On the frequency of oscillations in the pair plasma generated by a strong electric field

Alberto Benedetti

ROME UNIVERSITY "LA SAPIENZA"
&
ICRANET

IRAP Ph.D. Erasmus Mundus Workshop, Pescara, March 23, 2011
On the frequency of oscillations in the pair plasma generated by a strong electric field
Basic definitions

- **Critical electric field**

  \[ e \, E \, \lambda_c \simeq m \, c^2 \quad \Rightarrow \quad E_c \equiv \frac{m^2 \, c^3}{e \, \hbar}. \]

- **Rate of pair production** in a uniform electric field ($\hbar = c = 1$)

  \[ S \equiv \frac{dN}{dV \, dt} = \frac{m^4}{4 \, \pi^3} \left( \frac{E}{E_c} \right)^2 \exp \left( -\pi \frac{E_c}{E} \right), \]

  where the electric field can be rewritten in a covariant form

  \[ E = \sqrt{-\frac{1}{2} F_{\mu\nu} F^{\mu\nu}}. \]
Assumptions

- The formula for the pair production

\[ S \equiv \frac{dN}{dVdt} = \frac{m^4}{4\pi^3} \left( \frac{E}{E_c} \right)^2 \exp \left( -\pi \frac{E_c}{E} \right), \]

can be safely used also for **slowly time-varying electric fields**. This condition is fulfilled if the adiabaticity parameter is much larger than one

\[ \eta = \frac{m}{\omega} \frac{E_{peak}}{E_c} = \tilde{T} \tilde{E}_{peak} \gg 1. \]

- Interactions between electrons, positrons and photons are not taken into account (**collisionless approximation**).
Basic equations

Continuity eq. + eq. of motion + Maxwell eq.

\[
\frac{\partial (\bar{n} U^\mu)}{\partial x^\mu} = S, \quad \frac{\partial T^{\mu\nu}}{\partial x^\nu} = -F^{\mu\nu} J_\nu, \quad \frac{\partial F^{\mu\nu}}{\partial x^\nu} = -4\pi J^\mu,
\]

where \( U^\mu \) is the 4-velocity of the pairs, \( \bar{n} \) is the comoving number density. The total current is given by the sum of conducting and polarized currents

\[
J^\mu = J^\mu_{\text{cond}} + J^\mu_{\text{pol}},
\]

and the energy-momentum tensor is

\[
T^{\mu\nu} = m\bar{n} \left( U^{\mu}_{(+)} U^{\nu}_{(+)} + U^{\mu}_{(-)} U^{\nu}_{(-)} \right).
\]

Rescaling variables

With the new parameterization

\[ n = m^3 \tilde{n}, \quad \rho = m^4 \tilde{\rho}, \quad p = m^4 \tilde{p}, \quad E = E_c \tilde{E}, \quad t = m^{-1} \tilde{t}, \]

we rewrite the quantities defined before as follows

\[ \tilde{S} = \frac{1}{4\pi^3} \tilde{E}^2 \exp \left( -\frac{\pi}{\tilde{E}} \right), \quad \tilde{v} = \frac{\tilde{p}}{\tilde{\rho}}, \]
\[ \tilde{\gamma} = (1 - \tilde{v}^2)^{-1/2}, \quad \alpha = \frac{e^2}{\hbar c}, \]

in which each variable is expressed in units of \( \lambda_c \) and \( E_c \).
Four linear ODEs

Our initial system of equations becomes a system of linear ordinary coupled differential equations

\[
\begin{align*}
\diamond \quad \frac{d\tilde{n}}{d\tilde{t}} &= \tilde{S}, \\
\diamond \quad \frac{d\tilde{p}}{d\tilde{t}} &= \tilde{n}\tilde{E}\tilde{v} + \tilde{\gamma}\tilde{S}, \\
\diamond \quad \frac{d\tilde{\rho}}{d\tilde{t}} &= \tilde{n}\tilde{E} + \tilde{\gamma}\tilde{v}\tilde{S}, \\
\diamond \quad \frac{d\tilde{E}}{d\tilde{t}} &= -8\pi\alpha \left( \tilde{n}\tilde{v} + \frac{\tilde{\gamma}\tilde{S}}{\tilde{E}} \right).
\end{align*}
\]

where we considered **1D motion**.

