini et al., 1999, 2000) starts with Reissner-Nordström geometry, the numerical code is written for the case of flat space-time simply because curved space-time effects becomes insignificant soon after the fireshell reaches relativistic expansion velocities. The presence of rate equation in the model (Ruffini et al., 1999, 2000) has a deep physical ground and its luck in the other treatments means incompleteness of their models. Indeed, the number density of pairs influences the speed of expansion of the fireshell. However, in this section we neglect the rate equation and discuss the common points between all considered models.

First of all, let us come back to Fig. B.2. For almost all values of \( \eta \) parameter the gamma factor is determined by gas (i.e. plasma or baryons) admixture according to (B.4.2), consider this case below. For given initial energy and radius this temperature depends only on \( \eta \) only, so one can write:

\[
\gamma = \frac{\eta + 1}{(\eta_0/\eta) + 1} = \frac{\eta + 1}{a\eta^2 + 1}, \quad (B.4.17)
\]

where

\[
a = 2.1 \cdot 10^3 \eta_0^{-2.5} \eta_0^{-0.5}. \quad (B.4.18)
\]

From this formula we can get immediately the two asymptotic regimes, namely:

\[
\gamma = \begin{cases} 
\eta + 1, & \eta < \eta_{\text{max}}, \\
\frac{1}{a\sqrt{\eta}}, & \eta > \eta_{\text{max}}.
\end{cases} \quad (B.4.19)
\]

Notice, that the constant \( a \) is extremely small number, so that after obtaining precise value of \( \eta_{\text{max}} \) by equating to zero the derivative of function (B.4.17) one can expand the result in Taylor series and get in the lowest order in \( a \), that:

\[
\eta_{\text{max}} \simeq \left( \frac{2}{\lambda} \right)^{\frac{3}{2}} - 2, \quad (B.4.20)
\]

\[
\gamma_{\text{max}} \equiv \gamma(\eta_{\text{max}}) \simeq \frac{1}{3} \left[ 1 + \left( \frac{2}{\lambda} \right)^{\frac{3}{2}} \right]. \quad (B.4.21)
\]

In particular, in the case shown in Fig. B.3 one has \( \eta_{\text{max}} = 2.8 \cdot 10^5, \gamma_{\text{max}} = 9.3 \cdot 10^4 \) while according to (B.4.13) \( \Gamma_m = \eta_{\text{max}} = 1.75 \cdot 10^5 \). Clearly, our result
B. Hydrodynamics of the pair plasma

is much more accurate. Actually, the value $\Gamma_m$ in (B.4.13) is obtained from equating asymptotes in (B.4.19) and there exists the following relation:

$$a = (\tau_0 \eta)^{-1/2}. \tag{B.4.22}$$

Now we are ready to explain why the observed temperature (and consequently the observed energy) does not depend on $\eta$ in the region $\eta_{\text{max}} < \eta < \eta_p$. From the second line in (B.4.19) it follows that the gamma factor in this region behaves as $\gamma \propto \eta^{-1/2}$, while $T_{\text{esc}} \propto \eta^{1/2}$. These two exactly compensate each other leading to independence of the observed quantities on $\eta$ in this region. This remains the same for $\eta > \eta_p$ also, since here $T_{\text{esc}} = T_p = \text{const}$ and from (B.4.17) $\gamma = \text{const}$.

B.4.5. Nakar, Piran and Sari revision

Recently revision of the fireball model was made by Nakar et al. (2005). These authors presented new diagram for final Lorentz gamma factor and for energy budget of the fireball. Their work was motivated by observation of giant flares with the following afterglow spreading up to radio region with thermal spectrum. They concluded that the fireball have to be loaded by either baryons or magnetic field, and cannot be only pure $e^\pm, \gamma$ plasma in order to have $10^{-3}$ of the total energy radiated in the giant flare.

In analogy with cosmology authors define the number density of pairs which survives because expansion rate becomes larger than annihilation rate\(^4\) which gives condition

$$n_\pm \approx \frac{1}{\sigma_T R_0}. \tag{B.4.23}$$

Then, recalling (B.4.5), if we want to estimate number of pairs it turn out to be

$$N_\pm = \frac{4\pi R_0 c t}{\sigma_T} \left(\frac{T_0}{T_\pm}\right)^2, \tag{B.4.24}$$

where we identify $\Delta = ct$ in (B.4.10). In (Nakar et al., 2005) the authors obtained third power of the ratio of temperatures which influences all their subsequent results.

Obtaining the conclusion that the afterglow cannot be obtained as the result of interaction of $e^\pm, \gamma$ plasma with ISM authors turn to baryonic loading consideration. They attempt to define critical values of the loading parameter

\(^4\)This effect is accounted for automatically in our approach where rate equations for pairs include expansion term.
η finding in general 4 such values\(^5\), in particular:

\[
\eta_1 = \frac{E_0 \sigma_T}{4\pi R_0 c t m_e c^2} \left( \frac{T_\pm}{T_0} \right)^3,
\]

(B.4.25)

\[
\eta_2 = \frac{E_0 \sigma_T}{4\pi R_0 c t m_p c^2} \left( \frac{T_\pm}{T_0} \right)^3,
\]

(B.4.26)

\[
\eta_3 = \left( \frac{E_0 \sigma_T}{4\pi R_0 c t m_p c^2} \right)^{1/4}.
\]

We recall that the first two quantities are based on the formula for \(N_\pm\) and should contain factors \(\left( \frac{T_\pm}{T_0} \right)^2\) instead.

The first ‘critical’ value, \(\eta_1\), comes from the condition \(N_p m_p = N_\pm m_e\), where \(N_p \equiv \frac{E_0}{m_p c^2 \eta}\) is just the number of protons in plasma admixture. It does not correspond to any critical change in physics of the phenomena; for instance it cannot be interpreted as equality of masses (equal inertia) of pairs and baryons since the former is given mainly by their total energy \(E_\pm\), while the latter by their rest mass \(N_p m_p\). This value is however close to the one defined above \(\eta_1 \approx \eta_p\).

The second ‘critical’ value, \(\eta_2\), corresponds to the condition \(N_p = N_\pm\), namely equality of numbers of protons and pairs. It is also incorrectly interpreted as equal contribution to the Thompson scattering. In fact, cross-section for the Thompson scattering for protons contains additions factor \(\left( \frac{m_e}{m_p} \right)^2\) with respect to the usual formula for electrons.

Definition of the third ‘critical’ value, \(\eta_3\), is not clear, but important is its vicinity to the critical value \(\eta_c\) quoted above.

On the basis of adiabatic conditions (B.4.5) authors present the new diagram for the final gamma factor and energy budget of the pair-baryonic plasma at transparency. In fact, this diagram, shown in our Fig. B.6 by dashed curve for parameter \(B\), is very similar to the one, obtained by Grim- 
srud and Wasserman (1998), who considered hydrodynamics of relativistic \(e^\pm, \gamma\) winds. That problem is very different from ours, because of different boundary conditions\(^6\). In particular, in the wind energy conservation (B.1.23) does not hold; the reason is that constant energy (mass) supply takes place parametrized in (Grimsrud and Wasserman, 1998) by \(\dot{E} (M)\). In that paper in fact authors present the diagram for asymptotic value of the Lorentz gamma factor depending on the ratio \(\frac{\dot{E}}{M}\) which is very different from the quantity \(\eta\).

Surprisingly, the fundamental result about the presence of maximum in

---

\(^5\)The last value \(\eta_4\) corresponds to the case of heavy loading where spreading of the expanding shell is observed, and is not considered here.

\(^6\)Note, that the authors of (Grimsrud and Wasserman, 1998) also use rate equations describing decoupling plasma from photons.
B. Hydrodynamics of the pair plasma

the diagram for gamma factor on Fig. B.6 which was found by the same authors previously in (Shemi and Piran, 1990) (see Fig. B.3) that comes from the energy conservation (B.4.7) is ignored in (Nakar et al., 2005). It can be understood in the following way. For small loading (small $B$) the more baryons are present in the plasma the larger becomes the number density of corresponding electrons, the larger optical depth is. Therefore, transparency is reached later, which gives larger gamma factor at transparency. From the other hand, for heavy baryon loading (relatively large $B$) the more baryons are present, the more inertia has the plasma, and by energy conservation, the less final gamma factor has to be.

B.5. Significance of the rate equation

The rate equation describes the number densities evolution for electrons and positrons. In analytic models it is supposed that pairs are annihilated instantly when transparency condition is fulfilled. Moreover, the dynamics of expansion is influenced by the electron-positron energy density as can be seen from (B.3.1)-(B.3.5). Therefore, it is important to make clear whether neglect of the rate equation is a crude approximation or not.

Using eq. (B.4.7) one can obtain analytic dependence of the energy emitted at transparency point on parameter $B$ and we compare it in Fig. B.5.

![Figure B.5](image-url)

**Figure B.5:** Relative energy release in the form of photons emitted at transparency point of the GRB in terms of initial energy of the fireball depending on parameter $B$. Thick line represents numerical results and it is the same as in Fig. B.1. Normal line shows results for the analytic model of Shemi and Piran (1990). Dashed line shows the difference between exact numerical and approximate analytical results.

We also show the difference between numerical results based on integration of eqs. (B.3.1-B.3.5) and analytic results from Shemi and Piran model.
The values of parameters are: $\mu = 10^3$ and $\xi = 0.1$ (which correspond to $E_0 = 2.87 \cdot 10^{54}$ ergs and $R_0 = 1.08 \cdot 10^9$ cm). One can see that the difference peaks at intermediate values of $B$. The crucial deviations however appear for large $B$, where analytical predictions for observed energy are about two orders of magnitude smaller than the numerical ones. This is due to the difference in predictions of the radius of the fireshell at transparency moment. In fact, the analytical model overestimates this value at about two orders of magnitude for $B = 10^{-2}$. So for large $B$ with correct treatment of pairs dynamics the fireshell gets transparent at earlier moments comparing to the analytical treatment.

At the same time, the difference between numerical and analytical results for gamma factor is significant for small $B$ as illustrated at Fig. B.6. While both results coincide for $B > 10^{-4}$ there is a constant difference for the range of values $10^{-8} < B < 10^{-4}$ and asymptotic constant values for the gamma factor are also different. Besides, this asymptotic behavior takes place for larger values of $B$ in disagreement with analytical expectations. Thus the acceleration of the fireshell for small $B$ is larger if one accounts for pairs dynamics.

![Figure B.6:](image)

**Figure B.6:** Relativistic gamma factor when transparency is reached. The thick line denotes exact numerical results, the normal line corresponds to analytical estimate from Shemi and Piran model, the dotted line denotes the asymptotic value of the baryonic loading parameter. The dashed line shows results of Nakar, Piran and Sari.

It is clear that the error coming from neglect of the rate equation is significant. This implies that simple analytic model of Shemi and Piran gives only qualitative picture of the fireshell evolution and in order to get correct
B. Hydrodynamics of the pair plasma

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>conservation:</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>EM, bar. number.</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>rate equation</td>
<td>yes</td>
<td>not consider</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>const width approx.</td>
<td>justify</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>model $\gamma(r)$</td>
<td>num./analyt.</td>
<td>no</td>
<td>num./analyt.</td>
<td>num./analyt.</td>
</tr>
<tr>
<td>model $\gamma(\eta)$</td>
<td>num.</td>
<td>analyt.</td>
<td>not consider</td>
<td>analytic</td>
</tr>
</tbody>
</table>

Table B.1.: Comparison of different models for fireballs.

description of the fireshell one cannot neglect the rate equation.

Moreover, the difference between exact numerical model (Ruffini et al., 1999, 2000) and approximate analytical models (Shemi and Piran, 1990) becomes apparent in various physical aspects, namely in predictions of the radius of the shell when it reaches transparency, the gamma factor at transparency and the ratio between the energy released in the form of photons and the one converted into kinetic form. The last point is crucial. It is assumed in the literature, that the whole initial energy of the fireshell gets converted into kinetic energy of the shell during adiabatic expansion. Indeed, taking typical value of parameter $B$ as $10^{-3}$ we find that according to Shemi and Piran model we have only 0.2% of initial energy left in the form of photons. However, exact numerical computations (Ruffini et al., 1999, 2000) give 3.7% for the energy of photons radiated when the fireshell reaches transparency, which is a significant value and it cannot be neglected.

To summarize the above discussion, we present the result of this survey in the Table B.1. It is important to notice again that comparing to simplified analytic treatment, accounting for the rate of change of electron-positron pairs densities gives quantitatively different results on the ratio of kinetic versus photon energies produced in the GRB and the gamma factor at transparency moment, which in turn leads to different afterglow properties. Therefore, although analytical models presented in sections B.4.1 and B.4.4 agree and give correct qualitative description of the fireshell, one should use numerical approach described in sec. B.3 in order to compare the theory and observations.

B.6. The simple model for the afterglow

When electron-positron pairs are annihilated and the transparency condition $\tau \simeq 1$ is fulfilled photons escape and the only ingredient of the fireshell which is left is relativistically expanding shell of protons and electrons. The latter are not important kinematically and one can say that the shell consists of
baryons. This shell propagates in the interstellar medium sweeping up the cold gas. The constant width approximation is good enough in description of this process, however we do not restrict ourselves to this case. The only requirement is that the collision is inelastic and the energy released in the collision is shared within the whole shell for a short time.

