Non thermal radiation modeling of $\gamma$-ray burst afterglow within binary driven hypernova scenario

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If you want to find the secrets of the Universe, think in terms of energy, frequency and vibration.

- Nikola Tesla
Abstract

Major novelty presented in this thesis is the development of the understanding of \(\gamma\)-ray burst (GRB) afterglow within the binary driven hypernova (BdHN) scenario. Motivated with observed mildly to non relativistic expansion velocities in afterglow phase (notably in the case of GRB 130427A), this approach is in contrast with standard fireball approaches assuming ultra relativistic jetted expansion extending from prompt emission phase all the way to the afterglow.

Using BdHN paradigm it is possible to identify a number of independent processes forming the GRB phenomenon, each characterized by specific Lorentz factor \(\Gamma\) and space-time evolution. Comparison of observations with BdHN scenario lead toward the introduction of following time sequenced processes in recent publications: 1) ultra relativistic prompt emission within first \(\sim 10\) seconds having \(\Gamma \sim 500 - 1000\), 2) hard X-ray flares with \(\Gamma \sim 10\), 3) soft X-ray flares emerging within first hundreds of seconds characterized with mildly relativistic Lorentz factor \(\Gamma \sim 2 - 4\), and 4) late afterglow showing deceleration from mildly relativistic \(\Gamma \sim 2 - 4\) to non relativistic \(\Gamma \sim 1\) within approximately \(10^6\) seconds.

Main argument for the mildly to non relativistic character of GRB afterglow expansion comes from the signature of extended thermal X-ray emission, observed within soft X-ray flares signaling the conversion of supernova (SN) into a hypernova (HN), and the observations of SN emission lines indicating non relativistic expansion already at \(\sim 10^6\) seconds.

Another important ingredient in understanding of GRB afterglow within the BdHN scenario is the expected presence of the newly formed neutron star (\(\nu\)NS) within the HN ejecta. Using a well known GRB 130427A as a prototype and extending the results of its analysis for several other GRBs following important points were discussed:

1. The imprint which HN ejecta and the rotation of binary system progenitor have on the characteristics of \(\nu\)NS pulsar and its environment. The HN ejecta magnetized by the \(\nu\)NS provide necessary conditions for synchrotron emission to emerge as GRB afterglow.

2. The pulsar like behaviour of \(\nu\)NS and the specific dipole-quadrupole structure of its magnetic field necessary to explain the fits of X-ray and optical data with synchrotron model.

3. The development of a model for GRB afterglow consistent with the observed mildly relativistic expansion velocity determined through model independent procedure.
For that cause, a numerical procedure which would track the evolution of non thermal relativistic electrons radiating synchrotron radiation was developed in form of PESCARA code. The code provided solutions to kinetic equation in general time dependent framework without the limitation of analytical approaches and has shown potential of use in other areas where non thermal radiation processes play a crucial role.

This thesis is therefore organized as follows. In Chapter 1, a general overview of GRB observations and theoretical understanding is presented with special attention being given to the BdHN scenario and its background. Chapter 2 deals with concise derivation of non thermal radiation processes relevant in the understanding of GRB afterglow, most notably synchrotron radiation mechanism, which aside the radiation spectra signature play a crucial role in determining the magnitude of energy losses term for non thermal electrons. Emergence of non thermal particle distributions is covered in Chapter 3 which deals with the particle acceleration mechanisms relevant in GRB astrophysics and necessary for understanding of injection term in kinetic equation. In Chapter 4, a general derivation of the kinetic (i.e. diffusion-loss) equation is presented together with analytical and numerical approaches to its solution. A concise testing of numerical PESCARA scheme is presented together with interesting results emerging from following the evolution of Gaussian energy injection in decaying magnetic field and the fittings of Crab Nebula. Current understanding of GRB afterglow within BdHN scenario together with the synchrotron emission fittings of GRB 130427A, GRB 160625B and GRB 190114C using the PESCARA code are lastly presented in Chapter 5. Final conclusion is given in Chapter 6.
Contents

1 Gamma Ray Bursts - Theory and Observations 1
  1.1 Introduction ............................................. 1
  1.2 GRB origins .............................................. 6
  1.3 GRB progenitors ........................................... 7
    1.3.1 Collapsar model ...................................... 7
    1.3.2 Compact star mergers .................................. 8
    1.3.3 Induced gravitational collapse ........................... 8
  1.4 GRB central engine ........................................ 12
  1.5 Physical background of GRBs ............................... 15
    1.5.1 Fireball model ....................................... 15
    1.5.2 Fireshell model ...................................... 19
  1.6 Afterglow .................................................. 21
  1.7 BdHN overview ............................................. 22
    1.7.1 X-ray precursor ....................................... 22
    1.7.2 Onset of the GeV and the prompt emission ................. 24
    1.7.3 Early afterglow ...................................... 25
    1.7.4 Late afterglow ....................................... 25
    1.7.5 The optical SN ....................................... 26

2 Non Thermal Radiation Processes 29
  2.1 Introduction .............................................. 29
  2.2 Synchrotron Radiation .................................... 29
    2.2.1 Single Particle Synchrotron Spectra .......................... 31
    2.2.2 Synchrotron Emissivity ................................ 39
    2.2.3 Synchrotron Self Absorption ............................ 45
  2.3 Inverse Compton Scattering ............................... 48
    2.3.1 Inverse Compton Emissivity ............................ 51
    2.3.2 Synchrotron Self-Compton Radiation ....................... 52
3 Particle Acceleration in Astrophysics 55
   3.1 Introduction ............................................. 55
   3.2 Fermi Acceleration ....................................... 56
      3.2.1 Second Order Fermi Acceleration ...................... 56
      3.2.2 First Order Fermi Acceleration ..................... 59
      3.2.3 Nonlinear Theory of Particle Acceleration ........... 65
      3.2.4 Relativistic Shocks ................................ 68
   3.3 Unipolar Inductor ........................................ 72
   3.4 Magnetic Reconnection .................................... 74
      3.4.1 Particle Acceleration by Magnetic Reconnection .... 78

4 Kinetic Equation - Analytical and Numerical Approaches 83
   4.1 Introduction ............................................. 83
   4.2 Kinetic Equation ......................................... 83
   4.3 Method of characteristics ................................. 86
   4.4 Simple cases of kinetic equation ......................... 89
      4.4.1 Adiabatic expansion without injection .......... 90
      4.4.2 Adiabatic expansion with constant injection .... 92
      4.4.3 Synchrotron losses within static emitter without injection 93
      4.4.4 Synchrotron losses within static emitter with constant injection 96
      4.4.5 Synchrotron losses and adiabatic expansion .... 101
   4.5 Method using Green function ............................. 107
   4.6 Numerical approach to solving kinetic equation ....... 110
   4.7 Testing .................................................. 112
   4.8 Gaussian injection within decaying magnetic field .... 114
   4.9 Pulsar wind nebulae modelling ........................... 122

5 GRB Afterglow Modelling 133
   5.1 Introduction ............................................. 133
   5.2 GRB 130427A ............................................ 134
      5.2.1 Energetics and properties of fast-rotating \( \nu \)NS 136
      5.2.2 Model for the Optical and X-ray Spectrum of the Afterglow 138
      5.2.3 Initial Conditions for GRB 130724A ............... 141
      5.2.4 Results ........................................... 142
   5.3 GRB 160625B ............................................ 147
      5.3.1 Modelling of GRB 160625B .......................... 149
   5.4 GRB 190114C ............................................ 151
      5.4.1 Modelling of GRB 190114C .......................... 151
| 6 Conclusion            | 153 |
Chapter 1

Gamma Ray Bursts - Theory and Observations

1.1 Introduction

Gamma ray bursts (GRBs), since their serendipitous discovery by Vela satellites on July 2nd 1967 which was made public in 1973, constitute the brightest γ-ray sources in the Universe. Duration of GRBs has been found to be mostly between a few milliseconds till several hundreds of seconds, with some GRBs even lasting for \( \sim 10^5 \) seconds as extreme cases. Initially it was not known whether these sources were of galactic or extragalactic (cosmological) origin leading to the great inflation in number of proposed models trying to explain them. Indications of cosmological origins of GRBs were first presented by Hartmann and Epstein (1989) [117] who localised 88 GRB events using IPN network showing an isotropic distribution. Significant progress in GRB research started with the launch of the *Compton Gamma Ray Observatory* (CGRO) in 1991 carrying the *Burst and Transient Source Experiment* (BATSE) built specifically for the study of GRBs. BATSE being equipped with eight identical modules covering whole sky containing γ-ray detectors covering energy range from 20 keV till \( \sim 2 \) MeV was capable to localise GRB events with precision of 2-3 degrees (till 20 degrees for dimmer sources). With detection rate of approximately one GRB per day within nine years of operation BATSE managed to localise 2704 GRBs with further confirmation of their isotropic distribution as shown on figure [1.1] and strongly indicating cosmological origins.

Another major finding by BATSE came from measuring the duration of GRBs. In GRB analysis their duration is determined through value of \( T_{90} \) which is defined as time during which the burst emits from 5% till 95% of its total measured photon counts. Based on observations by BATSE, measured values of \( T_{90} \) show a bimodal distribution separating the bursts into two major groups: short GRBs (SGRB) lasting
2.1. Gamma Ray Bursts - Theory and Observations

Figure 1.1. Locations of 2704 GRBs in galactic coordinates localized by BATSE on board of NASA’s CGRO spacecraft during the nine year mission. Figure taken from NASA website: https://batse.msfc.nasa.gov/batse/grb/skymap/

Figure 1.2. The bimodal distribution of GRB durations.

for $T_{90} < 2$ s with harder spectra and long GRBs (LGRB) lasting for $T_{90} > 2$ s having generally softer spectra [130, 131] as it can be seen on figure 1.2.

Generally speaking GRB lightcurves are erratic and often exhibit coincidental emission episodes covering a wide range of variability times scales [30]. In cases of some GRBs, individual, clearly separated one or more 'pulses' could be identified, commonly having typical fast-rise and exponential-decay (FRED) shape.

Having an sufficient amount of detected photons, the GRB spectra from few keV till several MeV (depending on instrument) are commonly fitted with a phenomenological model called the "Band" function [20] represented by smoothly joined two
power law functions as follows

\[
f(E) = K \times \begin{cases} 
\left(\frac{E}{100}\right)^{\alpha} \exp \left[ -\frac{E(2 + \alpha)}{E_p} \right], & E \leq \left(\frac{\alpha - \beta}{2 + \alpha}\right) E_p \\
\left(\frac{E}{100}\right)^{\beta} \exp(\beta - \alpha) \left[\frac{(\alpha - \beta)E_p}{(2 + \alpha)}\right]^{\alpha - \beta}, & E > \left(\frac{\alpha - \beta}{2 + \alpha}\right) E_p
\end{cases}, \tag{1.1}
\]

where \( f(E) \) is the photon flux, \( \alpha \) is low energy power law index, \( \beta \) is high energy power law index and \( E_p \) is the peak energy. Common values obtained for GRBs observed by BATSE within \( 20\,\text{keV} \lesssim E \lesssim 2\,\text{MeV} \) band can be found in ranges between \(-2 \leq \alpha \leq 0 \) for \( \alpha \) and between \(-4 \leq \beta \leq -1 \) for \( \beta \). On the other hand, when possible, value of peak energy \( E_p \) has been generally estimated within range \( 100\,\text{keV} \leq E_p \lesssim 10\,\text{MeV} \). Anyway it is important to note that these spectral fit values come from time integrated spectra within time \( T_{90} \), while within each pulse migration of peak energy \( E_p \) from higher to lower values is observed.

Strong indications of cosmological origins of GRBs, due to positioning performed by BATSE showing isotropic distribution of events, has lead Meszaros and Rees (1997) \cite{178} to predict the existence of afterglow from soft X-ray, through optical till even radio wavelengths emerging respectively after GRB prompt hard X-ray emission within hours, days or weeks. Giant leap toward confirmation of cosmological origins of GRBs was taken with the launch of Dutch-Italian X-ray telescope BeppoSAX on April 30th 1996. With five instruments covering energy range \( 0.1\,\text{keV} \lesssim E \lesssim 300\,\text{keV} \) and significantly upgraded angular resolution of approximately \( 1'' \) inside \( 0.1\,\text{keV} \lesssim E \lesssim 10\,\text{keV} \) energy band, BeppoSAX managed to detect GRB 970228 as the first ever burst whose afterglow was observed. Due to upgraded precision in positioning of GRBs by BeppoSAX, host galaxy of GRB 970228 was also identified with corresponding redshift of \( z = 0.695 \) determined through ground based optical observations with Keck telescopes. Little more than a month later BeppoSAX detected GRB 970508, second GRB with observed afterglow, not only in X-ray and optical band, but also at radio frequencies. What makes GRB 970508 special is the fact it was the first GRB whose redshift \( z = 0.835 \) was directly measured using interstellar and intergalactic absorption models which finally coincided with the measured redshift of the host galaxy. These findings with subsequent further measurements of redshift from many later bursts confirmed beyond any doubt the cosmological origin of GRBs and put strict requirements on their energetics. Another major finding of BeppoSAX era was the confirmation of GRB-SNe associations for some long GRBs indicating crucial ingredients for theoretical understanding. First such instance was the detection of the famous GRB 980425 which was followed by a corresponding detection of a bright type Ic supernova SN 1998bw revealing the connection of GRBs and deaths of massive stars \cite{101}. Further confirmations
of GRB-SNe connection were established also due to observations by HETE-2 (e.g. GRB 030329 and SN 2003dh) and INTEGRAL (e.g. GRB 031203 and SN 2003lw) satellites.

Although all aforementioned missions gave significant contributions to the understanding of GRBs, scientific community saw the need to launch a dedicated GRB observatory carefully prepared for GRB observation. This came to fruition with the launch of *Swift* observatory in 2004. *Swift* observatory, still in use now, was designed as GRB dedicated observatory with three installed instruments aboard dedicated to different photon energy bands of interest: BAT dedicated to fast detection and positioning with accuracy within $1' - 4'$ range covering photon energy range $15\,\text{keV} \lesssim E \lesssim 150\,\text{keV}$; XRT whose purpose is GRB imaging and spectral analysis within $0.2\,\text{keV} \lesssim E \lesssim 10\,\text{keV}$ energy band with typical positioning accuracy of $2''$; and UVOT being 30 cm diameter telescope with a grism for spectral analysis within UV and optical bands $170\,\text{nm} < \lambda < 650\,\text{nm}$. Immense capabilities of *Swift*, still in use today, made detection of faint afterglows of short GRBs possible leading to the identification of host galaxies and their relative location to them (e.g. GRB 050509B, GRB 050709, GRB 050724) placing them in regions with little or no star formation like large elliptical galaxies or central regions of large galactic clusters. Compared with host galaxies of long GRBs with active star formation, this put a clear distinction between evolutionary paths of short and long GRBs. Although initial observations of GRB 990510 have suggested the existence of an 'achromatic' jet break in afterglow as a consequence of collimated relativistic jet, observations by *Swift* suggest either absence of jet breaks or more complex 'chromatic' behaviour suggesting more complicated afterglow physics then initial models offered.

Complementary to *Swift* observatory *Fermi* satellite was launched in 2008 with LAT instrument covering 20% of sky within energy range $20\,\text{MeV} \lesssim E \lesssim 30\,\text{GeV}$ using pair conversion detector and GBM instrument covering two energy ranges, one being $8\,\text{keV} \lesssim E \lesssim 1\,\text{MeV}$ and another being $150\,\text{keV} \lesssim E \lesssim 30\,\text{MeV}$ using scintillation detectors. Since both LAT and GBM cover more than 7 orders of magnitude in energy it was made possible to investigate broad band spectra of GRB prompt emission in great detail. Refined spectral analysis using *Fermi* data revealed presence of more complicated spectral features suggesting that GRB spectra are superposition of probably three different components: the traditional Band-function non thermal component given by equation (1.1), a quasi-thermal component present in some GRBs and sometimes an underlying power law component extending to the high energy range, e.g. like in cases of GRB 090902B and GRB 090510 (see [323] and references therein). Another major finding of *Fermi* was the delayed emergence of GeV emission in respect to MeV emission and existence of long-lasting GeV afterglow
1.1 Introduction

in some GRBs.

It is important to note that besides Swift and Fermi instruments collecting crucial data in mostly X-ray and \( \gamma \)-ray domain, Gamma-ray Coordinates Network also known as GCN, coordinates their observations with various other space based and ground based observatories. Optical observations are crucial for sub-arcsecond precision positioning of GRBs and detailed investigation of GRB afterglows in optical band and its host galaxy. Commonly used instruments in optical band include telescopes and telescope networks like PROMPT, ROTSE, MASTER, VLT, LBT, Keck telescopes etc. Observations of GRB afterglows at radio wavelengths, commonly emerging within weeks and often lasting for years after GRB prompt emission, are mostly performed by radio telescopes like VLBI, VLA, ATCA, WSRT, GMRT, AMI etc. Due to large baselines of many of these instruments, radio telescopes can offer unprecedented angular resolution within milliarcsecond range offering not just remarkable positioning of GRBs on sky but also the possibility of measuring the angular sizes of emitting region in afterglow phase.

Alongside electromagnetic emissions, GRBs are expected to emit copious amounts of high-energy neutrinos, ultra high energy cosmic rays (UHECR) and gravitational waves (GW). Presence of UHECR coming from GRBs can be inferred from the shape of cosmic ray energy spectra but due to significant deflection of charged particles by galactic and intergalactic magnetic fields it is not possible to assign them to a specific source. As for neutrinos, significant efforts were made within IceCube Neutrino Observatory at South Pole in search of coincidental neutrinos within high energy range from approximately \( 10^{12} \) till \( 10^{15} \) eV coming from GRBs. Owing to the lack of positive detection up till 2019, rigid upper limits of neutrino flux from GRBs have been placed posing additional constraints on GRB physics \[1, 2\].

First detection of binary black hole (BH) merger on September 14th 2015 (announced on February 11th 2016) by GW detectors of Virgo/LIGO collaboration paved the way to the new era in GW astronomy - making GWs not just a part of theoretical but also of observational astrophysics. As expected substantial attention went to possibility of detection of GWs coming from short GRBs formed by mergers of binary neutron stars (NS). A major announcement in this direction was the one presented by Fermi and Virgo/LIGO teams reporting the direct association between a GRB and a GW event in case of famous GRB 170817A \[3\]. Nevertheless there are still some open questions and doubts regarding this association between GRBs and GWs leaving the topic open for debate, see e.g. \[223, 242\] and references within.

While rest-frame energies of photons greater than 100 GeV were already confirmed by Fermi observatory for several GRBs (e.g. GRB 080916C, GRB 090510, GRB 130427A) posing significant constraints on GRB physics considering its dynamics,
particle acceleration and radiation mechanisms, detection of GRB radiation within TeV range although expected from theoretical considerations presented itself as a serious observational challenge due to EBL scattering. Decade long efforts by Major Atmospheric Gamma Imaging Cherenkov Telescopes (MAGIC) finally came to fruition with detection of 0.3 TeV photons coming from GRB 190114C, making it the first GRB ever detected in that energy band.

1.2 GRB origins

Confirmations of cosmological distances from GRBs through measurements of their redshift (0.0084 ≤ z ≤ 9.4) give us stringent limits on estimates of their energies and luminosities. Commonly in GRB physics luminosity and energy estimates are given in form of isotropic luminosity \(L_{\text{iso}}\) and isotropic energy \(E_{\text{iso}}\), which assume isotropic emission by the source. Since photon energies are generally in range from 10 keV till 100 MeV with typical observed photon fluxes being within \(0.01 \text{ cm}^{-2}\text{s}^{-1} \leq \Phi \leq 100 \text{ cm}^{-2}\text{s}^{-1}\) range subsequent estimates of isotropic luminosity can be found within \(10^{47} \text{ erg s}^{-1} \lesssim L_{\text{iso}} \lesssim 10^{54} \text{ erg s}^{-1}\) with corresponding values of isotropic energy within \(10^{49} \text{ erg} \lesssim E_{\text{iso}} \lesssim 10^{55} \text{ erg}\) range. It is very important to note that electromagnetic emission is not the only channel through which the energy of GRB is released.

Hints to the possible progenitors of GRBs can be found in the observation details of different types of GRBs. Short GRBs and their afterglows are notably fainter being within the energy range \(10^{49} \text{ erg} \lesssim E_{\text{iso}} \lesssim 10^{52} \text{ erg}\) and have no spectral lag (delay time between arrival of high energy photons relative to low energy photons) compared to the ones of long GRBs characterized by a positive spectral lag. Alongside it the median redshift of short GRBs \(z \sim 0.5\) is much smaller then the one of long GRBs being \(z \sim 2\). Another major point for understanding the progenitors of GRBs is the difference of environments where such different types of GRBs arise. Short GRBs are found to be connected with older stellar populations due to their association with both early and late-type host galaxies, often ellipticals, within regions with no or little star formation. On the other hand long GRBs were systematically associated with regions having active star formation, be it dwarf, blue, irregular galaxies of star forming regions of spiral galaxies and in many cases are linked with a corresponding unusually energetic supernova (SN) of type Ic (hypernova, HN) \[101\]. Type Ib and Ic SN are known to have unique spectral characteristics due to absence of hydrogen (H) and helium (He) lines in their spectra. Progenitors to such SNe are presumed to develop through two possible channels, either from a single massive Wolf-Rayet WR star losing it outer envelope through stellar winds \[21\] \[104\] or from
a star in close binary system losing its H/He envelope due to tidal stripping by a companion star or their common envelope phase \[195\]. Second channel, arising within binary systems, has been shown to be much more probable then the first one \[269\]. Moreover, presence of broad-lines in the spectra of HNs associated with GRBs implies energy injection beyond that of a normal type Ic SN \[166\].

In some cases distinction between short and long GRB can sometimes be blurred. Instances of GRBs having short and long GRB properties have been observed like in case of GRB 060614 which in spite of having long duration has all other characteristics of a regular short GRB, namely: a short-hard spike at the beginning of the lightcurve; location within a region with low star formation rate and very short spectral lag.

Existence of exceptionally uncommon ultra-long GRBs, with time \(T_{90}\) being in range from \(\sim1000\) s and going beyond \(\sim10\ 000\) s, has been confirmed through observations of notable examples like GRB 101025A, GRB 111209A and GRB 130925A. Nevertheless it is still unclear whether they form a separate class of GRBs emerging from a different kind of progenitor like a very massive blue giant star or are just the end-tail part of long GRB distribution.

### 1.3 GRB progenitors

The reported immense energetics of GRB events, making them the brightest explosions observed in the Universe, characterized by the release of approximately one solar mass-energy equivalent within the time of seconds - suggest a probable deep connection between formation and activity of stellar mass black holes with GRBs.

The progenitors of GRBs are not yet identified but aforementioned observational properties can give us numerous indications toward their understanding. On the other hand theoretical framework using toy models (e.g. collapse of massive star, compact star merger, mass transfer in close binaries) solved either analytically or numerically can give us significant clues whether or not GRBs can emerge from such systems.

#### 1.3.1 Collapsar model

*The collapsar model* is the most commonly used explanation for the long GRBs which states that they come from the deaths of single massive stars \[309\]. Proposed candidates for such scenario are massive WR stars cause of observed lack of H and He lines in the spectra of SNe connected with GRBs. According to this scenario at the final stage of their evolution massive core of WR star collapses to form a BH surrounded with high density accretion disc. As a consequence of the large
angular momentum stored in the core, bipolar relativistic jets are formed alongside rotation axis pushing matter away. Propagation of such relativistic jets through the outer stellar envelope is expected to form relativistic shock wave at the front. Time variability of GRB prompt emission within collapsar model is described as a consequence of jet modulation by surrounding matter leading to variable Lorentz factors necessary for creation of internal shocks. Another expected consequence of jet interacting with surrounding medium is the formation of so called "cocoon" on jets sides within the star, with possible outcome of jet collimation [53]. Also not all jets are expected to successfully drill through the stellar envelope, some might get choked by surrounding medium [54].

1.3.2 Compact star mergers

NS-NS and NS-BH mergers [188] were quite rapidly accepted by scientific community as the progenitors of short GRBs due to their connection with elliptical or early type galaxies low star formation rate and no SN association. Binary NS-NS systems, already observed in our Galaxy, have decaying orbits due to gravitational radiation emission and are destined to merge [122, 280]. On the other hand there is still no observation of NS-BH binary systems but in any case orbital decay like in case of NS-NS binaries is expected. Mergers of NS-NS binary systems are expected to produce approximately $5 \times 10^{53}$ erg of energy released mostly in the form of low energy neutrinos and gravitational waves. Nevertheless it is not yet clearly understood how this mergers generate relativistic winds necessary to power the bursts.

1.3.3 Induced gravitational collapse

Alongside short GRBs whose binary compact star progenitors have been widely and rapidly accepted by the scientific community, there are strong indications that also long GRBs originate from binary progenitors. Significant support for long GRB binary progenitor scenario can be found in systematic and spectroscopic analysis of GRBs associated with type Ib/c SNe, also known as GRB-SNe, starting with pioneering discovery of temporal and spatial correspondence between GRB 980425 [205] and SN 1998bw [101] and further confirmed by association of many other close-by GRBs with corresponding type Ib/c SNe [56, 79]. Motivation for introduction of such an alternative model invoking a binary star progenitor for long GRBs can be found in some of the known disadvantages of traditional collapsar scenario, namely:
• Absence of hydrogen and helium lines in SNe type Ic spectra indicates that type Ic SNe originate from either helium stars, carbon-oxygen cores (CO\textsubscript{core}) or Wolf-Rayet (WR) stars whose outer layers have been stripped away (see e.g., \cite{271}, and references therein). Contrary to the collapsar model which invokes single star evolution leading to stripped envelope, it has been shown that pre-SN star rather belongs to a tight binary with a compact companion (e.g. NS). In that case outer envelope is stripped away by a compact companion via mass transfer and tidal effects \cite{190, 123, 98, 315, 271}.

• While energetics of GRBs point toward the creation of stellar mass black hole through gravitational collapse, associated SNe most probably do not lead to the formation of BH but rather a new neutron star ($\nu$NS). Systematic observations of pre-SN stars disfavor the direct formation of BH by a core collapse SN as a result of their estimated masses $M \lesssim 18M_\odot$. According to the current theoretical understanding such pre-SN should not lead toward a direct collapse to a BH \cite{269, 270}.

• Another drawback of aforementioned scenario is the implication of dense wind-like circumburst medium (CBM) not in concordance with the ones observed for most GRBs \cite{125}. Notably, GRB afterglows place average baryon density of CBM within order of $\sim 1$ cm\textsuperscript{-3} \cite{224}. This baryonic matter component is described by the value of baryon load parameter $B$ \cite{234}. Non-thermal component of GRB’s prompt $\gamma$-ray emissions requires expansion at ultra-relativistic velocities with Lorentz factors $\Gamma \gtrsim 100$ placing a strict limit on amount of baryons immersed within GRB’s $e^+e^-$ plasma \cite{264, 208, 176}. Therefore the maximum value of the baryon-load parameter is placed within range $B = M_Bc^2/E_{e^+e^-}^{\text{tot}} \lesssim 10^{-2}$, with $M_B$ being the baryon matter mass and $E_{e^+e^-}^{\text{tot}}$ being the total energy of the $e^+e^-$ plasma \cite{234}.

• Study of the thermal component from X-ray flares occurring in the early afterglow phase, at rest-frame times $t \sim 10^2$ s, point out toward a smaller emitter ($R \sim 10^{12}$ cm) expanding at mildly relativistic velocity with Lorentz factor $\Gamma \lesssim 4$ \cite{243}. These findings do not comply with the “collapsar/fireball” scenario where an ultra-relativistic emitter (the jet) with $\Gamma \sim 10^2$–$10^3$ is present from the prompt emission till the afterglow.

Aforementioned arguments, challenging the paradigm of single star progenitors for long GRBs and associated SNe, lead to the introduction of \textit{induced gravitational collapse} (IGC) paradigm \cite{228, 238}. Initial theoretical formulation and approximate analytical computations within the IGC paradigm were introduced by Rueda &
Figure 1.3. Representation of induced gravitational collapse (IGC) scenario (as shown on Figure 1 in [96]). As the result of supernova (SN) explosion of the CO\(_{\text{core}}\), neutron star (NS) companion will accrete part of the SN ejecta arriving within its Bondi-Hoyle radius. Consequently, the NS reaching the critical mass due to accretion will gravitationally collapse into a black hole (BH), initiating the emission of a GRB. Infalling material is decelerated and shocked, piling over the surface of NS. Most of energy gained due to gravitational infall is radiated away in form of neutrinos within neutrino emission zone. Coalescence of infalling matter with NS is maintained through the reduction of matter’s entropy due to neutrino emission above NS surface. For further details, see [96, 23, 22].

Ruffini (2014) [219], followed by numerical simulations performed for the first time by Fryer et al. (2014) [96].

According to the current form of IGC scenario, clearly addressing the GRB-SN connection, IGC progenitor is a tight binary system (orbital period of the order of several minutes) composed of a CO\(_{\text{core}}\) and a magnetized NS companion (\(B \sim 10^{13}\) G) [238] (see Figure 1.3). IGC scenario posits that at the end of thermonuclear evolution the iron core of pre-SN star (CO\(_{\text{core}}\)) reaching Chandrasekhar limit gravitationally collapses forming a new neutron star (\(\nu\text{NS}\)). A strong outbound shock wave, carrying \(E_{\text{kin}} \sim 10^{51}\) erg of kinetic energy, is formed due to the recoil from \(\nu\text{NS}\) and further neutrino heating. Consequently the shock wave ejecting the outer layers of CO\(_{\text{core}}\) (hereafter SN ejecta) emerges as a type Ib/c SN [17] triggering an accretion process onto a NS companion.

Binary systems in question are very tight, having binary separation of the order of \(a \sim 10^{11}\) cm. Hence the initiated accretion of SN ejecta onto NS companion is hypercritical (i.e. highly super-Eddington). The hypercritical accretion is maintained through abundant emission of neutrino-antineutrino (\(\nu\bar{\nu}\)) pairs produced by the annihilation of electron-positron (\(e^+e^-\)) pairs thermally produced within the part of accretion region close to NS surface. Amount of mass NS companion accumulates due to accretion is dependant on several system specific parameters like mass of
### 1.3 GRB progenitors

<table>
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<tr>
<th>Class</th>
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<th>Previous Alias</th>
<th>Number</th>
<th>In-State</th>
<th>Out-State</th>
<th>$E_{p,i}$ (MeV)</th>
<th>$E_{iso}$ (erg)</th>
<th>$E_{iso,GeV}$ (erg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binary-driven</td>
<td>I</td>
<td>BdHN</td>
<td>329</td>
<td>CO$_{core}$-NS</td>
<td>νNS-BH</td>
<td>~0.2–2</td>
<td>$10^{50}$–$10^{51}$</td>
<td>$10^{34}$</td>
</tr>
<tr>
<td>Hypernova</td>
<td>II</td>
<td>XRF</td>
<td>(30)</td>
<td>CO$_{core}$-NS</td>
<td>νNS-NS</td>
<td>~0.01–0.2</td>
<td>$10^{50}$–$10^{52}$</td>
<td></td>
</tr>
<tr>
<td>(BdHN)</td>
<td>III</td>
<td>HN</td>
<td>(19)</td>
<td>CO$_{core}$-NS</td>
<td>νNS-NS</td>
<td>~0.01</td>
<td>$10^{50}$–$10^{50}$</td>
<td></td>
</tr>
<tr>
<td>Merger</td>
<td>IV</td>
<td>BH-SN</td>
<td>5</td>
<td>CO$_{core}$-BH</td>
<td>νNS-BH</td>
<td>≥2</td>
<td>$10^{54}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.1. Summary of the updated list of γ-ray burst (GRB) subclasses as presented in Wang et al. (2019) [300]. All the GRB subclasses are placed within two major groups of phenomena, namely *binary driven hypernovae* (BdHNe) and *binary mergers* (BMs). “Type” column denotes the newly adopted classification scheme of GRBs, while “Previous Alias” column presents the names used in previous literature. The fourth “number” column gives the number of GRBs with a known redshift identified in each subclass updated by the end of 2016 (the values in the brackets indicate the lower limits). System progenitors are represented by “in-state” while the “out-state” column represents the outcomes of different scenarios. Remaining columns give out the expected physical quantities like the peak energy of the prompt emission $E_{p,i}$, the isotropic γ-ray energy $E_{iso}$ radiated in the 1 keV $\leq E \leq 10$ MeV energy range, and the isotropic energy of ultra-high energy (UHE) photons, $E_{iso,GeV}$, emitted within 0.1 GeV $\leq E \leq 100$ GeV energy range. While kilonova (KN) can be an infrared/optical counterpart to NS-NS merger powered by the energy release from the decay of r-process heavy nuclei processed in the merger ejecta [159, 181, 279, 333], a *fallback-powered kilonova* (FB-KN) [223, 222] arises due to the binary white dwarf (WD) merger producing an infrared/optical transient powered by the fallback accretion of matter onto the newborn massive white dwarf (MWD).

Ensuing fates of the NS companions resulting from the hypercritical accretion are determined by whether or not they reach the critical mass to gravitationally collapse into a BH, therefore offering two possible scenarios [22, 224]. In case of tighter binaries, having shorter orbital periods of roughly $T \lesssim 5$ min, the NS companion accumulates enough matter to reach the critical mass leading toward the gravitational collapse of NS into a BH [25, 22, 224]. These events originally denoted as BdHNe are now classified as BdHNe of type I (see Table 1.1) [300]. On the other hand in the case of looser binaries, with longer orbital periods, hypercritical accretion onto NS companion is not sufficient enough to reach critical mass leading to the formation of more massive neutron star (MNS). This type of events originally denoted as *X-ray flares* (XRF) are now classified as BdHNe of type II [300].

Beside aforementioned CO$_{core}$-NS binaries there is another subclass of binary progenitors composed of CO$_{core}$ (or Helium or Wolf-Rayet star) and a BH. In that case a part of expanding SN ejecta is hypercritically accreted onto the BH already formed via prior evolution of the binary system. It has been suggested that this systems might be X-ray binaries like Cyg X-1 at their late evolutionary stages [106, 27].
microquasars [182] or the end result of the binary system’s evolution of following the evolutionary scenario XI presented by Fryer et al. (1999) [97]. Provided the binary system remains bound after the SN, either $\nu$NS-BH or BH-BH binary can be formed depending upon whether or not the SN central object forms $\nu$NS or collapses into a BH [269, 270]. These extremely energetic events $E_{\text{iso}} \gtrsim 10^{54}$ erg, previously named black hole-supernova (BH-SN), are now classified as BdHNe of type IV (see Table 1.1).

Alongside different types of BdHNe, several subclasses of binary merger (BM) scenarios were introduced to concisely describe differences among short $\gamma$-ray bursts. This lead toward the total of nine subclasses of GRBs. For further research details see Table 1.1 and [241, 246, 221].

1.4 GRB central engine

Notwithstanding identification of different types of GRB progenitors - there is still a possibility they may be powered by a common central engine. One of the leading candidates for GRB central engines are rapidly spinning BH surrounded by a torus of accreting material. The black hole-torus central engine was extensively discussed within the frameworks of both collapsar and compact stars (e.g. NS-NS and NS-BH) mergers. There are two major processes through which energy is released in the case of BH-torus central engine. First one is the accretion power due to the torus which can drive a hot jet along the spin axis coming from neutrino annihilation within the torus. Associated accretion powered jet luminosity $L_{\text{acc}}$ is consequently given by

$$L_{\text{acc}} = \zeta \dot{M} c^2 \sim 1.8 \times 10^{54} \text{erg} \text{s}^{-1} \zeta \dot{M},$$

(1.2)

where $\zeta$ is the efficiency factor of converting accretion power into jet power, $\dot{M}$ is the accretion rate and $\dot{M} = \dot{M} / (M_{\odot} \text{s}^{-1})$ is the accretion rate given in units of solar mass per second. Observed GRB luminosities require accretion rates in range $0.1 \lesssim \dot{M} \lesssim 1$ assuming reasonable values of efficiency $\zeta$. Another source of energy in BH-torus central engine paradigm is the spin energy of the black hole which can be extracted when magnetic fields thread through the ergosphere of Kerr BH via Blandford-Znajek (BZ) mechanism. The jet luminosity from BZ mechanism can be estimated as

$$L_{\text{BZ}} \simeq \frac{G^2 B^2 M_{\text{BH}}^2 \Omega_{\text{BH}}^2}{10^5 c^3} \simeq 3.3 \times 10^{49} \text{erg} \text{s}^{-1} \left(\frac{B}{10^{15} \text{G}}\right)^2 \left(\frac{M_{\odot}}{M_{\text{BH}}}\right)^2 a^2 f(a),$$

(1.3)

where $B$ is magnetic field, $M_{\text{BH}}$ is the BH mass and $\Omega_{\text{BH}}$ is the angular velocity, while $a$ is the dimensionless spin parameter of BH and $f(a)$ is the increasing function of
1.4 GRB central engine

Observed GRB luminosities require rapidly spinning BHs \((a \lesssim 1)\) with high accretion rate building up strong enough magnetic fields \((B \gtrsim 10^{15} \text{ G})\) near BH horizon. Certainly, due the expected strong magnetic fields within the jet, this central engine is likely Poynting-flux-dominated. Surely, it is expected that both neutrino annihilation within the torus and BZ mechanism near BH are active processes in case of BH-torus central engine. Variability in GRB behaviour can either come from interplay between magnetic field and accreting material or from interaction of the jet with instabilities within the stellar envelope creating variable jet outflow.

Further considerations have lead to the proposition of magnetars, highly magnetic NS with surface magnetic fields within \(10^{14} \text{ G} \lesssim B \lesssim 10^{15} \text{ G}\) range, as possible candidates for GRB central engines [287]. Major source of power in case of magnetar central engine is the spindown power which assuming dipole spindown model can be written as

\[
L_{\text{sd}} = \frac{B^2 \Omega^4 R^6}{6 c^5} \sim 3.7 \times 10^{50} \text{ erg s}^{-1} \left(\frac{B}{10^{15} \text{ G}}\right)^2 \left(\frac{\Omega}{10^4 \text{ s}^{-1}}\right)^4 \left(\frac{R}{10^6 \text{ cm}}\right)^6 , \tag{1.4}
\]

with \(B\) as magnetar’s surface magnetic field, \(\Omega\) as its angular velocity and \(R\) as its radius. Assuming such a system powers a GRB it is necessary that both surface magnetic field with \(B \gtrsim 10^{15} \text{ G}\) and angular velocity with \(\Omega \gtrsim 10^4 \text{ s}^{-1}\) attain quite high values, with angular velocity being quite close to the upper limit of NS stability. Although this mechanism seems capable to explain the bulk of GRB events, since total spin energy of magnetar is approximately \(E_{\text{rot}} \sim 5 \times 10^{52} \text{ erg}\), it is challenged by more energetic GRB events [291].

Another group of promising candidates for GRB central engines are strange quark stars (QS) since strange quark matter (SQM) could be more stable then the neutron matter of NS [61, 69, 193, 192]. In the framework of quantum chromodynamics (QCD) strange quark matter can form within high density/pressure environments, expected in massive compact stars, following phase transitions from neutron matter to 2-flavour quark matter (u and d quarks) and then from 2-flavour to 3-flavour strange quark matter (u, d and s quarks) finalizing with quark matter condensation. All this phase transitions introduce extra energy input capable to power GRB phenomena. Another benefit of QSs comes from the fact they are bound by strong interaction rather than gravity, by which they can release neutrinos and photons without contaminating the fireball contrary to magnetars and their proto-neutron star (PNS) phase.

As a consequence of the jet outflow paradigm frequently invoked in the aforementioned GRB central engine scenarios, a jet opening angle \(\theta_j\) is used in order to relax the necessary energy requirements. Subsequent intrinsic source energy
\[ E_s = f_b E_{iso} < E_{iso}, \text{ where } f_b = (1 - \cos \theta_j), \] can be significantly reduced compared to the observed isotropic energy \( E_{iso} \) due to extremely small suggested jet opening angles (e.g., \( 1^\circ \lesssim \theta_j \lesssim 5^\circ \)). As a result of this assumption observed isotropic energy of \( E_{iso} \sim 10^{54} \text{ erg} \) can be reduced to the expected intrinsic source energy of \( E_s \approx E_{iso} \theta_j^2 / 2 \sim 10^{51} \text{ erg} \). However it is important to mention that there is still no solid observational verification whether such narrow jet opening angles exist [68, 259, 55].

Likewise, energy budget of a BH was already diligently investigated by Christodolou and Ruffini (1971) [65] giving rise to a general mass-energy formula of a Kerr-Newmann BH characterized by the mass \( M_{BH} \), the charge \( Q \) and the angular momentum \( L \) as follows

\[
E_{BH}^2 = M_{BH}^2 c^4 = \left( M_{irr} c^2 + \frac{Q^2}{2\rho_+} \right)^2 + \frac{L^2 c^2}{\rho_+^2}, \tag{1.5}
\]

where \( E_{BH} \) is the energy of the BH, \( M_{irr} \) is the irreducible mass of the BH and \( \rho_+ \) is the horizon radius giving rise to the horizon surface \( S \) written as

\[
S = 4\pi \rho_+ = 16\pi \frac{G^2}{c^4} M_{irr}. \tag{1.6}
\]

Upper limits to the energy stored within the BH are fixed by following condition

\[
\frac{1}{\rho_+^4} \frac{G^2}{c^8} (Q^4 + 4L^2c^2) \leq 1. \tag{1.7}
\]

Afterwards Damour and Ruffini (1975) [74] within their pioneering work speculated about a mechanism, by which vacuum polarization within the critical electric field \( E_c = m_e^2 c^3 / \hbar e \) [200] around Kerr-Newmann BH should lead to the \( e^+e^- \) pair plasma creation, as a possible way to explain an immense release of energy in \( \gamma \)-rays as observed in GRBs delivering as much as

\[
E_{max} = 1.1 \times 10^{54} \frac{M_{BH}}{M_\odot} \text{ erg}, \tag{1.8}
\]

with \( E_{max} \) being the maximum extractable energy from the BH and \( M_\odot \) being the mass of the Sun. Compared to the energetics of previous central engine scenarios we can see that energy prescribed by equation (1.8) appears to be large enough to account for observed emission of GRBs avoiding the need to introduce an collimated emission from a jet in order to relax the energy requirements imposed by \( E_{iso} \).
1.5 Physical background of GRBs

From observations of GRBs it is quite clear that whatever causes GRBs releases an immense amount of energy within a small region and short timescales. While the possible sources of such energy releases can be concisely indicated, as noted in Section 1.4 dealing with GRBs central engines, details describing how this energy is transformed into radiation of the right fluence, spectra and time evolution needs still to be adequately addressed. Although many models have been proposed during the years, the model mostly used at present is the so-called 'standard' fireball model \[207\] which will be presented first in this Section. Later developments in GRB understanding lead toward the introduction of fireshell model \[227, 237\] as a valuable alternative to the fireshell model addressing several of its weaknesses. Fireshell model is presented in second part of this Section.