---

From the following fundamental relation

\[ \tilde{\rho}^2 = \tilde{p}^2 + \tilde{n}^2, \quad \text{and} \quad 16\pi \tilde{\rho} = \tilde{E}_0^2 - \tilde{E}^2, \]

and introducing the new velocity \( \tilde{u} = \tilde{\gamma} \tilde{v} = \gamma v / c \), we can obtain an expression for the number density \( \tilde{n} \) from which we can easily work out the maximum achievable number density \( \tilde{n}_{\text{max}} \)

\[ \tilde{n} = \frac{\tilde{E}_0^2 - \tilde{E}^2}{16\pi \alpha (1 + \tilde{u}^2)^{1/2}} \quad \Rightarrow \quad \tilde{n}_{\text{max}} = \frac{\tilde{E}_0^2}{16\pi \alpha}. \]

Besides, the equation of motion now reads

\[ \dot{\tilde{u}} = \tilde{E}. \]
Technical improvement

Inserting the previous equations into the full system, we get one ODE

\[ \ddot{\tilde{u}} + \frac{1}{2} \frac{\tilde{u}}{1 + \tilde{u}^2} \left( \tilde{E}_0^2 - \dot{\tilde{u}}^2 \right) + \frac{2\alpha}{\pi^2} \dot{\tilde{u}} \left(1 + \tilde{u}^2\right)^{1/2} \exp \left(-\frac{\pi}{|\dot{\tilde{u}}|}\right) = 0, \]

which can be rewritten symbolically as

\[ \ddot{\tilde{u}} + \tilde{\omega}_p^2 \tilde{u} + k \dot{\tilde{u}} = 0, \]

where

\[ \tilde{\omega}_p = \sqrt{\frac{8\pi\alpha\tilde{n}}{(1 + \tilde{u}^2)^{1/2}}}, \quad k = \frac{2\alpha}{\pi^2} \left(1 + \tilde{u}^2\right)^{1/2} \exp \left(-\frac{\pi}{|\dot{\tilde{u}}|}\right), \]

are the relativistic plasma frequency and the friction term.
Physical interpretation

- With **constant coefficients** the following equation
  
  \[ \ddot{u} + \omega_p^2 \dot{u} + k \dddot{u} = 0, \]

  describes **damped harmonic oscillation** for the velocity \( \dot{u} \).

- When electric field is small, \( k \) is **exponentially suppressed** and the master equation is reduced to classical plasma oscillations equation describing **Langmuir waves**
  
  \[ \ddot{u} + \omega_p^2 \dot{u} = 0. \]

For this reasons **we expect that the frequency of oscillations** \( \omega_{osc} \) **tends to the plasma frequency** \( \omega_p \).
Purposes of the numerical calculation

\[
\ddot{u} + \frac{1}{2} \frac{\ddot{u}}{1 + \ddot{u}^2} \left( \tilde{E}_0^2 - \ddot{u}^2 \right) + \frac{2\alpha}{\pi^2} \ddot{u} \left( 1 + \ddot{u}^2 \right)^{1/2} \exp \left( -\frac{\pi}{|\ddot{u}|} \right) = 0
\]

Initial conditions \( \Rightarrow \ddot{u}(0) = 0, \dot{\ddot{u}}(0) = \tilde{E}_0. \)

- **Oscillations frequency** \( \omega_{osc} \) for the cases \( E_0 = 0.5, 1, 2, 5, 10 \, E_c \).
- **Time evolution of the number density** \( n \) compared with the maximum achievable value \( n_{max} \).
- **Comparison** between \( \omega_{osc} \) and the averaged plasma frequency \( \omega_{pav} \).
- **Time-scale** needed by \( \omega_{osc} \) to attain \( \omega_{pav} \).
- **Power spectrum** due to the pairs oscillations.
\[ E_0 = E_c \implies \tilde{u} = \gamma v / c \]

