On Bose-Einstein condensation in relativistic plasma

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Outline

- Bose-Einstein condensation
- Condensation of photons
- Boltzmann code
- Plasma relaxation: kinetic vs thermal equilibrium
- BEC of photons in relativistic plasma
- Results of numerical simulations
- Conclusions
Bose-Einstein condensation (BEC)

The phenomenon of quantum condensation of bosons was predicted by Bose and Einstein. In physics textbooks BEC is associated with cooling to low temperatures, and it was indeed observed in:

- ultracold atoms at nanoK (Cornell, Wieman, and Ketterle received the Nobel Prize in Physics, 2001),
- quasi-particles in solids at room temperature (polaritons and excitons),
- photons in a dye microcavity at room temperature.

BEC is understood as a quantum phenomenon (phase transition), occurring in ideal Bose gas of massive particles, when the temperature decreases at constant density, or, alternatively, when the density of particles increases at fixed temperature, leading to condensation of a fraction of particles in the lowest energy state.

We show that **BEC may occur for photons at billion K!**
BEC of photons

BEC does not occur for blackbody radiation, because photons are massless particles, and their chemical potential (in thermal equilibrium) vanishes. One way to observe BEC for photons is to lock them in a cavity. The cavity mirrors provide both a confining potential and a non-vanishing effective photon mass, making the system formally equivalent to a two-dimensional gas of trapped, massive bosons (Klaers et al., 2010). Another possibility is to consider photons in a plasma, under such conditions, that the dominant interaction process is Compton scattering (Zeldovich and Levich, JETP 28 1287, 1969). As a practical realization, they proposed cooling of hot photon gas by cold electrons, i.e. optically thick plasma with $T_\gamma > T_e$. The process of condensation is non-equilibrium one, therefore use of kinetic equations is required.
Boltzmann code development and main results

*Kinetic and thermal equilibrium in electron-positron plasma*

*Proton admixture*

*Timescales*

*Reaction-oriented approach*

*Numerical scheme*

*Quantum degeneracy and exact three-body processes*

*Summary:*
Boltzmann equations

Relativistic Boltzmann equations for homogeneous and isotropic distribution functions of electrons, positrons and photons is

\[
\frac{1}{c} \frac{\partial f_i}{\partial t} = \sum_q (\eta_{i}^q - \chi_{i}^q f_i),
\]

where the emission and absorption coefficients for the particle \(I\) in a binary process \(I + II \leftrightarrow III + IV\) have the following form

\[
\eta_{I}^{2p} = \int d^3p_2d^3p_3d^3p_4 \ W_{(3,4|1,2)} \ f_{III} f_{IV} (1 + \zeta f_I) (1 + \zeta f_{II}),
\]

\[
\chi_{I}^{2p} f_I = \int d^3p_2d^3p_3d^3p_4 \ W_{(1,2|3,4)} \ f_I f_{II} (1 + \zeta f_{III}) (1 + \zeta f_{IV}),
\]

and transition rates are \(W_{(3,4|1,2)} d^3p_3d^3p_4 = V d_{w_{(3,4|1,2)}}\) and \(W_{(1,2|3,4)} d^3p_1d^3p_2 = V d_{w_{(1,2|3,4)}}\), \(V\) is normalization volume.
Boltzmann equations II

The differential reaction probability per unit time is

\[ dw = c (2\pi \hbar)^4 \delta^{(4)} (P_f - P_i) |M_{fi}|^2 V \left( \prod_{in} \frac{\hbar c}{2\epsilon_{in} V} \right) \left( \prod_{fin} \frac{d^3 p_{fin}}{(2\pi \hbar)^3} \frac{\hbar c}{2\epsilon_{fin}} \right), \]

with \( \xi = \psi h^3 / 2 \) and \( \psi \) is +1,-1,0 for Bose-Einstein, Fermi-Dirac or Boltzmann statistics, respectively. Analogously, the emission and absorption coefficients for the particle \( I \) in a triple process \( I + II \leftrightarrow III + IV + V \) have the following forms

\[ \eta_{3p}^I = \int d^3 p_2 d^3 p_3 d^3 p_4 d^3 p_5 \ W_{(3,4,5|1,2)} \ f_{III} f_{IV} f_{V} (1 + \bar{\xi} f_I) (1 + \bar{\xi} f_{II}), \]