1.5.1 Fireball model

Basic assumption of the fireball model is the deposition of an immense amount of energy by the GRB central engine of the order \(E_s \sim 10^{51}\) erg within the region around the central engine of approximate size \(r_0 \sim 10^6 - 10^7\) cm (e.g. inner radius of accretion disc around BH) leading toward the creation of fireball jet composed of photons, leptons (electrons and positrons) and few baryons (mostly protons). Compactness problem for such scenario, emerging from huge opacity of dense \(e^+e^-\) plasma for ambient photons, is circumvented by the introduction of low baryon loading leading toward necessary ultra-relativistic expansion of the fireball. Presence of baryons within fireball plasma is obligatory \[264\], since the original model of photon-lepton (baryon free) fireball plasma \[58, 112\] did not survive the observational scrutiny.

Dynamical evolution of the fireball and its observational footprint can be described in following five stages (see Figure 1.4):

1) **Energy injection**

   As already noted before the central engine of GRB deposits a huge amount of energy within a small volume. How exactly this happens is not addressed by the fireball model and can be a consequence of many different central engine models. Commonly used value to describe the energy deposition by the central source is photon-to-baryon ratio \(\eta\) defined as the ratio between the total internal energy of the fireball \(E_s\) and rest mass energy \(M_Bc^2\) of baryons engulfed within it

   \[
   \eta \equiv \frac{E_s}{M_Bc^2} = B^{-1}. \tag{1.9}
   \]

   Major requirement put upon the central engine is that it can inject enough
energy for $\eta \gtrsim 10^2$ and therefore enable the ultra-relativistic expansion of fireball.

2) **Ultra-relativistic expansion**

Optically thick fireball plasma accelerates to ultra-relativistic velocities under its own pressure. During expansion internal energy of the fireball plasma is converted into the bulk kinetic energy of baryons. Accelerated expansion lasts till the fireball attains coasting Lorentz factor $\Gamma_c \approx \eta$ at distance $r_c \approx r_0\eta$ from the central engine. Afterwards fireball coasts with the constant Lorentz factor. Invoking the conservation laws of energy and entropy following scaling laws emerge

\[
\begin{align*}
\Gamma(r) &\propto \begin{cases} r^1, & r \leq r_c \\ r^0, & r > r_c \end{cases}, \\
T(r) &\propto \begin{cases} r^{-1}, & r \leq r_c \\ r^{-2/3}, & r > r_c \end{cases}, \\
n(r) &\propto \begin{cases} r^{-3}, & r \leq r_c \\ r^{-2}, & r > r_c \end{cases},
\end{align*}
\]

with $\Gamma(r)$ being the fireball Lorentz factor, $T(r)$ being the temperature in the fireball rest-frame and $n(r)$ being the comoving baryon number density.

3) **Preburst (Photospheric emission)**

While the rest-frame temperature drops below $kT\approx 20$ keV last of electron-
1.5 Physical background of GRBs

Positron pairs will annihilate, leaving only electrons associated with baryons (protons). Amidst further expansion the optical depth inside the fireball decreases, leading to the preburst emission of a fraction of initial thermal energy stored in photons once it becomes optically thin. Though presence of thermal component was detected for some GRBs by Fermi, spectra of most GRBs are exclusively non-thermal. Likewise, since most of fireball energy till this phase is transferred into baryon kinetic energy, photospheric thermal spectra can account only for a small number of photons.

4) **Internal shocks (GRB prompt emission)**

Referring to the violent and erratic behaviour of GRBs in action, it is reasonable to assume that the fireball outflow launched by the central engine is unsteady - leading to the formation of several shells of relativistic particles having slightly different Lorentz factors. Internal shocks are formed when the faster shell overtakes the slower one \[212\]. The estimated distances from central engine at which this internal shocks form can be obtained using following expression

\[
r_{\text{int}} \sim \Gamma^2 ct_v \sim 3 \times 10^{13} \left( \frac{\Gamma}{316} \right)^2 \left( \frac{t_v}{0.01s} \right) \text{cm},
\]

where \( \Gamma \) is the Lorentz factor of the slower shell and \( t_v \) is the variability timescale of the order of 1 ms, connected with the size scale of \( r_0 \). Using common fireball model parameters, expected distances for formation of internal shocks are in range of \( r_{\text{int}} \sim 10^{14} - 10^{15} \) cm. Due to such large distances and low ambient densities, these semi-relativistic internal shocks are assumed to be collisionless with amplified ambient magnetic field. Charged particles (mostly electrons) accelerated through the Fermi mechanism at the shocks in the end radiate synchrotron radiation observed as the GRB prompt emission. Compton processes can also become significant in some special cases. Nevertheless, although this scenario can easily explain the strong variability of \( \gamma \)-ray lightcurves observed in GRBs, the kinetic-to-radiation energy conversion ratio is rather low since at most half of shell’s kinetic energy can be extracted due to dynamical limitations on relative shell velocities \[130\].

5) **External shock (GRB afterglow)**

As the relativistic outflow dissipates its kinetic energy within external circum-burst medium (CBM), being either a constant density interstellar medium (ISM) or a stratified stellar wind, two shocks are formed: a long-lasting forward shock propagating outwards into the CBM and a short-lived reverse shock returning into the outflow \[177\]. Electrons accelerated by these shocks via Fermi acceleration mechanism form a power law energy distribu-
tion and dominantly emit synchrotron radiation. Commonly it is assumed that fixed fractions of energy go into magnetic field energy, electron energy etc. - leading toward the synchrotron radiation spectra composed of several power law segments \[ 258 \]. While the long lasting broad-band afterglow emerges from the forward shock \[ 178, 258, 71, 52, 121 \], important early contribution to optical and radio emission can be provided by the reverse shock \[ 178, 180, 256, 135, 137, 138, 511, 85, 102, 214, 322 \]. Although in sense similar to the mechanism of SN remnants interacting with surrounding ISM, in case of fireball scenario ejecta are still assumed to be relativistic during the afterglow phase. Since the major focus of this thesis is understanding and modelling of afterglow within fireshell/BdHN scenario it is beyond the scope of this thesis to deal with all the details of afterglow within fireball external shock model. Therefore we direct the reader toward very comprehensive and concise review of external shock models by Gao et al. (2013) \[ 102 \] and general theoretical overviews presented in \[ 144, 320 \].

Although use of the internal shocks scenario is widely accepted as a way to interpret GRB prompt emission, there is still no general consensus regarding how GRB prompt emission properties should be addressed. This arises from uncertainties in the physical understanding of GRB prompt emission \[ 319 \], notably: the question of jet composition (matter-dominated fireball or Poynting-flux-dominated outflow), energy dissipation and particle acceleration mechanisms (internal shocks or magnetic reconnection), and radiation mechanism (synchrotron or comptonization of quasi-thermal photons). This lead toward the introduction of two additional types of GRB models alongside internal shock models according to the location of emission site \[ 323 \], namely:

- **The dissipative photosphere models** presented in \[ 213, 198, 107, 31, 149, 282, 187 \], where GRB prompt emission arises as the emission of comptonized quasi-thermal photons from the photosphere. Erratic nature of the GRB prompt emission is attributed to the generalized central engine consisting of the central BH (powering the jet) and the surrounding stellar envelope (through which the jet propagates).

- **The forced magnetic dissipation models**, for example the *internal collision-induced magnetic reconnection and turbulence model* (ICMART) introduced by \[ 325 \], where the GRB prompt emission originates from the synchrotron emission by electrons accelerated through magnetic reconnection at larger emission radii. This models operate within significantly magnetized environments, with magnetization in range \( \sigma \geq 1 \) compared to \( \sigma \ll 1 \) of other two model
types. In such an environment, according to ICMART model, slow variability is expected to arise from the activity of central engine while rapid variability emerges from the mini-jets created by local reconnection regions inside the jet [326].

Nevertheless, it is noteworthy to say that the aforementioned models don’t offer any additional details about the nature of central engine and its energy extraction process, neither they try to address them.

1.5.2 Fireshell model

The fireshell scenario [226,227,228], initially introduced to explain the totality of GRB phenomena as emerging from the gravitational collapse leading to the formation of a Kerr-Newman (charged, rapidly spinning) BH, certainly presents itself as an valuable alternative to the “traditional” fireball scenario. Although it is generally accepted that GRBs arise from the formation of a BH, till now no single model has addressed the issue of energy extraction out of the BH apart from the fireshell model [227,237].

Central to the fireshell paradigm is the quantum electrodynamic (QED) process of vacuum polarisation within an overcritical electric field surrounding the Kerr-Newman BH leading to the creation of an immense amount of electron-positron ($e^+e^-$) pair plasma. As indicated by equation (1.8) this impulsive discharge is expected to release energy around the order of $\sim 10^{54}$ erg [74].

This isotropic and optically thick $e^+e^-$ pair plasma of total energy $E_{e^+e^-}^{\text{tot}}$, also known as the fireshell, starts to expand and accelerate under its own internal pressure [233]. Adjacent baryonic matter, quantified by the already mentioned baryon-load parameter $B = M_B c^2 / E_{e^+e^-}^{\text{tot}}$ (see Section 1.3), is consequently engulfed within the expanding fireshell [234,10,11]. Both parameters $E_{e^+e^-}^{\text{tot}}$ and $B$ fully determine the subsequent self-accelerated expansion of the optically thick fireshell plasma, which lasts until the conditions of transparency are satisfied. At that point a flash of thermal radiation, termed Proper-GRB (P-GRB), is emitted [233,234].

Only a fraction of the initial $E_{e^+e^-}^{\text{tot}}$ is emitted via the P-GRB, while the rest of it is stored as kinetic energy of the remaining optically thin baryonic and leptonic matter fireshell. The inelastic collision of the remaining fireshell with the circumburst medium (CBM) gives rise to the multi-wavelength emission of the so called “afterglow” GRB (A-GRB) [227,37,38,39,237]. A-GRB lightcurve in the different energy bands is determined by three quantities associated with the CBM environment, namely parameters describing the CBM density profile $n_{\text{CBM}}$, the filling factor $\mathcal{R}$ that accounts for the filamentary structure, and an index $\alpha$ that accounts for the modification of the low-energy part of the thermal spectrum [197].
The “canonical” lightcurve of a GRB is composed of two components: P-GRB indicating the event of BH formation [40, 230] and A-GRB showing a clear hard-to-soft behaviour [229, 36, 225]. For this reason, two spectral components are usually identified in the P-GRB spectra: the thermal component arising from the transparency condition of P-GRB, and the “non-thermal” component from the early onset of the extended A-GRB arising from the convolution of a large number of modified thermal spectra with various Lorentz factors and temperatures [38, 39, 197]. The energy of P-GRB ($E^{\text{P-GRB}}_{e^+e^-}$) is solely determined from the thermal component. The peak of the A-GRB contributes to what is usually known as the γ-ray prompt emission (see Figure 1.5) [227, 225, 73].

GRB classification within the fireshell model according to the assigned values of $E^{\text{tot}}_{e^+e^-}$, $B$ and $n_{\text{CBM}}$, offers a clear theoretical solution to the existence of intermediate class of short GRBs with extended emission.

The “canonical long” GRBs, showing the predominance of A-GRB compared with P-GRB, typically occur within the star forming regions with CBM average density being $n_{\text{CBM}} \approx 1 \text{cm}^{-3}$, and are characterized by total energy $E^{\text{tot}}_{e^+e^-} \gtrsim 10^{52}$ erg and baryon-load $B$ in the range $3 \times 10^{-4} \lesssim B \lesssim 10^{-2}$. On the other hand, “disguised short” GRBs, although being in the same range of total energy and baryon-load as “canonical long” emerge within much more tenuous regions having $n_{\text{CBM}} \approx 10^{-3} \text{cm}^{-3}$. Due to this, A-GRB shows a “deflated” emission occasionally exceeded in peak luminosity
Afterglow

1.6 Afterglow

Major topic of this thesis is the understanding of the afterglow emission within the BdHN scenario. For that reason and due to vastness of existing literature covering the topic an overview of the current afterglow models will be briefly presented. Generally speaking, models of the afterglow emission of long GRBs can be divided in two groups depending on the expansion rate of ejecta: ultra-relativistic (traditional/external shock scenario) and non-relativistic/mildly relativistic (BdHN scenario). In case of the traditional approach the ultra-relativistic motion of ejecta emitting radiation persists from the prompt emission all the way to the late afterglow phase. Depending whether this ultra-relativistic ejecta propagates through the homogeneous ISM or the stellar wind environment, different lightcurves and spectral signatures can be achieved. Another major aspect in traditional understanding and modelling of GRB afterglows is the presence or lack of a central engine. Two versions of the traditional afterglow models emerged: synchrotron emission from an external ultra-relativistic shock wave, without and with energy injection from a central engine. The models having no energy injection follow the approach presented in while the ones with energy injection follow from the proposals by by adopting the presence of energy injection from a long-lasting spin-down millisecond pulsar or a magnetar (magnetic dipole strength of the order of $10^{15}$ G). This approach attributes the presence of shallow decays or the plateaus observed at times $\sim 10^2 - 10^4$ s to the energy injected by the magnetic dipole radiation from magnetar. With this, magnetar model is consistent with the so-called “internal plateaus”, specifically the ones which end with a very steep decay slope and cannot be explained by a solely expanding external shock waves. Therefore, steep drop in the lightcurve of GRB afterglow is explained as originating from the pulsar/magnetar engine at the characteristic lifetime of magneto-dipole emission, leading to a sharp decrease of the energy injection. However, basic assumption of ultra-relativistic models seems not to hold up against the scrutiny of observational data since the observations of the X-ray flares in the early afterglow indicate a mildly-relativistic expanding emitter. In line with this observational fact, rather

by the P-GRB. Nevertheless the integrated emission of the A-GRB is much larger compared to P-GRB, as is expected for long GRBs. Last class are the “genuine short” GRBs characterized by high energy range $E_{e^+e^-}^{\text{tot}} \gtrsim 10^{52}$ erg, but also very low baryon-load $B \lesssim 10^{-5}$. In this case peak emission and flux of the A-GRB are significantly smaller compared to the ones of the P-GRB. This is expected for objects coming from NS-NS mergers leading toward the formation of BH.
than a priori considering an ultra-relativistic shock wave emission \([258, 318, 103]\), BdHN scenario considers the afterglow to originate from the synchrotron emission of the expanding HN ejecta, which is mildly-relativistic. Energy injection into the HN ejecta by the magnetic dipole and, in addition, by the magnetic quadruple emission from the newborn millisecond pulsar at the SN center, the \(\nu\)NS, are the major elements of the BdHN model. Quadruple component of magnetic field, expected in a newborn \(\nu\)NS from the theoretical point of view \([274, 220]\), extends the capacity of the pulsar energy injection to fit the evolution of the plateau and the late time decay. With this, there is no differentiation between “internal plateaus” since the pulsar injection is able to produce nearly all types of plateaus. Detailed description of the background of BdHN afterglow model, together with results fortifying its assessments \([231, 300, 220]\), can be found in Chapter 5.

1.7 BdHN overview

Since the understanding of the GRB afterglow within the framework of BdHN paradigm is one of the major points of this thesis, a detailed outline of the underlying BdHN processes is necessary for better understanding. Evolution and the final fate of BdHN progenitors was already presented in Section 1.3 as an outcome of the IGC paradigm. Support for the BdHN paradigm is built upon the great amount and quality of the GRB data in all wavelengths, revealing the richness of the GRB phenomenon. In few seconds, a BdHN covers a wide array of different physical regimes leading to a variety of observables at different times and at different wavelengths (see Figure 1.6 for details).

1.7.1 X-ray precursor

X-ray precursor is the first observable signature of the forthcoming BdHN \([124, 96, 22, 300]\) resulting from the SN shock breakout and the hypercritical accretion onto the NS companion until it reaches the critical mass.

Owing to two underlying time separated physical processes, it is composed of two spikes in its lightcurve as seen in case of e.g. GRB 180728A (a BdHN II) \([300]\). With GRB 180728A it has been possible, for the first time, to identify in a X-ray precursor the emergence of the SN shock wave from the CO\(_{\text{core}}\), that is the SN shock breakout.

First spike, identified with the SN shock breakout, lasted for 2 s (cosmological rest-frame time). Within that time interval, observed energy was estimated to be \(8 \times 10^{49}\) erg in the 1 keV-10 MeV energy band. This is well in agreement with what can be expected from the conversion of shockwave’s kinetic energy into electromagnetic
1.7 BdHN overview

Figure 1.6. Schematic representation of the timeline and the corresponding physical processes within a type I BdHN.
energy $^{17}$.

The second spike coming from the hypercritical accretion of the SN ejecta onto the NS companion, emerged at 10.29 s after the starting time of the first spike (8.72 s since the trigger time) and lasted for 13.82 s, while emitting $3 \times 10^{51}$ erg in the 1 keV–10 MeV energy band.

The time difference between these two spikes, amounting to $\Delta t \approx 10$ s, was then used to estimate the distance of binary components $\sim 3 \times 10^{10}$ cm. Hypercritical accretion with the rate $\sim 10^{-3} M_\odot$ s$^{-1}$ is induced once the SN ejecta have reached the NS companion. The spectral analysis of the second spike shows the evolution of a thermal component with a temperature and radius in agreement with the expectation from the expanding convective bubble on top the NS triggered in the accretion process by the Rayleigh-Taylor instability $^{124, 96, 22, 300}$.

1.7.2 Onset of the GeV and the prompt emission

As already mentioned in Section $^{1.3}$ in the case of BdHN I, hypercritical accretion onto the NS finally leads toward the gravitational collapse of the NS into the BH. Due to this processes the newly formed $\nu$NS-BH binary will be immersed within asymmetric SN ejecta having a “cavity” of $\sim 10^{11}$ cm of very low-density matter around the newborn BH $^{22, 24}$. During the gravitational collapse magnetic field frozen into the inflowing plasma rapidly increases to values $B \sim 10^{14}$ G as a result of magnetic flux conservation. This enhanced magnetic field together with rotation of the BH triggers the “Wald” process by which an electric field $E$ is induced $^{297}$. Initial overcritical electric field $E \geq E_c = m_e^2 c^3 / (e \hbar)$ around the BH produces a copious amount of self-accelerating electron-positron ($e^- e^+$) pair plasma via the process of vacuum polarisation within the framework of the quantum electrodynamics (QED) $^{235}$. This leads toward the emergence of ultra-relativistic prompt emission (UPE) in $\gamma$-rays at instance when the self-accelerating $e^- e^+$ pair plasma reaches transparency. Another major imprint of the induced $E$-field is the onset of GeV radiation due to synchrotron emission of protons within surrounding magnetic field, which were previously accelerated by the induced electric field $^{245}$.

Activity of the Wald process is determined by the presence and morphology of the SN ejecta around the BH $^{245}$, causing specific features of the UPE $^{243}$, and providing explanation for the specific power-law properties of the GeV emission lightcurves $^{245, 247}$.

Highly asymmetric morphology of SN ejecta around the newborn BH, following the moment of BH formation and the emergence of GRB emission, has been demonstrated by three-dimensional numerical simulations of BdHNe $^{22, 24}$. This
implies that the out-flowing $e^+e^-$ pair plasma has different dynamics along different directions since it engulfs different amounts of baryonic mass, leading to different signatures for different viewing angles [243].

The cavity of low baryonic contamination (low-density matter) formed around the newborn BH is the result of prior NS accretion and BH formation. For that reason, the $e^+e^-$ pair plasma expanding out of the newborn BH gains a high Lorentz factor $\Gamma \sim 10^2$–$10^3$ alongside direction away from the $\nu$NS. This explains the UPE phase.

Detailed hydrodynamic equations describing the motion of the $e^+e^-$ pair plasma engulfed with a low baryonic mass (with respect to the plasma energy), its self-acceleration and impact on the circumburst (CBM) filaments have been presented in [209, 232, 234] and successfully applied on numerous GRBs, e.g. GRBs 050904, 080319B, 090227, 090618 and 101023 [200, 125, 197, 186].

In addition to the previous phenomena, the propagation of the $e^+e^-$ plasma through the ejecta leads toward further observable effects discussed in the following subsections.

1.7.3 Early afterglow

Excluding the direction leading to the emergence of UPE, the $e^+e^-$ plasma collides with the more dense part of SN ejecta. This occurs at the cavity border, being approximately $10^{10}$–$10^{11}$ cm away from BH. Carrying a large amount of baryons this plasma reaches transparency at radii of the order of $10^{12}$ cm with a mildly relativistic Lorentz factor of $\Gamma \lesssim 4$ [243].

The emission of such mildly-relativistic plasma has been shown in [243] using the analysis of the observed thermal radiation in the early X-ray afterglow, in agreement with the occurrence of X-ray flares [275, 298]. From the observed thermal radiation component the expansion velocity of the emitter can be inferred. For example, in the early few hundreds of seconds, GRB 090618 shows a velocity of $\beta = v/c \sim 0.8$ [240, 191], GRB 081008 shows $\beta \sim 0.9$ [243] and GRB 130427A shows $\beta \sim 0.9$ [236, 301, 231].

1.7.4 Late afterglow

Energy and momentum injection by the $e^+e^-$ plasma into the SN ejecta transforms the SN into a HN [240, 248].

It has been demonstrated in [231] that the synchrotron emission by relativistic electrons, injected by the pulsar emission of the $\nu$NS (at expenses of its rotational energy) into the expanding HN ejecta magnetized by $\nu$NS, can explain both the early part and the late power-law behavior of the X-ray afterglow. Significant finding
is also that the magnetic field of the $\nu_{NS}$ pulsar has to have a dipole+quadrupole structure.

The cases of GRB 130427A (a BdHN I) and 180728A (a BdHN II) have been analyzed in [300] using the above model [231]. Observed afterglow data required an initial 1 ms $\nu_{NS}$ pulsar period in case of GRB 130427A, and a longer initial period of 2.5 ms for GRB 180728A. A simple analysis can confirm that this result is in concordance with the BdHN I and II nature of GRB 130427A and GRB 180728A respectively. In close binary systems the binary components are very likely tidally locked. Rotation period of the CO core is therefore known since the orbital period can be inferred by following the analysis of the X-ray precursor and the prompt emission [300]. Knowing this, and assuming the conservation of angular momentum during the core-collapse SN process, the rotation period of the $\nu_{NS}$ formed at the SN center can be estimated. This independent method reported a binary components separation in remarkable agreement with the one implied by the precursor and the prompt emission, which demonstrates the self-consistency of this picture. Modelling of the non-thermal afterglow emission from GRB 130427A [231], and its use as a prototype in cases of GRB 160625B and GRB 190114C [220], played a crucial role in formation of the current BdHN afterglow paradigm, and will be presented in detail within Chapter 5.

1.7.5 The optical SN

Presence of the SN has been confirmed in a form of an excess in the optical lightcurve at about two weeks after the SN explosion for some cases of both BdHN I and BdHN II. Certainly, the detectability of such a signal is limited by the accurate localization of the source and absence of significant light absorption [50, 310]. The optical excess is powered by the radioactive decay of nickel and cobalt nuclei in the early several weeks after the SN explosion [116].

As already mentioned before, SNe associated with long GRBs (GRB-SNe) are of type Ic, with no hydrogen and helium lines present in their spectra [101, 310]. There is no significant difference among the optical spectra of GRB-SNe, even though the isotropic energies of GRBs cover a rather wide range from $\sim 10^{48}$ erg till $\sim 10^{54}$ erg [185].

The spectra of GRB-SNe indicate the presence of broad lines, implying larger expansion velocities of the ejecta than in the case of normal type Ic SNe. Alongside this, they are more luminous then normal type Ic SNe. GRB-SNe can be generally characterized by the averaged peak photospherical velocity $v_{ph} \simeq 2 \times 10^4$ km s$^{-1}$, the ejected mass $M_{ej} \simeq 6M_\odot$ including the nickel mass $M_{Ni} \simeq 0.4M_\odot$, and the kinetic energy $E_k \simeq 2.5 \times 10^{52}$ erg [56].
Identification of the BdHN type of long GRBs using their observations during the early days can be performed by utilizing the classification scheme of BdHNe. Consequently, it is possible to predict the appearance of the GRB-SN soon after the determination of its redshift. On 2013 May 2, GRB 130427A was classified as a BdHN I. Occurrence of the associated SN 2013cq was predicted in [239] and later observationally confirmed on May 13th 2013 [77, 314]. Likewise, GRB 180727A was classified as a BdHN II on July 31st 2018, leading to the prediction of the associated SN 2018fip [244], which got confirmed on August 18th 2018 [126, 262].

Recently, on 2019 January 14, a bright GRB 190114C occurred, it has a low redshift $z = 0.42$ measured by NOT and GTC [261, 57]. This GRB was soon classified as a BdHN I and its SN optical signal was predicted to peak $18.8 \pm 3.7$ days after the burst. An excess of the optical flux was measured at around 15 days after the burst, which was consistent with the emergence of the SN [172].
Chapter 2

Non Thermal Radiation Processes

2.1 Introduction

Even nowadays, with the observations of gravitational waves and neutrinos, electromagnetic radiation is still the most important messenger in astrophysics. Radiation processes can be divided into thermal and non thermal ones. Thermal radiation depends exclusively on emitter’s temperature such as in case of blackbody radiation and thermal bremsstrahlung where thermal equilibrium is achieved through Coulomb collisions among particles and it requires sufficient densities to happen. Nevertheless very often in many astrophysical settings, especially in the case of γ-ray bursts which are the topic of this thesis, we deal with tenuous collisionless plasmas in which thermal radiation mechanisms are not sufficient to explain the radiation excess. It was established that this non thermal radiation comes populations of accelerated charged particles mostly scattering on magnetic fields, leading to cyclotron and synchrotron radiation, or on background photon radiation, leading to inverse Compton radiation. In physics of γ-ray bursts and their afterglows synchrotron and inverse Compton mechanism are the most relevant radiation processes and require proper coverage - which is the goal of this chapter.

2.2 Synchrotron Radiation

Charged particles gyrating within magnetic fields will produce either cyclotron or synchrotron radiation, depending if their motion is nonrelativistic or relativistic. In the framework of high energy astrophysics this mechanism is one of the utmost importance. Within this thesis we follow the derivation of synchrotron radiation and its spectra as presented by Jackson (1998) \[127\] and Melia (2001) \[173\]. We start
with the motion of a charged particle with mass $m$ and charge $q$ moving through magnetic field $\vec{B}$ with velocity $\vec{v}$,

\[
\frac{d}{dt}(\gamma m \vec{v}) = q \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right),
\]

\[
\frac{d}{dt}(\gamma mc^2) = q \vec{v} \cdot \vec{E} = 0,
\]

where $\gamma$ is particles Lorentz factor and $c$ is the speed of light. We assume there is no electric field, $\vec{E} = 0$, so that the Lorenz factor $\gamma$ and the speed of particle $|\vec{v}|$ remain constant reducing the equations of motion (2.1a) into

\[
m\gamma \frac{d\vec{v}}{dt} = \frac{q}{c} \vec{v} \times \vec{B}.
\]  

(2.2)

Generally the absence of electric field in astrophysical settings is a reasonable assumption although there are cases were existence of electric field can not be neglected e.g. pulsars, reconnection sites etc. From equation (2.2) we see that we can separate the equation of motion into two components, one parallel and one orthogonal to magnetic field $\vec{B}$

\[
\frac{d\vec{v}_\parallel}{dt} = 0,
\]

\[
\frac{d\vec{v}_\perp}{dt} = \frac{q}{\gamma mc} \vec{v}_\perp \times \vec{B}.
\]

(2.3a)

(2.3b)

Equation (2.3b) describes a gyrating motion of particle in the plane perpendicular to the magnetic field $\vec{B}$ with a angular gyrofrequency $\omega_{\text{gyr}}$ given as

\[
\omega_{\text{gyr}} = \frac{|q|B}{\gamma mc},
\]

(2.4)

with corresponding gyration radius, also known as Larmor radius $r_B$

\[
r_B = \frac{\left|\vec{v}_\perp\right|}{\omega_{\text{gyr}}} = \frac{\gamma mc}{|q|B} v \sin \alpha,
\]

(2.5)

where $\alpha$ is the angle between velocity of the particle $\vec{v}$ and magnetic field $\vec{B}$.

Knowing this we can calculate the total power emitted, $P_e$, by one charged particle (e.g. electron or proton) using famous Larmor’s formula in Lorenz invariant general form

\[
P_e = -2 \frac{q^2}{3 m^2 c^3} \frac{dp_\mu}{d\tau} \frac{dp^\mu}{d\tau} = -\frac{2}{3} \frac{q^2}{c^3} a_\mu a^\mu
\]

(2.6)

where $p_\mu$ is energy-momentum four-vector, $d\tau = dt/\gamma$ is the proper time element and $a_\mu$ is acceleration four-vector. The scalar product of the proper time derivations
of energy-momentum four vector in (2.6) can be further evaluated as
\[- \frac{d\mu}{d\tau} \frac{d\mu}{d\tau} = \left( \frac{d\vec{p}}{d\tau} \right)^2 - \frac{1}{\beta^2} \left( \frac{dE}{d\tau} \right)^2 = \left( \frac{d\vec{p}}{d\tau} \right)^2 - \beta^2 \left( \frac{d\vec{p}}{d\tau} \right)^2. \quad (2.7)\]

Since \( E = \gamma mc^2 \) and \( \vec{p} = \gamma m \vec{v} \), by putting back equation (2.7) into equation (2.6), we can obtain the known Lienard’s result
\[ P_e = \frac{2}{3} q^2 c \gamma^6 \left( \hat{\beta}^2 - \left( \hat{\beta} \times \hat{\beta} \right)^2 \right), \quad (2.8) \]

where \( \hat{\beta} = \vec{v}/c \). Since energy \( E \) and time \( t \) transform exactly the same way with relativistic transformations, the ratio of energy over time \( E/t \) is Lorentz invariant. This means that total emitted power by a gyrating charged particle (e.g. electron) \( P_e \) is also Lorentz invariant. Knowing this we can implement equations (2.3) into Larmor’s formula (2.6) to obtain total emitted power by a gyrating charged particle
\[ P_e = \frac{2}{3} e^2 c \gamma^4 \omega_{\text{gyr}} v^2 \sin^2 \alpha, \quad (2.9) \]

where \( e \) denotes electron charge for further simplicity. Averaging equation (2.9) over all pitch angles \( \alpha \) we can calculate the average power emitted \( \langle P_e \rangle \)
\[ \langle \sin^2 \alpha \rangle = \frac{1}{4\pi} \int \sin^2 \alpha d\Omega = \frac{2}{3}, \quad (2.10a) \]
\[ \langle P_e \rangle = \frac{4}{3} \sigma_T \gamma^2 \beta^2 u_B, \quad (2.10b) \]

where \( \sigma_T = 8\pi e^4/3m^2c^4 \) is Thomson cross section and \( u_B = B^2/8\pi \) is magnetic field energy density.

### 2.2.1 Single Particle Synchrotron Spectra

In order to derive single charged particle synchrotron spectra it is necessary to understand the angular and consequently frequency spectral distribution of radiation field. For that we start with Lienart-Wiechert potentials for a single charged particle in vacuum
\[ \vec{A} = \left[ \frac{e\hat{\beta}}{r - \vec{r} \cdot \hat{\beta}} \right]_{\text{ret}}, \quad \phi = \left[ \frac{e}{r - \vec{r} \cdot \hat{\beta}} \right]_{\text{ret}}, \quad (2.11) \]

where \( \phi \) and \( \vec{A} \) are scalar and vector electromagnetic potentials determined by radius-vector of observer \( \vec{r}(t') \) and velocity \( \vec{\beta}(t') \) of charged particle corresponding to the retarded time \( t' = t - r/c \). Retarded time just tells us that a wave emitted by a particle at time \( t' \) travels with the speed of light \( c \) and arrives to the position \( \vec{r}(t') \)
Knowing this we can proceed to the calculation of electric field $\vec{E}$ and magnetic field $\vec{B}$ produced by a moving charged particle using the potentials from (2.11)

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi, \quad \vec{B} = \nabla \times \vec{A}. \quad (2.12)$$

Using all things mentioned we arrive to the following expression for electric field of a moving charge

$$\vec{E}(\vec{r}, t) = e \left[ \hat{n} - \frac{\hat{n} - \beta}{\gamma^2 r^2 (1 - \beta \cdot \hat{n})^2} \right]_{\text{ret}} + \frac{e}{c} \left[ \frac{\vec{n} \times ((\vec{n} - \beta) \times \dot{\beta})}{r (1 - \beta \cdot \hat{n})^3} \right]_{\text{ret}}, \quad (2.13)$$

where $\hat{n} = \vec{r}/r$ is the unit vector from the moving charge toward the observer in retarded time. We can notice that the first term is the near field component, also known as "velocity" component, of the equation (2.13) falls with $r^2$ and with $\beta = 0$ becomes like Coulombs static electric field. Second "acceleration" term of equation (2.13) represents the far reaching component which falls off with $r$ and is transverse to the radius-vector $\vec{r}$. This component is also known as emitted electric field $\vec{E}_\text{rad}$ and it has a corresponding emitted magnetic field component $\vec{B}_\text{rad}$

$$\vec{E}_\text{rad} = \frac{e}{c} \left[ \frac{\vec{n} \times ((\vec{n} - \beta) \times \dot{\beta})}{r (1 - \beta \cdot \hat{n})^3} \right]_{\text{ret}}, \quad (2.14a)$$

$$\vec{B}_\text{rad} = \frac{e}{c} \vec{E}_\text{rad}. \quad (2.14b)$$

Using the emitted electric and magnetic field from (2.14a) and (2.14b) it is possible to calculate the emitted fields Poynting flux

$$\vec{S}_\text{ret} = \frac{c}{4\pi} \left[ \vec{E}_\text{rad} \times \vec{B}_\text{rad} \right]_{\text{ret}} = \frac{c}{4\pi} \left[ \vec{E}_\text{rad} \times \left( \vec{n} \times \vec{E}_\text{rad} \right) \right]_{\text{ret}}, \quad (2.15)$$

$$[\vec{S} \cdot \hat{n}]_{\text{ret}} = \frac{c}{4\pi} \left[ \vec{E}_\text{rad}^2 \right]_{\text{ret}} = \frac{e^2}{4\pi c} \left\{ \frac{1}{r^2} \left| \vec{n} \times \left[ (\vec{n} - \beta) \times \dot{\beta} \right] \right|^2 \right\}_{\text{ret}}. \quad (2.16)$$

It is very important to say that although the Poynting flux from equations (2.15) and (2.16) represents the energy per unit area per unit time at the observation point, the time interval is the one of observation point $\Delta t$ not the one of the emitter $\Delta t'$. Power radiated per unit solid angle in the own time of emitter (i.e. moving
2.2 Synchrotron Radiation

charge) therefore becomes

\[
\frac{dP(t')}{d\Omega} = r^2 [\vec{S} \cdot \hat{n}] \frac{dt}{dt'} = r^2 \vec{S} \cdot \hat{n}(1 - \vec{\beta} \cdot \hat{n}).
\]  
(2.17)

Nevertheless to obtain the spectra it is necessary to use the power radiated through unit solid angle in observers frame given by equation (2.16) in the following form

\[
\frac{dP(t)}{d\Omega} = |\vec{A}(t)|^2,
\]  
(2.18)

where

\[
\vec{A}(t) = \sqrt{\frac{c}{4\pi}} [rE_{\text{rad}}(t)]_{\text{ret}}
\]  
(2.19)

is used for simplicity of notation. From equation (2.18) we can calculate total energy emitted per unit solid angle by integrating it over time

\[
\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{A}(t)|^2 dt.
\]  
(2.20)

We can now introduce the Fourier transform (i.e. frequency spectrum) \(\vec{A}(\omega)\) of \(\vec{A}(t)\)

\[
\vec{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{A}(t) \exp(i\omega t) dt,
\]  
(2.21)

while the inverse Fourier transformation is

\[
\vec{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{A}(\omega) \exp(-i\omega t) d\omega.
\]  
(2.22)

Based on mathematical restrictions put on the function \(\vec{A}(t)\) we can invoke Parseval’s theorem

\[
\int_{-\infty}^{\infty} |\vec{A}(t)|^2 dt = \int_{-\infty}^{\infty} |\vec{A}(\omega)|^2 d\omega
\]  
(2.23)

by which the total energy radiated through unit solid angle is also equal to the integral of square of frequency spectrum \(\vec{A}(\omega)\) over all frequencies

\[
\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{A}(\omega)|^2 d\omega.
\]  
(2.24)

Physically observed frequencies are only positive real values, therefore it is more
useful to write (2.24) as
\[ \frac{dW}{d\Omega} = \int_0^\infty \frac{d^2W}{d\omega d\Omega} d\omega, \] (2.25)
where
\[ \frac{d^2W}{d\omega d\Omega} = |\vec{A}(\omega)|^2 + |\vec{A}(-\omega)|^2 = 2|\vec{A}(\omega)|^2 \] (2.26)
since \( \vec{A}(t) \) is real.

Implementing equations (2.19) and (2.21) into (2.26) we can obtain the energy radiated per unit solid angle per unit angular frequency
\[ \frac{d^2W}{d\omega d\Omega} = \frac{c^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \left[ \hat{n} \times ((\hat{n} - \vec{\beta}) \times \vec{\beta}) \right] \exp(i\omega t) dt \right|^2, \] (2.27a)
and
\[ \frac{d^2W}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \left[ \frac{\hat{n} \times ((\hat{n} - \vec{\beta}) \times \vec{\beta})}{(1 - \vec{\beta} \cdot \hat{n})^2} \right] \exp(i\omega t) dt \right|^2, \] (2.27b)
evaluated at the retarded time \( t' = t - r/c \). Using the assumption that the observer at position \( \vec{x} \) is far from the source at position \( \vec{x}_e \) so that distance of the observer from the source can be approximated with \( r(t') \approx |\vec{x} - \hat{n} \cdot \vec{x}_e(t')| \) and by changing the integration variable to retarded time interval \( dt' = (1 - \hat{n} \cdot \vec{\beta}) dt \) the equation (2.27) takes the following form
\[ \frac{d^2W}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \frac{\hat{n} \times ((\hat{n} - \vec{\beta}) \times \vec{\beta})}{(1 - \vec{\beta} \cdot \hat{n})^2} \exp[i\omega(t' - \hat{n} \cdot \vec{x}_e(t')/c)] dt' \right|^2. \] (2.28)
It is useful to remind ourselves that
\[ \frac{d}{dt'} [\hat{n} \times (\hat{n} \times \vec{\beta})] = \frac{\hat{n} \times ((\hat{n} - \vec{\beta}) \times \vec{\beta})}{(1 - \vec{\beta} \cdot \hat{n})^2} \] (2.29)
so we can simplify equation (2.28) even further using partial integration
\[ \frac{d^2W}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n} \times \vec{\beta}) \exp[i\omega(t' - \hat{n} \cdot \vec{x}_e(t')/c)] dt' \right|^2. \] (2.30)
After deriving energy radiated by a accelerated point charge per unit solid angle per unit angular frequency in general case as shown with equation (2.27) and with far field approximation as in equation (2.30) we turn our attention to the specific situation of a point charged particle in relativistic gyrating motion i.e. synchrotron radiation as shown on Fig. 2.1.
2.2 Synchrotron Radiation

Figure 2.1. Schematic presentation of synchrotron radiation from a charged particle passing through the point \( x = 0 \) at retarded time \( t' = 0 \) where \( r_B \) is the Larmor radius, \( \vec{v} \) velocity of the charged particle, \( \psi = vt'/r_B \) phase of the particle’s motion, \( \hat{\epsilon}_\perp \) polarization vector perpendicular to \( \vec{v} \) at the moment \( t' = 0 \) while \( \hat{n} \) and \( \theta \) are respectively direction vector and angle toward the observer [173].

Having defined two polarization vectors \( \hat{\epsilon}_\perp \) and \( \hat{\epsilon}_\parallel = \hat{n} \times \hat{\epsilon}_\perp \), and assuming \( \beta \approx 1 \) from Fig. 2.1 we can derive that

\[
\hat{n} \times (\hat{n} \times \vec{\beta}) = -\hat{\epsilon}_\perp \sin(\frac{vt'}{r_B}) + \hat{\epsilon}_\parallel \cos(\frac{vt'}{r_B}) \sin \theta ,
\]

where \( r_B \) is the particle’s Larmor radius already defined in (2.5).