Consider the collision process in the lab reference frame where the ISM is initially at rest. Supposing the expanding baryons as well as ISM are cold, the total energy of the shell is \( E = M_B c^2 \gamma = M \gamma \) with \( \gamma \) being the Lorentz factor of the shell and \( M \) is the rest mass of the shell (together with its thermal energy) in units of energy. The mass-energy of the ISM swept up within infinitesimal time interval is \( dm \). The energy released during this process is \( dE \). In our definitions \( \gamma \beta = \sqrt{\gamma^2 - 1} \). Finally, the gamma factor after the collision becomes \( \gamma + d\gamma \). Rewrite energy and momentum conservation:

\[
M \gamma + dm = (M + dm + dE)(\gamma + d\gamma), \quad (B.6.1)
\]
\[
M \sqrt{\gamma^2 - 1} = (M + dm + dE) \sqrt{(\gamma + d\gamma)^2 - 1}. \quad (B.6.2)
\]

This set of equations is equivalent to the one used in (Ruffini et al., 2003). From (B.6.2) we get:

\[
dE = -dm - M + M \sqrt{\gamma^2 - 1} \frac{\gamma^2 - 1}{(\gamma + d\gamma)^2 - 1}. \quad (B.6.3)
\]

Substituting the last equality into (B.6.1) we arrive to

\[
d\gamma = \frac{dm}{M} + \gamma (1 - b), \quad (B.6.4)
\]

where

\[
b = \sqrt{1 + 2\gamma \left( \frac{dm}{M} \right) + \left( \frac{dm}{M} \right)^2}. \quad (B.6.5)
\]

Substituting this expressions into (B.6.3) we find

\[
dE = -dm - M(1 - b). \quad (B.6.6)
\]

Since \( dm \) is infinitesimally small we can expand \( b \) in series over \( dm/M \)

\[
b \simeq 1 - \frac{dm}{M} \gamma, \quad (B.6.7)
\]
so we finally obtain:

\[ dE = (\gamma - 1)dm, \quad (B.6.8) \]
\[ d\gamma = -\left(\gamma^2 - 1\right)\frac{dm}{M}. \quad (B.6.9) \]

These equations are used to describe collision of the baryonic remnant of the fireshell with ISM (Piran, 1999; Ruffini et al., 2000, 2003). Note that it is a mistake to suppose that \( dE = 0 \) and use only (B.6.2) to obtain differential equation for \( \gamma \) as was done in (Katz, 1994). The total mass-energy swept up during this process is given by

\[ dM = M + dm + (1 - \delta)dE - M = dm + (1 - \delta)dE, \quad (B.6.10) \]

where \( \delta \) denotes the portion of the energy that is radiated. There are basically two approximations which follow from this expression, namely fully radiative condition \( \delta = 1 \) and adiabatic condition \( \delta = 0 \). With adiabatic condition no energy is radiated and one have to assume that the kinetic energy of the decelerating baryons is converted into radiation in dissipation processes in shocks which are supposed to form during the collision (Rees and Meszaros, 1992; Narayan et al., 1992) see also (Piran, 1999). With the radiative condition no additional mechanisms are required to describe the afterglow since it results from the emission of part of the energy released during inelastic collision of accelerated baryonic pulse with ISM (Ruffini et al., 2003).

**B.7. Blandford and McKee radiative solution**

Equations (B.6.8),(B.6.9) can be found already in (Blandford and McKee, 1976). However the meaning of the quantities used there as well as the derivation are incorrect.

Following Blandford and McKee (1976) consider the relativistic shock (blast) wave resulting from an impulsive energy release, propagating into the homogeneous external medium with the Lorentz factor \( \gamma \). Authors suppose that the shock wave sweeps-up external matter. They treat this swept-up matter as being in a thin, cold shell.

Consider the reference frame where the shock is at rest. For an observer in this frame the external medium has relativistic Lorentz factor \( \gamma \). The shock front is a spherical massless surface. For our observer there is a flow of matter, energy and momentum through this surface. We can calculate these quantities using (B.1.23) and (B.1.3). Recall that the external medium is cold so
$p \ll \rho$. Thus we get

$$\frac{dE_{tot}}{dt} = -\int_S T^{01} dS = -\int_S \rho \gamma^2 \beta dS = -4\pi R^2 \rho \beta \gamma^2,$$

(B.7.1)

$$\frac{dP}{dt} = -\int_S T^{11} dS = -\int_S \rho (\gamma^2 - 1) dS = -4\pi R^2 \rho (\gamma^2 - 1),$$

(B.7.2)

$$\frac{dm}{dt} = -\int_S \rho U^1 dS = -4\pi R^2 \rho \beta \gamma,$$

(B.7.3)

where $P$ is a radial momentum of external medium, $E_{tot}$ is its total energy. The kinetic energy $E_k$ change is simply connected to its flux:

$$\frac{dE_k}{dt} = \frac{dE_{tot}}{dt} - \frac{dm}{dt} = -4\pi R^2 \rho \beta \gamma (\gamma - 1).$$

(B.7.4)

This formula coincides with eq. (84) in Blandford and McKee (1976). The next equation (85) is of the form of our (B.6.9). This implies that in order to derive this result these authors considered the full derivative of kinetic energy of the external medium with respect to expanding shock wave, namely:

$$\frac{dE_k}{dt} = (\gamma - 1) \frac{dm}{dt} + m \frac{d\gamma}{dt},$$

(B.7.5)

and equate these two expressions. However if to proceed in the same manner with the radial momentum equation, one arrives to a strange result: they become inconsistent. The reason is the following.

Equations (B.7.1), (B.7.2), (B.7.3) and (B.7.4) are of course correct. The next step is of doubt. To write down the total derivative means to suppose that the Lorentz factor changes with time. This means that the properties of the external medium change. However, the problem considered by Blandford and McKee is idealized one, namely the hydrodynamics of relativistic shocks without consideration of any microphysics\(^7\). In this context the damping and dissipation effect cannot be incorporated directly and the shock wave generally speaking will propagate with the constant velocity. Apart from difficulties in definition of the quantity $m$, the mass of external medium which can be thought infinite, the last term in (B.7.5) cannot exist and therefore there is no way to get (85) from (84) in (Blandford and McKee, 1976).

From the physical point of view instead of the massless shock (even if it is present) the massive shell have to be considered\(^8\). The only natural way to deal with this problem is to consider interaction of this shell with small shell of external medium. The problem is consequently equivalent to interaction of

---

\(^7\)Microphysical interactions responsible for changes in the blast wave can be Coulomb interactions within the plasma, ionization of external medium losses, interactions with magnetic field and various radiative processes.

\(^8\)In some approximation it can be considered as the relativistic piston problem.
two massive particles which is obvious to treat with the help of conservation equations, instead of shock equations.

Thus, equations (B.6.8),(B.6.9) can be derived only from energy and radial momentum conservation equations as was done in (Katz and Piran, 1997) and independently in (Ruffini et al., 2000).

**B.7.1. Reaching of transparency**

![Figure B.7:](image)

*Figure B.7:* The Lorentz factor as function of radius for selected values of the baryonic loading parameter. Filled circles show the radius at the moment of transparency. Squares show the moment of departure from thermal distributions of electron-positron pairs. The star denotes the moment when the temperature of the fireshell equals 511 keV.

To demonstrate the richness of physical phenomena associated with gamma-ray burst we provide below several illustrations. We calculate the temperature in comoving and laboratory frames of the plasma as well as the Lorentz factor with the code described in (Ruffini et al., 1999, 2000) for different values of the parameter $B$. We show our results in Fig. B.7 and B.8. From Fig. B.8 it is clear that during early phases of expansion the temperature decreases down to 0.511 MeV, as shown by the star in Fig. B.7, and the ratio between electron-positron pairs and photons becomes exponentially suppressed. However, because of the accelerated expansion the apparent temperature in the observer’s frame remains almost constant, see Fig. B.8. Then, after collision with a baryonic remnant after which the Lorentz factor decreases, the plasma continue to expand. At a certain moment, shown by squares in Fig. B.7 a departure from thermal distributions of electron-positron pairs occurs, due to the fact that the rate of the reaction $\gamma \gamma \rightarrow e^+e^-$ becomes smaller than expansion rate. From that moment electron-positron pairs freeze out, analogously to
**B.7. Blandford and McKee radiative solution**

**Figure B.8.** The temperature of the fireshell as functions of radius for selected values of the baryonic loading parameter. Filled circles show the radius at the moment of transparency.

**Figure B.9.** Total opacity of the fireshell.
the early Universe. Finally, transparency is reached, as denoted by circles in Fig. B.7. For high values of the baryonic loading $B > 10^{-4}$ the comoving temperature decreases at the late stages of expansion in the same way as the observed temperature.

![Figure B.10:](image)

**Figure B.10:** The ratio of the opacity due to electron-positron pairs to the opacity due to electrons associated with baryons in the fireshell.

Total opacity due to pair production and due to Compton scattering is shown in Fig. B.9, while the ratio of separate contributions, i.e. opacity due to pairs and baryons are shown in Fig. B.10. The expansion starts with $\tau \gg 1$ and the optical depth start to decrease. Then, after collision with the baryonic remnant, containing also associated electrons, the opacity may increase, but only for large baryonic loading $B > 10^{-4}$. The change of exponential decrease into a power law seen in Fig. B.9 corresponds to the departure of distributions of electrons and positrons from thermal ones. Finally, as one can see from Fig. B.10 at the late stage of expansion of the fireshell the opacity is dominated by pairs for $B < 10^{-7}$ and by Compton scattering for $B > 10^{-7}$.  

C. Cosmological structure formation

C.1. The Cosmological Principle

Modern Cosmology is based upon a fundamental principle, the so called cosmological principle, that can be stated in the following way:

*All positions in the Universe are equivalent.*

As long as we look at our ‘neighbour’ Universe, this statement is certainly false, because the distribution of matter is far from homogeneous: there are planets, stars, and, going to larger scales, galaxies and clusters of galaxies, separated by almost empty regions. However, when we average this distribution over a volume large enough to contain thousands of clusters, it appears to be very close to homogeneous (see fig C.1).

Homogeneous and isotropic solution of Einstein equations of general relativity was first obtained by Friedmann in 1922. A remarkable property of this solution is that it describes a non-static Universe. At that time, there were no observational evidences for the temporal evolution of the whole Universe; then, many decades passed before the Big Bang model, which is based on Friedmann solution, became the standard paradigm in cosmology, following the discovery of cosmic microwave background radiation by Penzias & Wilson in 1969.

In effect, one of the strongest predictions of Big Bang model is the presence of a background microwave radiation, relic of the early Universe. This radiation is highly isotropic, reflecting, through the coupling with matter, the high isotropy and homogeneity of the primeval plasma. This tells us that the cosmological principle, and then Friedmann picture, safely applies to the early Universe; but what about the present one?

Hubble was the first trying to study the spatial distribution of objects as large as the galaxies, at that time thought to be the largest self-gravitating systems to exist. His results, namely Hubble law, imply, that the galaxy distribution is close to homogenous on the large-scale average. Homogeneity on very large scales is confirmed by present day observations of, in particular:

- X-ray background
C. Cosmological structure formation

Figure C.1: The distribution of galaxies in the 2dFGRS, from (Peacock, 2002).

- radio sources
- gamma ray bursts
- galaxies and clusters of galaxies

On the other hand, on smaller scales, distribution of matter is far from homogeneous: galaxies tend to cluster, forming structures separated by large voids. These clusters of galaxies are themselves members of even larger structures, so called superclusters of galaxies. To study such a complicated distribution of matter, it is necessary to use a statistical approach. In the next section we will introduce the mathematical tools usually used to study large scale structure (LSS).

C.2. Two-point Correlation Function

The statistical description of clustering is based upon the concept of correlation, namely, in a more rigorous way, the probability of finding an object in the vicinity of another one. The standard way to quantify this probability is to define the two-point correlation function $\xi(\vec{x})$ (Peebles, 1993).

Let’s consider a distribution of objects in space, described by the number density function $n(\vec{x})$. The probability that an object is found in an infinitesimal volume $\delta V$ centered around the point $\vec{x}$ is proportional to the volume
C.2. Two-point Correlation Function

itself:

\[ \delta P \propto \delta V. \]  
(C.2.1)

In the absence of structure, the joint probability of finding two objects in two different infinitesimal volumes \( \delta V_1 \) and \( \delta V_2 \), centered respectively around \( \vec{x}_1 \) and \( \vec{x}_2 \) is given by the product of the two probabilities:

\[ \delta P = \delta P_1 \delta P_2 \propto \delta V_1 \delta V_2 \]  
(C.2.2)

On the other hand, if objects have a tendency to cluster, we will find an excess probability:

\[ \delta P \propto \delta V_1 \delta V_2 \cdot (1 + \xi(\vec{x}_1, \vec{x}_2)) \]  
(C.2.3)

According to the cosmological principle, we don’t expect the correlation function to depend on the position neither on the direction, but only on separation between volumes: \( \xi(\vec{x}_1, \vec{x}_2) = \xi(r_{12}) \), where \( r_{12} \equiv |\vec{x}_1 - \vec{x}_2| \).

An equivalent definition of the two-point correlation function is the following:

\[ \xi(r_{12}) = < \delta(\vec{x}_1) \delta(\vec{x}_2) >, \]  
(C.2.4)

where \(< ... > \) denotes averaging over all pairs of points in space separated by a distance \( r_{12} \), and \( \delta(\vec{x}) \equiv (n(\vec{x}) - \bar{n})/\bar{n} \).

C.2.1. Observed Galaxy Distribution

Observational data coming from galactic surveys are usually expressed in the form of correlation function in redshift space, \( \xi(\pi, \sigma) \), where \( \pi \) is a separation along the line of sight and \( \sigma \) is a angular separation on the plane of the sky between two galaxies. It is then possible to obtain the real-space correlation function \( \xi(r) \); this step is never a trivial one, but we are not going into details since it is beyond the purpose of this review.

Peebles (1993) have shown that distribution of galaxies can be described by a two point correlation function with a simple power law form:

\[ \xi_g(r) = \left( \frac{r}{r_g} \right)^{-1.77}, \quad r < 10h^{-1}\text{Mpc}, \]  
(C.2.5)

where \( h \) is a Hubble parameter today measured in \( 100 \left( \frac{\text{km}}{\text{s Mpc}} \right) \). The correlation length \( r_g \) determines the typical distance between objects. For galaxies, it was estimated to be \( \approx 5h^{-1}\text{Mpc} \).

Bahcall and Soneira (1983) and Klypin and Kopylov (1983) found the same power law for clusters of galaxies

\[ \xi_c(r) = \left( \frac{r}{r_c} \right)^{-1.8}, \quad 5h^{-1} < r < 150h^{-1}\text{Mpc}. \]  
(C.2.6)
with different correlation length, namely \( r_c \approx 25h^{-1}\text{Mpc} \). Finally, for super-clusters of galaxies Bahcall and Burgett (1986) have found correlation function with the same power law.

Recent observations support these conclusions. Results from the Sloan Digital Sky Survey (SDSS) on galaxy clustering (Zehavi et al., 2005) for about 200,000 galaxies give a real-space correlation function as

\[
\bar{\xi}_g(r) = \left( \frac{r}{r_0} \right)^{-1.8}, \quad 0.1h^{-1} < r < 10h^{-1}\text{Mpc},
\]

where \( r_0 \approx 5.0h^{-1}\text{Mpc} \), although the brightest subsample of galaxies has a significantly steeper \( \bar{\xi}(r) \). The geometry of samples in SDSS is quite close to Las Campanas Redshift Survey (Shectman et al., 1996) and the results are very similar, but with much better resolution.

