Mass, Radius and Moment of Inertia of Neutron Stars

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*From Nuclei to White Dwarfs and Neutron Stars*
Les Houches, France, 3-8 April 2011
In the standard picture the structure of neutron stars is given by:

- **Core:**
  - mainly neutrons and a small presence of protons and electrons in $\beta$-equilibrium
  - **local charge neutrality** $n_e = n_p$ in order to close the system of equations $\Rightarrow$ a priori neglecting of the electromagnetic interactions
  - $\rho \geq \rho_0$, where $\rho_0 \approx 2.7 \times 10^{14} \text{g cm}^{-3}$ is the nuclear saturation density for ordinary nuclei

- **Crust:**
  - inner crust: nuclei lattice (Coulomb lattice) in a background of electrons and neutrons; $\rho_0 \leq \rho \leq \rho_{\text{drip}}$, where $\rho_{\text{drip}} \approx 4.33 \times 10^{11} \text{g cm}^{-3}$ is the neutron-drip density
  - outer crust: nuclei lattice in a white-dwarf-like material (free electrons); $\rho \leq \rho_{\text{drip}}$

(Note that the core-crust transition happens when the uniform phase becomes energetically favorable with respect to the non-uniform Coulomb lattice)
The system of equation for such an object is given by local charge neutrality and β-equilibrium in addition to the Einstein equations:

\[
\frac{dM}{dr} = 4\pi r^2 \frac{\epsilon}{c^2}
\]

\[
\frac{dP}{dr} = - (\epsilon + P) \frac{4\pi r^3 P + Mc^2}{r \left( r - \frac{2GM}{c^2} \right)}
\]
New approach to neutron stars (a)

We follow a new description for neutron stars:

**Core:**

- degenerate gas of neutrons, protons and electrons in $\beta$-equilibrium

- global charge neutrality $N_e = N_p \Rightarrow$ the electromagnetic interactions are taken into account

- gravitational-electrodynamical-strong-weak equilibrium ensures the constancy of the general relativistic Fermi energy $E_{F}^{i}$ of all particles species

- $\rho \geq \rho_0$ as in the standard picture

**Crust:**

- inner crust: disappears in favor of a tiny transition layer (details in the next slides)

- outer crust: as in the standard description, nuclei lattice in a white-dwarf-like material (free electrons); $\rho \leq \rho_{drip}$
The new system of equation is:

\[ N_e = N_p \]

\[ E_e^F = e^{\nu/2} \mu_e - m_e c^2 - eV = \text{const} \]

\[ \frac{d^2 \hat{V}}{dr^2} + \frac{d \hat{V}}{dr} \left[ \frac{2}{r} - \frac{1}{2} \left( \frac{d \nu}{dr} + \frac{d \lambda}{dr} \right) \right] = -4\pi \alpha \hbar c e^{\nu/2} e^\lambda \times \]

\[ \times \left\{ n_p - \frac{e^{-3\nu/2}}{3\pi^2 \hbar^3 c^3} \left[ (\hat{V} + m_e c^2)^2 - m_e^2 c^4 e^\nu \right]^{3/2} \right\} \]

\[ E_n^F + m_n c^2 = E_p^F + m_p c^2 + E_e^F + m_e c^2 \]

\[ \frac{1}{r} \frac{d \nu}{dr} + \frac{1 - e^\lambda}{r^2} = \frac{8\pi G}{c^4} e^\lambda \left[ P - \frac{e^{-(\nu+\lambda)}}{8\pi \alpha \hbar c} \left( \frac{d \hat{V}}{dr} \right)^2 \right] \]

\[ \frac{dP}{dr} + \frac{1}{2} \frac{d \nu}{dr} (\epsilon + P) = -\sqrt{\alpha} (n_p - n_e) e^{\lambda/2} \frac{d \hat{V}}{dr} \]

\[ \frac{dM}{dr} = 4\pi r^2 \frac{\epsilon}{c^2} - 4\pi r^3 e^{-\nu/2} \frac{d \hat{V} / c^2}{dr} (n_p - n_e) \]

where \( \hat{V} = E_e^F + eV \).
How describe electromagnetic interaction between electrons and protons plus nuclear interaction??