Accordingly, within the same relativistic approximation \( \beta \approx 1 \), the argument of exponential in equation (2.30) can be approximated as

\[
t' - \frac{\hat{n} \cdot \vec{x}(t')}{c} = t' - \frac{r_B}{c} \cos \theta \sin \frac{vt'}{r_B} \approx \frac{1}{2\gamma^2} \left\{ (1 + \gamma^2 \theta^2)t' + \frac{c^2 \gamma^2 t'^3}{3r_B^2} \right\},
\]

using expansion for small arguments of sine and cosine functions which is founded on the fact that the radiation is very narrowly beamed along the trajectory of the charged particle at relativistic velocities. Further expansion of sine and cosine function in eq. (2.31) brings us to the synchrotron radiation spectra expressed by the contribution of two polarization states

\[
\frac{d^2W_\perp}{d\omega \,d\Omega} = \frac{e^2 \omega^2}{4\pi^2c} \left| \int_{-\infty}^{\infty} \frac{ct'}{r_B} \exp \left\{ \frac{i\omega}{2\gamma^2} \left[ (1 + \gamma^2 \theta^2)t' + \frac{c^2 \gamma^2 t'^3}{3r_B^2} \right] \right\} dt' \right|^2 , \quad (2.33)
\]

and

\[
\frac{d^2W_\parallel}{d\omega \,d\Omega} = \frac{e^2 \omega^2 \theta^2}{4\pi^2c} \left| \int_{-\infty}^{\infty} \exp \left\{ \frac{i\omega}{2\gamma^2} \left[ (1 + \gamma^2 \theta^2)t' + \frac{c^2 \gamma^2 t'^3}{3r_B^2} \right] \right\} dt' \right|^2 , \quad (2.34)
\]
whose sum gives total synchrotron radiation spectra

\[ \frac{d^2W}{d\omega\,d\Omega} = \frac{d^2W_\parallel}{d\omega\,d\Omega} + \frac{d^2W_\perp}{d\omega\,d\Omega}. \] (2.35)

Equations (2.34) and (2.33) can be significantly simplified by the introduction of the variable \( x \) given by

\[ x = \frac{\gamma c t'}{r_B(1 + \gamma^2 \theta^2)^{1/2}} \] (2.36)

and parameter \( \xi \) given by

\[ \xi = \frac{\omega r_B}{3\gamma c} \left(1 + \gamma^2 \theta^2\right)^{3/2}, \] (2.37)

leading to the spectral components written as

\[ \frac{d^2W_\perp}{d\omega\,d\Omega} = \frac{e^2\omega^2}{3\pi^2 c^3} \frac{r_B^2}{\gamma^4} (1 + \gamma^2 \theta^2)^2 x^2 \int_{-\infty}^{\infty} x \exp \left[\frac{3}{2} i \xi \left(x + \frac{1}{3} x^3\right)\right] dx, \] (2.38a)

\[ \frac{d^2W_\parallel}{d\omega\,d\Omega} = \frac{e^2\omega \theta^2}{3\pi^2 c^3} \frac{r_B^2}{\gamma^2} (1 + \gamma^2 \theta^2) \int_{-\infty}^{\infty} \exp \left[\frac{3}{2} i \xi \left(x + \frac{1}{3} x^3\right)\right] dx. \] (2.38b)

It is worth to notice that the integrals in equations (2.38) are Airy integrals which can be expressed through modified Bessel functions of order 2/3 and 1/3

\[ K_{2/3}(\xi) = \int_{-\infty}^{\infty} x \exp \left[\frac{3}{2} i \xi \left(x + \frac{1}{3} x^3\right)\right] dx, \] (2.39a)

\[ K_{1/3}(\xi) = \int_{-\infty}^{\infty} \exp \left[\frac{3}{2} i \xi \left(x + \frac{1}{3} x^3\right)\right] dx, \] (2.39b)

and provide a much more elegant way to write the spectral components of synchrotron radiation (2.38)

\[ \frac{d^2W_\perp}{d\omega\,d\Omega} = \frac{e^2\omega^2}{3\pi^2 c^3} \frac{r_B^2}{\gamma^4} (1 + \gamma^2 \theta^2)^2 K_{2/3}^2(\xi), \] (2.40a)

\[ \frac{d^2W_\parallel}{d\omega\,d\Omega} = \frac{e^2\omega \theta^2}{3\pi^2 c^3} \frac{r_B^2}{\gamma^2} (1 + \gamma^2 \theta^2) K_{1/3}^2(\xi). \] (2.40b)

Integrating equations (2.40) over whole solid angle we can obtain the energy radiated...
per angular frequency by a charged particle during a complete orbit:

\[
\begin{align*}
\frac{dW_{\perp}}{d\omega} &= \frac{2e^2\omega^2r_B^2\sin\alpha}{3\pi^2c^3\gamma^4} \int_{-\infty}^{\infty} (1 + \gamma^2\theta^2)^2 K_{2/3}(\xi) \, d\theta, \quad (2.41a) \\
\frac{dW_{\parallel}}{d\omega} &= \frac{2e^2\omega^2r_B^2\sin\alpha}{3\pi^2c^3\gamma^2} \int_{-\infty}^{\infty} (1 + \gamma^2\theta^2) \theta^2 K_{1/3}(\xi) \, d\theta. \quad (2.41b)
\end{align*}
\]

By introducing the critical frequency

\[
\omega_c = \frac{3}{2} \gamma^3 \omega_{\text{gyr}} \sin\alpha, \quad (2.42)
\]

equations (2.41) then can be written using modified Bessel functions of order 5/3 and 2/3 in the following manner

\[
\begin{align*}
\frac{dW_{\perp}}{d\omega} &= \frac{\sqrt{3} e^2 \gamma \sin\alpha}{2c} \left( \frac{\omega}{\omega_c} \right) \left[ \int_{\omega/\omega_c}^{\infty} K_{5/3}(y) \, dy + K_{2/3}(\omega/\omega_c) \right], \quad (2.43a) \\
\frac{dW_{\parallel}}{d\omega} &= \frac{\sqrt{3} e^2 \gamma \sin\alpha}{2c} \left( \frac{\omega}{\omega_c} \right) \left[ \int_{\omega/\omega_c}^{\infty} K_{5/3}(y) \, dy - K_{2/3}(\omega/\omega_c) \right]. \quad (2.43b)
\end{align*}
\]

By summation of equations (2.43a) and (2.43b) we obtain total radiated energy per unit angular frequency during a complete orbit of the charged particle

\[
\frac{dW}{d\omega} = \frac{dW_{\parallel}}{d\omega} + \frac{dW_{\perp}}{d\omega} = \frac{\sqrt{3} e^2 \gamma \sin\alpha}{2c} F(\omega/\omega_c), \quad (2.44)
\]

where \(F(x)\) is the synchrotron function defined as

\[
F(x) = x \int_{x}^{\infty} K_{5/3}(y) \, dy. \quad (2.45)
\]

Commonly in astrophysics synchrotron radiation spectra is expressed through number of emitted photons per unit time per unit of emitted photon’s energy which comes by dividing equation (2.44) with particle’s orbital period \(T_{\text{gyr}} = 2\pi/\omega_{\text{gyr}}\) and photon’s energy \(\mathcal{E}_\gamma = \hbar \omega\) giving

\[
\frac{dN_\gamma}{d\mathcal{E}_\gamma \, dt} = \frac{\sqrt{3} e^3 B \sin\alpha}{2\pi \, m c^2 \hbar \mathcal{E}_\gamma} F\left( \frac{\mathcal{E}_\gamma}{\mathcal{E}_c} \right), \quad (2.46)
\]

where \(N_\gamma\) is the number of emitted photons and \(\mathcal{E}_c = \hbar \omega_c\) is the critical photon energy corresponding to critical frequency \(\omega_c\).
While equation (2.46) holds in the case of a regular magnetic field, in the case of a chaotic magnetic field it needs to be averaged over all directions of magnetic field replacing the synchrotron function $F(x)$ with modified synchrotron function $G(x)$

$$G(x) = \int \sin \alpha F(\frac{x}{\sin \alpha}) \frac{d\Omega}{4\pi} = \frac{1}{2} \int_0^\pi F(\frac{x}{\sin \alpha}) \sin^2 \alpha d\alpha,$$

(2.47)

which by changing the order of integration can be written as single integral

$$G(x) = x \int_x^\infty K_{5/3}(z) \sqrt{1 - \frac{x^2}{z^2}} dz.$$  

(2.48)

Following equation (2.48) function $G(x)$ can be expressed using modified Bessel functions in a simple and compact way

$$G(x) = \frac{x}{20} \left[ (8 + 3x^2)(\kappa_{1/3}(x))^2 + x\kappa_{2/3}(x) (2\kappa_{1/3}(x) - 3x\kappa_{2/3}(x)) \right],$$

(2.49)

where $\kappa_{1/3}(x) = K_{1/3}(x/2)$ and $\kappa_{2/3}(x) = K_{2/3}(x/2)$.

Comparing functions $F(x)$ and $G(x)$ through Fig. 2.2 we can see that function $F(x)$ attains maximum value ($\max F(x) = 0.9180$) at $x = 0.2858$ while function $G(x)$ attains maximum value ($\max G(x) = 0.7126$) at $x = 0.2292$ showing us that the presence of chaotic magnetic field leads to smaller synchrotron emissivity with spectra shifted toward lower photon energies.

Although both $F(x)$ and $G(x)$ given in equations (2.45) and (2.49) are expressed in exact and compact way, for the necessity of numerical calculation of synchrotron

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Comparison of synchrotron function $F(x)$ and modified synchrotron function in case of chaotic magnetic field $G(x)$ in linear (left panel) and logarithmic (right panel) scale.}
\end{figure}
2.2 Synchrotron Radiation

\( \Delta F(x)/F(x) = (\tilde{F}(x) - F(x))/F(x) \) for the left panel and \( \Delta G(x)/G(x) = (\tilde{G}(x) - G(x))/G(x) \) for the right panel

Figure 2.3. Relative errors of synchrotron functions approximations given by

spectra it is very convenient to replace them with approximations which do not contain special functions \([7, 9]\)

\( \tilde{F}(x) \approx 2.15x^{1/3}(1 + 3.06x)^{1/6} \times \frac{1 + 0.884x^{2/3} + 0.471x^{4/3}}{1 + 1.64x^{2/3} + 0.974x^{4/3}} \exp(-x), \quad (2.50a) \)

\( \tilde{G}(x) \approx \frac{1.808x^{1/3}}{\sqrt{1 + 3.4x^{2/3}}} \frac{1 + 2.21x^{2/3} + 0.347x^{4/3}}{1 + 1.353x^{2/3} + 0.217x^{4/3}} \exp(-x), \quad (2.50b) \)

and which provide an accuracy better then 0.2% throughout whole range of variable \( x \) as shown on Fig. 2.3

2.2.2 Synchrotron Emissivity

In general case synchrotron emissivity should be obtained through summation of all charged particle’s contribution to the radiated electromagnetic field making a substitution within the integrand of equation \([127]\)

\( \vec{\beta} \exp \left[-\frac{i\omega \vec{n} \cdot \vec{x}_e(t')}{c}\right] \to \sum_{j=1}^{Z} \vec{\beta}_j \exp \left[-\frac{i\omega \vec{n} \cdot \vec{x}_e,j(t')}{c}\right], \quad (2.51) \)

where \( Z \) is the total number of charged particle indexed with \( j \). In this way we account for all possible interference effects between the electric fields emitted by a group of charged particles leading the way toward possible coherent effects of synchrotron radiation \([13]\).

In astrophysical settings coherent synchrotron radiation is fairly uncommon and unstable, especially at higher energies, therefore we can safely continue calculating...
synchrotron emissivity within an incoherent radiation regime in which radiated power
is calculated by direct summation of single particle power contributions \( P(\omega, E) \)
given through equation (2.46) as

\[
P(\omega, E) = \epsilon_{\gamma} \frac{dN_{\gamma}}{d\omega dt} = h \epsilon_{\gamma} \frac{dN_{\gamma}}{d\epsilon_{\gamma} dt} = \frac{\sqrt{3} e^2 B \sin \alpha}{2\pi mc^2} F \left( \frac{\omega}{\omega_c} \right).
\]

For further analysis of radiation by relativistic charged particles it is useful to
define their momentum distribution function \( f(\vec{p}) \) and distribution function over
energies \( n(E) \). Considering that we are operating in a relativistic regime where
\( E = pc \), and that by definition \( n(E) dE \) gives the number of particles per unit volume
of coordinate space in unit solid angle of momentum space with energies in range
from \( E \) to \( E + dE \), we can establish a simple connection between \( f(\vec{p}) \) and \( n(E) \)
given by

\[
n(E) = \frac{f(\vec{p}) p^2}{c}.
\]

Certainly the possible anisotropic angular dependence of this distributions is not
addressed by this definition but for sake of simplicity we will continue with isotropic
distributions since we are anyway dealing with randomly oriented particles. In that
case particle density \( n_s \) is given by

\[
n_s = 4\pi \int n(E) dE.
\]

Although rarely present in astrophysical settings monoenergetic particle distributions
given by

\[
n(E) = \frac{n_s}{4\pi} \delta(E - E_0)
\]

are often used for initial estimates of synchrotron radiation by astrophysical sources.
Since synchrotron radiation is very narrowly beamed along particle’s velocity \( \vec{v} \)
emissivity \( j_\omega \) of a particle distribution \( n(E) \) of can be written in very simple form

\[
j_\omega = \int_{E_{\min}}^{E_{\max}} P(\omega, E) n(E) dE,
\]

where \( E_{\min} \) and \( E_{\max} \) are minimum and maximum energy of particles accessible
to a given physical process. In case of monoenergetic distribution of particles this
becomes

\[
j_\omega = P(\omega, E_0) \frac{n_s}{4\pi}.
\]

It is important to notice that based on equations (2.43) it is also possible to calculate
the degree of linear polarization of radiation emitted by monoenergetic particle
distribution in regular magnetic field \[253, 329\]

\[
\Pi = \frac{j_{\omega_\perp} - j_{\omega_\parallel}}{j_{\omega_\perp} + j_{\omega_\parallel}} = \frac{P_\perp - P_\parallel}{P_\perp + P_\parallel} = \frac{K_{2/3}(\omega/\omega_c)}{\int_{\omega/\omega_c} K_{5/3}(x)dx}, \tag{2.58}
\]

where \(j_{\omega_\perp}\) and \(j_{\omega_\parallel}\) are two linearly polarized components of emissivity given by

\[
j_{\omega_\perp} = P_\perp(\omega, E_0)\frac{n_s}{4\pi} \quad \text{and} \quad j_{\omega_\parallel} = P_\parallel(\omega, E_0)\frac{n_s}{4\pi}, \tag{2.59}
\]

and where \(P_\perp(\omega, E)\) and \(P_\parallel(\omega, E)\) two linearly polarized components of single particle power spectra \(P(\omega, E)\) given by

\[
P_\perp(\omega, E) = \frac{1}{T_{\text{gyr}}} \frac{dW_\perp}{d\omega} \quad \text{and} \quad P_\parallel(\omega, E) = \frac{1}{T_{\text{gyr}}} \frac{dW_\parallel}{d\omega}. \tag{2.60}
\]

As we will see in Chapter 3 of this thesis astrophysical particle acceleration processes can build up a nonthermal ensemble of charged particles given by a power law energy spectrum

\[
n(E) = \mathcal{K}E^{-p}, \tag{2.61}
\]

where \(p\) is the power law index and \(\mathcal{K}\) is the normalization factor given by

\[
\mathcal{K} = \frac{n_s}{4\pi} \left[ \int_{E_{\text{min}}}^{E_{\text{max}}} E^{-p}dE \right]^{-1}. \tag{2.62}
\]

Except from theoretical considerations the power law particle spectra given with equation (2.61) has its justification in the observed power law spectra of cosmic rays and non thermal power law photon spectra of many astrophysical sources with radiating relativistic electrons.

Emissivity of such non-thermal charged particle distribution with an isotropic energy spectrum within an uniform magnetic field is then calculated by putting equation (2.61) into equation (2.56) giving back a power law photon spectra \[329\]

\[
j_\omega = \tilde{\mathcal{K}}\omega^{-\alpha_X}, \tag{2.63}
\]

where spectral index \(\alpha_X\) is given by

\[
\alpha_X = \frac{p - 1}{2}, \tag{2.64}
\]
and $\tilde{K}$ is normalization factor for photon spectra

$$
\tilde{K} = \frac{\sqrt{3}K}{2\pi(p+1)} \Gamma \left( \frac{3p-1}{12} \right) \Gamma \left( \frac{3p+19}{12} \right) \frac{e^3}{m^2c^3} \left( \frac{3e}{m^3c^5} \right)^{\frac{p+1}{2}} (B \sin \alpha)^{\frac{p+1}{2}}, \quad (2.65)
$$

with $\Gamma$ being the gamma function. In this case the degree of polarization depends only on the value of the power law index $p$

$$
\Pi = \frac{j_{\omega \perp} - j_{\omega \parallel}}{j_{\omega \perp} + j_{\omega \parallel}} = \frac{p + 1}{p + 7/3}. \quad (2.66)
$$

Since the magnetic field configuration of the emitting region is often chaotic rather then regular it is useful to calculate the synchrotron emissivity averaged over all possible magnetic field orientations. In such case the power law emissivity spectra stays of the same form as in equation (2.63) while the normalization factor $\tilde{K}$ changes to

$$
\tilde{K} = \frac{2^{p+1} \sqrt{3}K}{4(p+1)\sqrt{\pi}} \Gamma \left( \frac{3p-1}{12} \right) \Gamma \left( \frac{3p+19}{12} \right) \Gamma \left( \frac{p+5}{4} \right) \\
\times \Gamma^{-1} \left( \frac{p+7}{4} \right) \frac{e^3}{m^2c^3} \left( \frac{3e}{m^3c^5} \right)^{\frac{p+1}{2}} B^{\frac{p+1}{2}}. \quad (2.67)
$$

Certainly in the case of chaotic magnetic field configuration we can not talk about linear polarization since it vanishes by averaging over all directions of magnetic field lines.

Considering the synchrotron emissivity of charged particles it is necessary to remind ourselves of the limitations given by radiation process itself and the efficiency of the particle accelerator in action. Important role in this case is played by cooling and acceleration timescales. Synchrotron cooling time for a particle with mass $m$ and energy $E$ comes from dividing that energy with the averaged total radiated power $\langle P_e \rangle$ given by equation (2.10b)

$$
t_{\text{syn}} = \frac{E}{\langle P_e \rangle} = 6\pi \left( \frac{m}{m_e} \right)^4 \frac{m_e^2c^3}{\sigma_{Te}EB^2}, \quad (2.68)
$$

with $m_e$ being electron mass and $\sigma_{Te} = 8\pi e^4/3m_e^2c^4$ Thomson cross section of an electron. We can immediately notice the immense difference between synchrotron cooling time of a proton and an electron of same energy in same magnetic field, where proton cools $(m_p/m_e)^4 \approx 10^{13}$ times slower then electron [7]. With this we see electrons are very much dominant part of synchrotron radiation in astrophysics although at higher photon energies proton synchrotron radiation can also play a significant role.
Another important timescale is the particle acceleration time

\[ t_{\text{acc}} = \eta(E) \frac{E}{e c B}, \]  

(2.69)

where \( \eta(E) \) is a free parameter characterizing the acceleration rate \[ 7 \]. This parameter can be understood in sense of the ratio \( E_{\text{eff}} = B/\eta(E) \) giving the averaged projection of the electric field \( E_{\text{eff}} \) along particle’s trajectory and has shown great utility in discussing acceleration processes like most common diffusive shock acceleration. As we will see in the next chapter in majority of cosmic accelerators \( B \gg E_{\text{eff}} \) and therefore \( \eta(E) \gg 1 \), although in certain systems (e.g. pulsar magnetospheres) it is possible to have \( E_{\text{eff}} \geq B \) with \( \eta(E) \leq 1 \).

Coming back to the comparison of two forementioned timescales we see particles can not have cooling times shorter then acceleration ones. Therefore maximum energy of an accelerated charged particle \( E_0 \) radiating synchrotron radiation is the one at which those timescales are equal i.e. \( t_{\text{syn}} = t_{\text{acc}} \) giving out

\[ E_0 = \left( \frac{3}{2} \right)^{3/4} \frac{m^2 c^4}{\sqrt{\eta e d B}}. \]  

(2.70)

With this maximum particle energy comes an observed cutoff energy \( \mathcal{E}_0 \) of a photon in the synchrotron spectrum independent of magnetic field \( B \)

\[ \mathcal{E}_0 = \frac{9 m c^2}{4 \alpha_f \eta}, \]  

(2.71)

with \( \alpha_f = e^2/\hbar c \approx 1/137 \) being the fine structure constant. We can now obtain estimated cutoff photon energies for electrons \( \mathcal{E}_{0,e} \approx 160 \text{ MeV} \) and for protons being \( \mathcal{E}_{0,p} \approx 300 \text{ GeV} \) with parameter \( \eta = 1 \). Commonly observed values of cutoff energies are lower then the mentioned ones but in quite rare and special situations like the surroundings of relativistic compact objects it is possible to reach higher values which point toward \( \eta < 1 \) and \( E_{\text{eff}} > B \).

Concerning more ubiquitous situations when \( B \gg E_{\text{eff}} \) and \( \eta \gg 1 \) we turn our attention a bit to the case of non-relativistic diffusive shock acceleration with acceleration rate parameter being expressed through diffusion coefficient \( D(E) \) as following \[ 7 \]

\[ \eta(E) \approx 10 \frac{D(E)}{r_B c} \left( \frac{c}{v_s} \right)^2, \]  

(2.72)

where \( v_s \) is the shock speed. Most commonly it is assumed that the diffusion can be described with Bohm regime where diffusion coefficient becomes \( D(E) = r_B c/3 \) which included in equations \( 2.72 \) and afterwards \( 2.71 \) gives us the following
estimate of cutoff photon energy

\[ E_0 \approx \frac{27 m c^2}{40 \alpha f} \left( \frac{v_s}{c} \right)^2. \]  

(2.73)

With the maximum particle energy and cutoff energy of photon included the corresponding particle distribution \( n(E) \) and synchrotron emissivity \( j_\omega \) change into following forms

\[ n(E) = K_0 E^{-p} \exp \left[ - \left( \frac{E}{E_0} \right)^\kappa \right], \]  

(2.74)

and

\[ j_\omega = \tilde{K}_0 \omega^{-\alpha x} \exp \left[ - \left( \frac{\omega}{\omega_0} \right)^\lambda \right], \]  

(2.75)

where \( \lambda = \kappa / (\kappa + 2) \), \( \omega_0 = E_0 / \hbar \) is the cutoff frequency, with \( K_0 \) and \( \tilde{K}_0 \) being normalisation constants corrected for cutoff present in comparison to ones from equations (2.61) and (2.63) [7].

It is important to note that in all this analysis present we should not forget about possibility of charged particle’s backreaction since it is mostly assumed in literature that the time of Larmor’s rotation \( t_B = 2\pi E / qBc \) is much shorter than the cooling time \( t_{\text{syn}} \) giving the time ratio and energy requirement

\[ \frac{t_B}{t_{\text{syn}}} \ll 1 \rightarrow E \ll \frac{3 m^4 c^8}{4\pi e^3 B}, \]  

(2.76)

which in astrophysical settings is not necessarily present but also not very common requiring strong magnetic fields and/or very energetic particles [14]. For example in cases of electrons and protons synchrotron backreaction becomes significant at following energies

\[ E_e \gg 1.9 \times 10^4 B_0^{-1/2} \text{ GeV}, \]  

(2.77a)

\[ E_p \gg 6.5 \times 10^{10} B_0^{-1/2} \text{ GeV}, \]  

(2.77b)

with \( E_e \) and \( E_p \) are respectively electron and proton energies and \( B = B_0 \times 10^6 \) G is the magnetic field. The papers by Alosio and Blasi (2002) [13, 14] touch this in traditional literature commonly neglected point of synchrotron radiation derivation and certainly are of great use for further understanding of both coherent effects and synchrotron backreaction.
2.2 Synchrotron Radiation

2.2.3 Synchrotron Self Absorption

Following the principle of detailed balance synchrotron emission is accompanied by the corresponding absorption process in which a charged particle within magnetic field absorbs an incoming photon [253, 161]. Alongside absorption we can also have stimulated emission in which charged particle is emitting more radiation in a direction and at frequency where photons are already present. Both of this processes can be explained by existing formalism of the Einstein coefficients generalized to the case of continuum states of charged particles in magnetic field which are essentially treated as free particle states defined by particle’s momentum, position and some internal state.

Within the scope of statistical mechanics we divide the phase space of the particles by discrete elements with volume $h^3$ where $h$ is the Planck’s constant. Each discrete element within the phase space is then associated with a specific state of translational degree of freedom with our focus being on the transitions between them. In this picture for a photon with energy $h\nu$ being emitted or absorbed there are many possible transitions within continuum differing by the energy of photon and therefore it is necessary to sum all of them if we want to calculate the absorption coefficient [253, 161]

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [n(E_1)B_{12} - n(E_2)B_{21}] \phi_{21}(\nu), \quad (2.78)$$

where $\phi_{21}(\nu) = \delta(E_2 - E_1 - h\nu)$ is the delta function selecting only transitions with $h\nu = E_2 - E_1$ while $B_{12}$ and $B_{21}$ are Einstein coefficients respectively corresponding to absorption and induced/stimulated emission processes. From equations (2.52) and (2.78) we find more practical in current situation to use single particle synchrotron spectra given in frequency range $\nu$ instead of $\omega$ giving $P(\nu, E) = 2\pi P(\omega, E)$ which expressed through Einstein coefficients becomes

$$P(\nu, E) = h\nu \sum_{E_1} A_{21} \phi_{21}(\nu) = \frac{2h^2\nu^4}{c^2} \sum_{E_1} B_{21} \phi_{21}(\nu). \quad (2.79)$$

Remembering from Einstein relations that $B_{12} = B_{21}$ and invoking the result of equation (2.79) we can simplify the equation (2.78) into

$$\alpha_\nu = \frac{c^2}{8\pi h^3} \sum_{E_2} [n(E_2 - h\nu) - n(E_2)] P(\nu, E_2). \quad (2.80)$$

With isotropic momentum electron distribution from [2.53] the absorptivity of equation (2.80) can be transformed from discrete sum into integral with following
replacements
\[ \sum_{E_2} \rightarrow \frac{\tilde{w}}{h^3} \int d^3 p_2, \quad n(E_2) \rightarrow \frac{\hbar^3}{\tilde{w}} f(p_2), \quad (2.81) \]
where \( p_2 \) is the particle momentum corresponding to energy \( E_2 = \sqrt{p_2^2 c^2 + m^2 c^4} \) and \( \tilde{w} \) intrinsic statistical weight being \( \tilde{w} = 2 \) for spin \( s = 1/2 \) particles. Implementing aforementioned replacements from equation (2.81) into (2.80) we obtain the following expression for synchrotron absorptivity
\[ \alpha_\nu = \frac{c^2}{8\pi \hbar \nu^3} \int (f(p_2^*) - f(p_2)) P(\nu, E_2) d^3 p_2, \quad (2.82) \]
with momentum \( p_2^* \) corresponding to the particle energy \( E_2 - \hbar \nu \). One interesting situation to investigate is the one of thermal particle distribution given by
\[ f(p) = K \exp \left( -\frac{E(p)}{kT} \right), \quad (2.83) \]
where \( K \) is the normalization parameter, \( k \) is Boltzmann constant and \( T \) is temperature which leads the momentum difference of equation (2.82) to become
\[ f(p_2^*) - f(p_2) = f(p_2) \left( \exp \left( \frac{\hbar \nu}{kT} \right) - 1 \right), \quad (2.84) \]
giving back the synchrotron absorptivity of thermal particle distribution to be
\[ (\alpha_\nu)_{\text{thermal}} = \frac{c^2}{8\pi \hbar \nu^3} \left( \exp \left( \frac{\hbar \nu}{kT} \right) - 1 \right) \int f(p_2) P(\nu, E_2) d^3 p_2. \quad (2.85) \]
We can immediately notice that the integral part of equation (2.85) gives emissivity in frequency \( \nu \) space
\[ j_\nu = \frac{1}{4\pi} \int f(p) P(\nu, E) d^3 p, \quad (2.86) \]
which compared with emissivity in angular frequency \( \omega \) space is \( j_\omega = 2\pi j_\nu \), while part outside integral contains Planck’s blackbody intensity given by
\[ B_\nu(T) = \frac{2\hbar \nu^3}{c^2} \left( \exp \left( \frac{\hbar \nu}{kT} \right) - 1 \right)^{-1} \quad (2.87) \]
leading toward the final expression of synchrotron absorptivity of nonrelativistic thermal particles distribution
\[ (\alpha_\nu)_{\text{thermal}} = \frac{j_\nu}{B_\nu(T)}. \quad (2.88) \]
Since we commonly assume that particle distributions are relativistic and isotropic
it is easy to invoke equation (2.53) and instead of momentum particle distribution use their isotropic energy spectrum $N(E) = 4\pi n(E)$. In that case equation (2.82) can be written in following form

$$\alpha_\nu = \frac{c^2}{8\pi\hbar\nu} \int dE \frac{P(\nu, E)E^2}{E} \left[ \frac{N(E - \hbar\nu)}{(E - \hbar\nu)^2} - \frac{N(E)}{E^2} \right],$$

(2.89)

which since we operate in regime where based on our discussion about backreaction $\hbar\nu \ll E$ can be simply approximated by the expansion till the first order

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2} \int dE \frac{P(\nu, E)E^2}{E} \frac{\partial}{\partial E} \left[ \frac{N(E)}{E^2} \right].$$

(2.90)

At ultrarelativistic limit thermalized particles follow Maxwell-Jüttner distribution given by

$$N(E) = K E^2 \exp \left( -\frac{E}{kT} \right)$$

(2.91)

which included in equation (2.90) leads us toward Kirchoff’s law in Rayleigh-Jeans regime

$$\langle \alpha_\nu \rangle_{\text{thermal}} = \frac{j_\nu c^2}{2\nu^2 kT},$$

(2.92)

as it is expected since we assume that $\hbar\nu \ll E$.

Coming back to more common scenario of synchrotron sources, i.e. the presence of nonthermal power law energy spectrum of particles given by equation (2.61), the absorptivity can be easily calculated using (2.90)

$$\alpha_\nu = \frac{(p + 2)c^2}{2\nu^2} \int dE \frac{P(\nu, E)n(E)}{E},$$

(2.93)

which integrated gives the following analytical expression for homogenous magnetic field [253, 329, 161]

$$\alpha_\nu = \sqrt{\frac{3}{2m}} \left( \frac{3e}{2\pi m^3 c^5} \right)^{p/2} K(B \sin \alpha)^{(p+2)/2} \Gamma \left( \frac{3p + 2}{12} \right) \Gamma \left( \frac{3p + 22}{12} \right) \nu^{-(p+4)/2}.$$  

(2.94)

In the case of chaotic magnetic field with constant magnitude $B$ it is necessary to average the absorptivity over all angles $\alpha$ leading to [253, 329, 161]

$$\alpha_\nu = \sqrt{\frac{3\pi}{4m}} \left( \frac{3e}{2\pi m^3 c^5} \right)^{p/2} K B^{(p+2)/2} \Gamma \left( \frac{3p + 2}{12} \right) \Gamma \left( \frac{3p + 22}{12} \right) \times \Gamma \left( \frac{p + 6}{4} \right) \Gamma^{-1} \left( \frac{p + 8}{4} \right) \nu^{-(p+4)/2}.$$ 

(2.95)

Following this results lets us look at radiation transfer within a region with physical
thicknes \( l \). In that case radiation transfer equation

\[
\frac{dI_\nu}{dx} = j_\nu - \alpha_\nu I_\nu, \tag{2.96}
\]

where \( I_\nu \) is radiation intensity at the position \( x \) within the source, has the following solution

\[
I_\nu = S_\nu [1 - \exp(-\alpha_\nu l)], \tag{2.97}
\]

with \( S_\nu = j_\nu/\alpha_\nu \) being the source function. Knowing the synchrotron emissivity from equation (2.63) and absorptivity from equations (2.94) and (2.95) we can easily asses the radiation spectra in both optically thin regime, where \( \alpha_\nu l \ll 1 \),

\[
I_\nu = j_\nu l \propto B^{(p+1)/2} \nu^{-(p-1)/2}, \tag{2.98}
\]

and optically thick regime, where \( \alpha_\nu l \gg 1 \),

\[
I_\nu = S_n u \propto B^{-1/2} \nu^{5/2}. \tag{2.99}
\]

It is worth reminding ourselves that all this derivation is done within the paradigm of magnetic field having uniform magnitude. In many astrophysical objects chaotic magnetic fields of non uniform magnitude are to be expected leading to the broadening of transition region between optically thick and thin regime [42].

### 2.3 Inverse Compton Scattering

One of the most prominent radiation mechanisms involved in \( \gamma \)-ray production within astrophysics is the inverse Compton scattering. Its presence has been confirmed in many high energy astrophysical settings ranging from compact sources like pulsars and active galactic nuclei to extended sources like supernova remnants and galactic clusters.

Derivation of inverse Compton spectra and emissivity is pretty lengthy so we will rather refer to the results of classical works by [128], [51], [8], [67], while special approximation used in our calculations of inverse Compton energy losses is the one provided by Moderski et al. (2005) [184].

In the rest frame of the charged particle, most commonly electron, an incoming photon with energy \( \epsilon'_0 mc^2 \) is scattered at an angle \( \theta' \) relative to its initial momentum with the final energy \( \epsilon' mc^2 \). In general case the final energy of photon is given by

\[
\epsilon' = \frac{\epsilon'_0}{1 + \epsilon'_0(1 - \cos \theta')}. \tag{2.100}
\]
In special cases when rest frame photon energy is low, i.e. when $\epsilon'_0 \ll 1$, recoil of an electron can be easily neglected leading to well known Thomson regime where photon’s energy is conserved $\epsilon' = \epsilon'_0$. In another limit when $\epsilon'_0 \gg 1$ where electron recoil is significant photon’s end up with rest mass energy of the electron $\epsilon' \approx 1$ leading to so called Klein-Nishima regime.

In the case of inverse Compton mechanism an relativistic electron with initial energy $\gamma mc^2$ is scattering incoming photons with initial energy $\epsilon_0 mc^2$ increasing it to the final energy $\epsilon mc^2$ whose value can be calculated by employing relativistic transformations on equation (2.100) back to the lab frame [253]

$$\frac{\epsilon}{\epsilon_0} = \frac{1 - \beta \cos \phi}{1 - \beta \cos \phi' + (\epsilon_0/\gamma)(1 - \cos \theta)},$$

(2.101)

where $\phi$ and $\phi'$ are the angles between momenta of incoming and outgoing electron and photon respectively, $\theta$ is the angle between incoming and outgoing momenta of photon while $\beta$ is electron’s initial velocity in units of speed of light.

Corresponding cross section for this process can be approximated well inside 10% of accuracy within a broad set of values for $\gamma$ and $\epsilon_0$ obtained in the lab frame [67]

$$\sigma_{\text{IC}} = \frac{3 \sigma_T}{8 \kappa_0} \left(1 - \frac{2}{\kappa_0} - \frac{2}{\kappa_0^2}\right) \ln(1 + 2\kappa_0) + \frac{1}{2} + \frac{4}{\kappa_0} - \frac{1}{2(1 + 2\kappa_0)^2},$$

(2.102)

where $\kappa_0 = \gamma \epsilon_0$. In the rest frame of electron when $\gamma = 1$ expression (2.102) is becoming the exact Klein-Nishima cross section which accounts for the QED effects present in Compton scattering.

Let us now imagine a monochromatic beam of low energy photons entering and scattering inside a homogeneous and isotropic region filled with relativistic electrons. The resulting spectrum of scattered photons at specific angle $\theta$ is then given by Aharonian and Atoyan (1981) [8]

$$dN(\theta, \epsilon, \gamma) = \frac{3 \sigma_T}{16 \pi \epsilon_0 \gamma^2} \left[1 + \frac{z^2}{2(1 - z)} - \frac{2z}{b_\theta(1 - z)} + \frac{2z^2}{b_\theta^2(1 - z)^2}\right],$$

(2.103)

with $b_\theta = 2(1 - \cos \theta)\gamma \epsilon_0$ and $z = \epsilon/\gamma$. Spectra for a more common case when isotropically distributed photons permeate a region with isotropically distributed electrons can be easily obtained by integrating expression (2.103) over angle $\theta$ [51] [8]

$$dN(\epsilon, \gamma) = \frac{3 \sigma_T}{16 \pi \epsilon_0 \gamma^2} f_{\text{iso}}(\epsilon, \epsilon_0, \gamma),$$

(2.104)
2. Non Thermal Radiation Processes

where function $f_{\text{iso}}$ is given by

$$f_{\text{iso}}(\epsilon, \epsilon_0, \gamma) = 1 + \frac{z^2}{2(1 - z)} + \frac{z}{b(1 - z)} - \frac{2z^2}{b^2(1 - z)^2} + \frac{z^3}{b(1 - z)^2} - \frac{2z}{b(1 - z)} \ln \frac{b(z - 1)}{z},$$

(2.105)

with $b = 4\kappa_0 = 4\gamma\epsilon_0$. It is important to note that equation (2.104) refers to initially monochromatic photons.

Taking into account number density per energy $n_{\epsilon_0}$ of ambient photons we can calculate the electron energy losses with

$$|\dot{\gamma}|_\text{C} \simeq \frac{3}{4} \epsilon_0 \sigma_T \frac{1}{\gamma^2} \int \frac{n_{\epsilon_0}}{\epsilon_0} \left[ \int f_{\text{iso}}(\epsilon, \epsilon_0, \gamma) \epsilon d\epsilon \right] d\epsilon_0,$$

(2.106)

with inner integral having a known analytical solution [128, 8]

$$\int f_{\text{iso}}(\epsilon, \epsilon_0, \gamma) \epsilon d\epsilon = \frac{\gamma^2 g(b)}{b},$$

(2.107)

where function $g(b)$ is given as follows

$$g(b) = \left( \frac{1}{2}b^2 + 6 + \frac{6}{b} \right) \ln(1 + b) - \left( \frac{11}{12}b^3 + 6b^2 + 9b + 4 \right) \frac{1}{(1 + b)^2} - 2 + 2\text{Li}_2(-b),$$

(2.108)

with $\text{Li}_2(x)$ being the dilogarithm function

$$\text{Li}_2(x) = -\int_1^x \frac{\ln y}{1 - y} dy.$$

(2.109)

This leads to the final equation for the electron’s inverse Compton energy losses [184]

$$-\frac{dE}{dt} = 4 \sigma_T u_0 \gamma^2 F_{\text{KN}},$$

(2.110)

where $E$ is electron energy, $u_0$ total energy density of ambient photons and $F_{\text{KN}}$ being function describing inverse Compton scattering from Thomson to Klein-Nishima regime through integration over energies of ambient radiation

$$F_{\text{KN}} = \frac{1}{u_0} \int_{\epsilon_0,\text{min}}^{\epsilon_0,\text{max}} f_{\text{KN}}(b) u_\epsilon d\epsilon_0$$

(2.111)

with $f_{\text{KN}}(b) = \gamma^2 g(b)/b$ and $u_\epsilon$ being the energy distribution of ambient photons from their minimum energy $\epsilon_0,\text{min}$ to their maximum energy $\epsilon_0,\text{max}$. In monochromatic distribution of ambient photons it is clear that $F_{\text{KN}} = f_{\text{KN}}$. This can be also safely used as approximation for Planckian distribution which we did in this thesis. From
2.3 Inverse Compton Scattering

In the (2.110) we can attest two limiting inverse Compton regimes:

\[ f_{KN} \simeq 1 \text{ for } b \ll 1 \text{ (Thomson regime)}, \quad (2.112a) \]

\[ f_{KN} \simeq \frac{9}{2b^2}(\ln b - 11/6) \text{ for } b \gg 1 \text{ (Klein-Nishima regime)}. \quad (2.112b) \]

In case when \( b \lesssim 10^4 \), Moderski et al. (2005) provide a useful approximation

\[ f_{KN} \simeq \frac{1}{(1 + b)^{1.5}}, \quad (2.113) \]

which we will use in most of our numerical calculations later.

2.3.1 Inverse Compton Emissivity

Using the results coming from equations (2.103), (2.104), and (2.105) we can finally calculate inverse Compton emissivity from an isotropic electron distribution

\[ j_\epsilon = mc^2 \int_{\gamma_{\min}}^{\gamma_{\max}} \frac{dN(\epsilon, \gamma)}{d\epsilon d\Omega} \epsilon n_e d\gamma, \quad (2.114) \]

where \( \gamma_{\min} \) and \( \gamma_{\max} \) are respectively minimum and maximum energy of electron energy distribution \( n_e \). This leads to following emissivities for:

- monochromatic beam of photons incoming using the spectra given with equation (2.103)

\[ \epsilon j_\epsilon(\theta, \epsilon, \epsilon_0) = \frac{3}{16\pi} e\sigma_{\chi}u_0 \left( \frac{\epsilon}{\epsilon_0} \right)^2 \int_{\gamma_{\min}}^{\gamma_{\max}} \frac{n_e}{\gamma^2} f(\gamma, \epsilon, \epsilon_0, \theta) d\gamma, \quad (2.115) \]

with

\[ f(\gamma, \epsilon, \epsilon_0, \theta) = 1 + \frac{z^2}{2(1 - z)} - \frac{2 z}{b_\theta(1 - z)} + \frac{2 z^2}{b_\theta^2(1 - z)^2}, \quad (2.116) \]

where \( b_\theta = 2(1 - \cos \theta) \gamma \epsilon_0 \).

- monochromatic isotropic ambient photons using the spectra given by equation (2.104)

\[ \epsilon j_\epsilon(\epsilon, \epsilon_0) = \frac{3}{16\pi} e\sigma_{\chi}u_0 \left( \frac{\epsilon}{\epsilon_0} \right)^2 \int_{\gamma_{\min}}^{\gamma_{\max}} \frac{n_e}{\gamma^2} f_{iso}(\gamma, \epsilon, \epsilon_0) d\gamma. \quad (2.117) \]

Imagining a power law energy distribution of electrons, \( n_e \propto \gamma^{-p} \) from equation
subsequent inverse Compton photon spectra, within Thomson regime \( b \ll 1 \), will have spectral index \( \alpha_X = (p - 1)/2 \). On the other hand in ultrarelativistic Klein-Nishina regime \( b \gg 1 \) photon spectrum is significantly steeper \( \propto \epsilon^{-\alpha_X} \ln \left(b + \text{const}\right) \) with spectral index \( \alpha_X = p \).