**Smaller** is \( E_0 \), **larger** is the fraction of energy converted into **kinetic energy** of pairs!

---

A. Benedetti, W.-B. Han, R. Ruffini, G. V. Vereshchagin, paper accepted for publication on Phys. Lett. B
\( E_0 = E_c \implies \frac{n}{n_{\text{max}}} = (1 + \tilde{u}^2)^{-\frac{1}{2}}(1 - \tilde{u}^2/E_0^2) \)

**Larger** is \( E_0 \), **larger** is the fraction of energy converted into **rest mass energy** of pairs!

---

A. Benedetti, W.-B. Han, R. Ruffini, G. V. Vereshchagin, paper accepted for publication on Phys. Lett. B
Calculating $\omega_{osc}$

- We select a first set of roots $\{\tilde{T}_1^i\}$ of the function $\tilde{u}$ such that covers the entire evolution time.
- For each element $\tilde{T}_1^i$ we find the subsequent root $\tilde{T}_2^i$.
- The frequency at the time $\tilde{T}_1^i$ is equal to

$$\tilde{\omega}^i = \frac{\pi}{\tilde{T}_2^i - \tilde{T}_1^i},$$

where $\tilde{T}_2^i - \tilde{T}_1^i$ is the corresponding half period.
$E_0 = 2 E_c \implies \omega_p (\text{blue}), \omega_{pav} (\text{yellow}), \omega_{osc} (\text{red})$

$$\tilde{\omega}_p = \sqrt{\frac{\tilde{E}_0^2 - \dot{\tilde{u}}^2}{2 (1 + \tilde{u}^2)^2}}$$
$E_0 = 2E_c \quad \Rightarrow \quad \frac{\omega_{osc}}{\omega_p^{av}}$

The vertical line indicates $t_a$, which is the asymptotic time-scale needed to $\omega_{osc}$ to get $\omega_p^{av}$.

A. Benedetti, W.-B. Han, R. Ruffini, G. V. Vereshchagin, paper accepted for publication on Phys. Lett. B
The power spectrum is given by

\[ P(\omega) = 2 |\tilde{D}(\tilde{\omega})|^2, \]

where the amplitude \( \tilde{D}(\tilde{\omega}) \) is proportional to the Fourier transform of the electric current time derivative

\[ \tilde{D}(\tilde{\omega}) \propto \int \tilde{t} e^{i\tilde{\omega}\tilde{t}} \left( \frac{\partial \tilde{J}(\tilde{t})}{\partial \tilde{t}} \right). \]

The electric current is simply related to the new variables by the following expression

\[ \tilde{J} = 2 \tilde{n} \frac{\tilde{u}}{(1 + \tilde{u}^2)^{1/2}}. \]

\[ E_0 = 2 \tilde{E}_c \implies P(\omega) \]

\[ \tilde{\omega}_{peak} = \tilde{\omega}_p(\tilde{n}_{max}) \simeq \frac{\tilde{E}_0}{\sqrt{2}} \implies \hbar\omega_{peak} = 0.72 \tilde{E}_0 \text{ MeV} \]

A. Benedetti, W.-B. Han, R. Ruffini, G. V. Vereshchagin, paper accepted for publication on Phys. Lett. B
Validity of the collisionless approximation

- One can estimate the **optical depth for electron-positron annihilation** as

  \[ \tau(t) \sim \int_0^t \frac{\sigma_T}{\gamma^2} n v \, dt = \int_0^t \frac{8\pi\alpha^2}{3} \frac{|\tilde{u}|}{(1 + \tilde{u}^2)^{3/2}} \tilde{n} \, d\tilde{t}, \]

  where \( \sigma_T \) is the Thomson’s cross section.

