Equation of motion for test particle in modified gravity

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Contents

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Motivation

- Universe is not close with ordinary matter
- No definite dark matter candidate in standard model
- No quantum gravity
- Resolving dark matter problem (rotation curve)
- Quantizing gravity at same time
- Simple modification of General Relativity
Universe is unknown

• Composition of the Universe

https://www.lsst.org/science/dark-energy
Introduction

• Universe is in accelerating expansion, Riess & Perlmutter

• Apparent Bolometric Magnitude $m(z)$

$$m = M + 5 \log DT(z; \Omega_M, \Omega_\Lambda)$$

• $DT(z; \Omega_M, \Omega_\Lambda) = H_0 \delta L, M = M - 5 \log H_0 + 25, \Omega_\Lambda = \frac{\Lambda}{3H_0^2}$

• $d_L(z; \Omega_M, \Omega_\Lambda) = \frac{c(1+z)}{H_0 \sqrt{|k|}} \Xi \left( \sqrt{|k|} \int_0^z \frac{1}{\sqrt{(1+z')^2(1+\Omega_Mz')-z'(2+z')\Omega_\Lambda}} dz' \right)$
Introduction

• Universe is in accelerating expansion
  Riess & Perlmutter
Introduction

- Universe is in accelerating expansion
  
  Riess & Perlmutter
Evidence of dark energy
Galaxy rotation curve

- Observed mass is not enough

https://www.youtube.com/watch?v=hsjM_pCVSUw
http://www.astro.uwo.ca/~basu/teach/ast021/
Quantum Gravity

Unification of Quantum Mechanics and Gravitation
Candidates of QG

String theory

Loop quantum gravity

Supergravity
Regge calculus
Wheeler-DeWitt equation
Superfluid vacuum theory
Twistor theory
Horava-Lifshitz gravity
Emergent gravity
Etc.
Dark matter problem

- Composition of the Universe
- Concordance $\Lambda$CDM model

https://www.lsst.org/science/dark-energy
Observational evidences

http://www.particlecentral.com/mysteries_page.html

- image of a galaxy cluster
  \((\text{CL0024+17})\)
Models

- Cold dark matter: MACHOs, RAMBOs, WIMPs, axions
- Warm dark matter: sterile neutrino
- Hot dark matter: neutrino
Modified gravity

• MOND (Modified Newtonian Dynamics), M. Milgrom (1983)
• TeVeS (Tensor-vector-scalar gravity), Jacob Bekenstein (2004)
• STVG (Scalar-tensor-vector gravity), John Moffat (2006)
• Nonsymmetric gravity theory
• Dark fluid model
Main Starting point (old story)

QED

Quantum Gravity
Einstein Equation (standard)

- Einstein-Hilbert Action with matter Term

\[
S = \int \left[ \frac{1}{2\kappa} R + \mathcal{L}_M \right] \sqrt{-g} \, d^4 x
\]

\[
0 = \delta S
\]

\[
= \int \left[ \frac{1}{2\kappa} \frac{\delta (\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta (\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \, d^4 x
\]

\[
= \int \left[ \frac{1}{2\kappa} \left( \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta (\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \, \sqrt{-g} \, d^4 x
\]
Einstein Equation (standard)

• Equation of Motion
\[ \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -2\kappa \frac{1}{\sqrt{-g}} \frac{\delta (\sqrt{-g}L_M)}{\delta g^{\mu\nu}} \]

• Energy-Momentum Tensor
\[ T_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g}L_M)}{\delta g^{\mu\nu}} = -2 \frac{\delta L_M}{\delta g^{\mu\nu}} + g_{\mu\nu} L_M \]
Einstein Equations

• Lagrangian

\[ \mathcal{L} = c_0 + c_1 R + c_2 T \]

• Action

\[ S = \int (c_0 + c_1 R + c_2 T) \sqrt{-g} d^4 x \]

Metric is geometry and gravitational field

*Do not think matter without spacetime*
Variations

\[ \int \delta(\sqrt{-g})d^4x = \int \left\{ -\frac{1}{2}g_{\mu\nu} \right\} \delta g^{\mu\nu} \sqrt{-g}d^4x \]

\[ \int \delta(\sqrt{-gR})d^4x = \int \left\{ \left( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R \right) \delta g^{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) \delta g^{\mu\nu} \right\} \sqrt{-g}d^4x \]

\[ = \int \left( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R \right) \delta g^{\mu\nu} \sqrt{-g}d^4x, \]

\[ \int \delta(\mathcal{T})d^4x = \int \left\{ \left( T_{\mu\nu} + g_{\alpha\beta} \frac{\delta T^{\alpha\beta}}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu} \right\} d^4x \]

\[ := \int T_{\mu\nu} \delta g^{\mu\nu} d^4x = \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g}d^4x \]

\[ \mathcal{T}^{\mu\nu} = (-g)^{1/2}T^{\mu\nu}, \quad T^{\mu\nu} = (-g)^{-1/2}\mathcal{T}^{\mu\nu} \]
Equations of motion

\[-\frac{c_0}{2} g_{\mu\nu} + c_1 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + c_2 T_{\mu\nu} = 0\]

• $c_0 = -2\Lambda, c_1 = 1, c_2 = -\frac{8\pi G}{c^4} = -\kappa$

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \kappa T_{\mu\nu}$
Rotation speed in GR (Astrop. Phys. 29, 386(2008))

• Spherically symmetric metric

\[ ds^2 = -e^{\nu(r)} dt^2 + e^{\mu(r)} dr^2 + r^2 d\Omega^2 \]

• Lagrangian for massive test particle

\[ \mathcal{L} = \frac{1}{2} \left( -e^{\nu} \dot{t}^2 + e^{\lambda} \dot{r}^2 + r^2 \dot{\Omega}^2 \right) \]