### 2.3.2 Synchrotron Self-Compton Radiation

Alongside Compton scattering of radiation coming from external sources (ERC), like for example cosmic microwave background radiation or blackbody radiation from neighbouring stars etc, also synchrotron radiation produced within the source can scatter upon local relativistic electrons leading toward so called synchrotron self-Compton mechanism (SSC). It is another mechanism of synchrotron self interactions alongside synchrotron self absorption. This mechanism plays quite significant role in case of weak external radiation field or in case of compact sources changing significantly the electron distribution and subsequent radiation spectra. Electron energy losses are then given by

\[
|\dot{\gamma}|_{SSC} = \frac{4 \sigma_T}{3mc} u_{\text{syn}} \gamma^2 F_{KN} \tag{2.118}
\]

with \( u_{\text{syn}} \) being the energy density of background synchrotron radiation. Subsequent spectra calculation is done within same formalism as in case of external Compton radiation scenario with both target electrons and incoming photons following the expected power law like spectra described in former sections of this chapter dealing with synchrotron radiation. Derivation of general expression for SSC emissivity and spectral flux can be found in Finke et al. (2008) \[89\] where emissivity for isotropic and homogeneous photon and electron distributions in comoving frame \( j_{SSC}' \) is given by

\[
\epsilon'_{\text{SSC}}(\epsilon_s') = \frac{3}{4} \sigma_T \int_0^\infty d\epsilon' \frac{n_e'(\gamma')}{\gamma'^2} \int_{\gamma'_{\min}}^{\gamma'_{\max}} \frac{d\gamma}{\gamma'} n_e'(\gamma') F_C(q', \kappa') \tag{2.119}
\]

where \( \epsilon' mc^2, \epsilon_s' mc^2 \) and \( \gamma' mc^2 \) are respectively energies of ambient photons, scattered photons and electron, \( u' \) is radiation energy density and \( n_e'(\gamma') \) is the electron distribution, all in comoving frame. Function \( F_C(q', \kappa') \) is inverse Compton scattering kernel for isotropic photon and electron distributions as presented in \[128, 51\]

\[
F_C(q', \kappa') = \left[ 2q' \ln q' + (1 + 2q')(1 - q') + \frac{1}{2} \frac{(4\kappa q')^2}{(1 + 4\kappa q')} \right] H(q'; \frac{1}{4q'}, 1) \tag{2.120}
\]

where function \( H(x; x_1, x_2) \) is the rectangle function which has value 1 if \( x_1 \leq x \leq x_2 \) and 0 otherwise. Variables \( q' \) and \( \kappa' \) inside function \( F_c(q', \kappa') \) are on the other hand
2.3 Inverse Compton Scattering

given as

\[ q' \equiv \frac{\epsilon'_s / \gamma'}{4 \kappa'(1 - \epsilon'/\gamma')} \quad \text{and} \quad \kappa' = \epsilon' \gamma' , \tag{2.121} \]

which due to the limits of rectangle function in equation (2.120) define the limits of integration in equation (2.119) as

\[
\begin{align*}
\gamma'_{\text{min}} &= \frac{1}{2} \epsilon'_s \left( 1 + \sqrt{1 + \frac{1}{\epsilon'_s / \epsilon'_s}} \right) , \\
\gamma'_{\text{max}} &= \epsilon'_s H(\epsilon' - \epsilon'_s) + \gamma'_2 H(\epsilon'_s - \epsilon') ,
\end{align*} \tag{2.122a, 2.122b} \]

where \( H(x) \) is the Heaviside step function and \( \gamma'_2 \) is the maximum energy of accelerated electron dictated by particle acceleration mechanism. Since ambient photons scattering upon non thermal electrons in case of SSC come from synchrotron radiation, SSC emissivity can be calculated by using observed synchrotron radiation flux \( f_{\text{syn}}^{\epsilon} \) through following expression \[89\]

\[
\epsilon'_s j_{\text{SSC}}^{\epsilon}(\epsilon'_s) = \frac{9 \sigma_T c^2 L}{4 \pi^2 \delta_D^4 R_b^2} \int_0^\infty \frac{d\epsilon'}{\epsilon'^3} \int \frac{d\gamma'}{\gamma'^3} \int_{\gamma'_{\text{min}}}^{\gamma'_{\text{max}}} \frac{n'_e(\gamma')}{\gamma'^2} F_C(q, \Gamma) , \tag{2.123} \]

where \( d_L \) is the luminosity distance, \( R'_b \) is the size of the emitting blob and \( \delta_D \) is the Doppler boosting factor given as

\[ \delta_D = \frac{1}{\Gamma(1 - \beta \cos \theta)} , \tag{2.124} \]

with \( \Gamma \) being blob’s Lorentz factor in observer frame and \( \theta \) being the angle between direction of blob’s motion and line of sight of the observer. Consequent SSC spectral flux can be calculated using emissivity from equation (2.123) as follows \[89\]

\[
\begin{align*}
f_{\epsilon'}^{\text{SSC}} &= \frac{\delta_D^2 \epsilon'_s j_{\text{SSC}}^{\epsilon}(\epsilon'_s)V_b'}{4 \pi d_L^2} = \\
&= \frac{9}{16} \frac{(1 + z)^2 \sigma_T c^2 V_b'}{\pi \delta_D^2 \epsilon'_s^2 t_{\nu,\text{min}}^2} \int_0^\infty \frac{d\epsilon'}{\epsilon'^3} \int \frac{d\gamma'}{\gamma'^3} \int_{\gamma'_{\text{min}}}^{\gamma'_{\text{max}}} \frac{n'_e(\gamma')}{\gamma'^2} F_C(q, \Gamma) , \tag{2.125} \end{align*} \]

where \( V_b' \) is the size of the emitting blob in comoving frame, \( z \) is cosmological redshift and \( t_{\nu,\text{min}} \) is smallest possible variability scale observable from the emitting blob given by

\[ t_{\nu,\text{min}} = \frac{(1 + z) R'_b}{\delta_D c} . \tag{2.126} \]
For further research and details concerning the derivations leading to aforementioned expressions we direct the reader to the paper by Finke et al. (2008) [89] and the book by Dermer and Menon (2009) [80].
Chapter 3

Particle Acceleration in Astrophysics

3.1 Introduction

Observations of cosmic ray spectra together with radiation spectra of various astrophysical phenomena within heliosphere, active galaxies, supernova remnants, \(\gamma\)-ray bursts, pulsars, micro quasars and many others clearly confirm that non thermal particles are quite common in our Universe. Observed power law spectra of charged particles either radiating non thermal radiation or arriving as cosmic rays at Earth’s surface require specific acceleration mechanisms which can be divided in two following groups depending on the scales of induced electric fields \(\vec{E}\):

- \textbf{Regular acceleration mechanisms} having large scale induced electric fields \((\langle \vec{E} \rangle \neq 0)\) which require quite specific conditions considering the fact that astrophysical plasmas are highly conductive. Most famous examples of this type include the unipolar inductor, magnetic reconnection and betatron acceleration mechanism.

- \textbf{Stochastic acceleration mechanisms} having only small scale induced electric fields for which \(\langle \vec{E} \rangle = 0\) and \(\langle \vec{E}^2 \rangle \neq 0\). Most of cosmic accelerators fall into this group where charged particles gain energy from macroscopic kinetic energy of magnetized plasma through scattering on random magnetic oscillations within it. Within this group fall various types of Fermi acceleration mechanisms of first and second order.

This chapter will cover the most prominent particle acceleration mechanisms used to explain \(\gamma\)-ray burst observations. Starting from the theoretical background of Fermi acceleration mechanisms, through the unipolar inductors which are probably very
important for physics of central engines and finalizing with magnetic reconnection which has presented as excellent alternative to phenomena which can not be easily explained by the traditional shock acceleration model. Many of this mechanisms form quite immense fields of study themselves and it is certainly not possible to touch upon all of their details. Therefore we will refer to additional sources which the reader can check upon at each of the sections.

3.2 Fermi Acceleration

An acceleration mechanism recreating power law distribution of charged particles was already investigated by Fermi in 1949 [88]. According to Fermi’s original idea charged particles are accelerated through reflections off randomly moving magnetic clouds acting as magnetic mirrors. Although shown to be rather ineffective way of accelerating charged particles, original Fermi’s idea also known as second order Fermi acceleration has shown major points important for particle acceleration mechanisms in general and paved the way for development of much more efficient and in astrophysical sense way more present first order Fermi mechanism.

3.2.1 Second Order Fermi Acceleration

Following original Fermi’s calculation let us imagine a collision of a charged particle with a massive moving magnetic cloud where the particle is incoming at the angle $\theta$ relative to the normal vector of magnetic mirror’s surface. Since the mass of the single cloud $M$ is much larger than mass of the charged particle $m \ll M$ we consider the cloud to be infinitely massive and of unchanging typical velocity $\vec{V}$. For the same reason the centre of mass is within the cloud and moving with velocity $\vec{V}$ while the inbound energy of particle in the centre of momentum frame becomes

$$E_{in}' = \Gamma (E_{in} + V p_{in} \cos \theta),$$

(3.1)

where $\Gamma = (1 - V^2/c^2)^{-1/2}$ is the Lorentz factor of cloud’s bulk motion, $p_{in}$ inbound momentum and $E_{in}$ inbound energy of particle in observer’s frame of reference. The reflection from magnetic mirror does not change the energy of particle in centre of momentum frame $E_{in}' = E_{out}'$ but it reverses the direction of the momentum component along the normal vector of magnetic mirrors surface denoted as $x$-direction

$$p_{x,in}' = p_{x,in} \cos \theta = \Gamma \left( p_{in} \cos \theta + \frac{EV}{c^2} \right),$$

(3.2)

$$p_{x,in}' = -p_{x,out}'. \quad (3.3)$$
Transforming the outbound momentum $p'_{x,\text{out}}$ back into observer’s frame gives us also the outbound particle energy $E_{\text{out}}$ in observer’s frame of reference
\begin{equation}
E_{\text{out}} = \Gamma(E_{\text{out}} - Vp'_{x,\text{out}}) = \Gamma(E_{\text{in}} + Vp'_{x,\text{in}}).
\end{equation}

One of the assumptions made by Fermi, although also a problematic one leading toward the injection problem, is that the initial charged particle ensemble is already ultra relativistic having $p_x/E = v\cos\theta/c^2$ with $v$ being the particle’s velocity in observer’s frame. Including this assumption together with the results of equations (3.1) and (3.2) into equation (3.4) gives us following expression
\begin{equation}
E_{\text{out}} = \Gamma^2 E_{\text{in}} \left[ 1 + \frac{2 v \cos \theta}{c^2} + \frac{V^2}{c^2} \right],
\end{equation}

which by expanding $\Gamma$ till the second order in $V/c$ gives by back the change of particle’s energy in observer’s frame as
\begin{equation}
\Delta E = E_{\text{out}} - E_{\text{in}} = 2E_{\text{in}} \left[ \frac{V \cos \theta}{c^2} + \frac{V^2}{c^2} \right].
\end{equation}

Since the magnetic clouds accelerating/decelerating the charged particles are moving in random directions with a typical velocity $\vec{V}$ it is necessary to average over all values of angle $\theta$. Important part in averaging the increment of energy of the charged particle is to understand that head-on collisions of particles have a slightly larger probability $p(\mu) \propto v + V\mu$ than collisions in which particles come at the cloud from behind $p(\mu) \propto v - V\mu$ with $\mu = \cos \theta$. This means that in average particles gain energy although the increment comes only from head-on collisions while in overtaking ones particles are decelerated. Since we are dealing with relativistic particles in start with $v \approx c$ then collision probability becomes
\begin{equation}
p(\mu) \propto \Gamma \left[ 1 + \frac{V}{c} \mu \right],
\end{equation}

with $\mu$ now being from $-1$ to $1$ covering both head-on and overtaking collision types. With this in mind the average increment of particle’s energy is given by following expression
\begin{equation}
\left\langle \frac{\Delta E}{E} \right\rangle = 2 \left\{ \frac{V}{c} \int_{-1}^{1} \frac{\mu [1 + (V/c)\mu]d\mu}{\int_{-1}^{1} [1 + (V/c)\mu]d\mu} + \left( \frac{V}{c} \right)^2 \right\} = \frac{8}{3} \left( \frac{V}{c} \right)^2.
\end{equation}
Since the increment per collision of particle’s energy is only in second order of \( V/c \) this mechanism got the name \textit{second order Fermi acceleration mechanism}. For more palpable use in further derivations it is more convenient to calculate energy gain per unit time from equation (3.8). Time between collisions is given by

\[
t = \frac{L}{c \cos \phi},
\]

where \( L \) is mean free path between clouds along the field line and \( \phi \) is the pitch angle in respect to the magnetic field direction. Averaging equation (3.9) over all angles \( \phi \) we get the average time between collisions \( \langle t \rangle = 2L/c \) leading to increment of particle’s energy per unit time

\[
\frac{dE}{dt} = \frac{4}{3} \frac{V^2}{cLE} E = \frac{E}{\tau_{\text{acc}}},
\]

with \( \tau_{\text{acc}} \) being acceleration time. Assuming that each charged particle remains within the accelerator for a time \( \tau_{\text{esc}} \) we can calculate the accelerated charged particle spectra by finding the steady-state solution to \textit{diffusion-loss equation}

\[
\frac{dn}{dt} = D \nabla^2 n + \frac{\partial}{\partial E} \left[ b(E)n(E) \right] - n \frac{n}{\tau_{\text{esc}}} + q(E) = 0,
\]

in absence of diffusion \( D \nabla^2 n = 0 \) and sources \( q(E) = 0 \). On the other hand since energy gain/loss term \( b(E) \) through equation (3.10) becomes \( b(E) = E/\tau_{\text{acc}} \) equation (3.11) can be reduced into

\[
- \frac{d}{dE} \left[ E \frac{n}{\tau_{\text{acc}}} \right] - \frac{n}{\tau_{\text{esc}}} = 0,
\]

giving out as solution a power law spectra of particles

\[
n(E) = AE^{-p},
\]

with parameter \( p = 1 + \tau_{\text{acc}}/\tau_{\text{esc}} \). As shown in equation (3.13) in case of second order Fermi mechanism power law index is not unique as observed in cosmic ray spectra and can be as low as \( p = 1 \) in case of very efficient accelerators. Also in astrophysical framework this mechanism has been shown to be rather inefficient and insufficient to produce the observed spectra cause of its second order nature. Nevertheless second order Fermi acceleration was the first mechanism proposed and the cornerstone upon which more efficient \textit{first order Fermi acceleration mechanism} was built upon.
3.2 Fermi Acceleration

Cosmic plasmas being tenuous are significantly different environments from laboratory plasmas. There typical sound speeds are considerably smaller than observed bulk motion velocities which directly leads toward the creation of strong collisionless shocks. In late 1970s number of researchers independently came to a conclusion that particles through scattering on hydromagnetic waves can be confined near the shock leading to very efficient \textit{diffusive shock acceleration} mechanism \cite{143, 19, 44, 28}. Diffusive shock acceleration has been investigated through two equivalent theoretical approaches, either by employing an microscopic picture established by Bell (1978) \cite{28} or by solving the particle transport equation first used by Krymskii (1977) \cite{143} and Axford et al. (1977) \cite{19}. Following the derivation of second order Fermi mechanism here we will proceed with the microscopic physics treatment for the sake of clarity.

We start from a stationary propagation of shock in homogeneous medium where the shock front of plane geometry at \( x = 0 \) is creating two half spaces, as shown on figure \ref{fig:shock} with flow velocity \( v(x) \) given as a function of position \( x \) like

\[
v(x) = \begin{cases} 
  v_1 & x < 0 \\
  v_2 = r^{-1} v_1 & x > 0 
\end{cases},
\]

(3.14)

where \( v_1 \) is upstream and \( v_2 \) is downstream medium velocity, while \( r \) is the compression ratio. Also we will only consider here a parallel or nearly parallel MHD shock configurations. Hydromagnetic waves as particle’s scatterers are moving at

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{shock.png}
\caption{Diffusive shock acceleration illustration after Lee (2000) \cite{151}. Picture obtained from: \url{http://sprg.ssl.berkeley.edu/~pulupa/illustrations/}}
\end{figure}
Alfvén velocity in local plasma frame which in our case of shocks is much smaller than the bulk flow velocity of plasma $v_A \ll v(x)$. Cause of it hydromagnetic waves seem frozen into plasma flow for the energetic particles.

A particle entering from upstream to downstream region at an angle $\theta_1$ will transform its momentum from upstream to downstream reference frame as follows

$$p_2 = p_1 (1 + \mu_1 V/u_1),$$

(3.15)

where $p_1$ and $p_2$ are upstream and downstream momenta, $V = v_1 - v_2$ is shock velocity relative to the upstream region, $\mu_1 = \cos \theta_1$ with values in range $1 > \mu_1 > -v_1/u_1$ and $u_1$ is particle’s velocity in upstream medium. It is worth to note that in this scenario we are investigating a non-relativistic shock. Scattering in downstream region does not change particle’s energy and only changes the pitch angle bringing it back in upstream region with new momentum $\hat{p}_1$ given by

$$\hat{p}_1 = p_2 (1 - \mu_2 V/u_2),$$

(3.16)

where $\mu_2 = \cos \theta_2$ with values in range $-1 < \mu_2 < -v_2/u_2$ is given through after scattering pitch angle of the particle in downstream region $\theta_2$ and $u_2$ is particle’s velocity in downstream region after scattering.

The momentum gain $\Delta p = \hat{p}_1 - p_1$ given by equations (3.15) and (3.16) can be averaged within one scattering cycle upstream $\rightarrow$ downstream $\rightarrow$ upstream assuming particle distributions are isotropic $\langle \Delta p/p_1 \rangle = \frac{1}{\mu_1} \int_{-v_1/u_1}^{v_1} d\mu_1 \int_{-1}^{1} d\mu_2 |\mu_1 u_1 + v_1|^{-v_2/u_2} \mu_2 |\mu_2 u_2 + v_2| (\hat{p}_1 - p_1)/p_1.$

(3.17)

Considering that our investigation tackles only relativistic charged particles with $u_1, u_2 \approx c$ we can safely extend the solution of equation (3.17) only till the first order of $V/c$ as shown here

$$\langle \Delta p/p \rangle \approx \frac{V}{3} \frac{d\mu_1}{\mu_1 u_1} \int_{-1}^{0} d\mu_2 |\mu_1(\mu_1 - \mu_2)| = \frac{4}{3} \frac{V}{c}.$$

(3.18)

Since we are dealing with relativistic particles $E = pc$ and therefore the increment
3.2 Fermi Acceleration

of particles energy per cycle is

$$\langle \frac{\Delta E}{E} \rangle = \frac{4}{3} \frac{V}{c},$$

which compared with the second order in equation (3.8) is of first order therefore justifying the name of first order Fermi acceleration mechanism. Mayor difference in comparison with second order Fermi mechanism is that while in second order processes there are both head-on and overtaking collisions with scatterers in first order case collisions are only of head-on type. In astrophysical framework this acceleration mechanism has been shown to be much more efficient in producing cosmic rays and non thermal radiation. Nevertheless it is important to notice that particles need to perform many cycles around shock to increase their energy significantly. At the other hand at each cycle part of the particles wanders far into downstream region without returning back to upstream region.

From macroscopic solution to the shockwave configuration we can safely assume that downstream particle distribution is constant while upstream particle distribution falls exponentially with distance from the shock [134]. Following that statement the flux of particles escaping across the imaginary boundary far away from the shock is given as

$$\Phi_{\text{esc}}(p^2) = 2\pi p^2 \int_{-1}^{+1} d\mu_2(\mu_2 u_2 + v_2) f_2(p^2) = 4\pi p^2 f_2(p^2) v_2,$$

(3.20)

while the flux of particles crossing the shock front from upstream to downstream is

$$\Phi_{\text{down}}(p^2) = 2\pi p^2 \int_{-v_2/u_2}^{+1} d\mu_2|\mu_2 u_2 + v_2| f_2(p^2) \approx \pi p^2 f_2(p^2) c,$$

(3.21)

where $f_2(p^2)$ is the isotropic particle momentum distribution function. Particle’s escape probability per cycle is therefore given as the ratio of escaping particles flux $\Phi_{\text{esc}}$ and incoming particles flux $\Phi_{\text{cross}}$

$$P_{\text{esc}} = \frac{\Phi_{\text{esc}}}{\Phi_{\text{down}}} = \frac{4v_2}{c}.$$

(3.22)

With this in mind we can calculate the steady state spectrum of accelerated particles, total flux of particles crossing back into downstream region with momentum larger than $p + \langle \Delta p \rangle$ can be obtained by subtracting the flux of escaping particles after one cycle with momentum $p$ from the total flux of particles crossing the shock front
into downstream medium with momentum larger than \( p \) as shown here

\[
v_2 \int_{p + (\Delta p)}^{\infty} dp' \Phi_{\text{down}}(p') = (1 - P_{\text{esc}}) v_2 \int_{p}^{\infty} dp' \Phi_{\text{down}}(p'). \tag{3.23}
\]

Since both crossing flux \( \Phi_{\text{down}}(p) \) and escape probability \( P_{\text{esc}} \) are already known from equations \((3.21)\) and \((3.22)\) we can get the steady state particle spectra by incorporating them in equation \((3.23)\)

\[
\left\langle \frac{\Delta p}{p} \right\rangle = P_{\text{esc}} \int_{p}^{\infty} dp' \frac{p^2 f_2(p')}{p} , \tag{3.24}
\]

with the following solution for the downstream momentum distribution

\[
f_2(p) \propto p^{-\frac{3 - P_{\text{esc}}}{\left\langle \frac{\Delta p}{p} \right\rangle} - \frac{3r}{r - 1}}, \tag{3.25}
\]

which by using equation \((2.53)\), since momentum distribution function is assumed to be isotropic, gives back energy distribution of accelerated particles

\[
n(E) \propto E^{-\frac{(r+2)}{r-1}}. \tag{3.26}
\]

Another method of gaining this solution is, as mentioned in the beginning, by solving the transport equation

\[
\frac{\partial f}{\partial t} + v(x) \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial f}{\partial x} \right) - \frac{1}{3} \frac{\partial v}{\partial x} \frac{\partial f}{\partial p} + q_0(p) \delta(x), \tag{3.27}
\]

with \( D(x) \) being the diffusion coefficient for fast particles, \( v(x) \) flow velocity depending on position \( x \), \( f = f(x, p) \) momentum distribution function isotropized by strong scattering processes \([108, 252]\) and \( q_0(p) \) being injection of fresh suprathermal particles at the shock front.

Following the derivation from Blandford and Eichler (1987) \([43]\), a stationary case with \( \partial f/\partial t = 0 \) is considered in which equation \((3.27)\) is integrated around shock front giving

\[
\left[ D \frac{\partial f}{\partial x} \right]_2 - \left[ D \frac{\partial f}{\partial x} \right]_1 + \frac{1}{3} (v_2 - v_1) p f_0 dp + q_0(p) = 0, \tag{3.28}
\]

where \( f_0 \) is the momentum distribution at the shockfront \( x = 0 \).

Equation \((3.28)\) can be easily solved by applying the boundary conditions of:

- homogeneity of downstream particle momentum distribution \( \partial f_2/\partial x = 0 \), where we know that downstream momentum distribution is essentially constant
3.2 Fermi Acceleration

\[ f_2(p) = f_0(p) \]
\[ D \frac{\partial f}{\partial x} = -\frac{v_2 - v_1}{3} p \frac{\partial f_0}{\partial p} + q_0, \] (3.29)

- homogeneity of particle momentum distribution far upstream \( \partial f_- / \partial x = 0 \), where \( f_-(p) = f(-\infty, p) \), which implied in integration of equation (3.28) from \( x \to -\infty \) to \( x = 0 \) leads toward

\[ D \frac{\partial f}{\partial x} = v_1 (f_0 - f_-). \] (3.30)

An general solution can be found by combining equations (3.29) and (3.30)

\[ p \frac{\partial f_0}{\partial p} = \frac{3}{v_2 - v_1} [v_1 (f_0 - f_-) - q_0], \] (3.31)

which integrated gives the accelerated downstream particle distribution

\[ f_2(p) = \frac{3r}{r - 1} p^{-3r/(r-1)} \int_0^p \left[ \frac{q_0(p')}{v_1} + f_-(p') \right] p'(2r-1)/(r-1) dp'. \] (3.32)

In special case when far upstream distribution diminishes \( f_-(p) = 0 \) and where particle injection goes as \( q_0(p) = \delta(p - p_{inj}) \) with \( p_{inj} \) being injected particle’s momenta we can recreate the same results (3.25) and (3.26) as in the case of microscopic treatment. On the other hand if we already have an existing suprathermal power law distribution far upstream \( f_-(p) = C_0 p^{-q} \) with power law index \( q < 3r/(r - 1) \) consequent downstream momentum distribution will not change its shape but it will be amplified

\[ f_2(p) = \frac{3r}{3r - q(r - 1)} C_0 p^{-q}. \] (3.33)

Important value to be known when calculating the spectra of accelerated particles is the compression ratio \( r = v_1/v_2 \). For that purpose let us examine conservation laws in the rest frame of the shock front:

- Conservation of mass

\[ \frac{\partial}{\partial x} (\rho v) = 0, \] (3.34)

- Conservation of momentum

\[ \frac{\partial}{\partial x} (\rho v^2 + P) = 0, \] (3.35)
• Conservation of energy

\[
\frac{\partial}{\partial x} \left( \frac{1}{2} \rho v^3 + \frac{\gamma_{ad}}{\gamma_{ad} - 1} \rho v P \right) = 0,
\]

(3.36)

where \( \rho \) is mass density, \( P \) is pressure and \( \gamma_{ad} \) is the adiabatic constant of the gas. This set of equations provide us directly with following jump conditions and compression ratio

\[
\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma_{ad} + 1)M_1^2}{(\gamma_{ad} - 1)M_1^2 + 2} = r,
\]

(3.37a)

\[
\frac{P_2}{P_1} = \frac{2\gamma_{ad}M_1^2 - \gamma_{ad} - 1}{\gamma_{ad} + 1},
\]

(3.37b)

\[
\frac{T_2}{T_1} = \frac{[2\gamma_{ad}M_1^2 + \gamma_{ad}(\gamma_{ad} - 1)][(\gamma_{ad} - 1)M_1^2 + 2]}{(\gamma_{ad} + 1)^2M_1^2},
\]

(3.37c)

with \( M_1 \) as the upstream flow Mach number, while \( T_1 \) and \( T_2 \) together with \( P_1 \) and \( P_2 \) are respectively upstream and downstream temperature and pressure of gas. Assuming strong shocks with \( M_1^2 \gg 1 \) within mono-atomic non-relativistic gas with \( \gamma_{ad} = 5/3 \) compression ratio becomes \( r = (\gamma_{ad} + 1)/(\gamma_{ad} - 1) = 4 \) with consequent spectra of accelerated particles from equations (3.25) and (3.26) becoming

\[
f_2(p) \propto p^{-4} \quad \text{and} \quad n(E) \propto E^{-2}.
\]

(3.38)

As we can see spectral slope from both microscopic and macroscopic approach is not dependant on the details of diffusion process but on the other hand acceleration timescale \( \tau_{acc} \) is deeply connected with it \[294\] \[47\] \[145\]

\[
\tau_{acc} = \frac{3}{v_1 - v_2} \int_0^p \frac{dp'}{p'} \left( \frac{D_1(p')}{v_1} + \frac{D_2(p')}{v_2} \right) \sim \frac{3}{v_1 - v_2} \left( \frac{D_1}{v_1} + \frac{D_2}{v_2} \right),
\]

(3.39)

where \( D_1 \) and \( D_2 \) are upstream and downstream diffusion coefficients which are increasing with particle’s momentum. Maximum energy of accelerated charged particles is afterwards determined by comparison with dynamical age of event \( t_{age} \) and cooling time \( t_{cool} \) coming from the dominant cooling mechanism

\[
\tau_{acc} \leq \min[t_{age}, t_{cool}],
\]

(3.40)

by which lower time limit applies. In case of Bohmian diffusion and dominant synchrotron cooling, where event is sufficiently long lasting \( t_{age} \gg t_{cool} \), results from equations \[2.73\] already presented in Chapter 2 come forth from relation (3.40) \[7\]. Another important energy limit comes from the capability of the accelerator to
contain the accelerated particles within coming in the form of known Hillas limit

\[ E_{\text{max}} = \varepsilon BR, \quad (3.41) \]

where \( R \) is the size of accelerator and \( B \) is magnetic field.

During its development theory of Fermi acceleration encountered an issue known as "the injection problem" requiring presence of already energized suprathermal particles for acceleration process to proceed as shown with equation \((3.27)\). One of the most promising solutions to this issue is the "thermal leakage" scenario where protons from farther tail part of Maxwellian distribution enter acceleration the same way they are accelerated later. Part of particles within thermalized downstream plasma can overtake the shockfront and by leaking upstream generate Alfén waves through cyclotron resonance. Generation of this waves leads to two important effects:

1. Self-regulation of particle injection by trapping of particles downstream through increase of Alfvén waves amplitudes with rising particle leakage.

2. Diffusion confinement of particles around shock through scattering of accelerated particles on large amplitude Alfvén waves.

Wavelengths of Alfvén waves excited by protons are too long for electrons to partake in acceleration and instead interact with them adiabatically. Resolution to this problem was proposed either through self-generation and scattering on whistler waves or through extraction of electrons at the shock by proton-generated lower-hybrid waves. This processes either way invoke necessity for nonlinear treatment of particle acceleration mechanisms. In spite of negligible role electrons play in shock structure they are connected with dynamically important proton spectra - which is quite useful in cases of SNRs where there are direct proofs of electrons partaking in Fermi acceleration.

Another important aspect of both microscopic and transport equation approach is the assumption of particle distributions isotropy in both reference frames relying on efficient diffusion processes. Validity of this approach can be confirmed in case of Newtonian, i.e. nonrelativistic, shock waves where the anisotropic correction of order \( V/c \ll 1 \) is rather small. On the other hand this assumption is not valid in case of relativistic shocks since \( V/c \sim 1 \).

### 3.2.3 Nonlinear Theory of Particle Acceleration

*Test particle limit* presented in former section is the fruit of great early works done in 1970s upon first order Fermi acceleration. Although its merits are important, this approach neglects the dynamical impact accelerated particles play on shock
structure. An important quantity to consider are energy density and pressure of accelerated particles (i.e. cosmic rays) coming from upstream region

\[ E_{cr} = 4\pi mc^2 \int_{p_{\text{min}}}^{p_{\text{max}}} dp \frac{p^2 f(p)(\gamma - 1)}{p_{\text{max}}}, \quad (3.42a) \]

\[ P_{cr} = \frac{4}{3} \pi \int_{p_{\text{min}}}^{p_{\text{max}}} dp \frac{p^3 u(p)f(p)}{p_{\text{min}}}, \quad (3.42b) \]

where \( u(p) \) is particle’s velocity, \( p_{\text{min}} \) and \( p_{\text{max}} \) are minimum and maximum momenta of cosmic rays. Using particle distribution given by test particle limit treatment in equation (3.38) (or harder ones) we see that with increasing \( p_{\text{max}} \) cosmic ray pressure diverges, however in reality \( p_{\text{max}} \) is finite. While condition \( P_{cr} > \rho v^2 \) is possible solution of equation (3.42) it violates the conservation of momentum and is therefore unphysical.

With implementation of energy density \( E_{cr} \) and pressure \( P_{cr} \) of cosmic rays into energy and momentum conservation laws we are presented with two behaviour limits, in first cosmic ray pressure is negligible \( P_{cr} \ll \rho v^2 \) and therefore test particle limit approach is a valid approximation, in second cosmic ray pressure is comparable in magnitude with kinetic ram pressure of background plasma \( P_{cr} \approx \rho v^2 \) where non-linear approach is essential. Following second limit we encounter a system with significant relativistic particle fraction changing the adiabatic index closer to \( \gamma_{ad} \approx 4/3 \) characteristic for relativistic gasses. This also leads to higher compression ratios, stronger shocks and harder particle spectra.

Observations of shocks in SNRs and heliosphere give away acceleration efficiencies indicative of necessary development of nonlinear diffusive shock acceleration (see [15] and references therein). Initially an approach within hydrodynamic limit involving two fluid model was implemented, while later a more general and concise kinetic approach was derived (see [167] and references therein).

Including the cosmic ray energy and pressure component, as required by nonlinear diffusive shock acceleration paradigm, changes conservation laws given by equations (3.34), (3.35) and (3.36) into following stationary expressions

\[ \frac{\partial}{\partial x}(\rho v) = 0, \quad (3.43a) \]

\[ \frac{\partial}{\partial x} \left( \rho v^2 + P_{\text{gas}} + P_{cr} \right) = 0, \quad (3.43b) \]

\[ \frac{\partial}{\partial x} \left( \frac{1}{2} \rho v^3 + \frac{\gamma_{ad}}{\gamma_{ad} - 1} P_{\text{gas}} \right) = -v \frac{\partial P_{cr}}{\partial x}, \quad (3.43c) \]

with \( P_{\text{gas}} \) being the background gas pressure component. Presence of cosmic rays
Figure 3.2. Schematic representation of shock modification by cosmic rays in the rest frame of the shock. Plasma arriving with velocity $v_0$ far upstream ($x = -\infty$) is decelerated just before shock front ($x = 0^-$) to velocity $v_1 < v_0$ by cosmic ray pressure, forming the precursor region. Within completely gaseous subshock velocity is further reduced from $v_1$ upstream to $v_2$ downstream, leading to subshock compression ratio $R_{\text{sub}} = v_2/v_1 < 4$ available to low energy particles and total compression ratio $R_{\text{tot}} \sim 7$ available to high energy particles [47].

described by equations (3.43) leads toward formation of a shock precursor as shown on figure 3.2 where subshock compression ratio $R_{\text{sub}} < 4$ does not account for presence of cosmic rays, since cosmic ray pressure gradient is zero, while total compression ratio $R_{\text{tot}}$ can potentially become larger since more relativistic particles are diffusing farther away from shock front - giving out $R_{\text{tot}} \sim 7$. With this we notice particle acceleration is much more efficient for higher energy particle leading toward a concave accelerated particle spectra as shown on figure 3.3. Since acceleration of particles is also more efficient with the rising of shock’s Mach number $M$ less energy goes into heating of downstream plasma making gas behind the shock cooler as it can also be seen on figure 3.3 - but also leaving less particle’s available for injection into acceleration process. Another consequence of significant importance is the amplification of magnetic field within shock region by streaming cosmic rays leading toward even more efficient particle acceleration - leading toward self regulation of system. Proposed mechanisms of magnetic field amplification include: resonant streaming [28], non resonant streaming [29], shock corrugation - although it is not induced by cosmic rays [105], vorticity in the precursor [170] and firehose instability [263].

Research of nonlinear effects within diffusive shock acceleration is even today quite active field of research in astrophysics and its full coverage would be way out
3. Particle Acceleration in Astrophysics

Figure 3.3. Particle spectra (thermal + non-thermal) obtained within nonlinear diffusive shock acceleration model for various upstream Mach numbers $M_0$: solid line for $M_0 = 10$, dashed line for $M_0 = 50$ and dotted line for $M_0 = 100$. Vertical dashed line represents the position of thermal peak for a shock with no particle acceleration. Velocity $u_0 = 5 \times 10^8$ cm/s is the far upstream velocity at $x = -\infty$, $p_{\text{max}} = 10^5 m_p c$ maximum particle’s momentum (in this case of protons) while $\xi = p_{\text{inj}}/p_{\text{th}} = 3.5$ is injection parameter fixing the ratio between injected particle’s momenta $p_{\text{inj}}$ and thermal momenta $p_{\text{th}} = (2m_p k T_2)^{1/2}$ with $T_2$ being downstream plasma temperature [48].

the scope of this thesis trying to touch only some mayor results. Therefore we refer the reader toward some comprehensive reviews by Malkov and Drury (2001) [167], Blasi (2013) [47] and Bell (2013) [30].

3.2.4 Relativistic Shocks

Observations of a wide variety of astrophysical phenomena, most notably active galactic nuclei (AGNs), pulsar wind nebulae (PWNs) and early phases of $\gamma$-ray bursts (GRBs), point toward a presence of relativistic motion within plasma leading to formation of relativistic shock waves and associated particle acceleration.

Although the basic idea of particle gaining energy by crossing of shock front is still valid, in case of diffusive shock acceleration on relativistic shocks has significantly different crossing probabilities compared to nonrelativistic case. While at nonrelativistic shocks particle spectra is determined by compression ratio in case of relativistic shocks such universality can not be a priori expected. Alongside noted facts we need to remind ourselves that particle distributions we encounter at relativistic shocks are largely anisotropic both upstream and downstream and grow more anisotropic with rising bulk Lorentz factor $\Gamma_{\text{sh}}$ of the shock leading to quite different return probabilities for the particles.
In an extreme, but quite didactic, case of ultrarelativistic shock particle’s anisotropy is getting incrementally stronger by each crossing of the shockwave due to the relativistic beaming - since all shock reflected particles are funnelled within a cone with aperture size of $\sim \Gamma_{\text{sh}}^{-1}$. Increments in anisotropy are accompanied by increments in particle’s energy coming from Lorentz transformations between upstream and downstream regions. Initial crossing of shock leads to the particle’s energy increment of $\Delta E/E \sim 4\Gamma_{\text{sh}}^{-2}$, while consecutive crossings provide energy increments of order $\Delta E/E \sim 2^{[46]}$.

One of the most important parameters determining the efficiency of particle acceleration both in cases of nonrelativistic and relativistic shocks is its obliquity angle, i.e. the angle between the magnetic fields crossing the shock from either upstream or downstream relative to the shock front’s normal vector. Particles moving along magnetic field lines can not catch up the shock front, even if they are moving with the speed of light, if the upstream obliquity angle $\theta_1$ is larger then the critical value

$$\theta_{\text{cr},1} = \arccos \beta_1,$$

with $\beta_1 = v_1/c$ which transforms into critical obliquity angle downstream as follows

$$\theta_{\text{cr},2} = \arctan \left[ \frac{1}{\Gamma_1(\beta_{12} + \beta_1)} \right],$$

with $\Gamma_1$ being the bulk Lorentz factor of upstream flow relative to shock rest frame and $\beta_{12}$ being the relative velocity of upstream and downstream flow. In literature cases when particles approach shocks with angles $\theta < \theta_{\text{cr}}$ leading to further acceleration are called subluminal shocks, while cases when angles of particles’ approach is $\theta > \theta_{\text{cr}}$ are called superluminal shocks leading to no further acceleration of particles.

Spectral index $s_p$ describing the accelerated particle distribution as power law $f(p) \propto p^{-s_p}$ has been obtained semi-analytically by Keshet and Waxman (2005) $^{[133]}$ and using the assumption of small-angle scattering in the fluid rest frame

$$s_p = \frac{3\beta_1 - 2\beta_1\beta_2^2 + \beta_2^3}{\beta_1 - \beta_2},$$

with $\beta_2 = v_2/c$ coming from downstream flow velocity. In extreme case of ultrarelativistic shocks where $\beta_1 \approx 1$ and $\beta_2 \approx 1/3$ we are getting a converging value of spectral index from equation (3.46)

$$s_p(\beta_1 \to 1, \beta_2 \to 1/3) = \frac{38}{9} \approx 4.222...,$$

as shown on figure [3.4].
Figure 3.4. Spectral index \( s \equiv s_p \) shown as a function of upstream value \( \gamma_u \beta_u \equiv \gamma_1 \beta_1 \) for three different types of relativistic shocks. Curves are calculated using equation (3.46) while values presented with symbols are obtained through numerical simulations. Comparison was done for: a strong shock with the Jüttner-Synge equation of state (solid green curve versus crosses), a strong shock having fixed adiabatic index \( \gamma_{\text{ad}} = 4/3 \) (dashed blue curve versus x marks), and a shock within relativistic gas having \( \beta_1 \beta_2 = 1/3 \) (dash dotted red curve versus circles) \[133\].

Coinciding or very close values have been also obtained through Monte Carlo simulations within shocks whose upstream magnetic field is normal to the shock front \[26,82\]. On the other hand in case of oblique shocks with mildly relativistic Lorentz factors \( \Gamma_{\text{sh}} \sim O(1) \) significant deviations from the value presented in equation (3.47) are expected. One of mayor findings confirmed through various approaches, ranging from analytical, Monte Carlo to particle-in-cell (PIC) methods, is that operation of the first order Fermi acceleration mechanism at relativistic shocks requires the presence of intense micro-turbulences in the shock precursor \[153,152,189,268\].

Within last few years advances in numerical algorithms and the immense technological growth of computer capabilities have lead toward more frequent and sensible use of particle-in-cell (PIC) simulations as a first principles alternative to the methods used before based upon a priori assumptions on the diffusion coefficients. Power of PIC method lies in its capability to model astrophysical plasma behaviour at its most fundamental level, treating plasma as a system made of charged macro-particles whose motion is governed by the Lorentz force. Charge distributions and currents, being the consequence of particle’s motion, are calculated on computational grid and later used in solving Maxwell’s equations to obtain electromagnetic fields, which are then again extrapolated to the macro-particle’s positions for computation of Lorentz force - making the loop self-consistent.
3.2 Fermi Acceleration

Figure 3.5. Downstream particle energy spectra of accelerated electron-positron plasma at time $\omega_p t = 9000$ where $\omega_p = \sqrt{4\pi e^2 n_1 / \gamma_0 m}$ is upstream relativistic plasma frequency given through $n_1$ - upstream number density and $\gamma_0$ - injected plasma initial Lorentz factor. Particle acceleration was numerically calculated at different obliquity angles: $\theta = 0^\circ$ (blue), $\theta = 15^\circ$ (green), $\theta = 30^\circ$ (red) and $\theta = 45^\circ$ (black). Smaller panels within the figure from top to bottom denote: a) Power law slope at high energies as a function of obliquity angle $\theta$ with high energy tail being defined through intersection between Maxwellian and power law fittings. b) Percentage of particles in the high energy tail. c) Percentage of energy stored in high energy tail.

Magnetization parameter $\sigma$ given by

$$\sigma = \frac{B^2}{4\pi \Gamma_1 (\Gamma_1 - 1)n_1 mc^2}, \quad (3.48)$$

where $B$ is the background magnetic field in shock’s rest frame and $n_1$ is upstream particle density in proper frame, has been shown to be of significant importance for operation of particle acceleration at ultrarelativistic shocks by findings of Sironi and Spitkovsky (2009, 2011) and Sironi et al. (2013). PIC simulations of electron-ion and electron-positron pair plasma indicate that increase of magnetization $\sigma$ within shocks suppresses particle acceleration, namely acceleration is suppressed at $\sigma > 0.03$ for pair plasmas and at $\sigma > 0.03$ for electron-ion plasmas. Another interesting finding is that acceleration itself becomes more efficient as obliquity angle $\theta$ rises to be absolutely suppressed when overcoming value of critical angle $\theta_{cr}$ as shown on figure 3.5.

Findings from GRB observations require efficient particle acceleration within forward shocks. Forward shocks arising from initially relativistic flows entering weakly magnetized circumburst medium warrant efficient particle acceleration and can lead to luminous GRB afterglows. In different manner internal shocks and
external reverse shocks are probably strongly magnetized due to proximity to GRB central engine. Bright reverse shock emission requiring magnetic fields of moderate strength has been proposed for some GRBs (see [320] and references therein). Although implied values of magnetization $\sigma$ within reverse shock region are greater then $10^{-3}$ a way around has been found through proposition that reverse shocks are mildly relativistic - therefore avoiding supraluminal suppression of particle acceleration. On the other hand, case of GRB prompt emission is still open to debate whether is powered by internal shocks collisions or some other processes possible like magnetic reconnection. Within strong magnetic fields reconnection is a viable alternative for particle acceleration [325, 320] while in case of internal shocks efficient particle acceleration requires weaker magnetic field and/or lower values of relative Lorentz factors.

3.3 Unipolar Inductor

Beside stochastic acceleration processes described in former section, under specific situations direct acceleration mechanisms can be put to work. One of commonly encountered phenomena appearing near magnetized compact objects e.g. pulsars (or in another case a Kerr black hole immersed in a background magnetic field) is the unipolar inductor [32, 294, 15]. Let us imagine a conducting magnetized sphere or radius $R$ in vacuum rotating with angular velocity $\vec{\Omega}$ aligned with magnetic field $\vec{B}$ as shown in figure [3.6]. If a stationary wire is put between pole and equator, i.e. going from point A through C and ending in point B, in magnet’s rest frame it will rotate with angular velocity $-\vec{\Omega}$ leading to creation of an electromotive force $E_{\text{ind}}$. Neglecting the distortion of magnetic field due to wires, this electromotive force is given by an integral

$$E_{\text{ind}} = \frac{1}{c} \int_{\text{ACB}} \vec{B} \times (\vec{r} \times \vec{\Omega}) d\vec{l},$$

(3.49)

taken along the wire. Magnitude of consequent quadruple surface electric field $E$ coming from internal charge redistribution in case of a typical pulsar can be estimated in case of internal shocks efficient particle acceleration requires weaker magnetic field with

$$E \sim \frac{\Omega R}{c} B \sim 10^{10} - 10^{12} \text{ V cm}^{-1},$$

(3.50)

with longitudinal component parallel to local magnetic field lines $E_{\parallel}$ being of the same order as $E$ leading to particle (i.e. electron in this case) extraction from surface and acceleration along the magnetic field lines. Strong curved magnetic fields lead toward production of curvature $\gamma$-ray radiation which further in strong field is absorbed by magnetic field leading to a cascade of pair-creation $\gamma + B \rightarrow e^+ + e^-$.
forming an electron-positron plasma pulsar magnetosphere.