The 2dF Galaxy Redshift Survey (Peacock et al., 2001) (see fig. C.1) consists of approximately 250,000 galaxies redshifts. Hawkins et al. (2003) have found

\[
\bar{\xi}_g = \left( \frac{r}{r_0} \right)^{-1.67}, \quad 0.1h^{-1} < r < 12h^{-1}\text{Mpc},
\]

with \( r_0 = 5.05h^{-1}\text{Mpc} \)

Their measurements are in agreement with previous surveys. However, having much smaller statistical errors they were able to find a slight difference of the power law exponent as well as the correlation length on distances or redshifts, colors and types of galaxies. For a summary of measurements of \( \bar{\xi}(r) \) by different surveys other than the ones cited here, see Table 2 of Hawkins et al. (2003).

### C.2.2. Power Law Clustering and Fractals

It is clear that, once a correlation function is given, the density of objects around any randomly chosen member of the system is:

\[
n(r) \propto 1 + \bar{\xi}(r)
\]

If the correlation function has a power law behaviour with exponent \( \gamma \):

\[
\bar{\xi}(r) \propto r^{-\gamma}
\]

as for galaxies and clusters of galaxies, where \( \gamma \approx 1.8 \), then the number of objects in a given volume scales in a similar way:

\[
N(r) \propto r^{3-\gamma}
\]
So, for non integer $\gamma$, the number of objects scales with a fractional power of the radius of the volume under consideration. This behaviour is typical of fractal sets.

A fractal is a set in which ‘mass’ and ‘radius’ are linked by a fractional power law (Mandelbrot, 1983):

$$M(r) \propto r^{D_F}$$

(C.2.12)

where $D_F$ is the fractional or Hausdorff dimension of the set. So galaxies seem to show, at least up to scales of about 100 Mpc, a fractal distribution with $D_F \simeq 1.2$.

A crucial characteristic of a fractal distribution is the presence of fluctuations at all length scales and, consequently, impossibility of defining an average value for the density. It can be stressed that a fractal structure in a cosmological model, although not spatially homogeneous, is not in conflict with weaker form of the cosmological principle (Mandelbrot, 1983): in a homogeneous fractal set each observer at a matter point belonging to the set observes the same matter distribution as any other observer belonging to the set.

The question about fractality in galaxy distribution is still under debate (Kolb and Turner, 1990; Coleman and Pietronero, 1992; Luo and Schramm, 1992; de Gouveia dal Pino et al., 1995; Durrer and Labini, 1998; Gaite et al., 1999; de Bernardis et al., 2002; Joyce et al., 2005). There are two main problems that are to be faced with:

1. Most of the matter in the Universe is in the form of dark matter, while observations are about luminous matter. It is still unclear how (and even if) light traces mass: this is in particular related to the problem of matching the clustering of galaxies, that tells us about distribution of luminous (baryonic) matter, with the CMB anisotropies, that tell us about distribution of gravitating matter.

2. On the other hand, still little is known about fluctuations on intermediate scales between those of local galaxy surveys ($\sim 100h^{-1}\text{Mpc}$) and those probed by the observation of CMB anisotropies ($\sim 1000h^{-1}\text{Mpc}$). However, this gap is recently greatly reduced (de Bernardis et al., 2002; Peacock et al., 2001).

Assuming that the fractal framework is, at least up to some large scale, a good description of the real matter distribution, a consistent model of structure formation has been proposed by Ruffini in the eighties, see (Ruffini et al., 1988) and references therein. In this model fractality arises from successive fragmentations of primordial structures, so called ‘elementary cells’, formed via gravitational instability in the neutrino component of the matter in the Universe. In the following chapter we shall analyse in detail this model. First we are going to discuss the general idea of gravitational instability.
C. Cosmological structure formation

C.3. Gravitational instability

The gravitational instability is usually considered as the basic mechanism of structure formation in the Universe, see for example (Kolb and Turner, 1990). It is believed, that small inhomogeneities are already present at some initial time in the early Universe. Such small perturbations will grow due to gravitational attraction, because overdense regions will accrete matter from the neighbouring regions, raising a density contrast.

One of the simplest examples, showing the process of gravitational instability is a perfect fluid model. If density distribution in selfgravitating fluid is slightly nonuniform, i.e. small density perturbations exist, they will tend to grow. When the density contrast is small, linear approximation can be used. The main advantage of linear theory is that perturbations on different scales evolve independently.

It is the main result of this theory, that the growth of perturbations are damped by the Hubble expansion. It leads to a power law for the time dependence of density perturbations. For instance, in the Einstein-de Sitter model, that is thought to describe our Universe after recombination perturbations amplitude grow as \((1 + z)^{-1}\). Only during the nonlinear stage with large density contrast, the evolution becomes faster. At nonlinear stage, however, perturbations grow much faster, leading to formation of gravitationally bound objects.

The theory of linear density perturbations in a homogeneous medium was first developed by Jeans (1902, 1929). His study was motivated by intention to explain the mechanism of star formation. We describe this theory below. First, however, the validity range i.e. the evolution of cosmological horizon is discussed.

The linear perturbations in the expanding homogeneous and isotropic Friedman Universe were studied by Lishitz (1946) using completely relativistic treatment. Relativistic theory, however, is necessary only when the scale of perturbations lays outside the horizon, or when perturbations in ultrarelativistic matter are studied. In the most interesting cases, such as perturbations in dark matter well inside the horizon after equivalence epoch (when energy densities of radiation and other components are equal) it is sufficient to consider nonrelativistic theory, based on Newtonian gravity. Bonnor (1957), see also (Heath, 1991) was the first, who studied evolution of spherically symmetric perturbations in Newtonian cosmology.

The theory of linear density perturbations in Newtonian treatment is developed in details in some textbooks, see e.g. (Weinberg, 1972; Peacock, 1999; Padmanabhan, 1993; Zeldovich and Novikov, 1975).
C.3. Gravitational instability

C.3.1. Horizon scale and mass evolution

The Newtonian treatment is only applicable on scales smaller than horizon scale $\Lambda_H = cH^{-1}$. Associated mass scale, defined as the mass contained within the sphere of a radius $\Lambda_H/2$, where $H$ is the Hubble parameter, is given by

$$M_H = \frac{4}{3} \pi \rho \left(\frac{\Lambda_H}{2}\right)^3.$$  \hfill (C.3.1)

Beyond this scale events are causally disconnected and thus any correlation breaks outside horizon. Thus structures cannot form on scales larger than $\Lambda_H$. It monotonically increases with time, because the distance that light travels increases with time. There are different regimes, separated by the moment of equivalence in energy densities of radiation and nonrelativistic matter:

$$M_H \propto \begin{cases} a^3 & z > z_{eq} \\ a^{3/2} & z < z_{eq} \end{cases}.$$  \hfill (C.3.2)

Today the horizon scale is approximately 3000Mpc, that corresponds to a mass scale $M \sim 10^{22}M_\odot$ for $\Omega = 1$ Universe. At recombination the total mass inside the horizon, thus, was approximately $(1/z_{rec})^{-3/2} \sim 10^{17}M_\odot$, where $M_\odot = 2 \times 10^{30}$ kg is the mass of the sun.

C.3.2. Selfgravitating ideal fluid: linear theory

Fluid equations and background solutions

We consider a perfect fluid with density $\rho$ and pressure $p$, in the Newtonian space with Cartesian ("physical") coordinate system $r_i$. We assume that the fluid has a velocity field $v_i$. A gravitational potential $\Phi$ is induced in the fluid by its own mass density $\rho$ distribution. All these quantities are related through the continuity, Euler and Poisson equations respectively:

$$\frac{\partial \rho}{\partial t} + \partial_i (\rho v_i) = 0,$$  \hfill (C.3.3)

$$\frac{\partial v_i}{\partial t} + v_j \partial_j v_i + \frac{1}{\rho} \partial_i p + \partial_i \Phi = 0,$$  \hfill (C.3.4)

$$\partial^2 \Phi - 4\pi G \rho = 0,$$  \hfill (C.3.5)

where $\partial^2 = \partial_i \partial_i$. The cosmologically relevant solution of the above equations (C.3.3-C.3.5) is

$$v_i^0 = H(t) r_i.$$  \hfill (C.3.6)

\footnote{Greek indices denote comoving coordinates, latin indices denote physical coordinates, both take values 1,2,3; Einstein summation rule is adopted.}
C. Cosmological structure formation

\[ \frac{d\rho_0}{dt} + 3H\rho_0 = 0, \]  
(C.3.7)

\[ p_0(t) = 0, \]  
(C.3.8)

\[ \Phi_0 = \frac{2}{3}\pi G\rho_0 r^2, \]  
(C.3.9)

\[ \frac{dH}{dt} + H^2 = -\frac{4}{3}\pi G\rho_0, \]  
(C.3.10)

where \( r^2 = r_ir_i \) and density depends on time only: \( \rho_0 = \rho_0(t) \).

It is interesting to note, that both the Hubble law (C.3.6) and the continuity equation in expanding space (C.3.7) could be obtained by transition to a new coordinate system, namely the so called comoving system \( x_\alpha \) defined by

\[ r_\alpha = a(t)x_\alpha, \]  
(C.3.11)

where \( a(t) \) is a scale factor. This transformation implies the relation between coordinate differences \( \Delta r_i \) and \( \Delta x_\alpha \) in these two systems

\[ \frac{d\Delta r_\alpha}{dt} = a\frac{d\Delta x_\alpha}{dt} + \frac{da}{dt}\Delta x_\alpha = a\frac{d\Delta x_\alpha}{dt} + H(t)\Delta r_\alpha, \]  
(C.3.12)

where

\[ H(t) = \frac{1}{a}\frac{da}{dt}. \]  
(C.3.13)

The same relation holds for velocity fields

\[ v_\alpha(r_\beta, t) = u_\alpha(x_\beta, t) + Hr_\alpha = u_\alpha(x_\beta, t) + v_\alpha^0. \]  
(C.3.14)

Thus the solution (C.3.6-C.3.10) means uniform distribution of the fluid with zero peculiar velocity \( u_\alpha^0 = 0 \) and zero pressure \( p_0 = 0 \).

Usually, pressure and density are linked through equation of state \( p = p(\rho) \). The five equations (C.3.3-C.3.5) together with the equation of state form a complete set, allowing to study the temporal evolution of the density and velocity distributions as well as pressure and gravitational potential.

**Perturbed quantities**

As well known, solutions (C.3.6-C.3.10) represent isotropic and homogeneous distribution of matter. In order to study density perturbations in linear approximation suppose, that

\[ \rho(r_i, t) = \rho_0(t) [1 + \delta(r_i, t)], \]  
(C.3.15)

\[ v_i(r_j, t) = v_i^0(r_j, t) + \delta v_i(r_j, t), \]  
(C.3.16)

\[ \Phi(r_i, t) = \Phi_0(r_i, t) + \delta \Phi(r_i, t), \]  
(C.3.17)
p(r, t) = \delta p(r, t), \quad (C.3.18)

where \( \delta \equiv \frac{\rho - \rho_0}{\rho_0} \). Here all perturbed quantities \( \delta, \delta v_i, \delta p \) and \( \delta \Phi \) are assumed to be much smaller than the background quantities. All zero order values are given by \((C.3.6-C.3.10)\). It is also assumed that the spatial and temporal derivatives of perturbed quantities are of the same order of magnitude as the quantities themselves.

Note that the condition of perturbed quantities smallness is not necessary to hold in the whole space. In particular, it could be the region in space where relation \( |\delta v_i| > |v_i^0| \) takes place (Meszaros, 1974). In this case the standard linearization procedure leads to different perturbations equations and, consequently, to different solutions representing density contrast \( \delta(r, t) \) time dependence.

**Linearized perturbations equations**

We rewrite \((C.3.3-C.3.5)\) in comoving coordinates:

\[
\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a} \rho \partial_\alpha u_\alpha + \frac{1}{a} u_\alpha \partial_\alpha \rho = 0, \quad (C.3.19)
\]

\[
\frac{d^2 a}{dt^2} x_\alpha + \frac{\partial u_\alpha}{\partial t} + H u_\alpha + \frac{1}{a} \rho \partial_\beta u_\beta + \frac{1}{a \rho} \partial_\alpha p + \frac{1}{a} \partial_\alpha \Phi = 0, \quad (C.3.20)
\]

\[
\partial^2 \Phi - 4\pi G a^2 \rho = 0. \quad (C.3.21)
\]

Here all quantities, except for \( H^2 \), depend on comoving coordinates \( x_\alpha \) and time \( t \). Equations \((C.3.19-C.3.21)\) can be found for example in (Meszaros, 1993) written in physical coordinates. One arrives at the above from \((C.3.3-C.3.5)\) on using the transformation laws \((\partial/\partial t)_{phys} = (\partial/\partial t)_{com} - H x_\alpha \partial_\alpha \) and \((\partial_\alpha)_{phys} = (1/a)(\partial_\alpha)_{com} \).

With the goal to obtain equations for density contrast \( \delta \) in linear approximation we substitute \((C.3.15-C.3.18)\) into equations \((C.3.19-C.3.21)\). Taking into account that the spatial as well as temporal derivatives of perturbed quantities have the same order of magnitude as the perturbed quantities themselves, and using \((C.3.6-C.3.10)\), the perturbations equations read

\[
\frac{\partial \delta}{\partial t} + \frac{1}{a} \partial_\alpha \delta u_\alpha = 0, \quad (C.3.22)
\]

\[
\frac{\partial \delta u_\alpha}{\partial t} + H \delta u_\alpha + \frac{1}{a} \partial_\alpha \delta p + \frac{1}{a} \partial_\alpha \delta \Phi = 0, \quad (C.3.23)
\]

\(^2\)If one suppose in addition that Hubble parameter also can be disturbed (can have spatial dependence) then the system of equations becomes overdefined. There is another approach (Ellis and Bruni, 1989; Ellis, 1990), however, where \( \partial_\alpha \delta \) and \( \partial_\alpha H \) are taken as independent variables in order to study density perturbations.
\[ \partial^2 \delta \Phi - 4\pi G a^2 \rho_0 \delta = 0, \] (C.3.24)

where \( \delta u_\alpha(x_\beta, t) \) is a first order quantity, because its unperturbed value is \( u_\alpha^0(x_\beta, t) = 0 \).