We follow the so-called Walecka model, in which the strong interaction is modeled by meson-exchange through the $\sigma$, $\omega$ and $\rho$ meson-fields.

The total Lagrangian density of the system is given by

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_f + \mathcal{L}_\sigma + \mathcal{L}_\omega + \mathcal{L}_\rho + \mathcal{L}_\gamma + \mathcal{L}_{\text{int}}$$

- $\sigma$=isoscalar meson fields→ attractive long-range nuclear force
- $\omega$=massive vector field→ repulsive short-range nuclear force
- $\rho$=massive isovector field→ surface effects modeling a repulsive nuclear force
Einstein-Maxwell-Dirac Equations in Spherical Symmetry (a)

We consider non-rotating spherically symmetric neutron stars, so we introduce the spacetime metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

for which the Einstein-Maxwell-Dirac equations read

$$e^{-\lambda(r)} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = -8\pi G T^0_0$$
$$e^{-\lambda(r)} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = -8\pi G T^1_1$$

$$P' + \frac{\nu'}{2} (\epsilon + P) = -g_\sigma n_s \sigma' - \omega' g_\omega J^0_\omega - \rho' g_\rho J^0_\rho - V' e J^0_{ch}$$

$$V'' + V' \left[ \frac{2}{r} - \frac{(\nu' + \lambda')}{2} \right] = -e^\lambda e J^0_{ch}$$

$$\sigma'' + \sigma' \left[ \frac{2}{r} + \frac{(\nu' - \lambda')}{2} \right] = e^\lambda \left[ \partial_\sigma U(\sigma) + g_s n_s \right]$$

$$\omega'' + \omega' \left[ \frac{2}{r} - \frac{(\nu' + \lambda')}{2} \right] = -e^\lambda \left[ g_\omega J^0_\omega - m^2_\omega \omega \right]$$

$$\rho'' + \rho' \left[ \frac{2}{r} - \frac{(\nu' + \lambda')}{2} \right] = -e^\lambda \left[ g_\rho J^0_\rho - m^2_\rho \rho \right]$$

plus the constancy of electron Fermi energy and $\beta$-equilibrium.
Einstein-Maxwell-Dirac Equations in Spherical Symmetry (b)

- \( J_0^\omega = (n_n + n_p)e^{\nu/2}; \quad J_0^\rho = (n_p - n_n)e^{\nu/2}; \quad J_0^{ch} = (n_p - n_e)e^{\nu/2} \)

- \( T_0^0 = \epsilon + (E^2/8\pi) + U_\sigma + (1/2)C_\omega n_b^2 + (1/2)C_\rho n_3^2 \)

- \( T_1^1 = -P + (E^2/8\pi) + U_\sigma - (1/2)C_\omega n_b^2 + (1/2)C_\rho n_3^2 \)

- \( \epsilon = \sum_i \frac{2}{(2\pi)^3} \int_0^{P_i^F} 4\pi p^2 \sqrt{p^2 + \tilde{m}_i^2} \, dp \)

- \( P = \sum_i \frac{1}{3} \frac{2}{(2\pi)^3} \int_0^{P_i^F} 4\pi p^2 \frac{p^2}{\sqrt{p^2 + \tilde{m}_i^2}} \, dp \)

where \( \tilde{m}_p = \tilde{m}_n = m_N + g_\sigma \sigma \) and \( \tilde{m}_e = m_e \).
Due to the huge number of particles involved in the core ($\sim 10^{57}$ particles), and since the overlapping of the mesonic-fields profile, we follow the mean-field-approximation for the meson-fields