\[ \chi_{3p}^I f_I = \int d^3 p_2 d^3 p_3 d^3 p_4 d^3 p_5 \ W_{(1,2|3,4,5)} \ f_I f_{III} (1 + \bar{\xi} f_{III}) (1 + \bar{\xi} f_{IV}) (1 + \bar{\xi} f_{V}). \]

where \( W_{(3,4,5|1,2)} d^3 p_3 d^3 p_4 d^3 p_5 = V dw_{(3,4,5|1,2)} \) and

\[ W_{(1,2|3,4,5)} d^3 p_1 d^3 p_2 = V^2 dw_{(1,2|3,4,5)}. \]
# Interactions

<table>
<thead>
<tr>
<th>Binary interactions</th>
<th>Radiative and pair producing variants</th>
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| **Møller and Bhabha scattering**

\[
\begin{align*}
e^\pm e^\pm & \rightarrow e_1^\pm' e_2^\pm' \quad e^\pm e^\mp & \rightarrow e^\pm' e^\mp'
\end{align*}
\]

| **Bremsstrahlung**

\[
\begin{align*}
e_1^\pm e_2^\pm & \leftrightarrow e_1^\pm' e_2^\pm' \gamma \\
e^\pm e^\mp & \leftrightarrow e^\pm' e^\mp' \gamma
\end{align*}
\]

| **Single Compton scattering**

\[
\begin{align*}
e^\pm \gamma & \rightarrow e^\pm \gamma' 
\end{align*}
\]

| **Double Compton scattering**

\[
\begin{align*}
e^\pm \gamma & \leftrightarrow e^\pm' \gamma' \gamma''
\end{align*}
\]

| **Pair production and annihilation**

\[
\begin{align*}
\gamma \gamma' & \leftrightarrow e^\pm e^\mp 
\end{align*}
\]

| **Radiative pair production and three photon annihilation**

\[
\begin{align*}
\gamma \gamma' & \leftrightarrow e^\pm e^\mp \gamma'' \\
e^\pm e^\mp & \leftrightarrow \gamma \gamma' \gamma'' \\
e^\pm \gamma & \leftrightarrow e^\pm' e^\mp e^\pm''
\end{align*}
\]
Conservation laws

Energy conservation

\[ \frac{d}{dt} \sum_i \rho_i = 0. \]

Particle’s conservation for binary reactions

\[ \frac{d}{dt} \sum_i n_i = 0. \]

Charge conservation respectively

\[ n_- = n_+. \]

The condition for the chemical potentials

\[ \varphi_+ + \varphi_- = 2\varphi_\gamma. \]
Kompaneets equation

Boltzmann equation for photons in non-relativistic case with Thompson scattering only reduces to the Kompaneets equation. Two main approximations are adopted:

- electrons are assumed non-relativistic with kinetic energy $\varepsilon_k \equiv \varepsilon_e / m_e c^2 - 1 \ll 1$;
- photons have energies much less than electron rest mass energy $\varepsilon_\gamma \ll m_e c^2$.

Introducing the variable $x = \varepsilon_\gamma / k_B T_e = \varepsilon_\gamma / \theta_e$ and expressing the DF through the occupation number $\tilde{n}(x, p) = (2\pi \hbar)^3 f(x, p)$ we have

$$\frac{\partial \tilde{n}}{\partial y} = \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ x^4 \left[ \frac{\partial \tilde{n}}{\partial x} + \tilde{n} (\tilde{n} + 1) \right] \right\} .$$

(3)

Using this equation Zeldovich and Levich (1969) showed the possibility of BEC of photons in non-relativistic plasma.
Kinetic vs thermal equilibrium

Considering relaxation of non-equilibrium relativistic plasma (Aksenov et al., 2007-) it was found that this process occurs in two steps. First, detailed balance is established in binary interactions. This balance corresponds to a metastable state called kinetic equilibrium, which is characterized by the same temperature $T_k$ of all particles, and nonzero chemical potentials $\mu_i$. The DF of particles with energy $E$ is

$$f = \frac{2}{(2\pi \hbar)^3} \frac{1}{\exp \left( \frac{E-\mu_i}{kT_k} \right) \pm 1}. \tag{4}$$

Kinetic equilibrium is maintained as long as the rates of binary interactions exceed the ones of triple interactions. When triple interactions finally come into detailed balance, thermal equilibrium is established and the chemical potential of photons vanishes.
Kinetic equilibrium for Boltzmann gas