Longitudinal component of electric field in pulsars magnetosphere is screened \( E_\parallel \approx 0 \) by plasma present in magnetosphere when its charge density is given by

\[
\rho_{\text{GJ}} = -\frac{\Omega}{2\pi c} e_z \left[ \vec{B} - \frac{1}{2} \vec{r} \times (\nabla \times \vec{B}) \right],
\]

(3.51)

where \( \rho_{\text{GJ}} \) is well known Goldreich-Julian charge density which ensures magnetosphere’s corotation, while \( e_z \) is the unit vector along rotation axis and \( \vec{B} \) is magnetic field in magnetosphere at position \( \vec{r} \). Field equations in pulsar magnetospheres can be approximated to 1D electrodynamics following longitudinal components of electric field \( E_\parallel \) and current \( j_\parallel \) along magnetic field lines as follows

\[
\frac{\partial E_\parallel}{\partial x} = 4\pi (\rho - \rho_{\text{GJ}}),
\]

(3.52a)

\[
\frac{\partial E_\parallel}{\partial t} = 4\pi (j_\parallel - j_m),
\]

(3.52b)

with \( \rho \) being the local charge density and \( j_m \) being the current sustaining the twist of the field lines given by

\[
j_m = \frac{c}{4\pi} (\nabla \times \vec{B})_\parallel.
\]

(3.53)

Equations (3.52) essentially tell us that the regions with charge densities smaller then Goldreich-Julian charge density can accelerate charged particles. This led to proposal of two possible acceleration areas in pulsars magnetosphere, one being pulsar’s polar cap having an immense longitudinal electric field component where pair creation cascade just starts and from which sprout the open magnetic field lines going beyond the light cylinder, and the other being the outer gap in which local charge density drops bellow \( \rho_{\text{GJ}} \) leading to the reappearance of the longitudinal
Figure 3.7. Two major candidates for acceleration sites within the pulsar magnetosphere are indicated as hatched regions. Neutron star (NS) is rotating around the rotation axis indicated with light dashed line. At distance $R_{LC} = c/\Omega$, commonly within few hundreds neutron star radii, from rotation axis we approach the light cylinder marked here with the dotted line. The magnetic field component projected along the rotation axis, vanishes on the null surface denoted by a heavy dashed line while the closed field lines are in the shaded region. According to the CHR picture the outer-gap accelerator area is situated between null surface and the light cylinder.[119]

electric field component $E_\parallel$ accelerating particles. Although at first glance the principles upon which pulsar magnetosphere is built seem clean and simple, full treatment is rather quite complex and often requires complicated simulations either in form of magnetohydrodynamics or within a complex approach like particle-in-cell. For further investigation we refer to the comprehensive review by Venter (2016) [292].

3.4 Magnetic Reconnection

In section 3.2.4 we already mentioned that highly magnetized relativistic flows in astrophysics have an issue explaining non thermal particle distributions with relativistic shock acceleration processes. On account of that fact dissipation of magnetic field energy into plasma and non thermal particle energy through the process of magnetic reconnection has been often presented as most viable explanation. First theoretical model of magnetic reconnection was already proposed by ref Giovanelli (1946) [110] and established by Sweet (1958) [277] and Parker (1957) [196]. In this model reconnection was calculated for a steady state flow with the current sheet of small resistivity separating two oppositely oriented magnetic fields. Reconnection
3.4 Magnetic Reconnection

would then take place within a layer of thickness \( \delta \) and length \( \mathcal{L} \). In original nonrelativistic case, owing to conservation of mass, inflow approaching with speed \( v_{\text{in}} \) and outflow leaving with Alfvén speed \( v_A \) amount for the same mass flux leading to

\[
\frac{v_{\text{in}}}{v_A} = \frac{\delta}{\mathcal{L}} = \frac{1}{\sqrt{S}},
\]

(3.54)

with \( S \) being the Lundquist number given by

\[
S = \frac{\mathcal{L}v_A}{\eta},
\]

(3.55)

where \( \eta \) is magnetic diffusion coefficient. Since we expect relativistic flows in GRBs it would be useful to investigate Sweet-Parker model within relativistic framework. Therefore it is useful to introduce relativistic generalisation for Alfvén speed given by

\[
v_A = \frac{cB/\sqrt{4\pi\rho}}{\sqrt{c^2 + B^2/4\pi\rho}} = c\sqrt{\frac{\sigma}{1 + \sigma}},
\]

(3.56)

where \( \rho \) is rest mass density and \( \sigma \) is magnetisation. We can easily see that at low magnetisation Alfvén speed approaches nonrelativistic limit \( v_{A,\text{NR}} = B/\sqrt{4\pi\rho} \), while at high magnetisation when \( \sigma \gg 1 \) we approach relativistic case where Alfvén speed comes close to the speed of light \( v_A \sim c \). Consequently Alfvénic bulk Lorentz factor of the flow is directly connected with the magnetisation \( \sigma \) by following expression

\[
\gamma_A = \sqrt{1 + \sigma},
\]

(3.57)

and it is important to note that it is not the bulk Lorentz factor of the outflow \( \gamma_{\text{out}} \) due to plasma heating during reconnection as we will show later. In relativistic case we need to take into account both mass and energy flux conservation going through reconnection layer leading to

\[
\frac{n_1}{4\pi}v_{\text{in}} = n_2\delta v_{\text{out}}\gamma_{\text{out}},
\]

(3.58a)

\[
\frac{B^2}{4\pi}v_{\text{in}} = n_2\delta m_pc^2v_{\text{out}}\gamma_{\text{out}}\Theta,
\]

(3.58b)

with \( n_1 \) and \( n_2 \) being respectively densities in comoving frame for inflow and outflow, \( v_{\text{in}} \) and \( v_{\text{out}} \) being respectively inflow and outflow speeds, \( m_p \) as proton mass, \( \gamma_{\text{out}} \) being bulk outflow motion Lorentz factor and \( \Theta \) being the Lorentz factor of random particle, i.e. proton, motion in outflow. Based on equations (3.58a) and (3.58b) we can obtain a very important and didactic expression for relativistic Sweet-Parker reconnection

\[
\frac{B^2}{4\pi n_1 m_pc^2} \equiv \sigma \simeq \gamma_A^2 = \gamma_{\text{out}}\Theta,
\]

(3.59)
pointing clearly that during reconnection magnetic energy is transformed into kinetic and internal energy of plasma. Also equation (3.59) tells us that for relativistic reconnection high magnetisation $\sigma \gg 1$ is necessary. Within this relativistic regime of Sweet-Parker magnetic reconnection inflow speed $v_{in}$ can be estimated as follows [320]

$$v_{in} \sim \sqrt{\frac{v_A v_{out}}{S} \gamma_{out}}.$$ (3.60)

Although confirmed through various numerical simulations and experimentally at Magnetic Reconnection Experiment (MRX) at Princeton Plasma Physics Laboratory the major issue Sweet-Parker reconnection model has when compared with the observed astrophysical phenomena (e.g. solar flares) is that it is quite inefficient and slow. Reasons for this lie in fact that within astrophysical considerations Lundquist number tends to be immense, order of $10^{16}$ or even larger, leading to negligible inflow speeds even in relativistic cases $v_{in} \ll c$. During time various solutions to this issue were proposed, most notably the one of Petschek (1964) [204] and the one of Lazarian and Vishniac (1999) [148].

Petschek proposed that reconnection instead of taking place along whole resistive, i.e. diffusive, layer’s length $L$ should be confined to a much smaller part of it $L'$ resulting in $(L/L')^{-1/2}$ times faster reconnection than in Sweet-Parker model. In difference to planar geometry of Sweet Parker model, Petschek reconnection model requires an ‘X’-shaped geometry as shown on figure 3.8. Consequently in that case one would expect presence of slow shocks in area within boundary but outside
3.4 Magnetic Reconnection

diffusive region accelerating matter that did not pass through diffusive region to outflow velocity $v_A$. Acceleration itself would come due the magnetic tension from transverse magnetic field component which is different from Sweet-Parker model where acceleration comes mainly from pressure gradient. Upper limit for reconnection rate was put by limiting the current in the shocks so it does not seriously perturb incoming magnetic field leading to

$$v_{in, max} = \frac{\pi v_A}{8 \ln S},$$

(3.61)
giving back inflow speed at range from $10^{-2}$ to $10^{-1}$ of "outer" Alfvén speed. Although Petschek model gives much faster reconnection then Sweet-Parker model it is either unstable when imposed upon MHD simulations or requires an anomalous resistivity profile rising towards the "X"-point [289]. There is no clear reason why should that condition be satisfied and many numerical simulations show preference for Sweet-Parker model [41].

Both Sweet-Parker and Petschek model try to describe steady-state reconnection, nevertheless the circumstances within which reconnection happens were not addressed in their analysis. First major contribution in that direction was proposed by Furth, Killeen & Rosenbluth (1963) [99] who demonstrated that magnetic fields can be unstable to small perturbations leading to tearing modes, which reconnect the magnetic lines. Adler, Kulsrud & White (1980) [6] concluded in the investigation of tearing mode energetics that the driving energy comes from unstable current gradients within tearing layer and is transformed into kinetic energy of ions and thermal energy of electrons [331]. Also resistive instabilities can be implemented within modelling of reconnection but in difference to tearing mode they do not saturate [296] leading to faster reconnection. Many simulations confirm the role of tearing modes, either by showing formation of magnetic islands due to tearing modes in 2D simulations or of flux rope instabilities in 3D simulations, leading to creation of many smaller reconnection sites [162, 118, 254, 113, 114].

Trying to resolve the dilemma between slow Sweet-Parker model and unstable Petschek model of reconnection Lazarian and Vishniac (1999) [148] have shown that rapid reconnection may be achieved due to presence of turbulence. Turbulent motion, being ubiquitous among astrophysical phenomena, is certainly to be expected in case of GRBs due to immense Reynolds and magnetic Reynolds number. As shown on figure 3.8 in presence of turbulence magnetic reconnection will be constricted to happen stochastically over local length scales $\sim \lambda_T$ rather then uniformly over global length scale $L$. Cause of this reconnection efficiency will not be defined by a 'global'
3. Particle Acceleration in Astrophysics

Lundquist number $S$ but with a 'local' one given as

$$ s \equiv \frac{\lambda_{||} v_A}{\eta} . $$

(3.62)

Structure of turbulences lead to significantly smaller 'local' scales, $\lambda_{||} \ll \mathcal{L}$, so we can expect great rise in reconnection efficiency and great increase in inflow speed $v_{in}$. Turbulent reconnection has been shown to be a great and promising alternative to Sweet-Parker and Petschek model in resolving of their issues.

Besides the fact turbulences might be significantly important for reconnection efficiency, we need to remind ourselves that astrophysical plasmas in question are generally collisionless. Current research leads us to believe that collisionless magnetic reconnection are inherently much faster and thought to be predominantly driven by electron dynamics in plasma. Electron-ion plasma can also be subject to more significant influence of Hall effect leading to Petschek-like quadrupolar structure of magnetic reconnection which is not present in pair plasma (see [283] and references therein).

Magnetic reconnection is currently immensely active, promising and interesting topic of research with application ranging from nuclear fusion energy production to numerous number of astrophysical phenomena which certainly can not be covered within the scope of this thesis. Therefore for any further details and in depth research we recommend reviews by Zweibel & Yamada (2009) [331] and Treumann & Baumjohann (2013, 2015) [283, 284] dealing specifically with aspects of collisionless reconnection physics, and Kagan et al. (2015) [129] dealing specifically with the subject of relativistic reconnection in pair plasmas.

3.4.1 Particle Acceleration by Magnetic Reconnection

As shown in previous section fast rearrangement of magnetic field’s topology during magnetic reconnection leads to conversion of magnetic field energy into internal energy of electrons and ions (or electron and positrons depending on plasma composition) and bulk kinetic energy of plasma. Alongside this it is commonly thought that magnetic reconnection is responsible for particle acceleration within magnetically dominated systems [129].

Astrophysical phenomena like PWN, pulsars, AGNs, GRBs or magnetar flares commonly show high variability (e.g. some blazars with variability within TeV photon energies at $\sim 10$ min scales, prompt emission of GRBs, Crab Nebula $\gamma$-ray flares etc...) and very hard spectra ($p \lesssim 2$) within highly magnetized and relativistic environments which can not be explained by classical Fermi mechanism paradigm.

In introduction of this chapter we mentioned that magnetic reconnection belongs
3.4 Magnetic Reconnection

Figure 3.9. Diagram showing relativistic Speiser orbit of a positron marked with red line. Particle being accelerated along z-axis by reconnection electric field \( E \) is moving back and forth across the reconnection layer of thickness \( 2\delta \) due to magnetic fields \( \pm B_0 \) of opposite polarity along x-axis. During acceleration particle is getting more constrained within reconnection layer [59].

among regular acceleration mechanisms which is due to reconnection electric field \( \vec{E}_{\text{rec}} \) induced by reversing magnetic fields within reconnection layer. Cause of this electric field charged particles crossing the current sheet at reconnection layer are forced into current sheet following Speiser orbits [273] as shown on figure 3.9.

Energy gain due to presence of reconnection electric field is then given as

\[
\left( \frac{dW}{dt} \right)_{\text{rec}} = q \vec{E}_{\text{rec}} \cdot \vec{v} \sim q E_{\text{rec}} c,
\]

with the estimate being given for a relativistic particle. Certainly, this accelerated charged particle radiates synchrotron radiation due the presence of strong magnetic field, leading to particle’s cooling given by an estimate as

\[
\left( \frac{dW}{dt} \right)_{\text{syn}} \sim \frac{2q^2 B^2 \gamma^2}{3m^2 c^3}.
\]

In regions where we can confirm the validity of MHD we have the case that \( E_{\text{rec}} \lesssim B \) making an upper limit on particle’s energy due to synchrotron burn-out when \( E_{\text{rec}} = B \). Nevertheless, within reconnection layer, strong electric fields \( E_{\text{rec}} > B \) are expected to be present, leading to much more efficient particle acceleration and higher characteristic energy of emitted radiation [288].

Alongside regular acceleration mechanism within the reconnection layers, some other 'secondary' processes can contribute to particle acceleration. First of them is the first order Fermi acceleration mechanism due to reflections of charged particles of the boundaries of contracting magnetic islands. This process was already heavily
discusses for both non-relativistic reconnection by Drake et al. (2006) [81] and Oka et al. (2010) [191] - where particle gains energy after several reflections, and relativistic reconnection by Guo et al. (2014) [113] and Liu et al. (2011) [160] - where particle attains significant energy increment after just one reflection.

Second type of acceleration process is also a first order Fermi mechanism where accelerated particle scatters between multiple magnetic islands scattering upon reconnection outflow jets [120]. Offcourse in both cases particle is primarily accelerated by crossing the reconnection layer by electric field \( E_{\text{rec}} \) and this initial process is the one defining the particle spectra slope and future of incoming particles.

Due to complexity of particle dynamics most common tool used in modelling of the particle acceleration during magnetic reconnection are PIC simulations. For the framework of high energy astrophysics first PIC simulations were done for the case of relativistic pair plasma reconnection, initially by Zenitani & Hoshino (2001,2005) [316,317] and then later by Sironi & Spitkovsky (2014) [267], Guo et al. (2014, 2015) [113,115] and Werner et al. (2014,2016) [304,307].

These PIC simulations following relativistic reconnection in pair plasmas generally show formation of hard non thermal distribution of accelerated particles. For example,
2D simulations done by Sironi and Spitkovsky (2014) \cite{267} (see figure \ref{fig:3.10}) show hardening of particle spectra with time, which converges at later times toward slope $p \equiv -d\ln N/d\ln \gamma \sim 2$ fitted for energies $\gamma \gtrsim 1.5$ at magnetization $\sigma = 10$. Likewise the slope $p$ was shown to be harder with increasing magnetization $\sigma$ as shown on smaller panel in figure \ref{fig:3.10} reaching $p = 1.5$ at magnetization $\sigma = 50$.

Alongside Sironi and Spitkovsky (2014) \cite{267}, simulations done by both Guo et al. (2014) \cite{113} and Werner et al. (2014) \cite{304} have confirmed the trend of spectral hardening with rising magnetization, suggesting that for highly magnetized flows where $\sigma \gg 1$ spectral slope can go as low as $p \simeq 1$. Nevertheless more recent studies, running 2D simulations for longer reconnection timescales and within bigger boxes, have pointed out that reconnection event producing initially harder spectra at later times asymptotically converge toward $p \sim 2$ slopes (Petropoulou & Sironi (2018) \cite{203}).

Another major finding by Petropoulou & Sironi (2018) \cite{203} contrary to initial findings, which claimed harder particle distributions with $p \leq 2$ should terminate the growth of accelerated particle maximum energy $\gamma_{\text{max}}$ at some point, was the steady growth of $\gamma_{\text{max}}$ with time as long reconnection acceleration mechanism works. The major problem of former simulations was that slowing down of maximum energy rise might be misidentified for saturation.

Alongside 2D PIC simulations of pair plasma relativistic magnetic reconnection, several 3D simulations were done to investigate the range of validity for 2D approach, due to possible influence of (relativistic) drift-kink instability on particle acceleration. Simulations show that although drift-kink instability noticeably perturbs 3D reconnection current sheets, it does not diminish particle acceleration. Particle distributions produced by full 3D reconnection are non thermal power law distribution closely resembling the results of 2D simulations Werner & Uzdensky (2017) \cite{305}. This is cause the current sheet evolution in later times is anyway dominated by the tearing instability, making physics analogous to the one of 2D simulations \cite{267}. On the other hand it was found that strong guide magnetic field has a negative impact on particle acceleration \cite{305}.

As we can see most of numerical work done till now in case of relativistic reconnection was done for pair plasmas since electron and positron magnetization are assumed to be same and equally large $\sigma \gg 1$. On the other hand modelling of reconnection within electron-ion plasmas is rather complicated and requires a mixed semi-relativistic approach. This is due large difference in masses between ions and electrons, requiring PIC simulations to treat electrons as relativistic and ions as nonrelativistic particles since ion magnetization $\sigma_i = B_i^2/4\pi n_i m_i c^2$ is at least three orders of magnitude smaller then the one of electrons and frequently very small
\( \sigma_i \ll 1 \) \cite{303,302,174,175,114}. Preliminary studies have shown that total electron energy grows from \( 1/4 \) to \( 1/2 \) with rising ion magnetization \( \sigma_i \). Likewise, same simulations produce non thermal spectra of electrons with slopes going asymptotically from \( p \sim 4 \) at \( \sigma_i \sim 10^{-4} \) toward \( p \sim 2 \) at high magnetizations \( \sigma_i \gg 1 \). Reasons for that can be found in fact that at high ion magnetizations \( \sigma_i \gg 1 \) both electrons and ions are in ultrarelativistic regimes, resembling the behaviour of pair plasma. On the other hand there is no clear confirmation for a power law distributions of ions at semi-relativistic regimes \cite{306}.

Besides PIC simulations, there is significant recent progress in experimental studies of relativistic magnetic reconnection. Experimental setup at LFXE facility, by exposing a "micro-coil" \cite{139} to intense laser radiation, managed to produce relativistically magnetized plasma and induce magnetic reconnection with a maximum magnetic field of approximately \( 3 \times 10^7 \) Gauss. Both electron and proton spectra were consequently observed, with electron spectra being significantly harder with a slope \( p_e = 1.535 \pm 0.015 \) while proton spectra can be fitted with a steeper slope \( p_p \approx 3.013 \) till cutt off energy \( \sim 18 \) MeV where it drastically steepens to \( p_p \approx 17.98 \) \cite{147}. 


Chapter 4

Kinetic Equation - Analytical and Numerical Approaches

4.1 Introduction

After covering the topics of non thermal radiation mechanisms in chapter 2 and particle acceleration producing non thermal particle distributions in chapter 3, we now have all the necessary physical ingredients to compute the transient behaviour of non thermal radiation sources, in our case $\gamma$-ray bursts, by solving the kinetic equation (sometimes called also diffusion-loss equation). Therefore with section 4.2 we will start with introduction to the concept of kinetic equation and its derivation, afterwards in sections 4.3, 4.4, and 4.5 we present common analytical methods of solving some special cases of kinetic equation with their solutions. Since analytical methods are not generally applicable in section 4.6, we present a numerical kinetic equation solver scheme used as a backbone for PESCARA (abbreviation for 'Particle Spectra and Radiation') code written for our research purposes. Testing of PESCARA code is afterwards presented in section 4.7. After confirming the robustness and precision of the code we present its preliminary results for cases of Gaussian injection within decaying magnetic fields in section 4.8 and modelling of the Crab Nebula radiation spectra in section 4.9 based upon parameters given in papers by Martin et al. (2012) [168] and Tanaka & Takahara (2010) [278].

4.2 Kinetic Equation

For clarity and simplicity of the derivation of the kinetic equation let us imagine a coordinate space diagram as shown on figure 4.1 [161]. The change of number of particles within the small rectangle with sides $dx$ and $dE$ at point of coordinate space $(x, E)$, defined as $n(E, x, t)dE dx$, can be originating from the change of particle’s
position $x$ by diffusion and/or advection described by diffusive/advective flux $\Phi_x$ or from particle’s energy gains/losses described by energy flux $\Phi_E$. Total rate of particle number change in coordinate space rectangle is therefore given by

$$\frac{d}{dt}n(E, x, t)dE dx = (\Phi_x(E, x, t) - \Phi_{x+dx}(E, x + dx, t))dE$$

$$+(\Phi_E(E, x, t) - \Phi_{E+dE}(E + dE, x, t))dx$$

$$+q(E, x, t)dE dx,$$

(4.1)

with $q(E, x, t)$ being the injection rate of particles per unit volume of coordinate space. Employing the Taylor expansion equation (4.1) can be reduced into following form

$$\frac{\partial}{\partial t}n(E, x, t) = -\frac{\partial}{\partial x}\Phi_x(E, x, t) - \frac{\partial}{\partial E}\Phi_E(E, x, t) + q(E, x, t).$$

(4.2)

Adopting the scalar diffusion coefficient $D$ and using the flow velocity $\vec{v} = (v_x, v_y, v_z)$ to describe advection the diffusive/advective flux $\Phi_x(E, x, t)$ can be rewritten as

$$\Phi_x(E, x, t) = -D \frac{\partial}{\partial x}n(E, x, t) + v_x n(E, x, t),$$

(4.3)

while the energy flux $\Phi_E(E, x, t)$ can be specified using the particle’s energy gain $a(E, t) = (dE/dt)_{gain}$ and energy loss $b(E, t) = -(dE/dt)_{loss}$ rates as

$$\Phi_E(E, x, t) = n(E, x, t) \frac{dE}{dt} = [a(E, t) - b(E, t)]n(E, x, t),$$

(4.4)

by which equation (4.2) can be expressed as following

$$\frac{\partial}{\partial t}n(E, x, t) = D \frac{\partial^2}{\partial x^2}n(E, x, t) - \frac{\partial}{\partial x}(v_x(x, t)n(E, x, t))$$

$$+ \frac{\partial}{\partial E}(b(E, t)n(E, x, t)) - \frac{\partial}{\partial E}(a(E, t)n(E, x, t)) + q(E, x, t).$$

(4.5)

By extending equation (4.5) to three dimensions and with simplified notation we get

$$\frac{\partial n}{\partial t} = D \nabla^2 n - \nabla[\vec{v} n] + \frac{\partial}{\partial E}[b n] - \frac{\partial}{\partial E}[a n] + q.$$ (4.6)

Equation (4.6) can be even more extended by including the effects of statistical particle acceleration through random collisions. This is done by invoking the diffusion of particles in momentum or phase space within the Fokker-Planck equation leading toward following kinetic equation

$$\frac{\partial n}{\partial t} = D \nabla^2 n - \nabla[\vec{v} n] + \frac{\partial}{\partial E}[b n] - \frac{\partial}{\partial E}[a n] + \frac{1}{2} \frac{\partial^2}{\partial E^2} [d n] + q,$$ (4.7)
with \( d \) being mean square energy increment in unit time

\[
d = \frac{d}{dt} \langle (\Delta E)^2 \rangle.
\] (4.8)

Equation (4.8) could be further expanded for various species of particles, like for example atomic nuclei, by including spallation and catastrophic collisions - which are out of topic of interest of this thesis. For further information we refer to the classical book of Ginzburg and Syrovatsky (1964) [109] which tackles with many of related issues.

Let us now imagine a region in space with volume \( V \) filled with homogeneous distribution of charged particles (e.g. electrons, positrons, protons etc.) given as \( n(E, t) \) and homogeneous spatial distribution of magnetic field \( B \) into which new charged particles are injected homogeneously (over the whole volume) with the injection rate \( q(E, t) \). Additionally assuming that the expansion rate of this emitting region is also homogeneous leads us to conclude that nothing in this specific model will depend on position of charged particle. Our goal is to follow the evolution of particle distribution \( n(E, t) \) determined exclusively by particle energy losses, whether due adiabatic expansion, synchrotron radiation, inverse Compton process and particle injection \( q(E, t) \) directly connected with particle acceleration mechanisms in accelerator region injecting particles from outside of the emitter, assuming some initial conditions at time \( t = t_0 \). As already mentioned in the previous chapter, lower energy radiation is a consequence of synchrotron radiation mechanism defined by particle distribution \( n(E, t) \) and magnetic field \( B \) while higher energy radiation owes its existence to both synchrotron and inverse Compton processes.

With this in mind it is possible to integrate equation (4.8) over the whole volume of emitter leading to

\[
\frac{\partial}{\partial t} N(E, t) = \frac{\partial}{\partial E} \langle b(E, t) N(E, t) \rangle - \frac{N(E, t)}{\tau_{\text{esc}}(E, t)} + Q(E, t),
\] (4.9)

where

\[
N(E, t) = \int n(E, \vec{r}, t) d^3r \text{ is the total number of particles,}
\] (4.10a)

\[
Q(E, t) = \int q(E, \vec{r}, t) d^3r \text{ is the total particle injection,}
\] (4.10b)

while \( \tau_{\text{esc}}(E, t) \) is the particle escape time given as

\[
\tau_{\text{esc}} = \left( \frac{1}{\tau_{\text{diff}}} + \frac{1}{\tau_{\text{conv}}} \right)^{-1}
\] (4.11)

where \( \tau_{\text{diff}} \) and \( \tau_{\text{conv}} \) relate to diffusive and convective particle loss respectively. In
Figure 4.1. Diagram showing the coordinate space where positions are designated along the abscissa and energy is on the ordinate. Each point in this coordinate space has a corresponding particle density energy distribution $n(E, x, t)$ while $\Phi_E$ and $\Phi_x$ are particle fluxes in energy $E$ and position $x$ space respectively [161].

many occasions where the emitter’s size are sufficiently large escape time can be treated as immensely large or practically infinite $\tau_{esc} \rightarrow \infty$ shortening equation (4.9) into

$$\frac{\partial}{\partial t} N(E, t) = \frac{\partial}{\partial E} (b(E, t) N(E, t)) + Q(E, t) .$$

Equation (4.12) can be also written in the following form

$$\frac{\partial}{\partial t} N(E, t) = b(E, t) \frac{\partial N(E, t)}{\partial E} + \frac{\partial b(E, t)}{\partial E} N(E, t) + Q(E, t) ,$$

where we can immediately notice that this is linear partial differential equation (PDE) of the first order for which exists a powerful mathematical tool for solving called method of characteristics which we will cover in Section 4.3.

4.3 Method of characteristics

Referring to the form of equation (4.13) we will present the method of characteristics (see e.g. Fritz (1991) [94]) using the corresponding form of linear PDE given by

$$a(x, y) \frac{\partial u(x, y)}{\partial x} + b(x, y) \frac{\partial u(x, y)}{\partial y} + c(x, y) u(x, y) = q(x, y) ,$$

where $q(x, y)$ is the inhomogeneous term making (4.14) an inhomogeneous linear PDE of the first order. It is clear that this inhomogeneous term is related to the particle injection term from equations (4.12) and (4.13). We will focus our discussion
to the form presented with (4.14) although it is worth to notice that method of characteristics is applicable in more general cases, e.g. when \( q = q(x, y, u) \), which include nonlinear terms in \( u \).

The concept behind the method of characteristics is based on finding new variables \( \xi = \xi(x, y) \) and \( \eta = \eta(x, y) \) which would transform equation (4.14) into

\[
\frac{\partial u(\xi, \eta)}{\partial \xi} + h(\xi, \eta) u(\xi, \eta) = q(x, y). 
\]

(4.15)

One of the variables, and in our case we implicitly choose \( \eta \) is kept an arbitrary constant as a function of \( x, y \), meaning we are looking for curves \( \eta(x, y) = \text{const.} \) in \([x, y]\) space along which equation (4.15) is then solved as an ordinary differential equation (ODE) of the first order in variable \( \xi \). Changing of the of the constant value of \( \eta(x, y) \) by which \( \eta \) can be viewed as a parameter will move the curve in \([x, y]\) space making it possible to obtain the solution in the whole \([x, y]\) space or at least in some domain \( D \) of it.

Along the curve \( \eta(x, y) = \text{const.} \) the differential of variable \( \eta \) is always zero

\[
d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy = 0, 
\]

(4.16)

implying that along the curve we have

\[
\frac{dy}{dx} = -\frac{\frac{\partial \eta}{\partial x}}{\frac{\partial \eta}{\partial y}},
\]

(4.17)

provided that \( \frac{\partial \eta}{\partial y} \neq 0 \).

Knowing the new coordinate \( \eta(x, y) \) we can see that equation (4.17) gives us a whole family of curves on which the same new coordinate is constant. Equation (4.17) is therefore called characteristic equation while the family of curves defined through it are called characteristics. It is important to notice before proceeding to obtain the new coordinates satisfying aforementioned requirements that in equation (4.16) we should have used different symbol, e.g. \( w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) \), for \( u(\xi, \eta) \) due the functional dependence changes by changing variables. Notwithstanding it is much more convenient to continue using the notation we used till now.

Let us now focus our attention toward the derivatives in the transformed coordi-
nates \( \xi, \eta \) given by
\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x}, \quad (4.18a)
\]
\[
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y}, \quad (4.18b)
\]
and which when implemented into equation (4.14) give us
\[
a \left( \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right) \frac{\partial u}{\partial \xi} + b \left( \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial y} \right) \frac{\partial u}{\partial \eta} + cu = q. \quad (4.19)
\]
Equation (4.19) can be practically rearranged into
\[
\left( a \frac{\partial \xi}{\partial x} + b \frac{\partial \xi}{\partial y} \right) \frac{\partial u}{\partial \xi} + \left( a \frac{\partial \eta}{\partial x} + b \frac{\partial \eta}{\partial y} \right) \frac{\partial u}{\partial \eta} + cu = q \quad (4.20)
\]
By selecting \( \eta \) to satisfy the condition
\[
a(x, y) \frac{\partial \eta(x, y)}{\partial x} + b(x, y) \frac{\partial \eta(x, y)}{\partial y} = 0 \quad (4.21)
\]
meaning that
\[
\frac{\partial \eta}{\partial x} = -\frac{b(x, y)}{a(x, y)}, \quad (4.22)
\]
we come upon the condition requested by equation (4.17) and at the characteristic equation by which the new variable \( \eta \) is defined
\[
\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \quad (4.23)
\]
This means that characteristic equation is now completely defined in terms of coefficients we have in the original PDE (4.14) giving the solution in form of some function \( y = f(x) + \text{const.} \). This enables us to single out the new variable \( \eta \) with the constant value as \( \eta(x, y) = y - f(x) \). For further convenience variable \( \xi \) can be chosen to be \( \xi(x, y) = x \).

Now, as a second step we can proceed with solving the equation (4.15) as an ODE. This is fundamentally simpler task in which we will not use the general form given by equation (4.15) but rather a more specific one given in the framework of kinetic equation (4.13). We anyway need to be reminded that in difference to solving ordinary differential equation when we are solving equation (4.15) for each characteristic curve specified with value of \( \eta \) and the information about it needs to be paired with specific solution of ODE. Essentially constant coming with solution
of equation (4.15) is in fact a function of \( \eta \) determined from boundary or initial conditions.

### 4.4 Simple cases of kinetic equation

Since we are introduced to the method of characteristics presented in section 4.3, we will now proceed in applying it to the case of simplified kinetic equation given with (4.13) taking into account energy losses due to synchrotron radiation and adiabatic expansion which within a homogeneous emitter cloud expanding with constant velocity are given by

\[
b(E, t) = \frac{\sigma_T c B_0^2}{6\pi m c^2} \left( \frac{t_0}{t} \right)^{2\kappa} E^2 + t^{-1} E ,
\]

with the first term describing time dependent synchrotron losses where \( B_0 \) is the magnetic field at time \( t_0 \) and with the second term describing adiabatic energy losses due to constant expansion. Incorporating constant expansion magnetic field’s time dependence is taken as

\[
B(t) = B_0 \left( \frac{t_0}{t} \right)^\kappa ,
\]

where time power law index \( \kappa \) is chosen based on magnetic field configuration (from toroidal to poloidal) in range \( 1 \leq \kappa \leq 2 \). Kardashev (1962) \[130\] in his classical paper used \( m = 2 \) assuming poloidal magnetic field while Tanaka and Takahara (2010) \[278\] in their modeling of pulsar wind nebulae used \( \kappa = 1.5 \) based on equipartition. There are also some other models expecting even steeper time dependence of synchrotron losses as well as adiabatic energy losses.

Coming back to the method of characteristics applied on equation (4.13) we can immediately notice that variables \( t \) and \( E \) are respectively analogous to the variables \( x \) and \( y \) used in section 4.3. This implies that our characteristic equation is given by energy losses of equation (4.13)

\[
\frac{dy}{dx} = \frac{b}{a} \implies \frac{dE}{dt} = -b(E, t) ,
\]

which imposes new variable \( \eta \), while the other new variable \( \xi \) is chosen to be \( \xi = t \). Consequently equation (4.13) changes into

\[
\frac{\partial N(\xi, \eta)}{\partial \xi} - c(\xi, \eta)N(\xi, \eta) = g(\xi, \eta) ,
\]

where functions \( c(\xi, \eta) \) and \( g(\xi, \eta) \) are coming from known functions of equation
With this in mind we can proceed with some known simple examples presented in the classical paper of Kardashev (1962) \[130\].

### 4.4.1 Adiabatic expansion without injection

By ignoring terms with particle injection and synchrotron losses kinetic equation takes a following form

\[
\frac{\partial N(E,t)}{\partial t} - \frac{E}{t} \frac{\partial N(E,t)}{\partial E} - \frac{1}{t} N(E,t) = 0, \tag{4.29}
\]

with characteristic equation becoming

\[
\frac{dE}{dt} = -\frac{E}{t}. \tag{4.30}
\]

Solution to this characteristic equation is obtained through separation of variables

\[
\frac{dE}{E} = -\frac{dt}{t} \implies Et = \text{const.}, \tag{4.31}
\]

which according to the method of characteristics gives us \(\eta = Et\) and \(\xi = t\) as new variables. Keeping \(\eta\) constant as prescribed by method we can rewrite equation (4.29) by transforming the derivatives

\[
\begin{align*}
\frac{\partial N}{\partial t} & = \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial N}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{\partial N}{\partial \xi} + \frac{\partial N}{\partial \eta} \frac{\partial \eta}{\partial \xi}, \\
\frac{\partial N}{\partial E} & = \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial E} + \frac{\partial N}{\partial \eta} \frac{\partial \eta}{\partial E} = \frac{\partial N}{\partial \eta} \frac{\partial \xi}{\partial \eta}, \tag{4.32a}
\end{align*}
\]

leading to the form

\[
\frac{\partial}{\partial \xi} N(\xi, \eta) - \frac{1}{\xi} N(\xi, \eta) = 0. \tag{4.33}
\]

Now we proceed solving equation (4.33) along the curve \(\eta = \text{const.}\) as ODE while treating \(\eta\) as parameter. This means that in process of solving ODE through separation of variables

\[
\frac{dN}{N} = \frac{d\xi}{\xi} \implies \ln N = \ln \xi + C, \tag{4.34}
\]
we need to treat constant \( C \) as a function of \( \eta \). This comes from the fact that solution each (4.34) for each constant \( \eta \) has the same form, but the constant \( C \) may have different values, therefore to be dependent on \( \eta \). For convenience following the form of solution (4.34) we will write \( C(\eta) = \ln(G(\eta)) \) changing our solution into following form

\[
N(\xi, \eta) = \xi G(\eta),
\]

with \( G(\eta) \) being any differentiable function of \( \eta \). Since \( \eta = Et \) we can come back to original variables of our interest \( E, t \)

\[
N(E, t) = t G(Et),
\]

with function \( G(Et) \) being determined from initial conditions. Let us impose a common astrophysical initial condition, namely a power law particle distribution at time \( t_0 \)

\[
N(E, t_0) = KE^{-p},
\]

where \( p \) is the power law index. Since equation (4.36) is the solution to our case of adiabatic cooling with no injection of particles or synchrotron losses present this means that

\[
KE^{-p} = t_0 G(Et_0) \implies G(Et_0) = \frac{K}{t_0} t_0^p (Et_0)^{-p},
\]

from which we can obtain \( G(\eta) = G(Et) \) as

\[
G(Et) = K t_0^{p-1} (Et)^{-p},
\]

leading to time dependent total particle number distribution given as

\[
N(E, t) = KE^{-p} \left( \frac{t_0}{t} \right)^{p-1},
\]

with particle density distribution being

\[
n(E, t) = \frac{K}{V_0} E^{-p} \left( \frac{t_0}{t} \right)^{p+2},
\]

since we are considering a constant expansion of emitter whose volume goes as \( V(t) = V_0 (t/t_0)^3 \) where \( V_0 \) is emitter’s volume at time \( t_0 \).
4.4.2 Adiabatic expansion with constant injection

Kinetic equation with both adiabatic losses and particle injection included is formed by adding the inhomogeneous injection term into equation (4.29)

\[
\frac{\partial N(E,t)}{\partial t} - \frac{E}{t} \frac{\partial N(E,t)}{\partial E} - \frac{1}{t} N(E,t) = Q(E,t),
\]

(4.42)

with constant injection term being given with \( Q(E,t) = Q_0 E^{-\lambda} \) where \( Q_0 \) is the normalization constant and \( \lambda = s_p - 2 > 0 \) is the injection spectra power law index motivated by the theory of particle acceleration presented before in this chapter. As in the former case initial conditions are assumed to be given in with the equation (4.37). The procedure of finding solution is the same as the one in section 4.4.1 with just taking into account inhomogeneous injection term, since both the characteristic equation and new variables \( \xi = t \) and \( \eta = Et \). Using the same new variables the inhomogeneous term \( Q(E,t) \) becomes

\[
Q(E,t) = Q_0 t^\lambda (Et)^{-\lambda} \implies Q(\xi, \eta) = Q_0 \xi^\lambda \eta^{-\lambda},
\]

(4.43)

changing equation (4.42) into following form

\[
\frac{\partial}{\partial \xi} N(\xi, \eta) - \frac{1}{\xi} N(\xi, \eta) = Q_0 \xi^\lambda \eta^{-\lambda}.
\]

(4.44)

We can now proceed with solving this inhomogeneous ODE by using the method of integrating factors starting with multiplication of both side of equation with an unknown function \( \mu(\xi) \)

\[
\mu(\xi) \frac{\partial}{\partial \xi} N(\xi, \eta) - \frac{\mu(\xi)}{\xi} N(\xi, \eta) = \mu(\xi) Q_0 \xi^\lambda \eta^{-\lambda}.
\]

(4.45)

This new unknown function \( \mu(\xi) \) can be found requiring that

\[
\frac{\partial \mu(\xi)}{\partial \xi} = - \frac{\mu(\xi)}{\xi} \implies \mu(\xi) = \frac{1}{\xi},
\]

(4.46)

leading to

\[
\frac{\partial}{\partial \xi} (\mu(\xi) N(\xi, \eta)) = \mu(\xi) Q_0 \xi^\lambda \eta^{-\lambda},
\]

(4.47)

from which we can obtain the form of solution \( N(\xi, \eta) \), i.e. \( N(E,t) \), given as

\[
N(\xi, \eta) = Q_0 \eta^{-\lambda} \xi \int \xi'^{\lambda-1} d\xi' + \xi G(\eta) = \frac{Q_0}{\lambda} \eta^{-\lambda} \xi^{\lambda+1} + \xi G(\eta),
\]

(4.48a)

\[
N(E,t) = \frac{Q_0}{\lambda} (Et)^{-\lambda} t^{\lambda+1} + t G(Et),
\]

(4.48b)
where $G(\eta)$, i.e. $G(Et)$, is the unknown function of $\eta = Et$ used here for the same reasoning as in the one in case the without injection before. Since our initial condition at $t = t_0$ is $N(E, t_0) = KE^{-p}$ we can calculate the function $G(\eta) = G(Et)$ through following procedure

$$Kt_0^\lambda(Et_0)^{-\lambda} = \frac{Q_0}{\lambda}(Et_0)^{-\lambda}t_0^{\lambda+1} + t_0G(Et_0),$$  

(4.49a)

$$G(Et_0) = Kt_0^{\lambda-1}(Et_0)^{-\lambda} - \frac{Q_0}{\lambda}(Et_0)^{-\lambda}t_0^{\lambda},$$  

(4.49b)

leading to the final form of the function

$$G(\eta) = Kt_0^{\lambda-1}\eta^{-\lambda} - \frac{Q_0}{\lambda}t_0^{\lambda}\eta^{-\lambda}.$$  

(4.50)

Incorporating the results given with equation (4.50) into equations (4.48) we find that the solution is given by

$$N(E, t) = \frac{Q_0}{\lambda}E^{-\lambda}t \left[ 1 - \left(\frac{t_0}{t}\right)^{\lambda} \right] + KE^{-p}\left(\frac{t_0}{t}\right)^{p-1},$$  

(4.51)

in which we immediately can notice that in case of $Q_0 = 0$ the result given with equation (4.40) is recreated.