\begin{align*}
e^{-\lambda(r)} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} &= -8\pi G T_0^0, \quad (1) \\
e^{-\lambda(r)} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} &= -8\pi G T_1^1, \quad (2) \\
V'' + V' \left[ \frac{2}{r} - \left( \frac{\nu' + \lambda'}{2} \right) \right] &= -e^\lambda eJ_0^0 c_h, \quad (3) \\
\partial_\sigma U(\sigma) + g_s n_s &= 0, \quad (4) \\
g_\omega J_\omega^0 - m_\omega^2 \omega &= 0, \quad (5) \\
g_\rho J_\rho^0 - m_\rho^2 \rho &= 0, \quad (6) \\
e^{\nu/2} \mu_e - eV &= \text{constant}, \quad (7) \\
e^{\nu/2} \mu_p + eV + C_\omega n_B + C_\rho n_3 &= \text{constant}, \quad (8) \\
\mu_n - \mu_p - \mu_e - 2 C_\rho n_3 &= 0, \quad (9)
\end{align*}

where we have replaced the TOV equation with appropriate conservation laws for the generalized particle Fermi energies and the beta equilibrium condition.
In order to integrate the equilibrium equations we need to fix the parameters of the nuclear model, namely, fixing the coupling constants $g_s$, $g_\omega$ and $g_\rho$, and the meson masses $m_\sigma$, $m_\omega$ and $m_\rho$. Conventionally, such constants are fixed by fitting experimental properties of nuclei. Usual experimental properties of ordinary nuclei include saturation density, binding energy per nucleon (or experimental masses), symmetry energy, surface energy, and nuclear incompressibility. In the following table we present selected fits of the nuclear parameters. In particular, we show the following parameter sets: NL3, NL-SH, TM1, and TM2.

<table>
<thead>
<tr>
<th></th>
<th>NL3</th>
<th>NL-SH</th>
<th>TM1</th>
<th>TM2</th>
</tr>
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<tbody>
<tr>
<td>$m_\sigma$ (MeV)</td>
<td>508.194</td>
<td>526.059</td>
<td>511.198</td>
<td>526.443</td>
</tr>
<tr>
<td>$m_\omega$ (MeV)</td>
<td>782.501</td>
<td>783.000</td>
<td>783.000</td>
<td>783.000</td>
</tr>
<tr>
<td>$m_\rho$ (MeV)</td>
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<td>763.000</td>
<td>770.000</td>
<td>770.000</td>
</tr>
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<td>$g_s$</td>
<td>10.2170</td>
<td>10.4440</td>
<td>10.0289</td>
<td>11.4694</td>
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<tr>
<td>$g_\omega$</td>
<td>12.8680</td>
<td>12.9450</td>
<td>12.6139</td>
<td>14.6377</td>
</tr>
<tr>
<td>$g_\rho$</td>
<td>4.4740</td>
<td>4.3830</td>
<td>4.6322</td>
<td>4.6783</td>
</tr>
<tr>
<td>$g_2$ (fm$^{-1}$)</td>
<td>-10.4310</td>
<td>-6.9099</td>
<td>-7.2325</td>
<td>-4.4440</td>
</tr>
<tr>
<td>$g_3$</td>
<td>-28.8850</td>
<td>-15.8337</td>
<td>0.6183</td>
<td>4.6076</td>
</tr>
<tr>
<td>$c_3$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>71.3075</td>
<td>84.5318</td>
</tr>
</tbody>
</table>

**Table:** Selected parameter sets of the $\sigma$-$\omega$-$\rho$ model.
Figure: Mass-central density and mass-radius relations ($\rho_0 = 2.7 \times 10^{14}$ is the nuclear density).

Figure: Compactness-mass and compactness-radius relations.
the Coulomb potential energy inside the core of the neutron star is found to be \( \sim m_\pi c^2 \) and related to this there is an internal electric field of order \( \sim 10^{-14} E_c \) where \( E_c \) is the critical field for vacuum polarization.

at the core-crust boundary the continuity of all general relativistic particle Fermi energies guarantee a self-consistent matching of the core and the crust: a core-crust transition surface of thickness \( \geq \lambda_e = \hbar/(m_e c) \sim 100 \text{fm} \) (electron screening scale) is developed in which an overcritical electric field appears.

the continuity of the electron Fermi energy force the variation of electron chemical potential at the core-crust boundary to be of order \( \mu_e(\text{core}) - \mu_e(\text{crust}) \sim eV(\text{core}) \lesssim m_\pi c^2 \).