The simplest way to find the equation governing density perturbations is to take the time derivative of equation (C.3.22) and use the divergence of equation (C.3.23) together with equation (C.3.24). After some calculations one finds the final expression:

\[ \frac{\partial^2 \delta}{\partial t^2} + 2H \frac{\partial \delta}{\partial t} - \frac{v_s^2}{a^2} \partial^2 \delta - 4\pi G \rho_0 \delta = 0, \] (C.3.25)

where \( v_s \) denotes the sound speed in the fluid:

\[ v_s^2 = \frac{dp}{d\rho}. \] (C.3.26)

**The Jeans criterion**

Equation (C.3.25) governs dynamics of density perturbations. It is a wave-like second order partial differential equation. Thus, it is natural to perform Fourier transformation

\[ \delta = \sum_k h(t)e^{ik_\alpha x_\alpha} \] (C.3.27)

in order to split perturbations on different scales.

The equation (C.3.25) can be rewritten in \( k \)-space, taking into account that \( \partial_\alpha \delta \to ik_\alpha h \):

\[ \frac{d^2 h}{dt^2} = -2H \frac{dh}{dt} + (4\pi G \rho_0 - \frac{v_s^2 k^2}{a^2})h, \] (C.3.28)

where \( k_\alpha \) is a comoving wavevector and \( k = \sqrt{k_\alpha k_\alpha} \) is a corresponding wavenumber. The comoving wavelength of the perturbative mode is given by \( l = 2\pi/k \), while the proper (physical) wavelength is simply \( \lambda = al \).

Jeans criterion takes place for (C.3.28)

\[ \lambda_J = v_s \sqrt{\frac{\pi}{G \rho_0}}, \] (C.3.29)

where \( \lambda_J \) separates gravitationally stable scales from unstable ones. Fluctuations on scales well beyond \( \lambda_J \) grow via gravitational instability, while on scales smaller than \( \lambda_J \) pressure overwhelms gravity and perturbations do not grow.

The first term on the right-hand side of (C.3.28) comes from the general expansion. In the static world, initially considered by Jeans, such term is absent, leading to exponential growth of perturbations. In expanding space perturbations grow with time according to a power law.
A very important quantity is usually associated with the Jeans length (C.3.29), namely the Jeans mass

\[ M_j = \frac{4}{3} \pi \rho \left( \frac{\lambda_j}{2} \right)^3, \]  

(C.3.30)
defined as the mass contained within a sphere of radius \( \lambda_j/2 \), where \( \rho \) is density of the perturbed component.

**Multi-component system**

Perturbations for a given mode in a single component evolve according to (C.3.28). When several components such as Cold Dark Matter (CDM), Hot Dark Matter (HDM), baryons and radiation are present simultaneously, it is possible to generalize (C.3.25). Assuming gravitational interaction between components only, we arrive at

\[ \frac{d^2 h_i}{dt^2} = -2H \frac{dh_i}{dt} + \left( 4\pi G \rho_0 \sum_j \epsilon_i h_j - \left( \frac{v_j^2}{a^2} k^2 a^2 h_i \right) \right), \]  

(C.3.31)

where index \( i \) refers to the component under consideration, the sum is over all components and \( \epsilon_i = \rho_i / \sum_j \rho_j \). Notice, that any smoothly distributed component (like cosmological constant) does not contribute to the right-hand side of (C.3.31).

**C.3.3. Applications**

Some important cases of matter content of the Universe will be considered below. First we discuss perturbations dynamics in the dominant nonrelativistic component (baryonic or not). Second example is a dark matter perturbations in the presence of dominant radiation component.

**Einstein-de Sitter Universe**

First of all, consider the dust dominated \( \Omega = 1 \) Universe. This condition (\( \Omega \) is a density parameter) means flat-type cosmological model. In other words, curvature parameter \( k = 0 \) in Friedman solution of Einstein General Relativity equations. This model is thought to provide a good description of our Universe after recombination. To zero order \( a \sim t^{2/3} \), \( H = 2/3t \) and \( \rho_0 = 1/6\pi G l^2 \). For perturbations well inside horizon we have

\[ t^2 \frac{d^2 h}{dt^2} + 4t \frac{dh}{dt} - \frac{2}{3} \left[ 1 - \frac{\left( \frac{\lambda_j l}{\lambda} \right)^2}{\lambda} \right] h = 0. \]  

(C.3.32)
For modes well inside the horizon and still larger than the Jeans length the solution is

$$h(k, t) = h_1(k) \left( \frac{t}{t_0} \right)^{2/3} + h_2(k) \left( \frac{t}{t_0} \right)^{-1}.$$  \hspace{1cm} (C.3.33)

As expected there are two solutions, one growing and one decaying. At late time, however, only growing mode is important. Perturbations evolve proportionally to the scale factor or as $(1 + z)^{-1}$, where $z$ is the redshift defined by:

$$1 + z = \frac{a_0}{a(t)},$$  \hspace{1cm} (C.3.34)

and $a_0$ denotes the value of a scale factor today.

Perturbations on scales smaller than the Jeans length cease to grow and oscillate with time.

### Mixture of radiation and dark matter

Consider the radiation dominated Universe when $a \sim t^{1/2}$ and $H = 1/2t$. The second component to be considered is a collisionless dark matter with $v_{DM} = 0$. We still can use Newtonian treatment on scales much smaller than the horizon size.

Since the small scale photon distribution is smooth and the energy density is dominated by the radiation, the equation governing dark matter instability reduces to

$$t \frac{d^2 h_{DM}}{dt^2} + \frac{dh_{DM}}{dt} = 0.$$  \hspace{1cm} (C.3.35)

It possesses the solution

$$h_{DM}(k, t) = h_1(k) \log \left( \frac{t}{t_0} \right) + h_2(k).$$  \hspace{1cm} (C.3.36)

Perturbations in dark matter component inside the horizon experience a slow logarithmic growth. This is known as Meszaros effect (Meszaros, 1974).

### C.3.4. Initial spectrum of perturbations

In the first example we have shown, that perturbations on scales between horizon size and Jeans length $\lambda_J \ll \lambda \ll \lambda_H$ grow as $\delta \propto (1 + z)^{-1}$. In order to study perturbations dynamics one have to know, in addition, initial values of perturbations at some moment in early Universe.

One great possibility provide cosmic microwave background radiation anisotropy measurements, because density inhomogeneities when photons were coupled to baryons can be extracted from temperature fluctuations observed in
cosmic microwave background radiation. Since $\frac{\delta T}{T} \simeq 10^{-5}$ it follows that at the moment of recombination $\delta \simeq 10^{-4}$ (Kolb and Turner, 1990).

Perturbations amplitudes on different scales are usually represented by a power spectrum $P(k)$, that is a Fourier transform of a previously introduced correlation function (Peacock, 1999)

$$\xi(r) = \frac{V}{(2\pi)^3} \int P(k) \frac{\sin kr}{kr} 4\pi k^2 dk.$$  \hspace{1cm} (C.3.37)

There is no evidence, that initial spectrum contained some preferred scales, thus it should be a featureless power law

$$P(k) \propto k^n,$$  \hspace{1cm} (C.3.38)

where index $n$ governs the balance between perturbation amplitudes on large and small scales. The value $n = 0$ corresponds to a white noise, that have the same amplitude forevery mass scale. The value $n = 1$ corresponds to a so-called Harrison-Zel’dovich scale invariant spectrum. Term ‘scale invariance’ means that perturbations had the same amplitude at the moment of horizon crossing.

**C.3.5. Damping of Perturbations**

In addition to the Jeans scale some other cosmologically important scales appear in the theory of structure formation. They are related to physical processes, that cannot be described within perfect fluid approximation. However, fortunately, such processes take place on limited scale intervals and outside such intervals fluid description is still possible. We are going to discuss some dissipative effects, such as collisional damping of baryonic perturbations and free streaming of collisionless light particles.

**Silk damping**

Close to recombination the coupling between photons and baryons makes possible for the former to erase perturbations on the latter. This is because at that time the free mean path of photons becomes larger, so they can travel from overdense into underdense regions dragging baryons with them, thus smoothing inhomogeneities in primeval plasma. This effect was discovered by Silk (Silk, 1967). The physical scale, associated with it is (Kolb and Turner, 1990)

$$l_S \simeq 3.5(\Omega h^2)^{-3/4} \text{Mpc},$$  \hspace{1cm} (C.3.39)

that gives a mass scale

$$M_S \simeq 6.2 \times 10^{12}(\Omega h^2)^{-5/4} \text{M}_\odot.$$  \hspace{1cm} (C.3.40)
C. Cosmological structure formation

This scale is close the mass of a typical galaxy $10^{11} M_\odot$. However, Silk damping affects baryonic perturbations only. Moreover, it is important only around recombination when the coupling is still sufficiently strong to make photons drag baryons with them.

Free streaming

Another damping process is Landau damping or free streaming, that originates from free motion of collisionless particles on small scales. They can travel far if velocity dispersion is large. This is important after particles decouple from plasma and until they become nonrelativistic.

Free streaming discovery (Bond and Szalay, 1983) became a dramatic moment for structure formation scenarios based upon Hot Dark Matter (HDM) models. Name Hot Dark Matter means that particles, that constitutes such matter were ultrarelativistic at equivalence. Thus their velocity dispersion was near the light speed $c$. The maximum scale travelled by collisionless particles from decoupling can be estimated as (Padmanabhan, 1993)

$$l_{FS} \simeq 0.5 \left( \frac{m_{DM}}{1 \text{ keV}} \right)^{-4/3} \left( \Omega_{DM}^2 h^2 \right)^{1/3} \text{Mpc},$$

(C.3.41)

and a corresponding mass scale has an order of supercluster of galaxies or even larger if the particle mass is $m_{DM} < 30 \text{ eV}$.

Cold thermal relics, that constitutes Cold Dark Matter (CDM) content, are slow enough to allow to neglect the free streaming on cosmologically important scales. Therefore, Landau damping affects only light particles such as neutrinos with $m_\nu \sim 10 \text{ eV}$, and in the Universe dominated by HDM erase all perturbations on scales smaller than superclusters of galaxies. At the same time, after particles become nonrelativistic at $z_{nr}$, their velocity dispersion becomes small enough, making free streaming negligible.

C.3.6. Structure formation on late times

Nonlinear clustering

In previous sections we dealt with linear evolution of cosmological perturbations. During a long period of time from recombination and even earlier, until almost recent times such treatment allows to describe the growth of inhomogeneity in the Universe, because condition $\delta \ll 1$ were satisfied. On recent times, say, at $z \sim 10$, nonlinear behaviour of perturbations becomes important. During nonlinear stage gravitationally bound objects, such as galaxies, form. Nonlinear evolution is a rapid process, where not only gravitational effects are important, in particular during formation of galaxy different dissipation and relaxation processes take place (Peacock, 1999).
C.3. Gravitational instability

We are not going into details of galaxy formation here. The LSS formation is the subject of this section. Here the main interaction remains gravity. However, theoretical description, based on linear equations for ideal fluid becomes inadequate.

Usually N-body simulations are implemented in order to study nonlinear clustering (Peacock, 1999). However, some useful simplified models are still possible even on nonlinear stage, because numerical simulations provide limited physical insight into the physics of gravitational clustering. Among nonlinear approximations, the most famous are Zel’dovich approximation (Zeldovich, 1970) and spherical collapse model (see for example (Peebles, 1993)).

The key point in Zel’dovich model is that during collapse in almost spherically symmetric overdense region gravitational interaction amplifies asymmetry. Therefore, final structure acquires a preferred direction and finally collapsed body will look like a ‘pancake’.

In the spherical collapse model, on the contrary, it is assumed, that spherical symmetry preserves during the whole period of collapse. It allows to split the spherical overdensity region into concentric shells and study evolution of each shell separately, that sufficiently simplifies a problem. This model will be described in detail in the next chapter.

Structure formation scenarios

Historically two different pictures of structure formation were considered, namely HDM, see e.g. (Primack and Murdin, 2002) and CDM models, see e.g. (Primack, 2003). We will discuss them briefly below.

HDM models

The neutrino dominated Universe with $m_{\nu} \sim 10$ eV is a typical HDM model. The HDM model is associated with so-called “top-down” scenario, where structures form on large scales first. This is so, because the Jeans mass for HDM is of the order of supercluster mass or even higher. At the same time, free streaming erase perturbations on smaller scales. Thus, only when perturbations reach nonlinear regime on large scales they can induce fragmentation on smaller scales.

Usually it is assumed, that large scale perturbations become non-spherical according to the Zel’dovich model and thus LSS look like a “net” of density condensations separated by huge voids. Simulations agree with such picture.

The HDM model is in a good agreement with observational data on scales larger than 10 Mpc. On smaller scales, however, HDM simulations can agree with observed correlation function of galaxies only if the epoch of pancaking takes place at $z \simeq 1$ or less, which is too late, because we can see galaxies and quasars with much greater $z$. 

525
The crucial cosmological property of HDM with neutrinos, as was mentioned above, is the damping of perturbations on small scales due to free streaming. Neutrino dominated models ($\Omega_\nu \sim 1$) alone could not describe the real Universe because on scales smaller than $\sim 100$ Mpc no structure appears at all.

One important prediction of HDM models with neutrinos is existence of large smooth halos around galaxies. At the end of collapse, during formation of the galaxy, baryonic component can dissipate its energy via collisions, but the neutrino component cannot. Thus, neutrinos remain less condensed than baryons, forming a large galactic halos.

**CDM models**

CDM models do not have troubles with free streaming, because its particles have negligible velocities at decoupling. Moreover, the Jeans mass for typical CDM model lays well below $10^6 M_\odot$. Thus, perturbations start to develop on small scales simultaneously with perturbations on large scales.

In the CDM models the important feature is the weak growth experienced by perturbations between horizon crossing and equivalence (see §C.3.3). It means, that the density contrast increases when we move to smaller scales, or that the perturbations spectrum has more small-scale power.

After collapse first structures in CDM models virialize through violent relaxation (Lynden-Bell, 1967; Shu, 1978) into gravitationally bound objects that form galactic halos. Structures form in self-similar manner from small to large scales, in other words according to a ‘bottom-up’ scenario.

A pure CDM models, however, fail to predict the observed correlation function for galaxies on large scales. If one wants to retain the CDM hypothesis, the simplest way is to reduce the matter density. This shifts matter-radiation equivalence to a later epoch, resulting in redistribution of power in the spectrum of perturbations in favour of larger scales.