### 4.4.3 Synchrotron losses within static emitter without injection

Let us now imagine an emitter of constant size with inner magnetic field $B$ with no injection of particle. In that case kinetic equation (4.13) becomes

$$\frac{\partial N(E, t)}{\partial t} - k_{\text{syn}}E^2 \frac{\partial N(E, t)}{\partial E} - 2k_{\text{syn}}EN(E, t) = 0,$$  

(4.52)

with $k_{\text{syn}}$ coming from synchrotron energy losses equation (2.10b)

$$k_{\text{syn}} = \frac{\sigma_T}{6\pi mc} B^2.$$  

(4.53)

The characteristic equation is therefore

$$\frac{dE}{dt} = -k_{\text{syn}}E^2,$$  

(4.54)

whose solution is given through separation of variables as

$$\frac{dE}{E^2} = k_{\text{syn}}dt \implies \frac{1}{E} = k_{\text{syn}}t + C \implies C = \frac{1}{E} - k_{\text{syn}}t = \text{const.}.$$  

(4.55)
Following the method of characteristics this constant is identified as new variable \( \eta \) giving
\[
\eta = \frac{1}{E} - k_{\text{syn}} t ,
\]
(4.56)
while we again choose \( \xi = t \) for convenience. Rewriting derivatives from (4.52) in new variables gives us the following
\[
\begin{align*}
\frac{\partial N}{\partial t} &= \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial N}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{\partial N}{\partial \xi} - \frac{\partial N}{\partial \eta} k_{\text{syn}} , \quad (4.57a) \\
\frac{\partial N}{\partial E} &= \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial E} + \frac{\partial N}{\partial \eta} \frac{\partial \eta}{\partial E} = -\frac{\partial N}{\partial \eta} (\eta + k_{\text{syn}} \xi)^2 , \quad (4.57b)
\end{align*}
\]
by which we can rewrite equation (4.52) in the form
\[
\frac{\partial N}{\partial \xi} - \frac{2k_{\text{syn}}}{\eta + k_{\text{syn}} \xi} N = 0 . \quad (4.58)
\]
Solving this ODE along the \( \eta = \text{const.} \) curves as already mentioned in previous cases leads to
\[
\frac{dN}{N} = \frac{2k_{\text{syn}}}{\eta + k_{\text{syn}} \xi} d\xi \implies \ln N = 2 \ln(k_{\text{syn}} \xi + \eta) + k . \quad (4.59)
\]
Following the same reasoning as in former two cases with adiabatic losses we let constant \( k \) be a function of \( \eta \) given as \( k(\eta) = \ln(G(\eta)) \) giving out the solution of equation (4.58)
\[
N(\xi, \eta) = (k_{\text{syn}} \xi + \eta)^2 G(\eta) , \quad (4.60)
\]
which rewritten back with variables \( E \) and \( t \) gives us the general solution of equation (4.52) in following form
\[
N(E, t) = E^{-2} G \left( \frac{1}{E} - k_{\text{syn}} t \right) . \quad (4.61)
\]
In this case the initial conditions assumed to determine \( G(\eta) \) are still in the form of a power law like in former cases, but limited by a minimum \( E_{\text{min}} \) and maximum \( E_{\text{max}} \) particle energy as shown here
\[
N(E, t_0) = K E^{-p} \begin{cases} 
0, & E < E_{\text{min}} \\
1, & E_{\text{min}} \leq E \leq E_{\text{max}} \\
0, & E > E_{\text{max}}
\end{cases} , \quad (4.62)
\]
where \( p \) is the power law index and \( K \) is the normalization constant like in cases considered before. The procedure of determining \( G(\eta) \) is equivalent to finding the constant of integration along each curve of constant \( \eta \) as noted in former cases.
Therefore from initial condition at time \( t_0 \) we can write

\[
G\left(\frac{1}{E} - k_{\text{syn}}t_0\right) = KE^{p-2} \begin{cases} 
0, & E < E_{\text{min}} \\
1, & E_{\text{min}} \leq E \leq E_{\text{max}} \\
0, & E > E_{\text{max}}
\end{cases}
\]  

(4.63)

For simplicity of further calculation let us introduce variable \( x \) giving out

\[
x = \frac{1}{E} - k_{\text{syn}}t_0 \implies E = \frac{1}{x + k_{\text{syn}}t_0};
\]  

(4.64a)

\[
E_{\text{min}} = \frac{1}{x_{\text{max}} + k_{\text{syn}}t_0}, \quad E_{\text{max}} = \frac{1}{x_{\text{min}} + k_{\text{syn}}t_0},
\]  

(4.64b)

with \( x_{\text{min}} \) and \( x_{\text{max}} \) are determined from minimum \( E_{\text{min}} \) and maximum \( E_{\text{max}} \) energy of the initial particle spectrum. Implementing the results of equations (4.64) in equation (4.63) we obtain

\[
G(x) = K(x + k_{\text{syn}}t_0)^{p-2} \begin{cases} 
0, & x < x_{\text{min}} \\
1, & x_{\text{min}} \leq x \leq x_{\text{max}} \\
0, & x > x_{\text{max}}
\end{cases}
\]  

(4.65)

Since variable \( x \) is essentially defined as \( x = \eta(E,t_0) \) we can now expand function \( G(x) \) to \( G(\eta) \) using equation (4.65) leading to following expression

\[
G(\eta) = K(\eta + k_{\text{syn}}t_0)^{p-2} \begin{cases} 
0, & \eta < \eta_{\text{min}} \\
1, & \eta_{\text{min}} \leq \eta \leq \eta_{\text{max}} \\
0, & \eta > \eta_{\text{max}}
\end{cases}
\]  

(4.66)

Coming back to equation (4.66) we again substitute \( \eta \) with \( E \) and \( t \) in function \( G(\eta) \) given with equation (4.66). With consequent redefining of limits in space of variable \( E \) as shown here

\[
x_{\text{min}} \leq \eta \leq x_{\text{max}} \implies \frac{1}{E_{\text{max}}} - k_{\text{syn}}t_0 \leq \frac{1}{E} - k_{\text{syn}}t \leq \frac{1}{E_{\text{min}}} - k_{\text{syn}}t_0,
\]  

(4.67a)

\[
\frac{1}{E_{\text{max}}} + k_{\text{syn}}(t - t_0) \leq \frac{1}{E} \leq \frac{1}{E_{\text{min}}} + k_{\text{syn}}(t - t_0),
\]  

(4.67b)

\[
\frac{E_{\text{min}}}{1 + k_{\text{syn}}(t - t_0)E_{\text{min}}} \leq E \leq \frac{E_{\text{max}}}{1 + k_{\text{syn}}(t - t_0)E_{\text{max}}},
\]  

(4.67c)
and by putting back $G(\eta)$ into equation (4.61) we finally obtain the solution in following form

$$N(E, t) = KE^{-\eta}(1 - k_{\text{syn}}(t - t_0)E)^{\eta - 2} \Pi(E, t),$$

(4.68)

with $\Pi(E, t)$ being the boxcar function of energy with time dependent interval defined as

$$\Pi(E, t) = \begin{cases} 
0, & E < E_{\text{min}} \\
1, & \frac{E_{\text{min}}}{1 + k_{\text{syn}}(t - t_0)E_{\text{min}}} \leq E \leq \frac{E_{\text{max}}}{1 + k_{\text{syn}}(t - t_0)E_{\text{max}}} \\
0, & E > E_{\text{max}} 
\end{cases}$$

(4.69)

### 4.4.4 Synchrotron losses within static emitter with constant injection

Following the homogeneous kinetic equation (4.52) solved in section 4.4.3 we can now focus on the case where there is also injection, often known as "the source", term $Q(E, t)$ present within as given here

$$\frac{\partial N(E, t)}{\partial t} - k_{\text{syn}}E^2 \frac{\partial N(E, t)}{\partial E} - 2k_{\text{syn}}EN(E, t) = Q(E, t),$$

(4.70)

turning it into a corresponding inhomogeneous kinetic equation. In our special case injection term is not time dependent and is given by

$$Q(E, t) = Q_0E^{-\lambda} \begin{cases} 
0, & E < E_1 \\
1, & E_1 \leq E \leq E_2 \\
0, & E > E_2 
\end{cases}$$

(4.71)

with $E_1$ and $E_2$ being respectively lower and upper energy limit of particle injection in difference to injection form used in section 4.4.2. Since we already solved the homogeneous equation in section 4.4.3 using method of characteristics, finding the new variables to be $\eta = 1/E - k_{\text{syn}}t$ and $\xi = t$, we can proceed with solving the inhomogeneous equation written using the new variables as

$$\frac{\partial N}{\partial \xi} - \frac{2k_{\text{syn}}}{\eta + k_{\text{syn}}\xi} = Q(\xi, \eta),$$

(4.72)
4.4 Simple cases of kinetic equation

Figure 4.2. Time evolution of particle spectral distribution $N(E,t)$ for the case with synchrotron losses without injection for initial distribution given by equation (4.62) for various ranges of power law index: a) $p < 2$, b) $p = 2$ and c) $p > 2$. Values of parameters required are given in arbitrary units as $E_{\text{min}} = 10$, $E_{\text{max}} = 50$, $k_{\text{syn}} = 0.1$ and $t_0 = 1$. Particle distribution is followed at times: $t_1 = 1$ (solid line), $t_2 = 2$ (dashed line) and $t_3 = 3$ (dotted line) given in arbitrary units.
Figure 4.3. Time evolution of particle spectral distribution $N(E,t)$ for the case with synchrotron losses with null initial distribution and particle injection given by equation (4.71) for different ranges of injection power law index: a) $\lambda < 2$, b) $\lambda = 2$ and c) $\lambda > 2$. Values of parameters required are given in arbitrary units as $E_1 = 10$, $E_2 = 50$, $k_{\text{syn}} = 0.1$ and $t_0 = 0$. Particle distribution is followed at times: $t_1 = 0.25$ (black solid line), $t_2 = 0.5$ (red solid line), $t_3 = 0.75$ (green solid line), $t_4 = 1$ (black dashed line), $t_5 = 2$ (red dashed line) and $t_6 = 3$ (green dashed line) given in arbitrary units.
where $Q(\xi, \eta)$ represents the inhomogeneous term $Q(E, t)$ transformed into new variables coordinate space

$$Q(\xi, \eta) = Q_0(\eta + k_{\text{syn}}\xi)^{\lambda} \begin{cases} 
0, & \eta + k_{\text{syn}}\xi < \frac{1}{E_2} \\
1, & \frac{1}{E_2} \leq \eta + k_{\text{syn}}\xi \leq \frac{1}{E_1} \\
0, & \eta + k_{\text{syn}}\xi > \frac{1}{E_1} .
\end{cases} \tag{4.73}$$

Equation (4.72) can be solved using the method of integrating factors. For that reason we multiply both sides of the equation with an unknown function $\mu(\xi)$ as shown here

$$\mu(\xi) \frac{\partial N}{\partial \xi} - \frac{2k_{\text{syn}}}{\eta + k_{\text{syn}}\xi} \mu(\xi) N = Q(\xi, \eta) \mu(\xi) , \tag{4.74}$$

which needs to fulfill the following condition

$$\frac{\partial \mu(\xi)}{\partial \xi} = -\frac{2k_{\text{syn}}}{\eta + k_{\text{syn}}\xi} \mu(\xi) \implies \mu(\xi) = \frac{1}{(\eta + k_{\text{syn}}\xi)^2} . \tag{4.75}$$

Following equations (4.74) and (4.75) we can write

$$\frac{\partial (\mu(\xi)N(\xi, \eta))}{\partial \xi} = Q(\xi, \eta) \mu(\xi) , \tag{4.76}$$

which as already considered in section 4.4.2 leads toward the general solution given by

$$\mu(\xi)N(\xi, \eta) = \int_{\xi_0}^{\xi} Q(\xi', \eta) \mu(\xi') \, d\xi' + G(\eta) , \tag{4.77a}$$

$$N(\xi, \eta) = \frac{1}{\mu(\xi)} \int_{\xi_0}^{\xi} Q(\xi', \eta) \mu(\xi') \, d\xi' + \frac{G(\eta)}{\mu(\xi)} , \tag{4.77b}$$

$$N(\xi, \eta) = (\eta + k_{\text{syn}}\xi)^2 \int_{\xi_0}^{\xi} \frac{Q(\xi', \eta)}{(\eta + k_{\text{syn}}\xi')^2} \, d\xi' + (\eta + k_{\text{syn}}\xi)^2 G(\eta) . \tag{4.77c}$$
Integral in equation (4.77c) can be calculated by employing substitution $x = \eta + k_{\text{syn}} \xi'$ and $dx = k_{\text{syn}} d\xi'$ leading to following result

$$I(\xi, \eta) = \int_{\xi_0}^{\xi} \frac{Q(\xi', \eta)}{(\eta + k_{\text{syn}} \xi')^2} d\xi'$$

$$= Q_0 \int_{\xi_0}^{\xi} (\eta + k_{\text{syn}} \xi')^{\lambda - 2} \begin{cases} 0, & \eta + k_{\text{syn}} \xi < \frac{1}{E_2} \\ 1, & \frac{1}{E_2} \leq \eta + k_{\text{syn}} \xi \leq \frac{1}{E_1} \\ 0, & \eta + k_{\text{syn}} \xi > \frac{1}{E_1} \end{cases} d\xi'$$

(4.78)

$$= \frac{Q_0}{k_{\text{syn}}} \int_{\eta + k_{\text{syn}} \xi_0}^{\eta + k_{\text{syn}} \xi} x^{\lambda - 2} \begin{cases} 0, & x < \frac{1}{E_2} \\ 1, & \frac{1}{E_2} \leq x \leq \frac{1}{E_1} \\ 0, & x > \frac{1}{E_1} \end{cases} dx,$$

where we can see that integral $I(\xi, \eta)$ can be easily transformed back to the $E, t$ coordinate space as $I(E, t)$. For clear presentation of the aforementioned result we first need to introduce energies of particles $E_1'$ and $E_2'$ at time $t$ which were injected at time $t_0$ with initial energies $E_1$ and $E_2$ respectively

$$E_1'(t) = \frac{E_1}{1 + k_{\text{syn}}(t - t_0)E_1}, \quad (4.79a)$$

$$E_2'(t) = \frac{E_2}{1 + k_{\text{syn}}(t - t_0)E_2}. \quad (4.79b)$$

Including them in equation (4.78) leads us to the following expression describing particles’ distribution with particles’ energy $E$ at time $t$

$$I(E, t) = \frac{Q_0}{k_{\text{syn}}} \begin{cases} 0, & E > E_2 \\ \int_{1/E_2}^{1/E} x^{\lambda - 2} dx, & E_1 < E < E_2 \cap E > E_2'(t) \\ \int_{1/E_1}^{1/E} x^{\lambda - 2} dx, & E_1 < E < E_2 \cap E < E_2'(t) \\ \int_{1/E_1}^{1/E} x^{\lambda - 2} dx, & E < E_1 \cap E > E_2'(t) \\ 0, & E < E_1(t) \end{cases}, \quad (4.80)$$

which can be further written as

$$I(E, t) = \frac{Q_0}{k_{\text{syn}}(\lambda - 1)} W(E, t), \quad (4.81)$$
where $\mathcal{W}(E, t)$ is the piecewise function given as

$$\mathcal{W}(E, t) = \begin{cases} 
0, & E > E_2 \\
\left(\frac{1}{E}\right)^{\lambda-1} - \left(\frac{1}{E_2}\right)^{\lambda-1}, & E_1 < E < E_2 \cap E > E'_2(t) \\
\left(\frac{1}{E}\right)^{\lambda-1} - \left(\frac{1}{E_2} - k_{\text{syn}}(t - t_0)\right)^{\lambda-1}, & E_1 < E < E_2 \cap E < E'_2(t) \\
\left(\frac{1}{E_1}\right)^{\lambda-1} - \left(\frac{1}{E_2} - k_{\text{syn}}(t - t_0)\right)^{\lambda-1}, & E < E_1 \cap E > E'_1(t) \\
\left(\frac{1}{E_1}\right)^{\lambda-1} - \left(\frac{1}{E} - k_{\text{syn}}(t - t_0)\right)^{\lambda-1}, & E < E_1 \cap E'_1(t) < E < E'_2(t) \\
0, & E < E'_1(t) 
\end{cases}$$

(4.82)

Six cases defining piecewise function $\mathcal{W}(E, t)$ originate from different relative positions of integration limits employed and the limits imposed within particle injection $E_1$ and $E_2$. Clearly null results in case $E > E_2$ comes from the fact no particle is accelerated with energy above $E_2$ while the null result of case $E < E'_1(t)$ tells us that populating lower energies with particles through cooling takes specific time. Full solution to equation (4.70) is therefore given as

$$N(E, t) = \frac{1}{E^2} \left[I(E, t) + G(E, t)\right],$$

(4.83)

with function $I(E, t)$ already given by equation (4.82) and $G(E, t)$ given by equation (4.66) from homogeneous case. In special cases when there is no initial electron distribution, i.e. $N(E, t_0) = 0$, function $G(E, t)$ vanishes leading to the solutions coming only from particle injection given by equation (4.71) as shown in figure 4.3.

### 4.4.5 Synchrotron losses and adiabatic expansion

In case when the emitter is expanding, beside adiabatic energy losses, synchrotron losses term also becomes time dependent as already presented with equation (4.24). Following that and for the more simplified presentation of results in this analysis we will use description of energy losses in this form

$$\frac{dE}{dt} = - \left[k_{\text{syn},0} \left(\frac{t_0}{t}\right)^4 E^2 + t^{-1} E\right],$$

(4.84)

where we employ time power law behaviour with $\kappa = 2$ as presented in [130] for simplicity, although as noted there are many other scenarios, and where

$$k_{\text{syn},0} = \frac{\sigma_{\text{Tr}} B_0^2}{6\pi mc^2}.$$  

(4.85)
In this case the kinetic equation is given by
\[
\frac{\partial N(E,t)}{\partial t} - \left[ k_{\text{syn},0} \left( \frac{t_0}{t} \right)^4 E^2 + \frac{E}{t} \right] \frac{\partial N(E,t)}{\partial E} - \left[ 2k_{\text{syn},0} \left( \frac{t_0}{t} \right)^4 E + \frac{1}{t} \right] N(E,t) = Q(E,t),
\]
with injection term \( Q(E,t) \) and initial particle distribution \( N(E,t_0) \) prescribed as in former cases
\[
Q(E,t) = Q_0 E^{-\lambda} \begin{cases} 
0, & E < E_1 \\
1, & E_1 \leq E \leq E_2 \\
0, & E > E_2 \end{cases}, \quad \text{(4.87a)}
\]
\[
N(E,t_0) = K E^{-p} \begin{cases} 
0, & E < E_{\text{min}} \\
1, & E_{\text{min}} \leq E \leq E_{\text{max}} \\
0, & E > E_{\text{max}} \end{cases}. \quad \text{(4.87b)}
\]
We will continue with solving both cases, first the homogeneous case without injection with \( Q(E,t) = 0 \) and second case with injection constant with time \( Q(E,t) = Q(E) \).

As in all previously considered cases we start with solving the characteristic equation (4.84), which in difference to previous characteristic equations is nonlinear (i.e. containing \( E^2 \)) and is not separable due mixing of adiabatic and synchrotron energy losses. Equation (4.84) is a Bernoulli type differential equation which can be easily linearized implementing substitution \( z = 1/E \) leading to
\[
\frac{dz}{dt} = -\frac{1}{E^2} \frac{dE}{dt} \implies \frac{dz}{dt} - \frac{1}{t} z = k_{\text{syn},0} \left( \frac{t_0}{t} \right)^4. \quad \text{(4.88)}
\]
This ODE can be solved using method of integrating factors where we multiply both sides of equation (4.88) with function \( \zeta(t) \) requiring
\[
\frac{d\zeta(t)}{dt} = -\frac{\zeta(t)}{t} \implies \frac{d\zeta}{\zeta} = -\frac{dt}{t} \implies \zeta(t) = \frac{1}{t}, \quad \text{(4.89)}
\]
and enabling us to find the characteristic curves
\[
\frac{d}{dt} \left[ \zeta(t) z(t) \right] = k_{\text{syn},0} \left( \frac{t_0}{t} \right)^4 \zeta(t) \implies \frac{dz(t)}{dt} = k_{\text{syn},0} \frac{t_0^4}{t^5}, \quad \text{(4.90a)}
\]
\[
\frac{z(t)}{t} = -\frac{1}{4} k_{\text{syn},0} \left( \frac{t_0}{t} \right)^4 + \text{const.} \implies \frac{1}{Et} + \frac{1}{4} k_{\text{syn},0} \left( \frac{t_0}{t} \right)^4 = \text{const.}. \quad \text{(4.90b)}
\]
Based on this it is possible to continue solving the kinetic equation (4.86) through method of characteristics using the new variables

\[ \eta = \frac{1}{Et} + k_{\text{syn},0} \left( \frac{t_0}{t} \right)^4 \text{ and } \xi = t. \]  

With introduction of new variables given by (4.91) in equation (4.86) we get

\[ 2\kappa_0 t^{-4} E + t^{-1} = \frac{8\kappa_0}{\xi(4\eta^4 - \kappa_0)}, \]  

where \( \kappa_0 = k_{\text{syn},0} t_0^4 \) is introduced for simplicity of notation. The kinetic equation with this changes gains the following form

\[ \frac{\partial}{\partial \xi} N(\xi, \eta) - \left( \frac{8\kappa_0}{\xi(4\eta^4 - \kappa_0)} + \frac{1}{\xi} \right) N(\xi, \eta) = Q(\xi, \eta). \]  

We can now proceed with solving equation (4.93) through method of integrating factors multiplying both side of equations with unknown function \( \mu(\xi) \) requiring that

\[ \frac{d\mu(\xi)}{d\xi} = - \left( \frac{8\kappa_0}{\xi(4\eta^4 - \kappa_0)} + \frac{1}{\xi} \right) \mu(\xi), \]  

enabling us to write equation (4.93) as ordinary differential equation (with \( \eta \) kept as constant)

\[ \frac{d}{d\xi} [\mu(\xi) N(\xi, \eta)] = \mu(\xi) Q(\xi, \eta), \]  

with general solution given by

\[ N(\xi, \eta) = \frac{1}{\mu(\xi)} \int_{\xi_0}^{\xi} \mu(\xi') Q(\xi', \eta) d\xi' + \frac{G(\eta)}{\mu(\xi)}, \]  

where \( G(\eta) \) is any function of \( \eta \) as noted in former cases and \( \mu(\xi) \) is the solution of equation (4.94) gained through separation of variables

\[ \frac{d\mu}{\mu} = - \left( \frac{8\kappa_0}{\xi(4\eta^4 - \kappa_0)} + \frac{1}{\xi} \right) d\xi \implies \ln \mu = - \int \left( \frac{8\kappa_0}{\xi(4\eta^4 - \kappa_0)} + \frac{1}{\xi} \right) d\xi. \]
Integration of equation (4.97) is elementary and it becomes very clear when we introduce a substitution \( \alpha^4 = \kappa_0 / 4\eta \) (since both \( \kappa_0 \geq 0 \) and \( \eta \geq 0 \)) leading to

\[
\frac{8\alpha^2}{\xi(\xi^4 - \alpha^4)} = \frac{8\alpha^4}{\xi(\xi - \alpha)(\xi + \alpha)(\xi^2 + \alpha^2)} = \frac{-8}{\xi} + \frac{2}{\xi - \alpha} + \frac{2}{\xi + \alpha} + \frac{4}{\xi^2 + \alpha^2},
\]

significantly simplifying integration as follows

\[
\ln \mu = \int \left( \frac{7}{\xi} + \frac{2}{\xi - \alpha} + \frac{2}{\xi + \alpha} + \frac{4}{\xi^2 + \alpha^2} \right) d\xi
= \ln(\xi^7) - \ln[(\xi - \alpha)^2] - \ln[(\xi + \alpha)^2] - \ln[(\xi^2 + \alpha^2)^2]
= \ln \left( \frac{\xi^7}{(\xi - \alpha)^2(\xi + \alpha)^2(\xi^2 + \alpha^2)^2} \right) = \ln \left( \frac{\xi^7}{(\xi^4 - \alpha^4)^2} \right).
\]

By reversing the substitution \( \alpha^4 = \kappa_0 / 4\eta \) in equation (4.99) we get

\[
\mu(\xi) = \frac{16\eta^2 \xi^7}{(4\eta \xi^4 - \kappa_0)^2}.
\]

Knowing \( \mu(\xi) \) we continue with investigation of homogeneous kinetic equation, i.e. without injection, necessary to determine function \( G(\eta) \). In that case right hand side of equation (4.86) is equal to zero and its solution is given by

\[
N_h(\xi, \eta) = \frac{(4\eta \xi^4 - \kappa_0)^2}{16\eta^2 \xi^7} G(\eta),
\]

where \( N_h(\xi, \eta) \) denotes homogeneous part of solution and \( G(\eta) \) is an arbitrary function which can be determined from initial conditions, i.e. value of distribution \( N(E, t_0) \) at time \( t = t_0 \). If \( N(E, t_0) = 0 \) then also \( N(E, t) \) will remain zero and consequently \( G(\eta) = 0 \), but if our initial particle distribution is given by equation (4.87b) at \( t = t_0 \) we can write

\[
G(\eta_0) = \frac{16\eta_0^2 \xi_0^7}{(4\eta_0 \xi_0^4 - \kappa_0)^2} N(\xi_0, \eta_0)
= \frac{16K_0 \eta_0^2 \xi_0^7}{(4\eta_0 \xi_0^4 - \kappa_0)^2} \left( \frac{4\xi_0^3}{4\eta_0 \xi_0^4 - \kappa_0} \right)^{-p} \begin{cases} 
0, & \frac{4\xi_0^3}{4\eta_0 \xi_0^4 - \kappa_0} < E_{\text{min}} \\
1, & E_{\text{min}} \leq \frac{4\xi_0^3}{4\eta_0 \xi_0^4 - \kappa_0} \leq E_{\text{max}} \\
0, & \frac{4\xi_0^3}{4\eta_0 \xi_0^4 - \kappa_0} > E_{\text{max}}
\end{cases},
\]

(4.102)
with \( \xi_0 \) and \( \eta_0 \) being new variables corresponding to time \( t = t_0 \) and with condition \( E_{\text{min}} \leq E \leq E_{\text{max}} \) written in terms of \( \xi_0 \) and \( \eta_0 \). To obtain function \( G(\eta) \) it is necessary to replace \( \eta_0 \) in function \( G(\eta_0) \) with \( \eta \) starting from boxcar function conditions from equation (4.102)

\[
E_{\text{min}} \leq \frac{4\xi_0^3}{4\eta_0^4 - \kappa_0} \leq E_{\text{max}} \implies \frac{1}{E_{\text{max}}\xi_0} + \frac{\kappa_0}{4\xi_0^4} \leq \eta \leq \frac{1}{E_{\text{min}}\xi_0} + \frac{\kappa_0}{4\xi_0^4},
\]

leading to the final form of \( G(\eta) \) given by

\[
G(\eta) = \frac{16K\eta^2\xi_0^7}{(4\eta_0^4 - \kappa_0)^2} \left( \frac{4\xi_0^3}{4\eta_0^4 - \kappa_0} \right)^{-\nu} \begin{cases} 0, & \eta < \frac{1}{E_{\text{max}}\xi_0} + \frac{\kappa_0}{4\xi_0^4} \\ \frac{1}{E_{\text{max}}\xi_0} + \frac{\kappa_0}{4\xi_0^4} \leq \eta \leq \frac{1}{E_{\text{min}}\xi_0} + \frac{\kappa_0}{4\xi_0^4}, & 1, \\ \eta > \frac{1}{E_{\text{min}}\xi_0} + \frac{\kappa_0}{4\xi_0^4} \end{cases}
\]

Putting equation (4.104) back into equation (4.101) we get

\[
N_h(\xi, \eta) = \kappa \frac{\xi_0}{\xi} \left( \eta_0\xi - \frac{\kappa_0}{4\xi_0} \right)^2 \left( \eta_0\xi - \frac{\kappa_0}{4\xi_0} \right)^{p-2} \begin{cases} 0, & \eta < \frac{1}{E_{\text{max}}\xi_0} + \frac{\kappa_0}{4\xi_0^4} \\ \frac{1}{E_{\text{max}}\xi_0} + \frac{\kappa_0}{4\xi_0^4} \leq \eta \leq \frac{1}{E_{\text{min}}\xi_0} + \frac{\kappa_0}{4\xi_0^4}, & 1, \\ \eta > \frac{1}{E_{\text{min}}\xi_0} + \frac{\kappa_0}{4\xi_0^4} \end{cases}
\]

which when transitioned back into old variables \( E \) and \( t \) becomes

\[
N_h(E,t) = \kappa \frac{t_0}{t} E^{-p} \left[ \frac{t_0}{t} - \frac{k_{\text{syn},0}t_0}{4}E \left( 1 - \left( \frac{t_0}{t} \right)^4 \right) \right]^{p-2} \begin{cases} 0, & \frac{E_{\text{min}} (t_0)}{t} \\ \frac{E_{\text{min}} (t_0)}{t} \leq E \leq \frac{E_{\text{max}} (t_0)}{t} \end{cases} \begin{cases} 0, & \frac{E_{\text{min}} (t_0)}{t} \\ \frac{E_{\text{min}} (t_0)}{t} \leq E \leq \frac{E_{\text{max}} (t_0)}{t} \end{cases} \begin{cases} 0, & \frac{E_{\text{min}} (t_0)}{t} \\ \frac{E_{\text{min}} (t_0)}{t} \leq E \leq \frac{E_{\text{max}} (t_0)}{t} \end{cases}
\]

(4.106)
Now we turn our attention to the solution of the inhomogeneous equation while neglecting the homogeneous part

\[
N_{inh}(\xi, \eta) = \frac{1}{\mu(\xi)} \int_{\xi_0}^{\xi} \mu(\xi') Q(\xi', \eta) d\xi'
\]

(4.107)

with \(\mu(\xi)\) already being given by equation (4.100) and function \(Q(\xi, \eta)\) being given by equation (4.92b). Integral from equation (4.107) can therefore be written in following form

\[
I(\xi, \eta) = Q_0 \eta^2 \int_{\xi_0}^{\xi} \left( \eta \xi' - \frac{\kappa_0}{4\xi^3} \right)^{\lambda-2} \begin{cases} 
0, & \eta \xi' - \frac{\kappa_0}{4\xi^3} < \frac{1}{E_1} \\
0, & \frac{1}{E_2} \leq \eta \xi' - \frac{\kappa_0}{4\xi^3} \leq \frac{1}{E_2} \\
0, & \eta \xi' - \frac{\kappa_0}{4\xi^3} > \frac{1}{E_2} 
\end{cases} \lambda \left[ 1 + \frac{E \kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) \right] \lambda^{-1} \left( \frac{\xi'}{t} \right)^{\lambda-1} d\xi'.
\]

(4.108)

Integral itself can be expressed in terms of elementary functions for any value of \(\lambda\) although the boundaries of integration, determined by condition \(1/E_2 \leq \eta \xi' - \frac{\kappa_0}{4\xi^3} \leq 1/E_1\), can not generally be written in analytic form. Since they are determined through the polynomials of fourth order, they require numerical and generally rather complex procedure. An exception which we will discuss here is the case when \(E_1 \to 0\) and \(E_2 \to \infty\), considered in Kardashev (1962) [130], leading to a much simpler condition \(\xi' \geq \kappa_0/4\eta\). Following equation (4.108) and coming back to the old variables \(E\) and \(t\) inhomogeneous part of solution is given by

\[
N_{inh}(E, t) = Q_0 E^{-\lambda} \int_{t_0}^{t} \left( \frac{\xi'}{t} \right)^{\lambda-1} \begin{cases} 
0, & \frac{\xi'}{E t} \left[ 1 + \frac{E \kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) \right] \leq \frac{1}{E_1} \\
0, & \frac{1}{E_2} \leq \frac{\xi'}{E t} \left[ 1 + \frac{E \kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) \right] \leq \frac{1}{E_2} \\
0, & \frac{\xi'}{E t} \left[ 1 + \frac{E \kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) \right] > \frac{1}{E_2} 
\end{cases} \lambda \left[ 1 + \frac{E \kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) \right] \lambda^{-1} \left( \frac{\xi'}{t} \right)^{\lambda-1} d\xi'.
\]

(4.109)

which in case of \(E_1 \to 0\) and \(E_2 \to \infty\) leads to

\[
N_{inh}(E, t) = Q_0 E^{-\lambda} \int_{t_0}^{t} \left( \frac{\xi'}{t} \right)^{\lambda-1} \begin{cases} 
1, & \frac{\kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) \leq \frac{1}{E} \\
0, & \frac{1}{E} \leq \frac{\kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) > \frac{1}{E} 
\end{cases} \lambda \left[ 1 + \frac{E \kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi^4} \right) \right] \lambda^{-1} \left( \frac{\xi'}{t} \right)^{\lambda-1} d\xi'.
\]

(4.110)

becomes very close to the result obtained by Kardashev (1962) [130] but not exactly the same. Conditions in equation (4.110) needs to be checked for all pairs of energy
4.5 Method using Green function

$E$ and time $t$. If it is satisfied for all $\xi'$ in $t_0 < \xi' < t$ then a full integration going from $t_0$ till $t$ is employed, otherwise there exists $\xi_{\text{min}}$ in range $t_0 < \xi_{\text{min}} < t$ for which

$$\frac{\kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi_{\text{min}}^4} \right) = \frac{1}{E}, \quad (4.111)$$

meaning that for $\xi_{\text{min}}$ condition $t_0 < \xi' < \xi_{\text{min}}$ is not satisfied. In that case integration goes rather as follows

$$N_{\text{inh}}(E,t) = Q_0 E^{-\lambda} \int_{\max[t_0,\xi_{\text{min}}]}^{t} \left( \frac{\xi'}{t} \right)^{\lambda-1} \left[ 1 + \frac{E \kappa_0 t}{4} \left( \frac{1}{t^4} - \frac{1}{\xi'^4} \right) \right]^{\lambda-1}, \quad (4.112)$$

where $\max[t_0,\xi_{\text{min}}]$ is the bigger of two timescales, leading to the final result also obtained by Kardashev (1962) [130].

4.5 Method using Green function

Alongside already described method of characteristics in special cases of time independent energy losses $b = b(\gamma)$ in equation (4.9) we can also use a method involving Green’s function [18]. Green’s function $G(\gamma,t,t_0)$ is then a solution to the equation (4.9) for an arbitrary injection spectra given as

$$Q(\gamma,t) = N_0(\gamma) \delta(t - t_0), \quad (4.113)$$

where for practicality we use particle’s Lorentz factor $\gamma = E/mc^2$ instead of energy $E$. Within this method both before and after time $t_0$ it is necessary to solve only homogeneous part of kinetic equation (4.9) with major difference being that at times before $t < t_0$ Green’s function gives $G(\gamma,t,t_0) = 0$ while after at times $t \geq t_0$ homogeneous part of equation (4.9) has initial conditions given by $G(\gamma,t_0 + 0,t_0) = N_0(\gamma)$. Consequently using function $F = bG$ we can rewrite equation (4.9) in following form

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \zeta} - \frac{F}{\tau_1(\zeta,t)}, \quad (4.114)$$

where instead of particle’s energy (i.e. Lorentz factor) $\gamma$ we use new variable $\zeta$ defined as

$$\zeta = g(\gamma) = \int_{\gamma_s}^{\gamma} \frac{d\gamma'}{b(\gamma')}, \quad (4.115)$$

It is clear from definition (4.115) that variable $\zeta$ essentially indicates time necessary for a particle to cool down from energy $\gamma$ to energy $\gamma_s$ which for convenience may be chosen to be $\gamma_s = 1$. On the other hand $\tau_1$ is escape time expressed through use of new variables $(\zeta,t)$ as $\tau_1(\zeta,t) = \tau_{\text{esc}}[\epsilon(\zeta),t]$ with $\epsilon(\zeta)$ being the inverse function
of }g(\gamma)\text{ returning the value of energy }\gamma = \epsilon(\zeta).\text{ From initial conditions given by equation (4.113) we read that}

\[
F(\zeta, t_0) = b[\epsilon(\zeta)]N_0[\epsilon(\zeta)] = U(\zeta) .
\] (4.116)

Furthermore by transforming the equation (4.114) from variables \((\zeta, t)\) into new variables \((s = \zeta + t, u = t)\) we are presented with a partial differential equation for a function \(F_1(s, u) = F(\zeta, t)\) given by

\[
\frac{\partial F_1}{\partial u} = -\frac{F_1}{\tau_1(s - u, u)} ,
\] (4.117)

having initial condition \(F_1(s, u_0) = U(s - u_0)\). Solving equation (4.117) is quite straightforward from here through integration

\[
F_1(s, u) = U(s - s_0) \exp \left[ - \int_{t_0}^{t} \frac{du_1}{\tau_1(s - u_1, u_1)} \right] .
\] (4.118)

Having this it is now necessary to return back from \((s, u)\) to original variables of interest \((\gamma, t)\). For that reason it is worth noticing that function \(\epsilon(s - x)\) entering into equation (4.118) as part of function \(U\) is inverse function of \(g\) which for any value of \(z\) gives \(z = g[\epsilon(z)]\). Consequently, since we know that \(s = \zeta + t\) and \(\zeta = g(\gamma)\), for variable \(z = s - x\) we get \(t - x = g[\epsilon(s - x)] - g(\gamma)\). Remembering the definition of function \(g\) from equation (4.115) we arrive at following result

\[
t - x = \int_{\gamma}^{\gamma_x} \frac{d\gamma'}{b(\gamma')} ,
\] (4.119)

where \(\gamma_x = \gamma_x(\gamma, t)\) is a function giving back the energy of particle at time \(x\) and with that describing the trajectory of an particle in energy space leading to particle energy \(\gamma\) at time \(t\). Having this we obtain the Green’s function \(G = F/b\) as the solution of equation (4.9) with initial conditions given by (4.113)

\[
G(\gamma, t, t_0) = \frac{b(\gamma_{t_0})N_0(\gamma_{t_0})}{b(\gamma)} \exp \left[ - \int_{t_0}^{t} \frac{dx}{\tau(\gamma_x, x)} \right] ,
\] (4.120)

where \(\gamma_{t_0} = \gamma_x(\gamma, t_0)\). It is important to notice that \(G(\gamma, t, t_0)\) is not a standard Green’s function considering particle injection spectra \(Q(\gamma, t)\) is given through an arbitrary function of energy \(N_0(\gamma)\), not by a \(\delta\)-function in energy space. We see equation (4.120) describes evolution of relativistic particles with initial distribution \(N_0(\gamma)\) at time \(t_0\). This can be extended for an arbitrary continuous injection spectra
in time by inserting a substitution $N_0(\gamma) \rightarrow Q(\gamma, t_0)dt_0$ and integrating it over all injection times $t_0$ leading to final expression for particle distribution

$$N(\gamma, t) = \frac{1}{b(\gamma)} \int_{-\infty}^{t} b(\gamma_0)Q(\gamma_0, t_0) \exp \left[-\int_{t_0}^{t} \frac{dx}{\tau(\gamma_x, x)}\right] dt_0.$$ (4.121)

Having this we can now tackle some specific cases worth of brief discussion. First one is the case when escape time $\tau$ is only energy dependent but stationary, i.e. $\tau(\gamma, t) \rightarrow \tau(\gamma)$, while energy losses are negligible, i.e. $b(\gamma) \rightarrow 0$. Assuming particle injection’s form in energy space is not time dependent, making it a separable function $Q(\gamma, t) = Q_0(\gamma)q(t)$ with $q(t < 0) = 0$, equation (4.121) can be reduced to following simple expression

$$N(\gamma, t) = Q_0(\gamma)\tau(\gamma) \int_{0}^{t/\tau(\gamma)} q[t - \tau(\gamma)z]e^{-t}dz.$$ (4.122)

Hence in the case of stationary injection $q(t \geq 1) = 1$ integral in equation (4.122) becomes $(1 - e^{-t/\tau})$ leading to particle accumulation $N(\gamma, t) \approx Q_0(\gamma)t$ for short timescales $t < \tau(\gamma)$ while at longer timescales when $t \geq \tau(\gamma)$ particle escape becomes much more important modifying the particle distribution to $N(\gamma, t) \approx Q_0(\gamma)\tau(\gamma)$.

For the special situation where escape time can be described as a power law function of particle’s energy $\tau(\gamma) \propto \gamma^{-\Delta}$ it leads to the power law steepening of injection spectra by a factor $\Delta$. Modification of from injection spectra $Q(\gamma)$ to particle distribution $N(\gamma, t)$ changes if injection is not stationary. For example in case of impulsive injection, where $q(t) = \delta(t - t_0)$, particle distribution is reduced to an exponential cut-off above energy $\gamma_t$ found from requirement $\tau(\gamma_t) = t - t_0$.

Similarly to "radiative loss breaks" discussed in previous sections tackling method of characteristics when cooling time becomes equal to dynamical timescale $t_{cool}(\gamma) = t$ within emitters with continuous (not necessary stationary) injection, "escape loss break" occurs when escape time becomes equal to dynamical timescale $\tau(\gamma) = t$.

Considering stationary injection significantly simplifies equation (4.121) by transforming the integration from time $t$ to energy $\gamma$

$$N(\gamma, t) = \frac{1}{b(\gamma)} \int_{\gamma_0}^{\gamma} Q_0(\gamma') \exp \left[-\int_{\gamma}^{\gamma'} \frac{dz}{b(z)\tau(z)}\right] d\gamma',$$ (4.123)

using a general relation about energy losses

$$\frac{\partial\gamma_x}{\partial t} = \frac{\partial\gamma_x}{\partial x} = b(\gamma_x),$$ (4.124)

and implementing it in equation (4.121). Integration limit $\gamma_0(\gamma, t)$ denotes maximum
energy of injected particles removed through escape of cooled down through radiative cooling to energy $\gamma$ in time $t$. Approximation of an infinite medium with $\tau \to \infty$ at large $t$ leads to particle distributions in steady state given by

$$N(\gamma, t) = \frac{1}{b(\gamma)} \int_{\gamma}^{\infty} Q(\gamma') d\gamma', \quad (4.125)$$

cause $\gamma_0(\gamma, t) \to \infty$. It is useful to mention that in case of dominant synchrotron and/or inverse Compton losses energy losses where $b(\gamma) = k\gamma^2$ we have $\gamma_0(\gamma, t) = \gamma/(1 - bt\gamma)$ and radiative cooling break energy $\gamma_b(t) = 1/kt$. In that case particle distribution becomes $N(\gamma, t) \sim tQ_0(\gamma)$ for energies below break energy $\gamma \leq \gamma_b(t)$ while on higher energies it is steepening in relation to particle injection spectra by a power law factor of 1.