For Boltzmann particles (vanishing quantum degeneracy) the DF takes particularly simple form

$$f = \frac{2}{(2\pi \hbar)^3} \exp \left( - \frac{E}{kT_k} \right) \exp \left( \frac{\mu_k}{kT_k} \right).$$

The last multiplier appears in expressions for both number density and energy density. For non-degenerate electron-positron pairs one has

$$n_\pm = 8\pi \left( \frac{2\pi \hbar}{mc} \right)^{-3} j_1(\theta) \exp \left( \frac{\mu_k}{kT_k} \right), \quad \rho_\pm = n_\pm mc^2 j_2(\theta),$$

where

$$j_1(\theta) = \theta K_2(\theta^{-1}), \quad j_2(\theta) = \frac{3K_3(\theta^{-1}) + K_1(\theta^{-1})}{4K_2(\theta^{-1})}, \quad \theta = \frac{kT_k}{m_e c^2}.$$
The possibility of BEC for photons occurs if their initial number density $n_0^\gamma$ exceeds the one given by the Planck distribution with the temperature of kinetic equilibrium

$$n_0^\gamma > \frac{2\zeta(3)}{\pi^2} \left( \frac{\hbar}{mc} \right)^{-3} \left( \frac{kT_k}{m_e c^2} \right)^3,$$

where $\zeta(s)$ is the Riemann $\zeta$-function.

Necessary condition for BEC is photon number conservation (Zeldovich and Levich, 1969), i.e. BEC may form in kinetic equilibrium!
Initial conditions

We perform numerical simulations of plasma thermalization starting with different initial photon distributions:
- delta-function with some energy $E_0$;
- Gaussian distribution around $E_0$;
- Wien distribution.

In all cases we observe BEC formation when kinetic equilibrium is established.

We consider NR and UR cases separately. For NR case we present a particular result with total energy density $\rho_{tot} = 8.7 \times 10^{20} \text{ erg cm}^{-3}$ corresponding to a final equilibrium temperature $\theta = k_B T / m_e c^2 = 0.1$.

Total initial particle number density is $n_{tot}^{in} = 5 n_{tot}^{fin}$, where $n_{tot}^{fin} = 3.5 \times 10^{27} \text{ cm}^{-3}$ is the final total particle number density in thermal equilibrium. For UR case with a total energy density $\rho = 2.1 \times 10^{27} \text{ erg/cm}^3$, corresponding to the final equilibrium temperature $\theta = 3$. The initial state is pairless and photons are placed at the energy node with $\varepsilon = 1.06 m_e c^2$. 

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Thermalization dynamics

![Graph showing thermalization dynamics](image)

- t, sec
- n, cm$^{-3}$
- $\rho$, erg cm$^{-3}$

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Monochromatic photons

Figure: Time moments from left to right: $10^{-15}, 10^{-11}, 10^{-8}$ s.
Gaussian distribution

Figure: Time moments from left to right: $10^{-15}, 10^{-11}, 10^{-8}$ s.
Ultrarelativistic case

Figure: Time moments from left to right: $3.8 \times 10^{-20}$, $5 \times 10^{-20}$, $10^{-15}$ s.
In this work we presented the results of the first principles calculations demonstrating BEC of photons in relativistic plasma. This phenomenon was predicted in 1969 and still awaits confirmation in the laboratory.

We found that in order to favour BEC, sufficiently narrow initial distribution of photons (not broader than Wien spectrum) is required with the peak of the distribution located above the critical energy, below which triple interactions dominate over the binary ones.

Initial conditions proposed by Zeldovich and Levich (cooling of hot photon gas by cold electrons) are not suitable for BEC formation, essentially due to non-negligible triple interactions in low energy part of the spectrum, which change photon number prior to establishment of kinetic equilibrium.
Conclusions II

- BEC observed in a microcavity represents an optical analogy of photon condensation discussed in this work. Indeed, photon number in a microcavity is conserved and after photon thermalization via interaction with a solution, photons in excess over the thermal number density undergo condensation.

- Likewise fermion degeneracy manifests itself in relativistic systems such as white dwarfs and neutron stars, it is possible that BEC of photons might be as well observed in astrophysical conditions. Given a transient character of condensation and short time of its existence, the necessary condition is a supply of soft nonthermal photons, which might provide support for condensation on longer timescales.