Today $\Lambda$CDM model with $\Omega_{\text{tot}} = 1$ and $\Omega_\Lambda = 0.7$ is considered as the best fit to the full set of observational data.
D. Massive degenerate neutrinos in Cosmology

In Appendix C we have described the evolution of perturbations, and we saw that the nature of dark matter particles is crucial in determining the way structure formation goes. In spite of the fact that a lot of candidates for CDM particles are being considered (see Ref. (Bertone et al., 2005) for a review, there are no experimental detections of such particles at present. From the other hand, neutrinos are the only candidates for DM known to exist.

‘Light’ neutrinos \( m_\nu \ll 1 \text{ MeV} \) (Dolgov, 2002), namely neutrinos that decouple while still in their ultrarelativistic regime (see below), may provide a significant contribution to the energy density of the Universe \( \Omega_\nu \sim 1 \).

Models with light neutrinos were extensively studied in the eighties; a large literature exists on this subject (Bisnovatyi-Kogan and Novikov, 1980; Zeldovich and Syunyaev, 1980; Doroshkevich and Khlopov, 1981), see also (Peebles, 1982).

The key prediction of the cosmological model with neutrinos is a cellular structure on large scales (see Fig.3.1). The qualitative drawing of cellular structure of the Universe is represented at Fig.C.1.

Ruffini and collaborators have studied such models with particular attention to the problem of clustering on large scales and its relation to the fractal distribution of matter. In the following, we are going to describe their works in detail.

D.1. Neutrino decoupling

The cosmological evolution of a gas of particles can be split in two very different regimes. At early times, the particles are in thermal equilibrium with the cosmological plasma; this corresponds to the situation in which the rate \( \Gamma = \langle \sigma v \rangle \) of the reactions supposed to maintain the equilibrium (such as \( \nu_e + \bar{\nu}_e \leftrightarrow e^+ + e^- \leftrightarrow 2\gamma \) in the case of electronic neutrinos) is much greater than the expansion rate, given by the Hubble parameter. The gas evolves then through a sequence of thermodynamic equilibrium states, described by the usual Fermi-Dirac statistics:

\[
f(p) = \frac{1}{\exp ((E(p) - \mu)/k_BT) + 1}, \tag{D.1.1}
\]
where \( p, \mu \) and \( T \) are the momentum, chemical potential and temperature of neutrinos respectively, and \( k_B \) is a Boltzmann constant.

However, as the Universe expands and cools, the collision rate \( \Gamma \) becomes lower than the expansion rate; this means that the mean free path is greater than the Hubble radius, thus we can consider the gas as expanding without collisions. It is customary to describe the transition between the two regimes by saying that the gas has decoupled from the cosmological plasma.

### D.2. The redshifted statistics

Since in a spatially homogeneous and isotropic Universe, described by the Robertson-Walker metric, the product of the three-momentum \( p(t) \) of a free particle times the scale factor \( a(t) \) is a constant of the motion:

\[
p(t) \cdot a(t) = \text{const}, \quad (D.2.1)
\]

each particle in the gas changes its momentum according to this relation. This fact, together with Liouville’s theorem, implies that the distribution function after the decoupling time \( t_d \) (defined as the time at which \( \Gamma = H \)) is given by (Ruffini et al., 1983):

\[
f(p, t > t_d) = f\left(\frac{a(t)}{a_d} p, t_d\right) = \frac{1}{\exp\left[\frac{(E\left(\frac{a(t)}{a_d} p\right) - \mu_d)}{k_B T_d}\right] + 1} \quad (D.2.2)
\]

where the subscript \( d \) denotes quantities evaluated at the decoupling time.

Now let’s turn our attention to the special case of neutrinos with \( m_\nu \lesssim 10 \text{ eV} \). The ratio \( \Gamma / H \), as a function of the cosmological temperature, can be evaluated using quantum field theory (Kolb and Turner, 1990)

\[
\frac{\Gamma}{H} \simeq \left(\frac{T}{1 \text{ MeV}}\right)^3 \quad (D.2.3)
\]

as long as \( T \gg m \).

Therefore, neutrinos decouple from the cosmological plasma, when \( T = T_d \simeq 1 \text{ MeV} \). Since \( kT_d \gg mc^2 \), many of the particles obey \( pc \gg mc^2 \) and then, when performing integration over the distribution function (D.2.2), we can safely approximate:

\[
f(p, t > t_d) \approx f\left(\frac{a(t)}{a_d} p, t_d\right) \simeq \frac{1}{\exp\left[\frac{(a(t)}{a_d} pc - \mu_d}{k_B T_d}\right] + 1}, \quad (D.2.4)
\]

since the tail of the distribution function for which \( mc^2 \gg pc \) gives little contribution.
In the following, we will need to compute the mean value of physical quantities over this distribution. It will be useful to consider two limiting regimes, namely the nonrelativistic one and the ultrarelativistic one. They correspond to two approximations for the single particle energy (Ruffini et al., 1983):

\[ E \simeq mc^2 \quad kT \ll mc^2 \quad \text{NR} \]
\[ E \simeq pc \quad kT \gg mc^2 \quad \text{UR} \]

We stress the fact that this substitution has to be performed only in the function to be integrated, and not on the distribution function. The approximation (D.2.4) depends only on the fact that the particles are ultrarelativistic at the time of decoupling, and then it is valid even when \( kT \ll mc^2 \).

Then, with a suitable substitution of variables, all the relevant integrals can be recast in a very simple, dimensionless form:

\[
I_n(\xi) \equiv \int_0^\infty \frac{y^n dy}{\exp[(y-\xi)] + 1}, \quad \text{(D.2.5)}
\]

where \( \xi \equiv \mu_d/kT_d \) is the dimensionless chemical potential, or degeneracy parameter. These integrals can be expressed using Riemann zeta and related functions.

### D.3. Energy density of neutrinos

The present density parameter of neutrinos can be easily evaluated using the method outlined in the previous section. The energy density is given by:

\[
\rho_{\nu+\bar{\nu}}(t_0) = \frac{g}{h_P^2} \int_0^\infty E(p)f(p,t_0)\,d^3p \quad \text{(D.3.1)}
\]

where \( g \) is the number of helicity states and \( h_P \) is the Planck constant. By normalization with respect to the critical density \( \rho_c = 1.054 \times 10^4 \, \text{eV cm}^{-3} \), we obtain (Ruffini and Song, 1986; Ruffini et al., 1988):

\[
\Omega_{\nu+\bar{\nu}}h^2 \simeq 1.10 \cdot 10^{-1} g \frac{m}{10\,\text{eV}} A(\xi), \quad \text{(D.3.2)}
\]

where \( A(\xi) \) is defined as follows

\[
A(\xi) \equiv \frac{I_2(\xi) + I_2(-\xi)}{2I_2(0)} = \frac{1}{4\eta(3)} \left[ \frac{1}{3} |\xi|^3 + 4\eta(2)|\xi| + 4 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{e^{-k|\xi|}}{k^3} \right],
\]

and \( \eta(n) \) is the Riemann eta function of index \( n \).

The term \( I_2(-\xi) \) appears because we have to take into consideration the
D. Massive degenerate neutrinos in Cosmology

presence of antiparticles, for which the relation \( \xi_{\bar{\nu}} = -\xi_\nu \) holds. This result follows from the fact that, if we consider a reaction such as

\[
v + \bar{v} \longleftrightarrow ... \longleftrightarrow \gamma + \gamma
\]  

we get that, since the chemical potentials of the initial and final states have to be equal, and the chemical potential of the latter is equal to zero, it follows, that \( \xi_{\bar{\nu}} = -\xi_\nu \).

D.3.1. Neutrino mass

We now know from neutrino oscillation experiments that neutrinos do have mass, see (Maltoni et al., 2004) for a review. It is a remarkable fact that neutrino flavour and mass eigenstates do not coincide, but are instead related by a rotation in flavour space:

\[
|\nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i} |\nu_i\rangle
\]  

where \( \alpha = e, \mu, \tau \) labels flavour eigenstates, while \( i = 1, 2, 3 \) labels mass eigenstates. The “rotation” matrix \( U_{\alpha i} \) is called the neutrino mixing matrix. A great deal of effort is presently being put now in measuring the elements of the mixing matrix and the mass differences, which are the parameters actually probed in oscillation experiments. On the other hand, this kind of experiments do not give any information on the absolute scale of the neutrino mass. In this regard, useful information can be obtained by 1. tritium beta decay experiments, 2. neutrinoless beta decay experiments, 3. cosmological observations.

The tritium \( \beta \) decay experiments are sensitive to the “electron neutrino mass” (this is actually a misnomer since the electron neutrino is not a mass eigenstate and thus does not possess a well definite mass) \( m_e \):

\[
m_e = \left( \sum_{i=1}^{3} |U_{ei}|^2 m_i^2 \right)^{1/2}.
\]  

The present 95% CL bounds are:

\[
m_e < 2.05 \text{ eV} \quad \text{Troitsk experiment (Lobashev, 2003)}
\]
\[
m_e < 2.3 \text{ eV} \quad \text{Mainz experiment (Kraus et al., 2005)}
\]  

The upcoming KATRIN experiment (KATRIN collaboration, 2001) is expected to improve this bounds by nearly an order of magnitude, reaching a discovery potential for 0.3-0.35 eV masses.

At the same time, no direct measurements or constraints on muonic and
tauonic neutrino masses exist, although we know from oscillation experiments that the difference between masses should in the sub-eV range. Moreover, it is still unknown, whether neutrinos are Majorana or Dirac particles.

Experiments (Aalseth et al., 1999; Klapdor-Kleingrothaus et al., 2001), see also (Arnaboldi et al., 2005) on neutrinoless double $\beta$ decay are instead sensitive to the “Majorana mass” $m_{\beta\beta}$:

$$m_{\beta\beta} = \sum_{i=1}^{3} |U_{ei}^2 m_i|$$  \hspace{1cm} (D.3.8)

A recent paper (Strumia and Vissani, 2005) gives the following upper bound at 99% CL:

$$|m_{\beta\beta}| \lesssim 0.6\text{eV}. \hspace{1cm} (D.3.9)$$

Cosmology is mainly sensitive, at least to leading order, to the sum of neutrino masses $M_\nu$:

$$M_\nu = \sum_{i=1}^{3} m_i \hspace{1cm} (D.3.10)$$

We should stress that there is no single limit that can be obtained on $M_\nu$ by means of cosmological observables, since the exact result depends of several factor, like the datasets considered, and the theoretical assumptions that are made (“priors”). However, we can summarize the present status as follows:

$$M_\nu \lesssim 0.2 - 2.3\text{eV} \hspace{1cm} (D.3.11)$$

where of course the largest value should be taken as the most conservative one, i.e., the one that is obtained by using only the more robust pieces of data (basically the CMB spectrum) and without making any assumption other than the standard FRW cosmological model. It is worth noting that these bounds are competitive with the ones coming from particle physics experiments. They are also expected to improve by an order of magnitude with the next generation of cosmological observations. For a review on the current limits on neutrino mass from cosmology, and how these will be improved in the future, we refer the reader to the work of Lesgourgues and Pastor (2006).

### D.3.2. Chemical potential

First constraints on neutrino degeneracy parameter from BBN were obtained in (Doroshkevich et al., 1971; Beaudet and Goret, 1976). It was shown later (Bianconi et al., 1991) that a small value of $\xi_e$ and large values of $|\xi_{\mu,\tau}|$ simultaneously, can lead to BBN abundances which are consistent with observations. It is found in particular that

$$0 \leq \xi_e \lesssim 1.5, \hspace{1cm} (D.3.12)$$
D. Massive degenerate neutrinos in Cosmology

with the additional constraint $F(\xi_\mu) + F(\xi_\tau) \approx F(10\xi_e)$, where $F(\xi) \equiv \xi^2 + \xi^4/2\pi^2$. This, in particular, implies $|\xi_{\mu,\tau}| \lesssim 10\xi_e$.

Recent data both from BBN and CMBR (Orito et al., 2001; Kneller et al., 2001; Hansen et al., 2002; Orito et al., 2002; Hansen et al., 2002) strongly constrain neutrino degeneracy parameters. In the paper (Orito et al., 2001) these constraints are surprisingly wide, $\xi_e < 1.4$ and $|\xi_{\mu,\tau}| < 40$. Other papers give essentially stronger constraints using additional assumptions,

$$\xi_e < 0.3 \quad |\xi_{\mu,\tau}| < 2.6. \quad (D.3.13)$$

Recently, a very robust albeit less stringent limit has been obtained by the analysis of CMB data (Lattanzi et al., 2005):

$$|\xi| \leq 1.1 \quad (D.3.14)$$

where the same limit holds for every flavour.

D.3.3. Neutrino oscillations

When one starts with different chemical potentials for all neutrino flavors at the epoch prior to BBN, neutrino oscillations equalize chemical potentials (Savage et al., 1991), if there is enough time (Abazajian et al., 2002) for relaxation process. On the basis of large mixing angle solution of the solar neutrino problem, which is favored by recent data (Ahmad et al., 2001), the BBN consideration constrains degeneracy parameters of all neutrino flavors (Dolgov et al., 2002):

$$|\xi| \leq 0.07. \quad (D.3.15)$$

However the situation when flavor equilibrium is not achieved before BBN is also possible. Thus in the following we consider quite high values of the degeneracy parameter and assume it is positive without loss of generality.

The main result that comes from oscillations consideration is that masses of different neutrino species are nearly equal: $m_{\nu_e} \simeq m_{\nu_\mu} \simeq m_{\nu_\tau}$.

D.4. The Jeans mass of neutrinos

In neutrino dominated Universe the first possible structure occurs when these particles become nonrelativistic, since at earlier times free streaming erases all perturbations. At this epoch the cosmological redshift has the value (Ruffini et al., 1988)

$$1 + z_{nr} = 1.698 \times 10^4 \left(\frac{m_\nu}{10eV}\right) A(\xi) \frac{1}{2} B(\xi)^{-\frac{1}{2}}, \quad (D.4.1)$$
D.4. The Jeans mass of neutrinos

where

\[ B(\xi) \equiv \frac{I_3(\xi) + I_3(-\xi)}{I_3(0)} = \]
\[ = \frac{1}{48\eta_R(5)} \left[ \frac{1}{5}\xi^5 + 8\eta_R(2)\xi^3 + 48\eta_R(4)\xi + 48 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-n\xi}}{n^5} \right]. \quad (D.4.2) \]

The basic mechanism of fragmentation of the initial inhomogeneities in an expanding Universe is the Jeans instability described in the previous section.