therefore we obtain a suppression of the so-called inner crust of the neutron star if \( \mu_e(\text{crust}) \sim \mu_e(\text{core}) - eV(\text{core}) \lesssim 25 \text{ MeV} \), which is approximately the value of the electron chemical potential at the neutron drip point.
Figure: Left: $g_\rho \neq 0$. Right: $g_\rho = 0$

- Left region: on the proton-profile we can see a bump due to Coulomb repulsion while the electron-profile decreases. Such a Coulomb effect is indirectly felt also by the neutrons due to the coupled nature of the system of equations. However, the neutron-bump is much smaller than the one of protons and it is not appreciable due to the plot-scale.

- Central region: the surface tension due to nuclear interaction produces a sharp decrease of the neutron and proton profiles in a characteristic scale $\sim \lambda_\pi$. Moreover, it can be seen a neutron skin effect, analogous to the one observed in heavy nuclei, which makes the scale of the neutron density falloff slightly larger with respect to the proton one.

- Right region: smooth decreasing of the electron density similar to the behavior of the electrons surrounding a nucleus in the Thomas-Fermi model.
The system of equations for crust is:

\[
\frac{dP}{dr} = -\frac{G(\epsilon + P)(m + 4\pi r^3 P)}{r^2(1 - \frac{2Gm}{r})}, \quad \frac{dm}{dr} = 4\pi r^2 \epsilon
\]

Uniform nucleon and free-electron distribution model (Chandrasekhar approximation): no Coulomb interactions

\[
P = P_e = \frac{2}{3(2\pi \hbar)^3} \int_0^{p^F_e} \frac{c^2 p^2}{\sqrt{c^2 p^2 + m_e^2 c^4}} 4\pi p^2 dp
\]

\[
\epsilon = \frac{2}{(2\pi \hbar)^3} \int_0^{p^F_e} \sqrt{c^2 p^2 + m_e^2 c^4} 4\pi p^2 dp
\]

where assuming local charge neutrality \( \epsilon = (A/Z)m_n n_e \)

Point-like nucleus surrounded by uniformly distributed electrons (BPS EoS due to Bethe, Pethick and Sutherland): Coulomb interactions

\[
P = P_e + \frac{1}{3} W_L n_N
\]

\[
\frac{\epsilon}{n_b} = \frac{W_N + W_L}{A} + \frac{\epsilon_e (n_b Z / A)}{n_b}
\]

where \( W_N(A, Z) \) is the total energy of an isolated nucleus and \( W_L \) is the lattice energy per nucleus
Figure: Mass versus compactness and crust-thickness versus compactness for crust without Coulomb-interactions.

Figure: Mass versus compactness and crust-thickness versus compactness for crust with Coulomb-interactions.
In both cases we obtain as average nuclear composition \( ^{105}_{35} \text{Br} \)

Using these values for \( \langle A \rangle \) and \( \langle Z \rangle \) in the Chandrasekhar approximation we obtain:

\[
M_{\text{chandra}}^{\text{crust}} \sim 1.18 M_{\text{BPS}}^{\text{crust}} \quad \text{and} \quad \Delta R_{\text{chandra}}^{\text{crust}} \sim 0.95 \Delta R_{\text{BPS}}^{\text{crust}}
\]
Global comparison

Figure: Total mass versus total radius for crust without Coulomb-interactions.

Figure: Total mass versus total radius for crust with Coulomb-interactions.
Introducing the moment of inertia, in the not-rotating newtonian limit, we get:

\[ I_{\text{core}} = \int_0^{R_{\text{core}}} r^4 (\epsilon + P) e^{\frac{\lambda - \nu}{2}} dr; \quad I_{\text{crust}} = \int_{R}^{R_{\text{crust}}} r^4 (\epsilon + P) e^{\frac{\lambda - \nu}{2}} dr \]

**Figure:** Left: moment of inertia of the core vs central density in units of nuclear density. Right: moment of inertia of the core on the one of the crust vs central density in units of nuclear density. Both the plot are been made for the NL3 model.
End of the talk

Roma caput mundi

Thank you