- We define the time-scale \( t_\gamma \) as follows

  \[ \tau(t_\gamma) \sim 1 \]

  From that time moment interaction of pairs with photons can no longer be neglected. We expect that the **spectrum will be distorted** by these interactions.

---

The condition $t_\gamma < t_a$ means that photons produced by interaction of pairs will probably distort the spectrum.
Production of muons and pions

Muons and pions production from electron-positron collisions is suppressed because:

- **Kinematic threshold**: $\gamma \geq m_{\mu,\pi}/m_e \gtrsim 200$, restricting initial electric fields to be undercritical, $E_0 < E_c$, but ...

- ... the **number density** of pairs is exponentially suppressed for undercritical fields. Besides the **cross section** for all these processes decreases as $\sigma \propto \gamma^{-2}$ which further decreases the rate of electron-positron collisions.


Alberto Benedetti

On the frequency of oscillations in the pair plasma generated by a strong electric field
Does Pauli blocking inhibit pair production?

- $\dot{n} = \tilde{S} \implies$ when concentration of pairs reaches the maximum allowed value by the Pauli principle the pair production is blocked;
- $\dot{\tilde{u}} = \tilde{E} \implies$ however, particles produced at rest are accelerated by external electric field and thus leave the quantum state with zero momentum which can be subsequently filled by a new pair.

These effects are independent since they operate in orthogonal directions of the phase space, so one can estimate the value of external electric field at which phase space blocking occurs by comparing their rates. Pauli blocking begins to act if

$$\dot{n} \geq \dot{\tilde{u}} \implies \frac{\tilde{E}^2}{4\pi^3} \exp\left(-\frac{\pi}{\tilde{E}}\right) \geq \tilde{E} \implies \tilde{E} \geq 127$$

which is much higher than the electric fields considered.
Avalanche-like QED cascade I

It has been claimed recently that Schwinger field could never be reached in high power lasers due to occurrence of avalanche-like QED cascade operating mainly through nonlinear Compton scattering combined with nonlinear Breit-Wheeler process and via the trident process. As soon as one single pair is generated by the Schwinger process such electromagnetic cascade of secondary electron-positron pairs is expected to deplete the electromagnetic energy thus preventing further pair production from vacuum. The requirements for the avalanche to occur are twofold:

- the probability to emit photon should not be suppressed;
- the photon must be energetic enough to produce pair by interaction with another photon.

Avalanche-like QED cascade II

For circularly polarized standing electromagnetic wave, both these conditions may fulfill when $E < E_c$. For linearly polarized standing wave such electromagnetic cascade is not expected to dominate over the Schwinger process. It is easy to understand these results looking at the energy loss rate of charged particle in classical electrodynamics

$$\frac{dW}{dt} = \frac{2}{3} \frac{\alpha^2}{m^2} \gamma^2 \left[ (E + v \times H)^2 - (E \cdot v)^2 \right].$$

- $H = 0$, $E \parallel v \Rightarrow dW/dt \propto E^2 \Rightarrow$ radiation loss smaller than energy conversion into pairs;
- $E, v$ misaligned $\Rightarrow dW/dt \propto \gamma^2 \Rightarrow$ curvature radiation becomes more efficient.

Conclusions

- **Simplified study**: reduction to the analysis of a second order ODE. We can study the system for a long evolution time. Other physical quantities (*electric field*, *number density*, *plasma frequency*, *time-scales*, *spectrum*) can be obtained easily from its solution.

- **Physical interpretation**: $n_{\text{max}}, \omega_{\text{peak}}$ can be readily estimated without computation.

- **Unexpected result**: $\omega_{\text{osc}} \simeq \omega_p^\text{av}$ even at the beginning when the condition $S \to 0$ is not satisfied.

- The **collisionless approximation** is fulfilled for many oscillations, however it begins to be active before $t_a$. We expect a distortion of the spectrum computed within the collisionless approximation.

- We give a limiting value for the electric field under which the **Pauli blocking** does not affect the rate of pair production;

- Production of **muons** and **pions** is suppressed.