Figure D.1: The Jeans mass dependence on redshift for neutrinos with mass \( m_\nu = 2.5\text{eV} \) and degeneracy parameter \( \xi = 2.5 \).

However, in the calculation of Jeans’ length of nonrelativistic collisionless neutrinos, we cannot use the velocity of sound obtained by the classical formula (C.3.26). In fact, since particles are collisionless, their effective pressure is zero and this would lead to a vanishing Jeans length, meaning that even the smallest perturbation would be unstable. This is not the case, since, in the absence of pressure, another mechanism works against gravitational collapse, namely the free streaming of particles (see §C.3.5). The characteristic velocity associated with this process is simply the dispersion velocity \( \sqrt{\langle v^2 \rangle / 3} \), where the factor 3 comes from averaging over spatial directions. Thus, we have to make the substitution \( v_s^2 \rightarrow \langle v^2 \rangle / 3 \) (Ruffini and Song, 1986). The correct expression for \( \langle v^2 \rangle \) can be obtained using the method described above:

\[ \langle v^2 \rangle = \begin{cases} 
\frac{c^2}{12} \eta_R(5) \left( \frac{kT_\nu}{m_\nu} \right) \theta(\xi) & \text{if } z > z_{nr} \\
\frac{\eta_R(3)}{A(\xi)} & \text{if } z < z_{nr},
\end{cases} \quad (D.4.3) \]
where $T_{v0} = 1.97$ K is the present temperature of neutrinos. As a result, the

\[ T_{v0} = 1.97 \text{ K} \]

is the present temperature of neutrinos. As a result, the

1. Mass growth in UR regime and decreases in NR regime (Bond et al., 1980). The evolution of Jeans mass of neutrinos for $m_{v} = 2.5$ eV and $\xi = 2.5$ with redshift $z$ is represented at fig.D.1.

It is clear, that for such values of neutrino mass the peak of Jeans mass lay above $10^{17} M_{\odot}$ and the corresponding comoving Jeans length is $\lambda_0 > 100$ Mpc. From the other hand, Jeans mass today is still larger, than the mass of massive galaxy $10^{12} M_{\odot}$.

Finally, the maximum value of Jeans mass at the moment (D.4.1) is (Ruffini and Song, 1986)

\[
M_J(z_{nr}) = 1.475 \times 10^{17} M_{\odot} \xi^{-\frac{1}{2}} N_{v}^{-\frac{1}{2}} \left( \frac{m_{v}}{10 eV} \right)^{-2} A(\xi)^{-\frac{5}{4}} B(\xi)^{\frac{3}{4}} . \tag{D.4.4}
\]

The peak of Jeans mass depending on degeneracy parameter for different fixed values of energy density as well as with constant mass $m_{v} = 2.5$ eV is shown at Fig.D.2.

By comparing different curves with fixed value of $\xi$ one can find the well known result, that the Jeans mass increases with decreasing of neutrino mass. With growth of degeneracy parameter, however, neutrino mass decreases in the beginning, and its different values correspond to different points at the same curve.
The space above the dashed line at Fig.D.2 represents the region in which the neutrino mass is less than 2.5 eV. It is interesting to note, that this value of $m_\nu$ is still sufficient to obtain $\Omega_\nu = 1$ with $\xi \approx 4$. 
D. Massive degenerate neutrinos in Cosmology
E. Fermi’s approach to the study of hadronic interactions

E.1. Introduction

The knowledge of the radiation mechanisms is crucial for the correct understanding of many astronomical observations. In particular in gamma ray astronomy the observational data can be often explained by two or more production mechanisms. It is therefore important to model correctly the different interactions producing gamma rays. Among all mechanisms producing gamma rays hadronic interactions between nucleons which produce pions, which in turn decay into photons and neutrinos, are the most difficult to model. In fact, high energy collisions of nucleons cannot be treated perturbatively because of the large value of the interaction constant in nuclear interactions. Many authors have occupied themselves with this problem and already in 1950 Fermi developed a statistical method for computing the multiple production of particles in collisions of high energetic protons. In the meantime a very large set of data has been acquired from high energy accelerators. The aim of the paper is 1. to reduce the Fermi theoretical equations, 2. to compare them to the experimental data and 3. to outline novel theoretical approach.

E.2. Fermi’s approach to the study of hadronic interactions

In treating high energy collisions of nucleons, Fermi made the assumption that the possible final configurations of the system are determined by the statistical weights of the various possible final configurations and accordingly developed a statistical method to determine the final particles produced Fermi (1950). First of all one might think of many different final configurations for the system after the collision, but conservation laws of charge and of momentum, as well as the feasibility of the processes, have to be taken into account. Transitions in Yukawa’s theory, in which charged and neutral pions are emitted, are therefore the most probable processes taking place. So during the collisions of high energetic hadrons a large amount of energy is released in a small volume around the hadrons and used to form pions. In view of
E. Fermi’s approach to the study of hadronic interactions

the strong interactions between these pions, one can imagine that the energy available in the small volume will be rapidly distributed to the different pions having different energies. In other words the energy will be statistically distributed among all degrees of freedom of the system. Fermi himself said: “When two nucleons collide with very great energy in their center of mass system this energy will be suddenly released in a small volume surrounding the two nucleons. The event is a collision in which the nucleons with their surrounding retinue of pions hit against each other so that all the portion of space occupied by the nucleons and by their surrounding pion field will be suddenly loaded with a very great amount of energy. Since the interactions of the pion field are strong we may expect that rapidly this energy will be distributed among the various degrees of freedom present in this volume according to statistical laws. One can then compute statistically the probability that in this tiny volume a certain number of pions will be created with a given energy distribution. It is then assumed that the concentration of energy will rapidly dissolve and that the particles into which the energy has been converted will fly out in all directions.”

Fermi’s method resembles somehow Heisenberg’s approach (1949) to treat high energy collisions of nucleons, with the difference that Heisenberg used qualitative ideas of turbulence whereas, Fermi believed that in high energetic processes statistical equilibrium is reached.

According to Fermi the process proceeds as follows: in the laboratory frame a very energetic proton scatters off a proton target. For convenience the process is examined in the center of mass frame. The only parameter which has to be tuned in Fermi’s theory is the volume in which the energy is dumped. The value of this parameter can be modified to improve the agreement with the experiments. Fermi calls \( \Omega^{LF} \) the volume at rest in the laboratory frame that containing the energy of the colliding particles. Since the particles mediating the Yukawa interactions are the pions, the volume is taken as a sphere with radius of order of the pion Compton wavelength \( \frac{\hbar}{m_\pi c} \) (\( \Delta x \Delta p \geq \hbar \Rightarrow \Delta x = \frac{\hbar}{\Delta p} = \frac{\hbar}{m_\pi c} \)).

In the following we will indicate the laboratory frame with the index LF and the center of mass frame with cm.

We will work in c.m. frame, where we define the volume \( \Omega^{cm} \) containing the energy of the colliding particles. The volume \( \Omega^{LF} \) of the two pion clouds around the nucleons is Lorentz contracted in c.m. frame,

\[
\Omega^{cm} = \frac{1}{\gamma} \Omega^{LF}, \tag{E.2.1}
\]

where

\[
\Omega^{LF} = \frac{4}{3} \pi R^3, \tag{E.2.2}
\]
and
\[ R = \frac{\hbar}{m_\pi c} = 1.4 \times 10^{-13} \text{ cm}, \tag{E.2.3} \]
where \( \hbar \) is the Planck constant, \( c \) the light speed, \( \gamma = (1 - v^2/c^2)^{-1/2} \) is the Lorentz factor and \( \vec{v} \) the velocity of proton in center of mass frame. Fermi obtained that \( \gamma = W/2m_pc^2 \), where \( W \) is the c.m. energy, we here derive his formula.

We indicate the laboratory frame with an index \( LF \).

In the laboratory frame the incident proton (with energy \( E_{LF}^p \)) scatters off a proton at rest (with energy \( m_pc^2 = 0.938 \text{ GeV} \)). The 4-momentum at of each particle of the system in the laboratory frame is given as
\[ P^\mu_1 = \left( \frac{E_{LF}^p}{c}, \vec{p}_1 \right), \tag{E.2.4} \]
\[ P^\mu_2 = \left( m_pc, \vec{0} \right). \tag{E.2.5} \]

The total 4-momentum of the system in the laboratory frame is
\[ P^\mu_{\text{tot}} = P^\mu_1 + P^\mu_2 = \left( \frac{E_{\text{tot}}^L}{c}, \vec{p}_{\text{tot}}^L \right), \]
\[ P^\mu_{\text{tot}} = \left( \frac{E_{p}^L}{c} + m_pc, \vec{p}_1 \right), \tag{E.2.6} \]
where \( E_{\text{tot}}^L \) is the total energy of the system of two particles in laboratory frame and \( \vec{p}_{\text{tot}}^L \) the total momentum of the system of two particles in the laboratory frame.

In the c.m. frame by definition, the total momentum is always zero, \( \vec{p}_{\text{tot}}^{cm} = \vec{0} \). The 4-momentum of each particle is given to
\[ P^\mu_1 = \left( \frac{E_{p}^{cm}}{c}, \vec{p}_1 \right), \tag{E.2.7} \]
\[ P^\mu_2 = \left( \frac{E_{p}^{cm}}{c}, \vec{p}_2 \right), \tag{E.2.8} \]
then \( \vec{p}_1 + \vec{p}_2 = \vec{0} \) \( \Rightarrow \vec{p}_1 = -\vec{p}_2 \).

The total 4-momentum in c.m. frame is
\[ P^\mu_{\text{tot}} = \left( \frac{E_{\text{tot}}^{cm}}{c}, \vec{p}_{\text{tot}}^{cm} \right) = \left( \frac{2E_{p}^{cm}}{c}, \vec{p}_1 + \vec{p}_2 \right), \]
\[ P^\mu_{\text{tot}} = \left( \frac{2E_{p}^{cm}}{c}, \vec{0} \right), \tag{E.2.9} \]
where \( E_{\text{tot}}^{cm} \) is the total energy of the system of two particles in c.m. frame, \( \vec{p}_{\text{tot}}^{cm} \) the total momentum of the system of two particles in the c.m. frame and \( E_{p}^{cm} \) the energy of each proton in c.m. frame.

According to the Lorentz invariance, the square of the 4-momentum is frame invariance, then the square of the 4-momenta \( P^\mu_{\text{tot}} \) of Eq. (E.2.6) and
E. Fermi’s approach to the study of hadronic interactions

\( E.2.9 \) are equal, as

\[
P_{\text{tot}}^\mu P_{\text{tot}}_{\mu} = \left( \frac{E_{\text{p}}^{\text{LF}}}{c} + m_p c \right)^2 - p_1^2 = (2E_p^{\text{cm}}/c)^2 - \bar{v}^2,
\]

\[
2m_p E_{\text{p}}^{\text{LF}} + 2m_p^2 c^2 = (2E_p^{\text{cm}}/c)^2,
\]

where \( E_{\text{p}}^{\text{LF}} = \sqrt{m_p^2 c^4 + p_1^2 c^2} \). The energy of c.m. \( W \) is given as the square of the 4-momentum, then according to Eq. \( E.2.10 \), \( W \) is given as

\[
W = \sqrt{2m_p c^2 E_{\text{p}}^{\text{LF}} + 2m_p^2 c^4} = 2E_p^{\text{cm}}.
\]

In the literature, the c.m. energy is defined as \( \sqrt{s} \), then \( W = \sqrt{s} \), but for convenience we will use the notation of Fermi \( W \). From the Eq. \( E.2.11 \) we have

\[
W = \frac{\gamma m_p c^2}{2m_p c^2}.
\]

where from Eq. \( E.2.6 \) \( E_{\text{tot}}^{\text{LF}} = E_{\text{p}}^{\text{LF}} + m_p c^2 \).

The proton energy in the c.m frame is given as

\[
E_p^{\text{cm}} = \gamma m_p c^2,
\]

where the Lorentz factor \( \gamma \) is the same of Eq. \( E.2.1 \), because \( \bar{v} \) is the velocity of the proton ‘at rest’ in the c.m. frame.

Substituting Eq. \( E.2.13 \) in Eq. \( E.2.11 \),

\[
W = \gamma m_p c^2, \quad \gamma = \frac{W}{2m_p c^2}.
\]

Note that the velocity \( \bar{v} \) of the proton ‘at rest’ in c.m. frame is the same as the velocity of the c.m. in the laboratory frame. According to Landau and Lifshitz (1975), in this case the velocity of the c.m. is \( \bar{v} = \frac{p_{\text{tot}}^{\text{LF}} c^2}{E_{\text{tot}}^{\text{LF}}} = \frac{p_1 c^2}{(E_p + m_p c^2)} \). It is possible to get the Eq. \( E.2.15 \) using the Eq. \( E.2.11 \) and this definition of velocity of c.m. Landau and Lifshitz (1975).

Substituting Eq. \( E.2.15 \) in \( E.2.1 \), we get the expression of Fermi for the Lorentz contraction of the volume \( \Omega_{\text{LF}} \)

\[
\Omega_{\text{cm}} = \frac{2m_p c^2}{W} \Omega_{\text{LF}}.
\]

Note that if the c.m. energy \( W \) increase, the volume \( \Omega_{\text{cm}} \) decreases, as is predicted in special relativity. The parameter volume will be therefore energy dependent. Fermi calculated also the cross-section as the area available for
collisions around the pion cloud

\[ \sigma_{\text{tot}} = \pi R^2. \quad (E.2.16) \]

Substituting Eq. (E.2.3) in (E.2.16), we get

\[ \sigma_{\text{tot}} = 6 \times 10^{-26} \text{ cm}^2, \]

where \( \hbar = 1.054 \times 10^{-27} \text{ erg.s} \) and \( c = 2.9979 \times 10^{10} \text{ cm/s} \) a value close to the modern experimental value.

The probability to find \( n \) particles in a volume \( \Omega \) inside the possible physical volume \( V \) of the ‘container’ is

\[ P_n = \left( \frac{\Omega}{V} \right)^n, \quad (E.2.17) \]

where \( n \) is also the degrees of freedom number.