### 4.6 Numerical approach to solving kinetic equation

More often than not, while modelling evolution of nonthermal emitters, we encounter kinetic equations in forms which can not be solved analytically using aforementioned methods of characteristics and/or Green’s function. On this account we employ a fully implicit numerical scheme proposed by Chang and Cooper (1970) [60] and implemented by Chiaberge and Ghisellini (1999) [64]. For the sake of clear derivation let us again rewrite the kinetic equation for a homogeneous source using particle’s Lorentz factor $\gamma$ as indicator of energy:

$$\frac{\partial}{\partial t} N(\gamma, t) = - \frac{\partial}{\partial \gamma} (b(\gamma, t) N(\gamma, t)) - \frac{1}{\tau_{\text{esc}}} N(\gamma, t) + Q(\gamma, t) \quad (4.126)$$

with $N(\gamma, t)$ as particle distribution, $Q(\gamma, t)$ as particle injection rate, $b(\gamma, t)$ as particle energy losses and $\tau_{\text{esc}}$ as particle escape time. In that case injection power $P_{\text{inj}}(t)$, i.e. work done in particle acceleration region per unit time, is given by a following integral:

$$P_{\text{inj}}(t) = m_e c^2 \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} Q(\gamma, t) \gamma d\gamma, \quad (4.127)$$

where $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$ are respectively minimum and maximum energy of injected particles intricately connected with the type and regime of particle acceleration at work. Alongside it particle energy losses can be of different nature as already mentioned in chapter 2 but in our cases we investigated three dominant channels for relativistic electrons, namely

- adiabatic energy losses

$$b_{\text{ad}}(\gamma, t) = \frac{\dot{R}(t)}{R(t)} \gamma, \quad (4.128)$$
4.6 Numerical approach to solving kinetic equation

- synchrotron energy losses

\[ b_{\text{syn}}(\gamma, t) = \frac{4}{3} \frac{\sigma_T c}{m_e c^2} B^2(t) \frac{\gamma^2}{8\pi}, \quad (4.129) \]

- inverse Compton energy losses

\[ b_{\text{IC}}(\gamma, t) = \frac{4}{3} \frac{\sigma_T c}{m_e c^2} u_0 F_{KN} \gamma^2, \quad (4.130) \]

where \( \sigma_T \) is the Thomson cross section of an electron, \( B(t) \) is magnetic field, \( u_0 \) is background radiation energy density, \( F_{KN} \) is the function connecting Thomson and Klein-Nishima regimes of inverse Compton scattering given by equation (2.111), while \( R(t) \) and \( \dot{R}(t) \) are respectively emitter’s size and expansion velocity. Energy losses can be recalculated for any other charged elementary particle by applying appropriate particle mass and Thomson cross section in equations (4.129) and (4.130).

Now let us define energy and time mesh grid for our numerical scheme. We construct a logarithmic energy mesh grid with \( n_{\text{max}} \) points between minimum \( \gamma_{m,\text{min}} \) and maximum \( \gamma_{m,\text{max}} \) particle energy considered

\[ \gamma_j = \gamma_{m,\text{min}} \left( \frac{\gamma_{m,\text{max}}}{\gamma_{m,\text{min}}} \right)^{\frac{j-1}{n_{\text{max}}-1}}, \quad (4.131) \]

with energy difference \( \Delta \gamma_j \) being given as

\[ \Delta \gamma_j = \gamma_{j+1/2} - \gamma_{j-1/2}. \quad (4.132) \]

However time increments along which we follow the evolution of particle are kept constant with value \( \Delta t \) leading to time mesh points given as \( t_i = i\Delta t \). The mesh grid and difference scheme can be defined differently but this construction with care also gives good results. Corresponding values of particle distribution, energy losses, injection rate and energy flux on numerical grid are given as follows

\[ N^i_j = N(\gamma_j, i\Delta t), \quad (4.133a) \]

\[ b_{j+1/2} = b(\gamma_{j+1/2}, i\Delta t), \quad (4.133b) \]

\[ Q^i_j = Q(\gamma_j, i\Delta t), \quad (4.133c) \]

\[ F_{j+1/2}^{i+1} = b_{j+1/2} N^i_{j+1/2}, \quad (4.133d) \]

where in our special approach prescribed by Chang and Cooper (1970) [60] we use following equivalences \( N_{j+1/2} \equiv N_{j+1} \) and \( N_{j-1/2} \equiv N_j \).

By discretizing equation (4.126) using equations (4.131), (4.132) and (4.133) we
get
\[
\frac{N_j^{i+1} - N_j^i}{\Delta t} = \frac{F_{j+1/2}^{i+1} - F_{j-1/2}^i}{\Delta \gamma_j} + Q_j^i - \frac{N_j^{i+1}}{\tau_{esc}},
\] (4.134)

The numerical scheme becomes much more clear by defining the \( V \) coefficients as
\[
V_1^j = 0, \quad (4.135a) \\
V_2^j = 1 + \frac{\Delta t}{\tau_{esc}} + \frac{b_{j-1/2}\Delta t}{\Delta \gamma_j}, \quad (4.135b) \\
V_3^j = -\frac{b_{j+1/2}\Delta t}{\Delta \gamma_j}, \quad (4.135c)
\]
by which equation (4.134) can be written in following form
\[
V_3^j N_j^{i+1} + V_2^j N_j^{i+1} + V_1^j N_{j-1}^{i+1} = N_j^i + Q_j^i \Delta t. \tag{4.136}
\]

Retrieved system of equations (4.136) forms a tridiagonal matrix
\[
\begin{bmatrix}
V_2^1 & V_3^1 & \cdots & 0 \\
V_1^2 & V_2^2 & V_3^2 & \cdots \\
\ddots & \ddots & \ddots & \ddots \\
0 & \cdots & V_1^{n-1} & V_2^{n-1} & V_3^{n-1}
\end{bmatrix}
\begin{bmatrix}
N_1^{i+1} \\
N_2^{i+1} \\
\vdots \\
N_{n-1}^{i+1} \\
N_n^{i+1}
\end{bmatrix}
= \begin{bmatrix}
N_1^i \\
N_2^i \\
\vdots \\
N_{n-1}^i \\
N_n^i
\end{bmatrix}
+ \begin{bmatrix}
Q_1^i \\
Q_2^i \\
\vdots \\
Q_{n-1}^i \\
Q_n^i
\end{bmatrix} \Delta t, \tag{4.137}
\]

which uses \( n \equiv n_{\text{max}} \) for clarity of presentation and can be solved numerically using "tridag" subroutine presented in Press et al. (1992) [210].

### 4.7 Testing

The PESCARA code (abbreviation for "Particle Spectra and Radiation" code) written using the numerical method described in previous section was initially tested against Kardashev’s [130] analytical solutions for synchrotron losses within static emitter with constant injection and null initial particle spectra given by equations (4.81) and (4.82). Particle injection was given by
\[
Q(\gamma, t) = \left( \frac{\gamma}{\gamma_{\text{min}}} \right)^{-p} \theta(\gamma - \gamma_{\text{min}}) \theta(\gamma_{\text{max}} - \gamma), \tag{4.138}
\]
with minimum injected particle energy being fixed at \( \gamma_{\text{min}} = 10^4 \), maximum injected particle energy being fixed at \( \gamma_{\text{max}} = 10^6 \) and magnetic field \( B = 1 \text{ G} \). Function \( \theta(x) \)
used in equation (4.138) is Heaviside step function given by

\[
\theta(x) = \begin{cases} 
1, & x \geq 0 \\
0, & x < 0 
\end{cases}.
\]

(4.139)

First thing we can notice from panels in figures 4.4, 4.5, 4.6 and 4.7 is that although numerical solutions settle very fast toward analytical solutions at higher particle energies near upper energy injection cut-off, at lower particle energies there is noticeable divergence between numerical and analytical solutions. While analytical solutions clearly show clear movement of the lower energy cut-off in energy space, numerical solutions produce an overestimation of the particle distribution at lower particle energies below the expected analytical cut-off. Above mentioned figures also show us first clear remedy to this problem through increasing the number of mesh grid points \( n_{\text{max}} \). This can be seen very clearly from figure 4.9. On the other hand changing minimum energy of the energy mesh grid \( \gamma_{\text{min, min}} \) for a fixed \( n_{\text{max}} \), as shown on figure 4.8 does not significantly merge numerical solution to analytical one. Another way to reach reasonable convergence between numerical and analytical solutions near lower energy cut-off is through control of time interval \( \Delta t \) as shown on figure 4.10, where we see that by decreasing \( \Delta t \) at fixed \( n_{\text{max}} \) we reach better correspondence with analytical solutions. Therefore optimal results can be reached with increasing of mesh grid points number \( n_{\text{max}} \) and by decreasing time step interval \( \Delta t \), but due to computational limitations another way to solve overestimation at low energy cut-off might be more reasonable through estimation and/or calculation of lower energy cut-off position with time as the solution of energy loss equation

\[
\frac{d\gamma_{\text{min,0}}}{dt} = -b(\gamma_{\text{min,0}}, t),
\]

(4.140)

where \( \gamma_{\text{min,0}}(t) \) is the energy of particle injected at energy \( \gamma_{\text{min}} \) at start of injection \( t_0 \). Solving equation (4.140) gives us the final position of lower energy cut-off which can be then externally imposed on the numerical solution. Of course this treatment holds only if the minimum energy of injected particles is constant with time. Also it is worth noting that realistic injection spectra do not really come with such sharp cut-offs and that introducing a steep power law tails below minimum and above maximum injection energy can also resolve this issue as proposed by Böttcher and Chiang (2002) [52].

Physically speaking \( \gamma \)-ray bursts are in pronounced fast cooling regimes as we will see in chapter 5 meaning that the lower energy cut-off of particle distribution is anyway deep below the energies of interest or even \( \gamma_{\text{min,0}} = 1 \). In that case all aforementioned treatment of the lower energy cut-off in particle distribution is
essentially not needed.

Lastly we have figure 4.11 where numerical solution was compared with analytical one as the function of maximum energy on mesh grid $\gamma_{m,\text{max}}$ clearly only for didactic purposes. As we can see reducing $\gamma_{m,\text{max}}$ can create artificial numerical upper cut-offs on injection function and by that significantly changing the solution. For time independent cut-offs in injection spectra when $\gamma_{\text{max}} \neq \gamma_{\text{max}}(t)$ that does not pose much problem since a solid choice of $\gamma_{m,\text{max}} = \gamma_{\text{max}}$ gives a good solution. In cases when maximum injection spectra energy is time dependent, i.e. $\gamma_{\text{max}} = \gamma_{\text{max}}(t)$ special care needs to be taken and $\gamma_{m,\text{max}}$ needs to expanded for several orders of magnitude to cover for possible time dependent changes of $\gamma_{\text{max}}$ providing us with a sensible numerical solution.

4.8 Gaussian injection within decaying magnetic field

Quite often due to limitations of the numerical methods, by which they essentially can not work directly with $\delta$-functions, $\delta$-functions are approximated with either a narrow Gaussian distributions or a narrow boxcar functions. In our code this is certainly the case when dealing with time pulsed injections, monoenergetic initial particle distributions and monoenergetic particle injections. One very interesting case to examine is the evolution of time independent Gaussian injection written as

$$Q(\gamma) = \frac{1}{\sqrt{2\pi} \sigma_\gamma} \exp \left[ -\frac{1}{2} \left( \frac{\gamma - \gamma_0}{\sigma_\gamma} \right)^2 \right], \quad (4.141)$$

where $\gamma_0$ is the central value and $\sigma_\gamma$ is the width of distribution, within time dependent magnetic field. In our case time dependent decaying magnetic field $B(t)$ is described by a form of power law avoiding divergent behaviour

$$B(t) = B_0 \left( 1 + \frac{t}{t_0} \right)^{-b}, \quad (4.142)$$

where $B_0$ is the magnetic field strenght at $t = 0$, $t_0$ is characteristic decay timescale of magnetic field and $b$ is the power law index of magnetic field decay. Time decay of magnetic field given with equation (4.142) was later also used to model $\gamma$-ray burst afterglow as it will be shown in chapter 5.

In cases of both constant (when $b = 0$) and time decaying magnetic fields (when $b > 0$) to define the time independent Gaussian injection we used the same set of parameters which are given in table 4.1. For simplicity we also assumed negligible particle escape by putting $\tau_{\text{esc}} \simeq 10^{20}$ s and absence of inverse Compton energy losses.
4.8 Gaussian injection within decaying magnetic field

Figure 4.4. Comparison of numerical solutions obtained by PESCARA code with analytical solutions for injection with power law index $p = 2.0$ at shorter timescales and its dependence on mesh grid point number $n_{\text{max}}$. Numerical parameters used are: $\gamma_{\text{m,min}} = 1$, $\gamma_{\text{m,max}} = 10^8$, $\Delta t = 1$ s except for the magenta line where $\Delta t = 10$ s. Evolution of distribution was tracked at: $t_1 = 5 \times 10^0$ s (black line), $t_2 = 5 \times 10^1$ s (red line), $t_3 = 5 \times 10^2$ s (green line), $t_4 = 5 \times 10^3$ s (cyan line) and $t_5 = 5 \times 10^4$ s (magenta line). Analytic lines are compared with numerical results given with dots of corresponding colours.
Figure 4.5. Comparison of numerical solutions obtained by PESCARA code with analytical solutions for injection with power law index $p = 1.5$ at larger timescales and its dependence on mesh grid point number $n_{\text{max}}$. Numerical parameters used are: $\gamma_{\text{m.min}} = 1$, $\gamma_{\text{m.max}} = 10^8$, $\Delta t = 10^3$ s. Evolution of distribution is tracked at: $t_1 = 1 \times 10^5$ s (black line), $t_2 = 5 \times 10^5$ s (red line), $t_3 = 1 \times 10^6$ s (green line) and $t_4 = 5 \times 10^6$ s (cyan line). Analytic lines are compared with numerical results given with dots of corresponding colours.
4.8 Gaussian injection within decaying magnetic field

Figure 4.6. Comparison of numerical solutions obtained by PESCARA code with analytical solutions for injection with power law index $p = 2.0$ at larger timescales and its dependence on mesh grid point number $n_{\text{max}}$. Numerical parameters used are: $\gamma_{\text{m,min}} = 1$, $\gamma_{\text{m,max}} = 10^8$, $\Delta t = 10^3$ s. Evolution of distribution is tracked at: $t_1 = 1 \times 10^5$ s (black line), $t_2 = 5 \times 10^5$ s (red line), $t_3 = 1 \times 10^6$ s (green line) and $t_4 = 5 \times 10^6$ s (cyan line). Analytic lines are compared with numerical results given with dots of corresponding colours.
Figure 4.7. Comparison of numerical solutions obtained by PESCARA code with analytical solutions for injection with power law index $p = 2.5$ at larger timescales and its dependence on mesh grid point number $n_{\text{max}}$. Numerical parameters used are: $\gamma_{\text{m.min}} = 1$, $\gamma_{\text{m.max}} = 10^8$, $\Delta t = 10^3$ s. Evolution of distribution is tracked at: $t_1 = 1 \times 10^5$ s (black line), $t_2 = 5 \times 10^5$ s (red line), $t_3 = 1 \times 10^6$ s (green line) and $t_4 = 5 \times 10^6$ s (cyan line). Analytic lines are compared with numerical results given with dots of corresponding colours.
4.8 Gaussian injection within decaying magnetic field

Injection spectra power law index is $p = 2.0$. Fixed numerical parameters used are: $n_{\text{max}} = 10^3$, $\gamma_{m,\text{max}} = 10^8$, $\Delta t = 10^3$ s. Minimum energy on numerical grid $\gamma_{m,\text{min}}$ is changed for investigation purposes. Evolution of distribution is tracked at: $t_1 = 1 \times 10^5$ s (black line), $t_2 = 5 \times 10^5$ s (red line), $t_3 = 1 \times 10^6$ s (green line) and $t_4 = 5 \times 10^6$ s (cyan line). Analytic lines are compared with numerical results given with correspondingly coloured dots.
4. Kinetic Equation - Analytical and Numerical Approaches

Figure 4.9. Correspondingly to figures 4.5, 4.6 and 4.7 in this figure we compare analytical solutions with numerical results obtained by PESCARA code for much higher values of mesh grid point number $n_{\text{max}}$. Situation investigated was for an injection spectra with power law index $p = 2.0$. Numerical parameters used at fixed values are: $\gamma_{\text{m.min}} = 1$, $\gamma_{\text{m.max}} = 10^8$, $\Delta t = 10^3$ s. Evolution of distribution is tracked at: $t_1 = 1 \times 10^5$ s (black line), $t_2 = 5 \times 10^5$ s (red line), $t_3 = 1 \times 10^6$ s (green line) and $t_4 = 5 \times 10^6$ s (cyan line). Analytic lines are compared with numerical results given with correspondingly coloured dots.
Figure 4.10. Influence of the size of time step $\Delta t$ on quality of numerical solutions. Parameters used for modelling are: $p = 2.0$ and $t = 10^6$ s. Numerical parameters used for calculations are: $\gamma_{m,\min} = 1$, $\gamma_{m,\max} = 10^8$, $n_{\max} = 1000$. Time step $\Delta t$ is of course changed for investigation purposes.

Figure 4.11. Comparison of analytical solutions with numerical ones where maximum mesh energy $\gamma_{m,\max}$ is changed for investigation purposes. Parameters used for modelling are: $p = 2.0$ and $t = 10^6$ s. Numerical parameters used are: $\gamma_{m,\min} = 1$, $\Delta t = 10^3$ s, $n_{\max} = 1000$. 
### Table 4.1. Parameters used in modelling the evolution of Gaussian injection within decaying magnetic field.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial magnetic field (G)</td>
<td>$B_0$</td>
<td>10</td>
</tr>
<tr>
<td>Magnetic field decay timescale (s)</td>
<td>$t_0$</td>
<td>$1.0 \times 10^3$</td>
</tr>
<tr>
<td>Central energy of particle injection</td>
<td>$\gamma_0$</td>
<td>$1.0 \times 10^6$</td>
</tr>
<tr>
<td>Particle injection energy width</td>
<td>$\sigma_\gamma$</td>
<td>$2.0 \times 10^5$</td>
</tr>
</tbody>
</table>

In the first simulation we deal with the case of the constant magnetic field presented in figure 4.12. Considering this in first simulation we took the value of constant magnetic field to be $B = B_0$ while adiabatic losses where ignored, while in all other cases with decaying magnetic fields as a part of reasonable assumption we also included adiabatic losses due to constant expansion, given by $\dot{\gamma}_{ad} = -\gamma/t$, alongside synchrotron energy losses. From first simulation we can see that synchrotron cooling of time independent Gaussian injection within constant magnetic field leads to a particle distribution following quite clearly a power law behaviour with $N(\gamma) \propto \gamma^{-2}$ where its minimum energy $\gamma_{\text{min}}(t)$ is getting lower with time as

$$\gamma_{\text{min}}(t) \sim \frac{\gamma_0}{\sigma_\gamma B_0^{3/2} \gamma_0 t + 1}. \quad (4.143)$$

On the other hand as we can see on figures 4.13, 4.14 and 4.15, time independent Gaussian particle injection in decaying magnetic fields will lead towards harder particle distributions with $N(\gamma) \propto \gamma^{-p}$ where $p < 2$. Essentially what we see here is that with steeper time decay of magnetic fields achieved by increasing parameter $b$ the particle distributions become harder. This comes due to fact that with decaying magnetic field cooling efficiency decreases with time leading to more stacking of particles around central energy of injection which can be specifically seen in case when $b \geq 1.5$ as shown on figure 4.15. Similar findings in the paper by Uhm and Zhang (2014) [286] were used to explain hard low energy synchrotron spectra, having photon index $\alpha \approx -1$, observed during $\gamma$-ray burst’s prompt phase.

### 4.9 Pulsar wind nebulae modelling

Following the testing of PESCARA and investigation of some interesting numerical solutions, we wanted to make a calculation using already known model with fixed parameters to see how our code fits to existing procedures. For that case we directed our attention toward modelling of pulsar wind nebulae (PWN) which presented itself as a perfect example. Pulsars, besides emitting electromagnetic radiation themselves, dissipate their rotational energy through relativistic wind of particles. Since the
Figure 4.12. Particle distributions at various times arising from synchrotron cooling of time independent Gaussian particle injection within a constant magnetic field \( B_0 = 10 \) G.

Figure 4.13. Particle distributions at various times arising from synchrotron and adiabatic cooling of time independent Gaussian particle injection within a time decaying magnetic field with \( b = 0.5 \).
Figure 4.14. Particle distributions at various times arising from synchrotron and adiabatic cooling of time independent Gaussian particle injection within a time decaying magnetic field with $b = 1.0$.

Figure 4.15. Particle distributions at various times arising from synchrotron and adiabatic cooling of time independent Gaussian particle injection within a time decaying magnetic field with $b = 1.5$. 
bulk velocity of such particle flows are supersonic they will form a termination shock when encountering ambient medium. Charged particles from pulsar are furthermore accelerated at the termination shock. This leads toward the formation of PWN since charged particles now emit electromagnetic radiation due to scattering on magnetic field and background photon field. PWN morphology is quite dependent on the proper motion of the pulsar relative to the interstellar medium - leading to either cases of toroidal shape of PWN around pulsars equator for slow moving pulsars or bullet like morphology for fast moving pulsars. PWNs are one of most prominent high energy photon sources and are known to emit electromagnetic radiatation in range from radio to TeV energies. One of the best studied and most famous PWN is the Crab nebula radiating at almost all wavelengths together with its central object, i.e. pulsar.

We followed two quite similar but in sense different approaches. First one formulated by Tanaka and Takahara (2010) \[278\] assumes no significant particle escape losses leading to $\tau_{\text{esc}} \to \infty$, while the one of Martin et al. (2012) \[168\] included particle escape losses due to Bohm diffusion as formulated by Zhang et al. (2008) \[328\]

$$\tau_{\text{esc}} \simeq 6.7 \times 10^5 \left( \frac{B}{10 \, \mu G} \right) \left( \frac{\gamma}{10^6} \right)^{-1} \left( \frac{R_{\text{PWN}}}{1 \, \text{pc}} \right)^2 \text{yr}, \quad (4.144)$$

where $R_{\text{PWN}}$ is the radius of PWN. PWN is freely expanding into surrounding medium leading to adiabatic losses of particles inside. While \[278\] assumes simple constant expansion equation for PWN radius since for young PWNs this is a valid approximation

$$R_{\text{PWN}}(t) = v_{\text{PWN}} t, \quad (4.145)$$

where $v_{\text{PWN}}$ is PWN expansion velocity, \[168\] on other hand takes the approach of freely expanding PWN presented in \[290\] where PWN radius $R_{\text{PWN}}$ is given by

$$R_{\text{PWN}}(t) = C \left( \frac{L_0 t}{E_0} \right)^{1/5} v_0 t, \quad (4.146)$$

where $L_0$ is the initial luminosity, $E_0$ is the energy of SN explosion and $v_0$ is the velocity of the front of the ejecta

$$v_0 = \sqrt{\frac{10}{3} \frac{E_0}{M_{\text{ej}}}}, \quad (4.147)$$

with $M_{\text{ej}}$ being the ejected mass. Constant $C$ in equation \[4.146\] is intricately connected with gas equation of state and determined through following expression

$$C = \left( \frac{6}{15(\gamma_{\text{PWN}} - 1)} + \frac{289}{240} \right)^{-1/5}, \quad (4.148)$$
where, since we are dealing with the relativistically hot gas of PWN, adiabatic constant $\gamma_{PWN}$ takes the value of $4/3$. Due to the differences in formulation of PWNs free expansion the associated form of adiabatic energy losses of charged particles (i.e. electrons) also become different. In the case of Tanaka and Takahara (2010) $[278]$ they can be written as

$$\dot{\gamma}_{ad}(\gamma, t) = -\frac{\gamma}{t},$$  \hspace{1cm} (4.149)

while the free expansion form adopted by Martin et al. (2012) $[168]$ leads to adiabatic losses given as

$$\dot{\gamma}_{ad}(\gamma, t) = -\frac{6}{5}\frac{\gamma}{t}.$$  \hspace{1cm} (4.150)

Both models include synchrotron energy losses which are more dominant in young PWNs and/or at higher electron energies and inverse Compton energy losses due to scattering of ambient CMB photons off relativistic electrons. Model by Martin et al. (2012) $[168]$ also includes bremsstrahlung energy losses which from our perspective can be easily neglected for practical purposes due to very low particle densities within PWN.

In both models energetics of PWN arise from the spin-down power $L(t)$ of the central pulsar presented by following expression

$$L(t) = L_0 \left(1 + \frac{t}{\tau_0}\right)^{-\frac{n+1}{n-1}},$$  \hspace{1cm} (4.151)

with $L_0$ being the initial spin-down luminosity, $\tau_0$ being the initial spin-down timescale of the pulsar and $n$ being the breaking index of the pulsar given through

$$n = \frac{\Omega \ddot{\Omega}}{\dot{\Omega}^2} \simeq \frac{P \ddot{P}}{\dot{P}^2},$$  \hspace{1cm} (4.152)

where $\Omega = 2\pi/P$ is the angular velocity of the pulsar with period $P$, $\dot{\Omega}$ and $\ddot{\Omega}$ are first order time derivatives, while $\dot{P}$ and $\dddot{P}$ are second order time derivatives of pulsar’s angular velocity and period respectively. Alongside $n$ all other parameters from equation (4.151) can be easily obtained from observations $[100, 155]$. Spin-down luminosity of pulsar can be obtained using observations of pulsar’s period $P$ and its first order time derivative $\dot{P}$ through following expression

$$L(t) = 4\pi^2I \frac{\dot{P}}{P^3},$$  \hspace{1cm} (4.153)

with $I$ being the pulsar’s moment of inertia commonly assumed to be of the order $\sim 10^{45}$ g cm$^2$. On the other hand initial spin-down timescale $\tau_0$ of the pulsar can be estimated by having an additional information of the age of pulsar $t_{age}$ by using the
following expression [100]

\[ \tau_0 = \frac{P_0}{n - 1} \dot{P}_0 = \frac{2\tau_c}{n - 1} - t_{\text{age}}, \quad (4.154) \]

where \( P_0 \) and \( \dot{P}_0 \) are respectively the initial period of pulsar and its first order time derivative, while \( \tau_c \) is the characteristic age of the pulsar given as

\[ \tau_c = P/2\dot{P}. \quad (4.155) \]

Although there are other approaches to follow the evolution of magnetic field within PWN present in the literature [211, 132, 293, 75], both [278] and [168] follow the prescription presented by Tanaka and Takahara (2010) [278] in which magnetic energy conservation is assumed. Major assumption of this approach is that energy injection coming from spin-down of pulsar is split between magnetic field energy \( E_B \) and energy of relativistic particles (i.e. electrons) \( E_e \) using a time independent parameter \( \eta \) \((0 \leq \eta \leq 1)\) as follows

\[ L(t) = \dot{E}_e + \dot{E}_B = (1 - \eta) \cdot L(t) + \eta \cdot L(t). \quad (4.156) \]

Before we continue it is important to say that this approach as many others is quite simplistic compared with the reality of magnetic field structure evolution around a pulsar since it follows magnetic field evolution within an uniform PWN. Realistic treatment of magnetic field’s evolution requires solving the equations of relativistic magnetohydrodynamics (RMHD) to account for effects of twisting and winding of magnetic field lines [78]. Continuing on track of the magnetic field energy conservation initiated by equation (4.156) we can assert that build up of magnetic field within PWN comes from energy injection as follows

\[ \frac{4\pi}{3} (R_{\text{PWN}}(t))^3 \frac{(B(t))^2}{8\pi} = \int_0^t \eta L(t')dt' = \eta E_{\text{spin}}(t), \quad (4.157) \]

with \( E_{\text{spin}}(t) \) being the total energy injected from pulsar spin-down into PWN at time \( t \). Magnetic field within PWN can now be calculated using equations (4.151), (4.156) and (4.157) as follows

\[ B(t) = \sqrt{\frac{3(n - 1)\eta L_0\tau_0}{(R_{\text{PWN}}(t))^3} \left[ 1 - \left(1 + \frac{t}{\tau_0}\right)^{-\frac{2}{n - 1}} \right]}. \quad (4.158) \]

As already mentioned in equation (4.156) remaining part of spin-down luminosity of pulsar goes into relativistic particles which we use to normalize the injection spectra
of relativistic electrons $Q(\gamma, t)$ as follows

$$(1 - \eta)L(t) = \int_{1}^{\infty} \gamma m_e c^2 Q(\gamma, t) d\gamma. \quad (4.159)$$

Injection spectra shape in both models was assumed to follow a broken power law as proposed by Venter and de Jager (2007) [293]

$$Q(\gamma, t) = \begin{cases} 
Q_0(t) \left(\frac{\gamma}{\gamma_b}\right)^{-p_1} & \text{for } \gamma_{\text{min}} \leq \gamma \leq \gamma_b, \\
Q_0(t) \left(\frac{\gamma}{\gamma_b}\right)^{-p_2} & \text{for } \gamma_b \leq \gamma \leq \gamma_{\text{max}},
\end{cases} \quad (4.160)$$

with time independent parameters $\gamma_{\text{min}}, \gamma_b, \gamma_{\text{max}}, p_1$ and $p_2$ respectively denoting minimum, break and maximum particle Lorentz factors together with low and high energy power law indices of injection spectra. On the other hand time dependent normalization parameter $Q_0(t)$ can be easily calculated using equation $4.159$. It is worth noting that in cases of most PWNs injection spectra power law indices take values in ranges $p_1 < 2$ and $p_2 > 2$ leading to following expression for the normalization parameter $Q_0(t)$

$$Q_0(t) = \frac{L_0(1-\eta)}{m_e c^2} \left(1 + \frac{t}{\tau_0}\right)^{-\frac{n+1}{n-1}} \left(\frac{\gamma_b^2 (p_1 - p_2)}{(2 - p_1)(2 - p_2)} + \frac{\gamma_b^{p_2} - p_2}{2 - p_2} + \frac{\gamma_b^{p_1} - p_1}{2 - p_1}\right)^{-1}. \quad (4.161)$$

Having presented all the important ingredients included in aforementioned models of PWN by using the parameters given in table 4.2 we proceeded modelling of most famous PWN, the Crab Nebula. The major motivator for this certainly lies in astonishing quality of its spectral data being persistently acquired and expanded since 1960s. Particle distributions at currently observed age of Crab Nebula were calculated using PESCARA code and as presented on figure 4.16 they show solid concordance with the ones presented in papers of both groups [168, 278].

After obtaining the distribution of electron at currently observed age of PWN we can also calculate the synchrotron radiation spectra. Crab Nebula can be essentially treated as optically thin object within whole range of its observed radiation spectra. In that case synchrotron spectral luminosity can be easily calculated using the following equation (e.g. see Rybicki and Lightman (1979) [253])

$$L_{\text{syn}}(\nu, t) = \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} N(\gamma, t) P_{\text{syn}}(\nu, \gamma, B(t)) d\gamma, \quad (4.162)$$

with $P_{\text{syn}}(\nu, \gamma, B(t))$ being the spectral power emitted by single electron already
Table 4.2. Comparison of parameters used by models of Tanaka and Takahara (2010) \cite{278} (here denoted as TTM) and Martin et al. (2012) \cite{168} (here denoted as MTR) to reproduce the currently observed spectrum of Crab Nebula.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>TTM</th>
<th>MTR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current period (ms)</td>
<td>$P$</td>
<td>33.1</td>
<td>33.4033474094</td>
</tr>
<tr>
<td>Current period derivative (s s(^{-1}))</td>
<td>$\dot{P}$</td>
<td>$4.21 \times 10^{-13}$</td>
<td>$4.209599 \times 10^{-13}$</td>
</tr>
<tr>
<td>Braking index</td>
<td>$n$</td>
<td>2.51</td>
<td>2.509</td>
</tr>
<tr>
<td>Pulsar age (yr)</td>
<td>$t_{age}$</td>
<td>950</td>
<td>940</td>
</tr>
<tr>
<td>Initial spin-down luminosity (erg s(^{-1}))</td>
<td>$L_0$</td>
<td>$3.4 \times 10^{39}$</td>
<td>$3.1 \times 10^{39}$</td>
</tr>
<tr>
<td>Initial spin-down timescale (yr)</td>
<td>$\tau_0$</td>
<td>700</td>
<td>740</td>
</tr>
<tr>
<td>Magnetic energy fraction</td>
<td>$\eta$</td>
<td>0.005</td>
<td>0.012</td>
</tr>
<tr>
<td>Minimum energy at injection</td>
<td>$\gamma_{\text{min}}$</td>
<td>$1.0 \times 10^2$</td>
<td>1.0</td>
</tr>
<tr>
<td>Break energy at injection</td>
<td>$\gamma_b$</td>
<td>$6.0 \times 10^5$</td>
<td>$7 \times 10^5$</td>
</tr>
<tr>
<td>Maximum energy at injection</td>
<td>$\gamma_{\text{max}}$</td>
<td>$7.0 \times 10^9$</td>
<td>$7.9 \times 10^9$</td>
</tr>
<tr>
<td>Low energy power law index</td>
<td>$p_1$</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>High energy power law index</td>
<td>$p_2$</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>Constant expansion velocity (km s(^{-1}))</td>
<td>$v_{\text{PWN}}$</td>
<td>1800</td>
<td>N/A</td>
</tr>
<tr>
<td>Ejected mass (M(_{\odot}))</td>
<td>$M_{\text{ej}}$</td>
<td>N/A</td>
<td>9.5</td>
</tr>
<tr>
<td>SN explosion energy (erg)</td>
<td>$E_0$</td>
<td>N/A</td>
<td>$10^{51}$</td>
</tr>
</tbody>
</table>

Figure 4.16. Comparison of particle (i.e. electron) distributions within Crab Nebula computed at currently observed age using parameters from models of Tanaka and Takahara (2010) \cite{278} (TTM) and Martin et al. (2010) \cite{168} (MTR) indicated in table 4.2.
discussed in detail within chapter 2 given by

\[ P_{\text{syn}}(\nu, \gamma, B(t)) = \sqrt{3} e^3 B(t) \frac{\nu}{m_e c^2} F \left( \frac{\nu}{\nu_c(\gamma, B(t))} \right), \]  

(4.163)

where \( \nu_c \) is the characteristic synchrotron frequency and \( F(x) \) is the synchrotron function. In equation (4.163) special care needs to be given to the limits of integration since at very low particle energies \( \gamma \) we are not talking about simple synchrotron approximation but rather a synchro-cyclotron regime as presented in Katarzynski et al. (2006) \[131\]. Also it is important to mention that the limits of integration in (4.163) should not be confused with minimum \( \gamma_{\text{min}} \) and maximum \( \gamma_{\text{max}} \) energy at injection, although they have been treated that way both in case of Tanaka and Takahara (2010) \[278\] and Martin et al. (2012) \[168\] as an relatively acceptable approximation.

Consequently particle distributions obtained through aforementioned PWN models shown on figure 4.16 were implemented into equation (4.162) leading to the calculated synchrotron radiation spectra of Crab Nebula as shown on figure 4.17. Other mechanisms (i.e. IC, SSC etc.) important for the modelling of high energy peak at energies around \( 10^6 \) MeV were not incorporated in this analysis but the reader is invited to consult the corresponding papers of each model where it has been shown they coincide pretty well to the spectral data in photon energy range above \( 10^3 \) MeV. On the other hand as it can be seen on figure 4.17 the fitting of calculated
synchrotron radiation spectra corresponds rather remarkably to the spectral data of Crab Nebula at photon energies below $10^3$ MeV.

This result gave us a significant impetus to apply our code to the case of some well-observed $\gamma$-ray bursts as it will be shown in following chapter [5].
Chapter 5

GRB Afterglow Modelling

5.1 Introduction

With robustness of the PESCARA code being confirmed through testing against the analytical solutions from Kardashev (1962) [130] and by direct application in case of Crab Nebula using the parameters from both Tanaka and Takahara (2010) [278] and Martin et al. (2012) [168], it is now possible to use it for modelling of γ-ray burst afterglows. GRB 130427A, outstanding for the excellent quality of observed multi-wavelength data clearly endorsing the application of BdHN approach, certainly presents itself as a perfect candidate for such an procedure. Therefore in Section 5.2, a detailed overview of BdHN afterglow theory is presented together with the observational confirmation of mildly-relativistic to non-relativistic expansion regimes present during the afterglow GRB 130427A. Consequently, following the observational constrictions (mildly- to non-relativistic expansion) and implications by BdHN model (presence of dipole-quadrupole magnetic field of the νNS), the non thermal afterglow spectra arising from synchrotron radiation were modelled and compared with the data of GRB 130427A. Results of this modelling were published in Ruffini et al. (2018) [231]. Using the same expansion velocity evolution as in case of GRB 130427A as a conservative assumption due to lack of data, afterglow spectra of GRB 160625B was also modelled (see Section 5.3). Finally, in Section 5.4, we indicate a clear similarity, arising from the expectations of BdHN scenario, between the afterglows of GRB 1301427A and GRB 190114C through scaling of the modelled afterglow light curve of GRB 130427A. Above-mentioned analysis of GRB 160625B and GRB 190114C within BdHN paradigm is presented in Rueda et al. (2019) [220].
5.2 GRB 130427A

GRB 130427A is well-known for its high isotropic energy $E_{iso} \simeq 10^{54}$ erg, SN association and high quality multi-wavelength observations. It triggered Fermi-GBM at 07:47:06.42 UT on April 27 2013, when it was within the field of view of Fermi-LAT. A long-lasting burst (~ $10^4$ s) of ultra-high energy (100 MeV–100 GeV) radiation was observed. Swift started to follow up from 07:47:57.51 UT, 51.1 s after the GBM trigger, observing a soft X-ray (0.3–10 keV) afterglow for more than 100 days. NuStar joined the observation during three epochs, approximately at ~ 1.2, 4.8 and 5.4 days after the Fermi-GBM trigger, providing rare hard X-ray (3–79 keV) afterglow observations. Ultraviolet, optical, infrared, radio observations were performed by more than 40 satellites and ground-based telescopes, among which Gemini-North, NOT, William Herschel, and VLT confirmed the redshift of $z \approx 0.34$.

Radio, optical and the GeV data were obtained from various published articles and GCNs. The soft and hard X-rays were analyzed by handling the original data obtained from Swift repository and NuStar archive. Standard data reduction procedure was employed using Heasoft 6.22 with relevant calibration files and generating the spectra by XSPEC 12.9. During the data reduction, the pile-up effect in the Swift-XRT was corrected for the first 5 time bins (see Fig. 5.5) before $10^5$ s. The NuStar spectrum at 388800 s was inferred from the closest first 10000 s of the NuStar third epoch at ~ 5.4 days, by assuming the spectra at these two times have the same cutoff power-law shape but different amplitudes. The amplitude at 388800 s was computed by fitting the NuStar light-curve. K-correction was implemented for transferring observational data to the cosmological rest frame.

The GRB afterglow emission within the BdHN model comes from a mildly relativistic expanding supernova ejecta. This has been confirmed by measuring the expansion velocity $\beta \sim 0.6 – 0.9$ (corresponding to the Lorentz gamma factor $\Gamma < 5$) within the early hundreds of seconds after the trigger from the observed thermal emission in the soft X-ray. For instance, finds a velocity of $\beta \sim 0.8$ for GRB 090618, and in GRB 081008 is found to have a velocity $\beta \sim 0.9$. The optical signal at tens of days also implies a mildly relativistic velocity $\beta \sim 0.1$.

The expansion velocity can be directly calculated from the observable X-ray spectrum.
The left term is a function of velocity $\beta$, the right term is from observables, $D_L(z)$ is the luminosity distance for redshift $z$. From the observed thermal flux $F_{bb,\text{obs}}$ and temperature $T_{\text{obs}}$ at time $t_1$ and $t_2$, the velocity $\beta$ can be inferred. This model independent equation valid in Newtonian and relativistic regimes is general. The results inferred do not agree with the ones of the fireball model [72, 199], coming from a ultra-relativistic shockwave.

Indeed, GRB 130427A is a well-known example of a GRB associated with SN [314]. For this GRB an X-ray thermal emission has been found between 196–461 s [236]. The spectral evolution of this source is shown on Figure 5.1. Using the best fit, it can be clearly shown that temperature in the observer’s frame that drops in time from 0.46 keV to 0.13 keV. The thermal flux also diminishes in time.

Applying equation (5.1), a radius in the laboratory frame that increases from $1.67^{+0.43}_{-0.29} \times 10^{13}$ cm to $1.12^{+0.49}_{-0.33} \times 10^{14}$ cm can be easily identified. The velocity inferred from the first and second spectra is $\beta = 0.85^{+0.06}_{-0.10}$, from the second and third spectra increases to $\beta = 0.96^{+0.02}_{-0.03}$. The average velocity of the entire duration of

Figure 5.1. Spectral fitting [236] of three time intervals (196s - 246s, 246s - 326s, 326s - 461s) in the Swift-XRT band (0.3 keV - 10 keV). Black points present the spectral data with H absorption, green dashed line is the fitted thermal component, blue long-dashed line is the power-law component, and red line is the sum of two components. Clearly the temperature and the thermal flux drop along the time.
thermal emission is $\beta = 0.94^{+0.03}_{-0.05}$, corresponding to a Lorentz factor $\Gamma = 2.98^{+1.20}_{-0.79}$, at an average radius $3.50^{+1.46}_{-0.97} \times 10^{13}$ cm. At later observer’s time around 16.7 days after the GRB trigger, the mildly relativistic velocity $\sim 32,000$ km s$^{-1}$ ($\beta \sim 0.1$) of the afterglow is measured from the line of Fe II 5169. Both the mildly relativistic velocities and the small radii are inferred directly from the observations and agree with the required properties of the BdHN model.

The above data are in contrast with the traditional fireball model [e.g. 206], which involves a shockwave with a high Lorentz factor $\Gamma \sim 500$ continuously expanding and generating the prompt emission at a radius of $\sim 10^{15}$ cm, and then the afterglow at a lab-frame radius of $> 10^{16}$ cm. Therefore, any model of the afterglow with ultra relativistic velocity following after the UPE does not conform to the stringent observational constraints.

One is left, therefore, with the task of developing a consistent afterglow model with a mildly relativistic expansion that is compatible with this clear observational evidence that the afterglow arises from mildly relativistic ejecta.

