Developments in Stellar Turbulence Theory (2)

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Rotation and magnetic fields are key to understanding gravitational collapse to form pulsars and black holes.

Rotating, magnetized neutron star.

Crab Nebula Supernova Remnant composite from Chandra and Hubble Space Telescopes.
Hydrodynamic Equations

Mass (nucleon) conservation
\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \]  

Momentum conservation with zero viscosity, in tensor form
\[ \frac{\partial (\rho \mathbf{v}_i)}{\partial t} = -\frac{\partial (P \delta_{ik} + \rho \mathbf{v}_i \mathbf{v}_k)}{\partial x_k} \]  

Energy conservation
\[ \frac{\partial \left( \frac{1}{2} \rho \mathbf{v}^2 + \rho E \right)}{\partial t} = -\nabla \cdot [\rho \mathbf{v} \left( \frac{1}{2} \mathbf{v}^2 + W \right)], \]  

where \( W = E + PV \) is the enthalpy. Note that enthalpy appears to account for \( PdV \) work done.

Using the mass conservation equation, the momentum conservation equation is often written in the form of the Euler equation,
\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - \mathbf{g} \]  

where \( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \) is the co-moving time derivative of the velocity vector with respect to time.

Nonlinear equations may have discontinuous solutions. These nonlinearities are related to shocks and turbulence!
Bethe (1942): shock dissipation is a function of shock width and velocity difference

\[ \varepsilon \approx \Delta u^3 / \Delta r \]

Kolmogorov (1942): turbulent kinetic energy dissipates as

\[ \varepsilon \approx u_t^3 / \ell \]

in terms of mean turbulent velocity and the linear size of the turbulent region.
Les Houches Lectures (2)

- Analysis of Turbulence and Implications
- The Lorenz model and strange attractor
- Multiple shell burning and core collapse progenitors
Lorenz model and its strange attractor
The Lorenz model (E. Lorenz, 1963, J. Atmos. Sci., 20, 130) is a simple but mathematically precise model of a single mode convective roll, which has a strange attractor and captures a key element of turbulence.

It is the simplest representative of a class of dynamical systems which can be projected down to low order but preserve nonlinear aspects of fluid flow.
Fig. 3.— The Lorenz Model of Convection: Convection in a Loop.
Fig. 4.— The Lorenz Model extended: Convection in a shell composed of cells. Notice the alternation of the sign of rotation. This may be thought of as a cross sectional view of infinitely long cylindrical rolls, or of a set of toroidal cells, with pairwise alternating vorticity. Each cell can exhibit random fluctuations in time.
The Lorenz Equations

\[
\begin{align*}
\frac{dX}{d\tau} &= -\sigma X + \sigma Y \\
\frac{dY}{d\tau} &= -XZ + rX - Y \\
\frac{dZ}{d\tau} &= XY - bZ
\end{align*}
\]

Three equations allow chaos (remember orbits and the unrestricted 3-body problem).

X, Y, and Z are amplitudes of velocity, and of horizontal and vertical temperature variations.
KE bursts

3D simulation

Lorenz model
• The largest cells in turbulent convection seem to behave somewhat like the Lorenz model

• The turbulent cascade implies a multi-mode representation
$X = \text{convective speed}$
$Y = \text{horizontal } T \text{ fluctuation}$

$XY = \text{enthalpy flux} = \text{buoyancy driving}$

$s=10.00$
$r=28.00$
$b=2.667$
$Y = \text{horizontal T fluctuation}$

$Z = \text{vertical T fluctuation}$
Turbulence causes Luminosity fluctuations
Turbulence provides a source of broad-band noise in stellar variability (e.g., Betelgeuse)
• calculate the largest laminar mode which will fit in the convective region

• calculate the global buoyancy power from the convective luminosity, and equate to Kolmogorov damping to get velocity scale

• recalculate boundary physics for acoustic luminosity and entrainment

• do an evolution step
Supernova Progenitors
• Preliminary analysis: no MHD or rotation
• Does include dynamical turbulence
• The most extended computations are 2D
2D simulation of convective shell with overlying radiative region (velocity)
3D simulation of convective shell with overlying radiative region (velocity)
2D-3D differences

- velocities are too large in 2D relative to 3D
- cyclones dominate in 2D, break up in 3D
- qualitatively similar otherwise (e.g., convective boundary, wave generation), use 2D as 3D surrogate with caution
• 20 Msun core collapse progenitor

• 2D fluid dynamics

• C, Ne, O and Si burning shells

• one octant sector on top of Fe core, below He and H layers
Model: si.2d.a  Time = 345 sec

Casey Meakin & David Arnett (2006) – Steward Observatory
Model: si.2d.a  Time = 345 sec

Casey Meakin & David Arnett (2006) – Steward Observatory
Porter and Woodward: red giant
Coupled convection and pulsation
Woodward and Porter: Big Star Rotation and convection