If \( W \) is the total energy of the system, the density of states of energies possible is

\[ G(W) = \frac{dN(W)}{dW}, \quad (E.2.18) \]

where \( dN(W) \) is the number of states in the phase space lying between the energies \( W \) and \( W + dW \). For ‘\( n \)’ particles we have Reif (1965)

\[ N_n(W) \propto \int_{W}^{W+\delta W} \frac{d^3 \vec{r}_1 \ldots d^3 \vec{r}_n d^3 \vec{p}_1 \ldots d^3 \vec{p}_n}{(h^3)^n}, \quad (E.2.19) \]

where the numerator is the volume of the phase space of particles, and the denominator ‘\( (h^3)^n \)’ is the volume element in phase space (according to Heisenberg uncertainty principle). Note that \( N \) is dimensionless. Since each integral over \( \vec{r}_i \) extends over the volume \( V \) of the container, \( \int d^3 \vec{r}_i = V \), but there are \( n \) such integrals. Then the Eq. (E.2.19) becomes,

\[ N_n(W) = \left( \frac{V}{8\pi^3 h^3} \right)^n Q_n(W), \quad (E.2.20) \]

where \( Q_n(W) \propto \int_{W}^{W+\delta W} d^3 \vec{p}_1 \ldots d^3 \vec{p}_n \), independent of \( V \), is the sum over all possible state in the momentum space. This case describes \( n \) completely independent particles.

Substituting the Eq. (E.2.20) in (E.2.18),

\[ G_n(W) = \left( \frac{V}{8\pi^3 h^3} \right)^n \frac{dQ_n(W)}{dW}. \quad (E.2.21) \]

The probability to form this state is given via product of the probability \( P_n \).
E. Fermi’s approach to the study of hadronic interactions

(Eq. E.2.17) with the density of states (E.2.21).

\[ S_n(W) = P_n G_n(W), \]
\[ S(n) = \left( \frac{\Omega}{8\pi^3\hbar^3} \right)^n \frac{dQ_n(W)}{dW} \]  (E.2.22)

For the case of two particles \((n = 1)\), which produce no pion, we have that the volume of momentum space is \(Q = \frac{4}{3} \pi p^3\). In the classical case we have \(T' = \frac{p^2}{2\mu}\) (kinetic energy), where \(\mu\) is the reduced mass of the two protons, \(\mu = \frac{m_p m_p}{m_p + m_p} = m_p / 2\), then \(p = \sqrt{m_p T'}\).

The number of constraints is 1, due to the fact that the impulse in the c.m. is zero. The classical case, Eq. (E.2.22), is rewritten,

\[ S_{n=1} = \frac{\Omega m_p^{3/2}}{4\pi^2 \hbar^3} \sqrt{T'} \]  (E.2.23)

where \(T' = W - 2m_p c^2\). In the case of extremely high energy \(p = \frac{1}{c} \sqrt{W^2 - m_p^2 c^4} \simeq W / c\) (because \(W \gg m_p c^2\)), then \(Q = \frac{4}{3} \pi (W / c)^3\), and

\[ S_{n=1} = \frac{\Omega}{2\pi^2 \hbar^3 c^3} W^2. \]  (E.2.24)

To account for \(n + 1\) particles, the statistical weights of the classical and relativistic cases, respectively, are

\[ S_n = \frac{(m_1 m_2 ... m_n)^{3/2}}{2^{3n/2} \pi^{3n/2} h^{3n}} \frac{T'^{3n/2 - 1}}{(3n/2 - 1)!}, \]  (E.2.25)
\[ S_n = \frac{\Omega^n}{\pi^{2n} \hbar^{3n} c^{3n}} \frac{W^{3n - 1}}{(3n - 1)!}. \]  (E.2.26)

It can be verified that considering \(n = 1\) in Eq. (E.2.25) and (E.2.26) we get, respectively, the Eqs. (E.2.23) and (E.2.24), where \((1/2)! = \sqrt{\frac{2}{\pi}}\) (property of Gamma Function). The factors \(\pi^{3n/2}(3n/2 - 1)!\) and \(\pi^{2n}(3n - 1)!\) are of the dimension of the sphere, it is analog of the factor \(4\pi/3\) in the volume of an ordinary sphere. In treating the collisions of extremely high energy nucleons Fermi make use of thermo-dynamic laws, instead of considering a derailed statistical treatment. The density of energy around the colliding nucleons is so high that multiple pions will be produced.

From the Stefan-Boltzmann law of the black-body (radiation flux \(R(T) = \sigma T^4\), dimension \(dE/dt dA\)), the density of energy \(\rho\) (dimension \(dE/dV\)) is
E.2. Fermi’s approach to the study of hadronic interactions

given as

\[ \rho = \frac{1}{c} \int I_\nu d\nu d\Omega = \frac{4\pi}{c} I = \frac{4}{c} R, \Rightarrow \rho(T) = \frac{4}{c} \sigma T^4, \]

\[ \rho = \left( \frac{\pi^2}{15c^3\hbar^3} \right) (kT)^4 = \left( \frac{6.494}{\pi^2c^3\hbar^3} \right) (kT)^4, \quad (E.2.27) \]

where \( \pi^4/15 = 6 \sum_{n=1}^{\infty} 1/n^4 = 6.494 \), \( I_\nu(T, \nu) \) is the Planck’s law or spectral intensity of the black-body (dimension \( I(\nu) \to dE/dt dA d\Omega d\nu \), and \( I(T) = \int I_\nu(T, \nu) d\nu \)), \( k \) the Boltzmann constant, \( \sigma \) the Stefan-Boltzmann constant, \( \nu \) the frequency and \( T \) the temperature. According to Fermi: “Consequently the Stefan’s law for the pions will be quite similar to the ordinary Stefan’s law of the black-body radiation. The difference is only in a statistical weight factor. For the photons the statistical weight is the factor, \( 2 \), because of the two polarization directions. if we assume that the pions have spin zero and differ only by their charge \( \pm e \) or 0, their statistical weight will be 3. Consequently, the energy density of the pions will be obtained by multiplying the energy density of the ordinary Stefan’s law (E.2.27) by the factor \( 3/2 \).” Then the energy density via pions is

\[ \rho_\pi = \frac{3}{2}\rho = \frac{3 \times 6.494(kT)^4}{2\pi^2\hbar^3c^3}. \quad (E.2.28) \]

The total energy of the system is divided among pions (E.2.11) and protons and anti-protons. “The contribution of the nucleons and anti-nucleons to the energy density is given by a similar formula. The differences are that the statistical weight of the nucleons is eight since we have four different types of nucleons and anti-nucleons and for each, two spin orientations. A further difference is due to the fact that these particles obey the Pauli principle.” The process is \( pp \to \pi + X \), where \( X \) is the protons and anti-protons. Where the energy density is

\[ \rho_X = \frac{4 \times 5.682}{\pi^2\hbar^3c^3}(kT)^4 \quad (E.2.29) \]

where \( 6 \sum_{n=1}^{\infty} (-1)^{n+1}/n^4 = 5.682 \).

The total energy density \( \rho_{tot} \) of the system a little after the collision is given for the sum \( \rho_{tot} = \rho_\pi + \rho_X \), but also it is the energy of c.m. divided per volume, \( \rho_{tot} = W/\Omega^{cm} \). Then

\[ \rho_{tot} = \frac{W}{\Omega^{cm}} = \rho_\pi + \rho_X. \quad (E.2.30) \]
Substituting (E.2.28) and (E.2.29) in (E.2.30)

\[
\rho_{\text{tot}} = \frac{W}{2m_p c^2 \Omega_{LF}} = \frac{3 \times 3.494 (kT)^4}{2\pi^2 \hbar^3 c^3} + \frac{4 \times 5.682}{\pi^2 \hbar^3 c^3} (kT)^4,
\]

\[
(kT)^4 = 0.152 \frac{\hbar^3 c^3 W^2}{m_p c^2 \Omega_{LF}}. \tag{E.2.31}
\]

Note that the density of energy is frame invariant, \(\rho_{\text{tot}} = E_{\text{tot}}^{\text{cm}} / \Omega_{\text{cm}} = E_{\text{tot}}^{\text{LF}} / \Omega_{\text{LF}}\).

Analogously it is possible to get the classical case

\[
(kT)^4 = 0.152 \frac{\hbar^3 c^3 (W - 2m_p c^2)^2}{m_p c^2 \Omega_{LF}}. \tag{E.2.32}
\]

where the kinetic energy is \(T' = W - 2m_p c^2\).

If we divide the density of energy (E.2.28) by thermic energy of a pion \(\frac{3}{2} kT\), we get the numerical density \(n_\pi \rightarrow \frac{\text{# pions}}{\text{volume}}\) of pions

\[
n_\pi \approx \frac{(kT)^3}{\hbar^3 c^3}. \tag{E.2.33}
\]

Substituting Eq. (E.2.32) in (E.2.33), we get, respectively, for the relativistic and classical cases,

\[
n_{\pi}^{\text{HE}} \approx \left( \frac{W^2}{\hbar c m_p c^2 \Omega_{\text{cm}}} \right)^{3/4}, \tag{E.2.34}
\]

\[
n_{\pi}^{\text{ME}} \approx \left( \frac{T'^2}{\hbar c m_p c^2 \Omega_{\text{cm}}} \right)^{3/4}. \tag{E.2.35}
\]

We can define the multiplicity of pions for high energy \(N_{\pi}^{\text{HE}} \rightarrow \# \text{ pions}\) being

\[
n_{\pi}^{\text{HE}} = \frac{N_{\pi}^{\text{HE}}}{\Omega_{\text{LF}}} \Rightarrow N_{\pi}^{\text{HE}} = \frac{2m_p c^2}{W} \Omega_{\text{cm}}^{\text{cm}} n_{\pi}^{\text{HE}}. \tag{E.2.36}
\]

Substituting Eq. (E.2.34) in (E.2.36) we get

\[
N_{\pi}^{\text{HE}} \approx \left( \frac{m_p c^2 \Omega_{\text{cm}}^{\text{cm}}}{\hbar^3 c^3} \right)^{1/4} W^{1/2}. \tag{E.2.37}
\]

According Eqs. (E.2.2) and (E.2.3), we have

\[
\Omega_{\text{cm}}^{\text{cm}} = \frac{4\pi \hbar^3 c^3}{3(m_\pi c^2)^3}. \tag{E.2.38}
\]
Substituting Eq. (E.2.38) in (E.2.37)

\[ N_{\pi}^{HE} \approx \left( \frac{m_{\pi}W^2}{m_{\pi}^3c^6} \right)^{1/4}, \]

\[ N_{\pi}^{HE} = 0.54 \sqrt{\frac{W}{m_pc^2}}, \tag{E.2.39} \]

where Fermi considered \( m_{\pi}/m_p = 0.15 \Rightarrow m_{\pi} = 0.15m_p \). As the center of mass energy is according Eq. (E.2.11), \( W = \sqrt{s} \). Substituting Eq. (E.2.11) in (E.2.39), we get the equation of Fermi to the multiplicity of pions in extreme high energies

\[ N_{\pi}^{HE}(E_{pF}) = 0.54 \left[ 2 \left( 1 + \frac{E_{pF}}{m_pc^2} \right) \right]^{1/4}. \tag{E.2.40} \]

For the intermediate energy range (E.2.35), classical treating, it is known \( T' = W - 2m_pc^2 \). Doing the same procedure from Eq. (E.2.37) to (E.2.39), give us

\[ N_{\pi}^{ME} = 1.34 \frac{(W/m_pc^2 - 2)^{3/2}}{W/m_pc^2}. \tag{E.2.41} \]

Substituting Eq. (E.2.11) in (E.2.41), we get the equation of Fermi to the multiplicity of pions in intermediate energy range

\[ N_{\pi}^{ME}(E_p) = \frac{1.34}{\sqrt{2}} \left( \frac{\sqrt{2(1 + E_{pF}^L/m_pc^2)} - 2}{\sqrt{2(1 + E_{pF}^L/m_pc^2)}} \right)^{3/2}. \tag{E.2.42} \]

### E.3. Modern approach

Currently the modeling of pp interactions is done through computational codes, Monte Carlo codes such as SIBYLL, PHYTHIA, Dpmjet. (Kelner et al., 2006) presented new parameterizations of energy spectra of secondary particles, \( \pi \) and \( \eta \) mesons, gamma rays, electrons, and neutrinos produced in inelastic proton-proton collisions based on the SIBYLL code by (Fletcher et al., 1994). These parameterizations have very good accuracy in the energy range above 100 GeV (see figure E.1 and E.2).

The fit to the pp cross section obtained by (Kelner et al., 2006) is

\[ \sigma_{pp}(E_p) = (34.3 + 1.88L + 0.25L^2) \left[ 1 - \left( \frac{E_{th}}{E_p} \right)^4 \right]^2 \text{mb}, \tag{E.3.1} \]

where \( E_p \) is the incident proton energy in laboratory frame (the same that
$E_p$ in the last section), $L = \ln[E_p(\text{TeV})]$ and $E_{th}$ is the minimum threshold energy of the incident proton for production of a pion ($E_{th} = 1.22 \text{GeV}$) and $1 \text{ mb} = 1 \text{ mbarn} = 10^{-27} \text{ cm}^2$.

**Figure E.1.** The graphic compares the experimental data (black points) with the SIBYLL code (open points) and with the Kelner’s expression Kelner et al. (2006) for the cross section (solid curve).

The multiplicity of pions is defined by (Kelner et al., 2006) as

$$dN_\pi \equiv F_\pi(x, E_p) dx,$$

$$F_\pi(x, E_p) = \frac{d}{dx} \phi(x, E_p),$$

$$\phi_{\text{SIBYLL}} = -B_\pi \left( \frac{1 - x^{\beta\gamma}}{1 + k\gamma x^{\beta\gamma} (1 - x^{\beta\gamma})} \right)^4,$$

where $x = E_\pi/E_p$, $E_\pi$ is the ratio of the energy of incident proton $E_p$ transferred to the secondary $\pi^0$ meson, and $B_\pi$, $\beta\gamma$ and $k\gamma$ are functions dependent only of $E_p$ (see graphic E.2). Figure E.2 to shows the percentage $x$ of the energy of the incident proton which is transferred to the pions. Note in Fig. E.2 that it exists the possibility to transfer just 0.15% ($x = 0.0015$) of $E_p$ in $E_\pi$, also to transfer 70% ($x = 0.7$), but the plot shows that these situations are not very probable. The maximum probability is $x = 0.03$, so approximately 3% of incident proton energy is transferred via neutral pion $\pi^0$. Since the experimental data say that the production of neutral pions $\pi^0$ is practically the same to the productions of positive $\pi^+$ and negative $\pi^-$ charged pions, then we get that approximately 10% of $E_p$ is transferred via pions ($E_\pi$).
The number of $\pi^0$ produced per collision depending on $E_p$ in the Kelner-SIBYLL’s approach is

$$N_{\pi^0}(E_p) = 3.92 + 0.83L + 0.075L^2.$$  \hfill (E.3.5)

![Energy spectra of $\pi$ and $\eta$ mesons from the numerical simulations of the SIBYLL code (histograms) and from the analytical presentations given by Eqs. (E.3.3) and (E.3.4) for energy 0.1 TeV.](image)

**Figure E.2.** Energy spectra of $\pi$ and $\eta$ mesons from the numerical simulations of the SIBYLL code (histograms) and from the analytical presentations given by Eqs. (E.3.3) and (E.3.4) for energy 0.1 TeV.