• Conserved quantities

\[-e^{\nu(r)} \dot{t} = E, \quad r^2 \dot{\Omega} = L\]
Rotation speed in GR

- Geodesic equation
  \[ e^{\nu + \lambda} \dot{r}^2 + e^\nu \left( 1 + \frac{L^2}{r^2} \right) = E^2 \]

- For circular motion, (a) \( \dot{r} = 0 \) (b) \( \frac{\partial V_{\text{eff}}}{\partial r} = 0 \) (c) \( \frac{\partial^2 V_{\text{eff}}}{\partial r^2} \big|_{\text{extr}} > 0 \)
  \[ E^2 = \frac{e^\nu}{1 - r v'/2}, \quad L^2 = \frac{r^3 v'/2}{1 - r v'/2} \]

- Rotation speed
  \[ v_{tg}^2 = e^{-\nu} r^2 \left( \frac{d\Omega}{dt} \right)^2 = e^{-\nu} r^2 \dot{\Omega}^2 / t^2 = e^\nu \frac{L^2}{r^2 E^2} \]
  \[ v_{tg}^2 = \frac{r v'}{2} \]
Properties of rotation speed

\[ v_{tg}^2 = \frac{rv'}{2} \]

- Exact general relativistic expression for static spherically symmetric spacetimes
- Tangential velocity is only sensitive to \( g_{tt} \) component
- Independent of modification of Einstein-Hilbert action, since it is obtained from geodesic equation

\[ ds^2 = -e^{\nu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2d\Omega^2 \]

We need to find modified Einstein-Hilbert action which is compatible to assumed metric.
Schwarzschild metric

\[
d s^2 = - \left(1 - \frac{2M}{r}\right) \, dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} \, dr^2 + r^2 \, d\Omega^2
\]

\[
e^{\nu(r)} = \left(1 - \frac{2M}{r}\right) \quad \nu(r) = \ln \left(1 - \frac{2M}{r}\right)
\]

\[
\nu'(r) = \frac{2M/r^2}{1 - 2M/r} = \frac{2M}{r^2} \left\{ 1 + \frac{2M}{r} - \left(\frac{2M}{r}\right)^2 + \cdots \right\}
\]

\[
v_{tg}^2 = \frac{r \nu'}{2} = \frac{M}{r - 2M} = \frac{M}{r} \frac{2M^2}{r^2} + \frac{4M^3}{r^3} - \frac{8M^4}{r^4} + \cdots
\]
Schwazschild metric

\[ \frac{1}{\sqrt{r-2M}} \]

Rotation Velocity

Rotation Velocity at Near Horizon

\[ \frac{1}{\sqrt{r}} \]
Modification to E-H action

Weyl tensor is an appropriate Propagating degree of freedom
Modification to E-H action

• Lagrangian

\[ \mathcal{L} = \sqrt{-g} \left( R - 2\Lambda - \kappa T + \lambda C_{\alpha\mu\beta\nu} T^{\alpha\beta} T^{\mu\nu} \right) \]

• Action

\[ S = \int \left( 2\Lambda + R + \kappa T + \lambda C_{\alpha\mu\beta\nu} T^{\alpha\beta} T^{\mu\nu} \right) \sqrt{-g} d^4 x \]
Field Equations

\[ L = \sqrt{-g} \left( R - 2\Lambda - \kappa T + \lambda C_{\alpha\mu\beta\nu} T^{\alpha\beta} T^{\mu\nu} \right) \]

\[ C_{\alpha\mu\beta\nu} = R_{\alpha\mu\beta\nu} - \frac{1}{2} (g_{\alpha\beta} R_{\mu\nu} + g_{\mu\nu} R_{\alpha\beta} - g_{\alpha\nu} R_{\mu\beta} - g_{\mu\beta} R_{\alpha\nu}) + \frac{1}{6} (g_{\alpha\beta} g_{\mu\nu} - g_{\mu\beta} g_{\alpha\nu}) R \]

\[ T^{\mu\nu} = (\rho + p) u^\mu u^\nu + pg^{\mu\nu} \]
Various Tensors

\[ g = -e^{\nu + \mu} r^4 \sin^2 \theta, \]

\[ R_{tt} = \frac{1}{4r} e^{\nu - \mu} \left( 2rv'' + 2v'^2 - rv' \mu' + 4v' \right), \]

\[ R_{rr} = -\frac{1}{4r} \left( 2rv'' + 2v'^2 - rv' \mu' - 4\mu' \right), \]

\[ R_{\theta\theta} = \frac{1}{2} e^{-\mu(r)} \left( r \mu' - rv' + 2e^\mu - 2 \right), \]

\[ R_{\phi\phi} = \frac{\sin^2 \theta}{2} e^{-\mu(r)} \left( r \mu' - rv' + 2e^\mu - 2 \right) = \sin^2 \theta R_{\theta\theta}, \]

\[ R = -\frac{e^{-\mu}}{2r^2} \left( 2r^2v'' + r^2v'^2 - r^2v' \mu' + 4rv' - 4r \mu' - 4e^\mu + 4 \right) \]
Newtonian case

\[ \rho_h \sim \frac{1}{r^2} \]

Can we explain without dark matter?

http://www.astronomy.ohio-state.edu/~thompson/1101/lecture_darkmatter_darkenergy.html
What to do

• Calculate Einstein-Hilbert action with metric functions
• Get Euler-Lagrange equations of motion
• Calculate the correction terms caused by extra term for Rotation curve
  • How it change with scale?
  • What is the proper range of $\lambda$?
Further works

• Vacuum condition?
• Calculate expected rotation curve
• Comparison with observed rotation curves
Thanks