**E.4. Conclusion**

We reviewed Fermi’s work of 1950 concerning the multiplicity of pions emitted in hadronic interactions and the experimental curves known currently. In the forthcoming work we will show a detailed comparison between the Fermi and modern approach.
E. Fermi’s approach to the study of hadronic interactions
Bibliography


ABAZAJIAN, K.N., BEACOM, J.F. AND BELL, N.F.


AKSENOV, A.G., MILGROM, M. AND USOV, V.V.

AKSENOV, A.G., RUFFINI, R. AND VERESHCHAGIN, G.V.

AKSENOV, A.G., RUFFINI, R. AND VERESHCHAGIN, G.V.
«Thermalization of Electron-Positron-Photon Plasmas with an application to GRB».

ARBOLINO, M.V. AND RUFFINI, R.
«New Limit on the Neutrinoless $\beta\beta$ Decay of $^{130}$Te».

BAHCALL, N.A. AND BURGETT, W.S.
«Are superclusters correlated on a very large scale?»

BAHCALL, N.A. AND SONEIRA, R.M.
«The spatial correlation function of rich clusters of galaxies».

BEAUDET, G. AND GORET, P.
«Leptonic numbers and the neutron to proton ratio in the hot big bang model».

BEYAEV, S. AND BUDKER, G.
«Relativistic Kinetic Equation».

BERESTETSKII, V.B., LIFSHITZ, E.M. AND PITAEVSKII, V.B.
Quantum Electrodynamics (Elsevier, 1982).

BERTONE, G., HOOPER, D. AND SILK, J.
«Particle dark matter: evidence, candidates and constraints».

BIANCO, C.L., RUFFINI, R., VERESHCHAGIN, G. AND XUE, S.S.
«Equations of motion, initial and boundary conditions for GRB».

BIANCONI, A., LEE, H.W. AND RUFFINI, R.
«Limits from cosmological nucleosynthesis on the leptonic numbers of the universe».

BISNOVATYI-KOGAN, G.S. AND NOVIKOV, I.D.
«Cosmology with a Nonzero Neutrino Rest Mass».

BLANDFORD, R.D. AND McKEE, C.F.
«Fluid dynamics of relativistic blast waves».
BOND, J.R., Efstathiou, G. and Silk, J.
«Massive neutrinos and the large-scale structure of the universe». 

BOND, J.R. AND SZALAY, A.S.
«The collisionless damping of density fluctuations in an expanding universe». 

BONNOR, W.B.
«Jeans’ formula for gravitational instability». 

CALZETTI, D., GIVALISCO, M. AND RUFFINI, R.
«The normalization of the correlation functions for extragalactic structures». 

CALZETTI, D., GIVALISCO, M., RUFFINI, R., EINASTO, J. AND SAAR, E.
«The correlation function of galaxies in the direction of the Coma cluster». 

CAVALLO, G. AND REES, M.J.
«A qualitative study of cosmic fireballs and gamma-ray bursts». 

COLEMAN, P.H. AND PIETRONERO, L.
«The fractal structure of the universe.» 

DE BERNARDIS, P., ADE, P.A.R., BOCK, J.J., BOND, J.R., BORRILL, J., 
BOSCALERI, A., COBLE, K., CONTALDI, C.R., CRILL, B.P., DE TROIA, G. 
ET AL.
«Multiple Peaks in the Angular Power Spectrum of the Cosmic Microwave 
Background: Significance and Consequences for Cosmology». 

DE GOUVEIA DAL PINO, E.M., HETEM, A., HORVATH, J.E., DE SOUZA, 
C.A.W., VILLELA, T. AND DE ARAUJO, J.C.N.
«Evidence for a very large scale fractal structure in the universe from COBE measurements». 

DOLGOV, A.D.
«Neutrinos in cosmology». 
Bibliography

DOLGOV, A.D., HANSEN, S.H., PASTOR, S., PETCOV, S.T., RAFFELT, G.G. AND SEMIKOZ, D.V.
«Cosmological bounds on neutrino degeneracy improved by flavor oscillations».

DOROSHKEVICH, A.G. AND KHLOPOV, M.Y.
«The formation of structure in the neutrino universe».

DOROSHKEVICH, A.G., NOVIKOV, I.D., SUNYAEV, R.A. AND ZELDOVICH, Y.B.
«Helium Production in the Different Cosmological Models».

DURRER, R. AND LABINI, F.S.
«A fractal galaxy distribution in a homogeneous universe?»

EHLERS, J.
«Survey of general relativity theory».

ELLIS, G.F.R.
«The evolution of inhomogeneities in expanding Newtonian cosmologies».

ELLIS, G.F.R. AND BRUNI, M.
«Covariant and gauge-invariant approach to cosmological density fluctuations».

FERMI, E.
«High Energy Nuclear Events».
Progress of Theoretical Physics, 5, pp. 570–583 (1950).

FLETCHER, R.S., GAISER, T.K., LIPARI, P. AND STANEV, T.
«sibyll: An event generator for simulation of high energy cosmic ray cascades».

GAITE, J., DOMÍNGUEZ, A. AND PÉREZ-MERCADER, J.
«The Fractal Distribution of Galaxies and the Transition to Homogeneity».
Gao, J.G. and Ruffini, R.
«Relativistic limits on the masses of self-gravitating systems of degenerate neutrinos.»

Giavalisco, M.

Goodman, J.
«Are gamma-ray bursts optically thick?»

Greiner, W. and Reinhardt, J.

Grimsrud, O.M. and Wasserman, I.
«Non-equilibrium effects in steady relativistic $e^-e^-$-gamma winds».

Hall, G. and Watt, J.M.

Hansen, S.H., Mangano, G., Melchiorri, A., Miele, G. and Pisanti, O.
«Constraining neutrino physics with big bang nucleosynthesis and cosmic microwave background radiation».

Haug, E.
«Energy loss and mean free path of electrons in a hot thermal plasma».

«The 2dF Galaxy Redshift Survey: correlation functions, peculiar velocities and the matter density of the Universe».

Heath, D.J.
«Gravitational instability».

Heisenberg, W.
«Über die Entstehung von Mesonen in Vielachprozessen».
Zeitschrift fur Physik, 126, pp. 569–582 (1949).
JEANS, J.H.
*The universe around us* (New York, The Macmillan company; Cambridge, Eng., The University press [c1929], 1929).

JEANS, J.
«The Stability of a Spherical Nebula».

JOYCE, M., SYLOS LABINI, F., GABRIELLI, A., MONTUORI, M. AND PIETRONERO, L.
«Basic properties of galaxy clustering in the light of recent results from the Sloan Digital Sky Survey».

KATRIN COLLABORATION.
«KATRIN: A next generation tritium beta decay experiment with sub-eV sensitivity for the electron neutrino mass».

KATZ, J.I.
«Two populations and models of gamma-ray bursts».

KATZ, J.I. AND PIRAN, T.
«Persistent Counterparts to Gamma-Ray Bursts».

KELNER, S.R., AХARONIAN, F.A. AND BUGAYOV, V.V.
«Energy spectra of gamma rays, electrons, and neutrinos produced at proton-proton interactions in the very high energy regime».

KLAPDOR-KLEINGROTHAUS, H.V., DIETZ, A., BAUDIS, L., HEUSSE, G., KRIVOSHEINA, I.V., MAJOROVITS, B., PAES, H., STRECKER, H., ALEXEEV, V., BAILYSH, A. ET AL.
«Latest results from the HEIDELBERG-MOSCOW double beta decay experiment».

KLYPIN, A.A. AND KOPYLOV, A.I.
«The Spatial Covariance Function for Rich Clusters of Galaxies».

KNELLER, J.P., SCHERRER, R.J., STEIGMAN, G. AND WALKER, T.P.
«How does the cosmic microwave background plus big bang nucleosynthesis constrain new physics?»
KOLB, E.W. AND TURNER, M.S.

«Final results from phase II of the Mainz neutrino mass search in tritium \( \beta \) decay».

LANDAU, L.D. AND LIFSHITZ, E.M.
*Fluid Mechanics (Course of Theoretical Physics)* (Pergamon, New York, 1987).

LANDAU, L.D. AND LIFSHITZ, E.M.

LATTANZI, M., RUFFINI, R. AND VERESHCHAGIN, G.
«On the possible role of massive neutrinos in cosmological structure formation».

LATTANZI, M., RUFFINI, R. AND VERESHCHAGIN, G.V.
«Joint constraints on the lepton asymmetry of the Universe and neutrino mass from the Wilkinson Microwave Anisotropy Probe».

LATTANZI, M., RUFFINI, R. AND VERESHCHAGIN, G.V.
«Do WMAP data constraint the lepton asymmetry of the Universe to be zero?»

LEE, B.W. AND WEINBERG, S.
«Cosmological lower bound on heavy-neutrino masses».

LESGOURGUES, J. AND PASTOR, S.
«Massive neutrinos and cosmology».

LISHITZ, E.
«».
LOBASHEV, V.M.
«The search for the neutrino mass by direct method in the tritium beta-decay and perspectives of study it in the project KATRIN».  

LUO, X. AND SCHRAMM, D.N.
«Fractals and cosmological large-scale structure».  

LYNDEN-BELL, D.
«Statistical mechanics of violent relaxation in stellar systems».  

MALTONI, M., SCHWETZ, T., TÓRTOLA, M. AND VALLE, J.W.F.
«Status of global fits to neutrino oscillations».  

MANDELBROT, B.B.

MESZAROS, A.
«A possible fast growth of adiabatic cosmological perturbations».  

MESZAROS, P.
«The behaviour of point masses in an expanding cosmological substratum».  

MÉSZÁROS, P., LAGUNA, P. AND REES, M.J.
«Gasdynamics of relativistically expanding gamma-ray burst sources - Kinematics, energetics, magnetic fields, and efficiency».  

MIHALAS, D. AND MIHALAS, B.W.

NAKAR, E., PIRAN, T. AND SARI, R.
«Pure and Loaded Fireballs in Soft Gamma-Ray Repeater Giant Flares».  

NARAYAN, R., PACZYNSKI, B. AND PIRAN, T.
«Gamma-ray bursts as the death throes of massive binary stars».  
ORITO, M., KAJINO, T., MATHEWS, G.J. AND BOYD, R.N.

ORITO, M., KAJINO, T., MATHEWS, G.J. AND WANG, Y.

PADMANABHAN, T.

PEACOCK, J.A.

PEACOCK, J.A.


Peebles, P.J.E.

Peebles, P.J.E.

PILLA, R.P. AND SHAHAM, J.

Piran, T.
Bibliography

Piran, T., Shemi, A. and Narayan, R.
«Hydrodynamics of Relativistic Fireballs».

Primack, J. and Murdin, P.
«Hot Dark Matter».

Primack, J.R.
«Status of cold dark matter cosmology».

Rees, M.J. and Meszaros, P.
«Relativistic fireballs - Energy conversion and time-scales».

Reif, F.

Ruffini, R.
«On the de Vaucouleurs density-radius relation and the cellular intermediate large-scale structure of the universe».

«The Blackholic energy and the canonical Gamma-Ray Burst».

Ruffini, R., Bianco, C.L., Chardonnet, P., Fraschetti, F., Vitagliano, L. and Xue, S.
«New perspectives in physics and astrophysics from the theoretical understanding of Gamma-Ray Bursts».

Ruffini, R., Salmonson, J.D., Wilson, J.R. and Xue, S.S.
«On the pair electromagnetic pulse of a black hole with electromagnetic structure».

Ruffini, R., Salmonson, J.D., Wilson, J.R. and Xue, S.S.
«On the pair-electromagnetic pulse from an electromagnetic black hole surrounded by a baryonic remnant».
RUFFINI, R. AND SONG, D.J.
«On the Jeans mass of weakly interacting, neutral, massive leptons.»

RUFFINI, R., SONG, D.J. AND STELLA, L.
«On the statistical distribution off massive fermions and bosons in a Fried-
mann universe».

RUFFINI, R., SONG, D.J. AND TARAGLIO, S.
«The ‘ino’ mass and the cellular large-scale structure of the universe».

SAVAGE, M.J., MALANEY, R.A. AND FULLER, G.M.
«Neutrino oscillations and the leptonic charge of the universe».

SHECTMAN, S.A., LANDY, S.D., OEMLER, A., TUCKER, D.L., LIN, H., KIR-
SHNER, R.P. AND SCHECHTER, P.L.
«The Las Campanas Redshift Survey».

SHEMI, A. AND PIRAN, T.
«The appearance of cosmic fireballs».

SHU, F.H.
«On the statistical mechanics of violent relaxation».

SILK, J.
«Fluctuations in the Primordial Fireball».

STEPNEY, S.
«Two-body relaxation in relativistic thermal plasmas».

STRUMIA, A. AND VISSANI, F.
«Implications of neutrino data circa 2005».

SVENSSON, R.
«Steady mildly relativistic thermal plasmas - Processes and properties».
Taub, A.H.
«Relativistic Rankine-Hugoniot Equations».

Tremaine, S. and Gunn, J.E.
«Dynamical role of light neutral leptons in cosmology».

Vereshchagin, G.V.
Pair Plasma and Gamma Ray Bursts.

Weinberg, S.

«The Luminosity and Color Dependence of the Galaxy Correlation Function».

Zeldovich, I.B. and Novikov, I.D.
Structure and evolution of the universe (Moscow, Nauka, 1975).

Zeldovich, Y.B.
«Gravitational instability: An approximate theory for large density perturbations.»

Zeldovich, Y.B. and Sunyaev, R.A.
«Astrophysical implications of the neutrino rest mass. I - The universe».