### 5.2.1 Energetics and properties of fast-rotating $\nu$NS

The angular momentum conservation implies that the $\nu$NS should be rapidly rotating. For instance, it is expected that the gravitational collapse of an iron core of radius $R_{Fe} \sim 5 \times 10^{8}$ cm of a carbon-oxygen star leading to a type Ic SN, rotating with a period $P \sim 5$ min, leads to an initial $\nu$NS rotation period $P_0 = (R_{NS}/R_{Fe})^2 P_{CO} \sim 1$ ms. Consequently the $\nu$NS has a large amount of rotational energy available to power the SN remnant. Understanding of the structure properties within fast rotating NSs, here based upon considerations presented in 66, is necessary in order to evaluate such a rotational energy.

The structure of NSs in uniform rotation is obtained by numerical integration of the Einstein equations in axial symmetry and the stability sequences are described by two parameters, e.g.: the baryonic mass (or the gravitational mass/central density) and the angular momentum (or the angular velocity/polar to equatorial radius ratio). The stability of the star is bounded by (at least) two limiting conditions 276. The first is the mass-shedding or Keplerian limit: for a given mass (or central density) there is a configuration whose angular velocity equals the one of a test particle in circular orbit at the stellar equator. Thus, the matter at the stellar surface is marginally bound to it and any small perturbation causes mass loss/mass shedding to bring the star back to stability or otherwise to bring it to a point of dynamical instability. The second is the secular axisymmetric instability: in this limit the star becomes unstable against axially symmetric perturbations and is expected to evolve first quasi-stationarily to then find a dynamical instability point where gravitational
5.2 GRB 130427A

Table 5.1. Critical mass (and corresponding radius) obtained in [66] for selected parameterizations of nuclear EOS. Last column reports the rotation period of the fastest possible configuration which corresponds to the one of the critical mass configuration (i.e. secularly unstable) that intersects the Keplerian mass-shedding sequence.

<table>
<thead>
<tr>
<th>EOS</th>
<th>$M_{J=0}^{\text{crit}} (M_\odot)$</th>
<th>$R_{J=0}^{\text{crit}}$ (km)</th>
<th>$M_{J=0}^{\text{max}} (M_\odot)$</th>
<th>$R_{J=0}^{\text{max}}$ (km)</th>
<th>$p$</th>
<th>$k$</th>
<th>$P_{\text{min}}$ (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NL3</td>
<td>2.81</td>
<td>13.49</td>
<td>3.38</td>
<td>17.35</td>
<td>1.68</td>
<td>0.006</td>
<td>0.75</td>
</tr>
<tr>
<td>GM1</td>
<td>2.39</td>
<td>12.56</td>
<td>2.84</td>
<td>16.12</td>
<td>1.69</td>
<td>0.011</td>
<td>0.67</td>
</tr>
<tr>
<td>TM1</td>
<td>2.20</td>
<td>12.07</td>
<td>2.62</td>
<td>15.98</td>
<td>1.61</td>
<td>0.017</td>
<td>0.71</td>
</tr>
</tbody>
</table>

Figure 5.2. Rotational energy and period of NSs along the Keplerian sequence for the NL3 EOS.

collapse takes place. This instability sequence thus leads to the NS critical mass and it can be obtained via the turning-point method by [93]. In [66] the values of the critical mass were obtained for the NL3, GM1 and TM1 EOS and the following fitting formula was found to describe them with a maximum error of 0.45%:

$$M_{\text{NS}}^{\text{crit}} = M_{J=0}^{\text{crit}} (1 + k J_{\text{NS}} p), \quad (5.2)$$

where $J_{\text{NS}} \equiv c J_{\text{NS}} / (G M_\odot^2)$ is a dimensionless angular momentum parameter, $J_{\text{NS}}$ is the NS angular momentum, $k$ and $p$ are parameters that depend on the nuclear EOS, and $M_{J=0}^{\text{crit}}$ is the critical mass in the non-rotating case (see Table 5.1).

The configurations lying along the Keplerian sequence are also the maximally rotating ones (given a mass or central density). The fastest rotating NS is the configuration at the crossing point between the Keplerian and the secular axisymmetric instability sequences. Fig. 5.2 shows the minimum rotation period and the rotational energy as a function of the NS gravitational mass for the NL3 EOS.

Another major point to address are properties of the magnetosphere. Within the traditional model of pulsars [111], in a rotating, highly magnetized NS, a corotating magnetosphere is enforced up to a maximum distance $R_{lc} = c/\Omega = c P/(2\pi)$, where $c$ is the speed of light and $\Omega$ is the angular velocity of the star. This defines the so-
called light cylinder since corotation at larger distances imply superluminal velocities of the magnetospheric particles. The last $B$-field line closing within the corotating magnetosphere is located at an angle $\theta_{pc} = \arcsin(\sqrt{R_{NS}/R_{lc}}) \approx \sqrt{R_{NS}/R_{lc}} = \sqrt{R_{NS}\Omega/c} = \sqrt{2\pi R_{NS}/(cP)}$ from the star’s pole. The $B$-field lines that originate in the region between $\theta = 0$ and $\theta = \theta_{pc}$ (referred to as magnetic polar caps) cross the light cylinder and are called “open” field lines. Charged particles leave the star moving along the open field lines and escaping from the magnetosphere passing through the light cylinder.

At large distances from the light cylinder the magnetic field lines becomes radial and thus the magnetic field geometry is dominated by the toroidal component which decreases with the inverse of the distance. For typical pulsar magnetospheres it is expected to be related with the poloidal component of the field at the surface, $B_s$, as

$$B_t \sim \left(\frac{2\pi R_{NS}}{cP}\right)^2 \left(\frac{R_{NS}}{r}\right) B_s, \quad (5.3)$$

up to a factor of order unity. Thus, as the SN remnant expands it finds a magnetized medium with a different value of the $B$-field. Magnetic field can be written as

$$B(t) = B_0 \left(\frac{R_0}{r}\right)^{-m}, \quad (5.4)$$

with $1 \leq m \leq 2$. Consequently the value of $m$ which fits best the data was examined as shown in Sections 5.2.2–5.2.4.

Following the previous agreement found between BdHN model and GRB data [22, 243], values for $R_0$ and the expansion velocity $\dot{R}$ are settled by model independent observations (see below Sections 5.2.2, 5.2.4), while the parameter $B_0$ is left to be set by the fit of the afterglow data. The results are finally compared and contrasted with the ones expected from the NS theory.

5.2.2 Model for the Optical and X-ray Spectrum of the Afterglow

The origin of the observed afterglow emission is interpreted as due to the synchrotron emission of electrons accelerated in an expanding magnetic HN ejecta. A fraction of the kinetic energy of the ejecta is converted, through a shockwave, to accelerated particles (electrons) above GeV and TeV energies — enough to emit photons up to the X-ray band by synchrotron emission. Depending on the shock speed, number density, magnetic field, etc., different initial energy spectra of particles can be formed. In the most common cases, the accelerated particle distribution function can be

---

4Synchrotron emission of electrons in fast cooling regime has been previously employed for GRBs but to explain the prompt emission [286].
described by a power law in the form of

\[ Q(\gamma, t) = Q_0(t) \gamma^{-p} \theta(\gamma_{\text{max}} - \gamma) \theta(\gamma - \gamma_{\text{min}}), \]

(5.5)

where \( \gamma = E/mc^2 \) is the electron Lorentz factor, \( \gamma_{\text{min}} \) and \( \gamma_{\text{max}} \) are the minimum and maximum Lorenz factors, respectively. \( Q_0(t) \) is the number of injected particles per second per energy, originating from the remnant impacted by the \( e^+e^- \) pair plasma of the GRB.

After the electrons are injected with the spectrum given by Eq. (5.5), the evolution of the particle distribution at a given time can be determined from the solution of the kinetic equation that describes the time-evolution of the electrons taking into account the particle energy losses \[ \frac{\partial N(\gamma, t)}{\partial t} = \frac{\partial}{\partial \gamma} (\dot{\gamma}(\gamma, t) N(\gamma, t)) - \frac{N(\gamma, t)}{\tau} + Q(\gamma, t), \]

(5.6)

where \( \tau \) is the characteristic escape time and \( \dot{\gamma}(\gamma, t) \) is the cooling rate. In the present case, for electrons, the escape time is much longer than the characteristic time of the cooling rate (fast cooling regime). The term \( \dot{\gamma}(\gamma, t) \) includes various electron energy loss processes, such as synchrotron and inverse-Compton cooling as well as adiabatic losses due to the expansion of the emitting region. For the magnetic field considered here, the dominant cooling process for higher energy electrons is synchrotron emission (the electron cooling timescale due to inverse-Compton scattering is significantly longer) while adiabatic cooling can dominate for the low energy electrons at later phases. By introducing the expansion velocity of the remnant \( \dot{R}(t) \) and its radius \( R(t) \), the energy loss rate of electrons can be written as

\[ \dot{\gamma}(\gamma, t) = \frac{\dot{R}(t)}{R(t)} \gamma + \frac{4}{3} \frac{\sigma_T}{m_e c} \frac{B(t)^2}{8\pi} \gamma^2, \]

(5.7)

where \( \sigma_T \) is the Thomson cross section and \( B(t) \) is the magnetic field strength. Early X-ray data indicate that the initial expansion velocity of GRB 130427A at times \( \sim 10^2 \) s is \( 0.8c \) \[236\], which then decelerates to \( 0.1c \) at \( 10^6 \) s, as inferred from the SN optical data \[314\]. Conservative assumption that the ejecta initially linearly decelerates until \( 10^6 \) s and later expands with a constant velocity of \( 0.1c \) was used within the modelling. In that case expansion velocity and radius of the ejecta are given by

\[ \dot{R}(t) = \begin{cases} v_0 - a_0 t & t \leq 10^6s \\ v_f & t > 10^6s \end{cases}, \]

(5.8)
and

\[ R(t) = \begin{cases} 
  v_0 t - a_0 t^2 / 2 & t \leq 10^6 s \\
  1.05 \times 10^{16} \text{ cm} + v_f t & t > 10^6 s 
\end{cases} \]  

(5.9)

where \( v_0 = 2.4 \times 10^{10} \text{ cm s}^{-1} \), \( a_0 = 2.1 \times 10^4 \text{ cm s}^{-2} \), and \( v_f = 3 \times 10^9 \text{ cm s}^{-1} \).

Due to the above decelerating expansion of the emitting region, the magnetic field decreases. As already shown in equation (5.4) magnetic field scales as \( B(t) = B_0(R(t)/R_0)^{-m} \) with \( 1 \leq m \leq 2 \). It is shown below (see Section 5.2.4) that the model fits best the data with \( m = 1 \). This corresponds to conservation of magnetic flux for the longitudinal component.

The initial injection rate of particles, \( Q_0(t) \), depends on the energy budget of ejecta and on the efficiency of converting from kinetic to non-thermal energy. This can be defined as

\[ L(t) = Q_0(t) m_e c^2 \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \gamma^{1-p} d\gamma, \]  

(5.10)

where it is assumed that \( L(t) \) varies in time, based on the recent analyses of BdHNe which show that the X-ray light curve of GRB 130724A decays in time following a power-law of index \( \sim -1.3 \) (see Fig. 5.3). Within BdHN interpretation, the emission in the optical and X-ray bands is produced from synchrotron emission of electrons: if one assumes the electrons are constantly injected \( (L(t) = L) \), this will produce constant synchrotron flux. Therefore it is necessary to assume that the luminosity of electrons changes from an initial value \( L_0 \) as follows:

\[ L(t) = L_0 \times \left(1 + \frac{t}{\tau_0}\right)^{-k}, \]  

(5.11)

where the index \( k \) will be set in order to obtain a synchrotron luminosity (see Eq. 5.12) that fits the afterglow light-curve (see details below in Secs. 5.2.3 and 5.2.4).

The kinetic equation given in Eq. (5.6) has been solved numerically. The discretized electron continuity equation (5.6) is re-written in the form of a tridiagonal matrix which is solved using the implementation of the “tridiag” routine in [210] within tested PESCARA code (see Chapter 4).

The synchrotron luminosity temporal evolution is calculated using \( N(\gamma, t) \) with

\[ L_{\text{syn}}(\nu, t) = \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} N(\gamma, t) P_{\text{syn}}(\nu, \gamma, B(t)) d\gamma, \]  

(5.12)

where \( P_{\text{syn}}(\nu, \gamma, B(t)) \) is the synchrotron spectra for a single electron which is calculated using the parameterization of the emissivity function of synchrotron radiation presented in [9].
5.2.3 Initial Conditions for GRB 130724A

In [243] an analysis was completed for seven subclasses of GRBs including 345 identified BdHNe candidates, one of which is GRB 130724A that was seen in the Swift-XRT data and analyzed in detail in [236]. From the host-galaxy identification it is known that this burst occurred at a redshift $z = 0.334$. After transforming to the cosmological rest-frame of the burst and properly correcting for effects of the cosmological redshift and Lorentz time dilation, one can infer a time duration $t_{90} = 162.8$ s for 90% of the GRB emission. The isotropic energy emission in the range of $1 \text{–} 10^4$ keV in the cosmological rest-frame of the burst is also deduced to be $E_{\text{iso}} = (9.3 \pm 1.3) \times 10^{53}$ erg and the total emission in the power-law afterglow can be inferred [236].

Fig. 5.3 shows the slope of the light-curve, defined by the logarithmic time derivative of the luminosity: slope = $d \log_{10}(L)/d \log_{10}(t)$. This slope is obtained by fitting the luminosity light-curve in the cosmological rest-frame, using a machine learning, locally weighted regression (LWR) algorithm. The corresponding technical details and codes to perform this calculation are publicly available at: https://github.com/YWangScience/AstroNeuron. The green line is the slope of the soft X-ray, in the $0.3$–$10$ keV range, and the blue line corresponds to the optical R-band, centered at 658 nm. The solid line covers the time when the data are well observed, while the dashed line, corresponds to an epoch in which observational data are missing. The rapid change of the slope implies variations of the energy injection, different emission mechanisms or different emission phases. The slope of the soft X-ray varies sharply at early times when various complicated GRB components (prompt emission, gamma-ray flare, X-ray flare) are occurring. Therefore this early part is not addressed with the synchrotron emission model defined above. Times later than $10^3$ s are only considered. At times later than $10^5$ s, the slopes of the X-ray and R bands reach a common value of $-1.33$, indicated by the red line.

Furthermore, GeV emission observed in most of BdHNe (when LAT data are available) is not addressed within the synchrotron radiation model proposed here. Such emission has been explained in [236] as emerging from the further accretion of matter onto the newly-formed BH. This explanation is further reinforced by the fact that a similar GeV emission, following the same power-law decay with time, is also observed in the authentic short GRBs (S-GRBs; short bursts with $E_{\text{iso}} \gtrsim 10^{52}$ erg; see [241]) which are expected to be produced in NS-NS mergers leading to BH formation [241].

Regarding the model parameters, the initial velocity of the expanding ejecta is expected to be $v_0 = 2.4 \times 10^{10}$ cm s$^{-1}$ [236] from the thermal black body emission. Similarly, the radius at the beginning of the X-ray afterglow should be
Figure 5.3. The slope of the afterglow light-curve of BdHN 130427A, defined by the logarithmic time derivative of the luminosity: slope = $d\log_{10}(L)/d\log_{10}(t)$. This slope is obtained by fitting the luminosity light-curve in the cosmological rest-frame, using a machine learning, locally weighted regression (LWR) algorithm. Corresponding technical details and codes can be found at: https://github.com/YWangScience/AstroNeuron. The green line is the slope of the soft X-ray emission, in the 0.3–10 keV range, and the blue line corresponds to the optical R-band, centered at 658 nm.

$R_0 \approx 2.4 \times 10^{12}$ cm. This corresponds to an expansion timescale of $t_0 = 100$ s. These values are consistent with previous theoretical simulations of BdHNe [22]. Within simulation of this burst all expected energy losses were included (dominantly synchrotron and adiabatic energy losses). However, the escape timescale was assumed to be large so that its effect could be neglected.

5.2.4 Results

Modelling results of the broadband spectral energy distribution (SED) for GRB 130724A during different periods are shown in Fig. 5.4. The corresponding parameters are given in Table 5.2. The radio emission is due to low-energy electrons that accumulate for longer periods. That is why the radio data are not fully included within the model. Only the optical and X-ray emissions due to synchrotron emission of electrons are concisely followed. Such emission, for instance at 604 s, is produced in a region with a radius of $1.4 \times 10^{14}$ cm and a magnetic field of $B = 8.3 \times 10^4$ G. For such magnetic field strength synchrotron self-absorption can be significant as estimated following [253]. At the initial phases, when the system is compact and the magnetic field is large, synchrotron-self absorption can be neglected for the photons with frequencies above $10^{14}$ Hz. Otherwise it is important. Thus, it is effective in reducing the radio flux predicted by the model, but not the optical and X-ray emission.
5.2 GRB 130427A

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic field at distance ( R_0 ) (G)</td>
<td>( B_0 )</td>
<td>( 5.0 \times 10^5 )</td>
</tr>
<tr>
<td>Initial injection power (erg s(^{-1}))</td>
<td>( L_0 )</td>
<td>( 2.0 \times 10^{51} )</td>
</tr>
<tr>
<td>Size of emitter at time ( \tau_0 ) (cm)</td>
<td>( R_0 )</td>
<td>( 2.4 \times 10^{12} )</td>
</tr>
<tr>
<td>Time decay power law index</td>
<td>( k )</td>
<td>1.58</td>
</tr>
<tr>
<td>Characteristic expansion timescale (s)</td>
<td>( \tau_0 )</td>
<td>( 1.0 \times 10^2 )</td>
</tr>
<tr>
<td>Injection spectra power law index</td>
<td>( p )</td>
<td>1.5</td>
</tr>
<tr>
<td>Minimum energy at injection</td>
<td>( \gamma_{\text{min}} )</td>
<td>( 4.0 \times 10^4 )</td>
</tr>
<tr>
<td>Maximum energy at injection</td>
<td>( \gamma_{\text{max}} )</td>
<td>( 5.0 \times 10^5 )</td>
</tr>
</tbody>
</table>

The optical and X-ray data can be well fit by a single power-law injection of electrons with \( Q \propto \gamma^{-1.5} \) and with initial minimum and maximum energies of \( \gamma_{\text{min}} = 4 \times 10^3 \) \( (E_{\text{min}} = 2.0 \text{ GeV}) \) and \( \gamma_{\text{max}} = 5 \times 10^5 \) \( (E_{\text{max}} = 255.5 \text{ GeV}) \), respectively. Due to the fast synchrotron cooling, the electrons are cooled rapidly forming a spectrum of \( N(\gamma, t) \sim \gamma^{-2} \) for \( \gamma \leq \gamma_{\text{min}} \) and \( N(\gamma, t) \sim \gamma^{-2.5} \) for \( \gamma \geq \gamma_{\text{min}} \). The slope of the synchrotron emission \( \nu F_\nu \propto \nu^{1-s} \) below the frequency defined by \( \gamma_{\text{min}} \) \( (\text{e.g., } h \nu_{\text{min}} \simeq 3 \, e \, B(t) \, \gamma_{\text{min}}^2 / (4 \pi m_e c)) \) is \( s = 2 - 1/2 = 0.5 \). This explains well both the optical and X-ray data.

For frequencies above \( \nu_{\text{min}} \), the slope is \( \nu F_\nu \propto \nu^{0.25} \) which continues up to \( h \, \nu_{\text{max}} \simeq 3 \, e \, B(t) \, \gamma_{\text{max}}^2 / (4 \pi m_e c) \). Since \( \nu_{\text{min}} \) and \( \nu_{\text{max}} \) depend on the magnetic field, they decrease with time, e.g. at \( t = 5.2 \times 10^6 \) s, \( \nu_{\text{min}} \simeq 6.5 \times 10^{14} \) Hz and \( \nu_{\text{max}} \simeq 1.0 \times 10^{19} \) Hz. Due to the changes in the initial particle injection rate and magnetic field, the synchrotron luminosity also decreases. This is evident from Fig. 5.5 where the observed optical and X-ray light-curves of GRB 130427A are compared with the theoretical synchrotron emission light-curve obtained from Eq. (5.12). In this figure the electron injection power \( L(t) \) given by Eq. (5.11) is also shown. Here, it can be seen how the synchrotron luminosity fits the observed decay of the afterglow luminosity with the correct power-law index \(-1.3\) (see also Fig. 5.3).

The SN ejecta is expected to become transparent to the \( \nu NS \) radiation at around \( 10^5 \) s. Therefore, it is necessary to address the pulsar emission that might power the late \( (t \gg 10^5 \) s) X-ray afterglow light-curve.

The late X-ray afterglow also shows a power-law decay of index \( \sim -1.3 \) which, as presented below, if powered by the pulsar implies the presence of a quadrupole magnetic field in addition to the traditional dipole one.

Thus, a dipole+quadrupole magnetic field model is adopted \[202\]. The luminosity
Figure 5.4. Model evolution of synchrotron spectral luminosity at various times compared with measurements in various spectral bands for GRB 130724A.

Figure 5.5. Optical, X-ray and GeV radiation light-curves for GRB 130427A compared with the theoretical synchrotron lightcurve from Eq. (5.12). Electron injection power $L(t)$ given by Eq. (5.11) is also shown. As it can be seen, X-ray data fit very well with the BdHN afterglow model but not the GeV data. This is in agreement with the BdHN paradigm since the GeV emission is expected to be emerging from the activity of newborn BH and not from the one of $\nu$NS.
from a pure dipole \((l = 1)\) is

\[
L_{\text{dip}} = \frac{2}{3c^3} \Omega^4 B_{\text{dip}}^2 R_{\text{NS}}^6 \sin^2 \chi_1,
\]  

(5.13)

where \(\chi_1 = 0\) degrees gives the axisymmetric mode \(m = 0\) alone whereas \(\chi_1 = 90\) degrees gives the \(m = 1\) mode alone. The braking index, following the traditional definition \(n \equiv \Omega \ddot{\Omega} / \dot{\Omega}^2\), is in this case \(n = 3\).

On the other hand, the luminosity from a pure quadrupole field \((l = 2)\) is

\[
L_{\text{quad}} = \frac{32}{135c^5} \Omega^6 B_{\text{quad}}^2 R_{\text{NS}}^8 \sin^2 \chi_1 (\cos^2 \chi_2 + 10 \sin^2 \chi_2),
\]  

(5.14)

where the different modes are easily separated by taking \(\chi_1 = 0\) and any value of \(\chi_2\) for \(m = 0\), \((\chi_1, \chi_2) = (90, 0)\) degrees for \(m = 1\) and \((\chi_1, \chi_2) = (90, 90)\) degrees for \(m = 2\). The braking index in this case is \(n = 5\).

Thus, the quadrupole to dipole luminosity ratio is:

\[
R_{\text{dip}}^{\text{quad}} = \frac{16}{45} \frac{R_{\text{NS}}^2 \Omega^2}{B_{\text{quad}}^2 / B_{\text{dip}}^2},
\]  

(5.15)

where

\[
\eta^2 = (\cos^2 \chi_2 + 10 \sin^2 \chi_2) \frac{B_{\text{quad}}^2}{B_{\text{dip}}^2}.
\]  

(5.16)

It can be seen that \(\eta = B_{\text{quad}} / B_{\text{dip}}\) for the \(m = 1\) mode, and \(\eta = 3.16 \times B_{\text{quad}} / B_{\text{dip}}\) for the \(m = 2\) mode. For a 1 ms period \(\nu_{\text{NS}}\), if \(B_{\text{quad}} = B_{\text{dip}}\), the quadrupole emission is about \(\sim 10\%\) of the dipole emission; if \(B_{\text{quad}} = 100 \times B_{\text{dip}}\), the quadrupole emission increases to 1000 times the dipole emission; and for a 100 ms pulsar, the quadrupole emission is negligible when \(B_{\text{quad}} = B_{\text{dip}}\), or only \(\sim 10\%\) of the dipole emission even when \(B_{\text{quad}} = 100 \times B_{\text{dip}}\). From this result one infers that the quadrupole emission dominates in the early fast rotation phase, then the \(\nu_{\text{NS}}\) spins down and the quadrupole emission drops faster than the dipole emission and, after tens of years, the dipole emission becomes the dominating component.

The evolution of the \(\nu_{\text{NS}}\) rotation and luminosity are given by

\[
\frac{dE}{dt} = -I \dot{\Omega} \ddot{\Omega} = -(L_{\text{dip}} + L_{\text{quad}})
\]  

\[
= -\frac{2}{3c^3} \Omega^4 B_{\text{dip}}^2 R_{\text{NS}}^6 \sin^2 \chi_1 \left(1 + \eta^2 \frac{16}{45} \frac{R_{\text{NS}}^2 \Omega^2}{c^2} \right),
\]  

(5.17)

where \(I\) is the moment of inertia. The solution is

\[
t = f(\Omega) - f(\Omega_0)
\]  

(5.18)
where
\[ f(\Omega) = \frac{3Ic\left(\frac{16}{45}\eta^2R_{NS}^2\Omega^2 \left(2\ln \Omega - \ln(c^2 + \frac{16}{45}\eta^2R_{NS}^2\Omega^2)\right) + c^2\right)}{4B_{dip}^2\sin^2 \chi R_{NS}^3\Omega^2} \] (5.19)

and
\[ f(\Omega_0) = \frac{3Ic\left(\frac{16}{45}\eta^2R_{NS}^2\Omega_0^2 \left(2\ln \Omega_0 - \ln(c^2 + \frac{16}{45}\eta^2R_{NS}^2\Omega_0^2)\right) + c^2\right)}{4B_{dip}^2\sin^2 \chi R_{NS}^3\Omega_0^2} \] (5.20)

The first and the second derivative of the angular velocity are
\[ \dot{\Omega} = -\frac{2B_{dip}^2\sin^2 \chi R_{NS}^6\Omega^3}{3Ic^3}(1 + \eta^2\frac{16}{45c^2}R_{NS}^2\Omega^2) \] (5.21)
\[ \ddot{\Omega} = -\frac{2B_{dip}^2\sin^2 \chi R_{NS}^6\Omega^2\dot{\Omega}}{Ic^3}(1 + \eta^2\frac{16}{27c^2}R_{NS}^2\Omega^2) \] (5.22)

Therefore the braking index is
\[ n = \frac{\Omega \ddot{\Omega}}{\dot{\Omega}^2} = \frac{135c^2 + 80\eta^2R_{NS}^2\Omega^2}{45c^2 + 16\eta^2R_{NS}^2\Omega^2} \] (5.23)

that in the present case ranges from 3 to 5. From Eqs. 5.18, 5.21, the evolution of total pulsar luminosity can be computed as
\[ L_{tot}(t) = I\dot{\Omega}. \] (5.24)

Therefore, the spin-down energy of the νNS powers the energy injection of the late-time afterglow. The spin period and the magnetic field of the νNS can be inferred by fittings the observed emission through the synchrotron model. This approach was initially applied on GRB 130427A and GRB 180728A [200], and more recently on GRB 190114C, GRB 160509A and GRB 160625B [220]. The fitting parameters for all aforementioned GRBs are summarized in Table 5.3 and their fitting results are shown on Figure 5.6. One can easily note that the νNSs in the BdHN I systems spin faster, with periods ≲ 2 ms, and carry more rotational energy \(E_{rot} \geq 10^{52}\) erg. This is attributed to the tidal locking between tightly bound CO core and NS before the emergence of SN and GRB [300, 220]. GRB 160625B especially stands out among analyzed GRBs by having the shortest initial spin period of \(P_0 = 0.5\) ms, putting it on a very limit of the rotational period of a NS at the Keplerian sequence. For a NS of mass \(1.4 M_\odot\) and radius 12 km, its Keplerian frequency \(f_K \simeq 1900\) [146, 215], corresponding to the spin period of \(P \simeq 0.5\) ms.

All GRB fittings show a strong presence of quadrupole magnetic field component with parameter \(\eta\) being around ∼100. If choosing the harmonic mode \(m = 2\), the
Table 5.3. Observational properties of the GRB and inferred physical quantities of the νNS of the corresponding BdHN model that fits the GRB data \([300, 220]\). Column 1: GRB name; column 2: identified BdHN type; column 3: the isotropic energy released \(E_{\text{iso}}\) in gamma-rays; column 4: cosmological redshift \(z\); column 5: νNS rotation period \(P_{\text{NS}}\); column 6: νNS rotational energy \(E_{\text{rot}}\); columns 7 and 8: strength of the dipole \(B_{\text{dip}}\) and quadrupole \(B_{\text{quad}}\) magnetic field components of the νNS. The quadruple magnetic field component is given in a range that the upper limit is three times than the lower limit, this is brought by the freedom of inclination angles of the magnetic moment. During the fitting, the NS mass of 1.4\(M_\odot\) and the NS radius of 10\(^6\) cm was consistently assumed in all cases. The fitted light-curves are shown in figure 5.6.

<table>
<thead>
<tr>
<th>GRB</th>
<th>Type</th>
<th>Redshift</th>
<th>(E_{\text{iso}}) (erg)</th>
<th>(P_{\text{NS}}) (ms)</th>
<th>(E_{\text{rot}}) (erg)</th>
<th>(B_{\text{dip}}) (G)</th>
<th>(B_{\text{quad}}) (G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>130427A</td>
<td>BdHN I</td>
<td>0.34</td>
<td>(1.40 \times 10^{54})</td>
<td>0.95</td>
<td>(3.50 \times 10^{52})</td>
<td>(6.0 \times 10^{12})</td>
<td>(2.0 \times 10^{14} \sim 6.0 \times 10^{14})</td>
</tr>
<tr>
<td>160509A</td>
<td>BdHN I</td>
<td>1.17</td>
<td>(1.06 \times 10^{54})</td>
<td>0.75</td>
<td>(5.61 \times 10^{52})</td>
<td>(4.0 \times 10^{12})</td>
<td>(1.3 \times 10^{14} \sim 4.0 \times 10^{14})</td>
</tr>
<tr>
<td>160625B</td>
<td>BdHN I</td>
<td>1.406</td>
<td>(3.00 \times 10^{54})</td>
<td>0.5</td>
<td>(1.26 \times 10^{53})</td>
<td>(1.5 \times 10^{12})</td>
<td>(5.0 \times 10^{13} \sim 1.6 \times 10^{14})</td>
</tr>
<tr>
<td>190114C</td>
<td>BdHN I</td>
<td>0.42</td>
<td>(2.47 \times 10^{53})</td>
<td>2.1</td>
<td>(7.16 \times 10^{51})</td>
<td>(5.0 \times 10^{12})</td>
<td>(1.5 \times 10^{15} \sim 5.0 \times 10^{15})</td>
</tr>
<tr>
<td>180728A</td>
<td>BdHN II</td>
<td>0.117</td>
<td>(2.73 \times 10^{44})</td>
<td>3.5</td>
<td>(2.58 \times 10^{43})</td>
<td>(1.0 \times 10^{13})</td>
<td>(3.5 \times 10^{48} \sim 1.1 \times 10^{49})</td>
</tr>
</tbody>
</table>

The quadrupole magnetic field is about 30 times stronger than the dipole magnetic field. Consequently the luminosity of the pulsar νNS before 10\(^6\) s is mainly powered by the quadrupole emission, which is approximately tens of times higher than the dipole emission. At about 10 years the dipole emission starts to surpass the quadrupole emission and afterwards continues to dominate.

One additional self-consistency check of this picture can be very useful. Namely, a cross-check of the estimated νNS parameters obtained first from the early afterglow via synchrotron emission, and then from the late X-ray afterglow via the pulsar luminosity, with respect to expectations from NS theory.

GRB 130427A is chosen for this check due to its high quality data and studious analysis presented in this thesis. Using Eqs. (5.4) and (5.3), via the values of \(B_0\) and \(R_0\) from Table 5.2 and for \(P_{\nu\text{NS}} = P_0 = 1\) ms, an estimate of the dipole field at the νNS surface from the synchrotron emission powering the early X-ray afterglow can be obtained, \(B_s \approx 2 \times 10^{13}\) G. This value is then compared with the one obtained from the request that the pulsar luminosity powers the late afterglow, \(B_{\text{dip}} = 6 \times 10^{12}\) G (see Table 5.3). An even better agreement can be obtained by using a more accurate value of the νNS radius than the fiducial value \(R_{\nu\text{NS}} = 10^6\) cm utilized in these estimates.

5.3 GRB 160625B

At redshift \(z = 1.406\) [313], GRB 160625B was a bright γ-ray burst whose polarisation has been detected. Fermi-LAT has detected more than 300 photons with energy above 100 MeV [163]. The γ-ray light curve has three distinct pulses [156, 327]. The first totally thermal short pulse lasted for \(\sim 2\) s; the second bright pulse started at
Figure 5.6. Afterglow powered by the νNS pulsar [300]: colored points representing bolometric lightcurves (around five times brighter than the soft X-ray observed by Swift XRT) of GRB 160625B (brown), GRB 160509A (deep blue), GRB 130427A (orange), GRB 190114C (green) and GRB 180728A (light blue) are compared with the fittings of the energy injection from the rotational energy of the pulsar, indicated by the solid lines of corresponding colors. Fitting parameters for each GRB are shown within the legend of the figure and are also listed in the Table 5.3. The quadruple field is given in a range, with its upper value three times larger than the lower one due to the oscillation angle $\chi_2$, which is a free parameter.
Table 5.4. Parameters used for simulation of GRB 160625B.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of emitter at time $\tau_0$ (cm)</td>
<td>$R_0$</td>
<td>$1.2 \times 10^{11}$</td>
</tr>
<tr>
<td>Magnetic field at distance $R_0$ (G)</td>
<td>$B_0$</td>
<td>$1.0 \times 10^{6}$</td>
</tr>
<tr>
<td>Initial injection power (erg s$^{-1}$)</td>
<td>$L_0$</td>
<td>$8.44 \times 10^{52}$</td>
</tr>
<tr>
<td>Time decay power law index</td>
<td>$k$</td>
<td>1.42</td>
</tr>
<tr>
<td>Characteristic expansion timescale (s)</td>
<td>$\tau_0$</td>
<td>5.0</td>
</tr>
<tr>
<td>Injection spectra power law index</td>
<td>$p$</td>
<td>1.5</td>
</tr>
<tr>
<td>Minimum energy at injection</td>
<td>$\gamma_{\text{min}}$</td>
<td>$4.0 \times 10^3$</td>
</tr>
<tr>
<td>Maximum energy at injection</td>
<td>$\gamma_{\text{max}}$</td>
<td>$1.0 \times 10^6$</td>
</tr>
</tbody>
</table>

$\sim 180$ s and ended at $\sim 240$ s, while the last pulse emerged at $\sim 330$ s and lasting for $\sim 300$ s. The total isotropic energy reached $E_{\text{iso}} \sim 3 \times 10^{54}$ erg \cite{12, 163}. Swift-XRT started the observation at late time ($\gtrsim 10^4$ s), identifying a time power-law behaviour with decaying index $\sim -1.25$. There was no supernova confirmation, most probably due to the larger redshift $z \gtrsim 1$ \cite{310}.

5.3.1 Modelling of GRB 160625B

Following the ample spectral similarity between GRB 130427A and GRB 160625B the same modelling procedure was applied to the later GRB. In the lack of direct measurements of expansion velocity the same expansion velocity behaviour as in case of GRB 130427A was conservatively assumed. Remainder of the parameters were obtained from the fitting procedure and are listed in Table 5.4. Significant change in this calculation is the value of time scale parameter $t_0$ which accounts for the explanation of the early optical band observations of GRB 160625B. Consequent time resolved spectral luminosity evolution can be seen on figure 5.7.

Modelled lightcurve was obtained by integrating within the band of Swift satellite from 0.3 keV to 10 keV and compared with the observed Swift data as shown on figure 5.9.

The correspondence of X-ray afterglow data of noted GRBs with modelled values leads us to believe BdHN model is more in line with observational expansion velocities and more capable to describe GRB afterglows than traditional model. Since velocities of expansion measured go from early mildly relativistic values at hundreds of seconds to nonrelativistic values of few weeks after the event. Also within this regimes it is not unheard of to have harder injection spectra, either due to non-linearity of diffusive shock acceleration mechanism, or as a consequence of magnetic reconnection operating within highly magnetized optically thin plasma (see Chapter 3). Surplus of modelled radio luminosity can be easily explained with synchrotron self absorption processes being more pronounced bellow $10^{12}$ Hz.
Another observation necessary to be explained was the very early 'spike'-like behaviour of optical V band light curve since G2 trigger with initially very steep decay. In that case fitting with fixed values of $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$ used for later afterglow phase does not produce enough flux for early stages. It is worth noting that although $\gamma_{\text{min}}$ is quite stable during whole acceleration process coming from suprathermal ensemble of particles being accelerated - $\gamma_{\text{max}}$ is certainly time dependent - be it either from transitive behaviour of accelerator coming to the steady state acceleration or due to changing conditions defined by synchrotron burnout $t_{\text{acc}} = t_{\text{syn}}$. Considering that a very simple model of rising $\gamma_{\text{max}} = \gamma_{\text{max,0}} + a_{\gamma} t$ was applied with very interesting correspondence to the values observed in V-band as shown on figure 5.8.

Like in the case of GRB 130427A, a self-consistency check can be also performed for GRB 160625B. Parameters $B_0$ and $R_0$ for GRB 160625B listed in Table 5.4 and the corresponding $P_{\nu\text{NS}} = P_0 \approx 0.5$ ms, give out surface magnetic field of $B_s \approx 6.8 \times 10^{11}$ G which fits closely within the order of magnitude compared with $B_{\text{dip}} \approx 10^{12}$ G shown on Table 5.3.
Figure 5.8. Early optical band observations of GRB 160625B since G2 trigger compared with modelled values using time evolving maximum energy of injected electrons $\gamma_{\text{max}}(t)$

5.4 GRB 190114C

GRB 190114C, at redshift $z \approx 0.42$ [201], is the first GRB with TeV photon detection by MAGIC [183]. Having twin features with GRB 130427A [299], it caught great attention as well. The prompt emission of GRB 190114C starts as a multi-peaked pulse, with initial pulse at $\sim 1.5$ s being non-thermal, followed by a possible thermal emission till $\sim 1.8$ s. The confident thermal emission exists during the peak of the pulse, from $2.7 - 5.5$ s. The GeV emission started from $2.7$ s, commencing with a spiky structure and followed by a time power-law decay with index $\sim -1.2$ [251]. The GeV emission was very luminous, since more than 200 s photons with energy above 100 MeV were detected. The X-ray afterglow observed by Swift-XRT showed a persistent time power-law decay behaviour, with decaying index $\sim 1.35$ [299]. An continuous observational campaign lasting $\sim 50$ days unveiled the SN emergence at $\sim 15$ days after the GRB [172], which is consistent with the prediction of $18.8 \pm 3.7$ days after the GRB given by [250].

5.4.1 Modelling of GRB 190114C

Similarities between GRB 130427A and GRB 190114C, the first ever to be observed by MAGIC observatory in ultra high energy band $\geq 250$ GeV, both in temporal behaviour and expected magnetic field configuration invite for the comparison of
Figure 5.9. Swift XRT lightcurves of GRB 160625B, GRB 130427A, GRB 190114C and GRB 160509A (black, red and green diamonds and blue stars with error bars respectively) compared with the model-based lightcurves integrated within Swift XRT energy range (0.3-10 keV) of GRB 160625B (black line) and GRB 130427A (red line) together with the scaled down (by a factor of 0.2) fitting of GRB 130427A compared to data of GRB 190114C (green line).

Assuming everything else as similar, from magnetic field strength and structure to expansion velocity time dependence, there is a striking similarity of GRB 190114C and GRB 130427A which can be normalized just through the simple scaling of the injection power from equation 5.11 by a factor of 0.2. This scaling is very close to the ratio between corresponding rotational energies of pulsars $\approx 0.2$ shown on Table 5.3.
Chapter 6

Conclusion

Modeling of non-thermal particle distributions and their radiation spectra coming from astrophysical sources, although in some cases performed using approximate analytical solutions, overall requires the use of an exact and robust numerical procedure. This arises from the general time variability of the parameters describing energy injection, particle losses, energy losses, particle acceleration and radiation mechanisms. For that reason PESCARA code, written initially in Mathematica and later in Python programming language, was tested against the analytical solutions given by Kardashev (1962) \[130\]. Numerical solutions obtained using PESCARA code showed clear correspondence with analytical ones, even though there were small deviations in proximity of sharp low energy cut-offs which can be easily resolved, either by a priori optimized choice of mesh grid density, time step size and cutoff softening, or a posteriori by implementing an external cutoff by following the evolution of lowest energy particles. Later, following the approximation of a homogeneous source, PESCARA code was used to model the non-thermal radiation arising from the Crab Nebula using the parameters provided by models of Tanaka and Takahara (2010) \[278\] and Martin et al. (2012) \[168\] with great success. As such, this code has shown great future potential in investigation of non-thermal astrophysical phenomena like AGNs, SNRs, PWNs, microquasars, GRBs etc. Certainly for each of mentioned phenomena a detailed understanding of underlying physical parameters, particle acceleration and radiation processes is prerequisite. With this, a essential tool was provided to address the question of the nature of $\gamma$-ray burst afterglows investigated in this thesis.

Major contribution of this thesis is the successful construction of a model for a broad frequency range of the observed spectrum in the afterglow of BdHNe. This was achieved through a specific fit to the GRB 130427A (BdHN type I) as a representative example. Crucial motivation for such a procedure arose from the early X-ray afterglow observations of GRB 130427A indicating a mildly- to non-relativistic
emitter in agreement with BdHN paradigm. Further model parameters used for the fit were also consistent with the BdHN interpretation for this class of GRBs.

Using GRB 130427A as a prototype, it was shown that the optical and X-ray emission of the early \(10^2 s \lesssim t \lesssim 10^6 s\) afterglow can be explained via the synchrotron emission by relativistic electrons within the magnetic field of the \(\nu NS\) threading the expanding HN ejecta. At later times the HN becomes transparent and the electromagnetic radiation from the \(\nu NS\) dominates the X-ray emission. It was determined that in case of GRB 130427A the \(\nu NS\) possesses an initial rotation period of 1 ms and a dipole magnetic field of \((5–7) \times 10^{12} G\). The magnetic field found by the synchrotron emission fit was consistent with the toroidal/longitudinal magnetic field component of the \(\nu NS\), as expected from the dominance of this component at distances much larger \((\sim 10^{12} \text{ cm})\) than the light cylinder radius in which this emission occurs. Independently derived strength of the magnetic dipole field both from the synchrotron emission model at early times \((t \lesssim 10^6 s)\) and from the magnetic braking model powering the late \((t \gtrsim 10^6 s)\) X-ray afterglow show that they are in full agreement. Following the mildly- to non-relativistic expansion timeline, prescribed by the observations and fit of GRB 130427A, additional fits were made for GRB 160625B and GRB 190114C. Modelled time resolved afterglow emission spectra of GRB 160625B was obtained with similar level of data correspondence as in case of GRB 130427A, indicating a smaller dipole magnetic field \((6–8) \times 10^{11} G\) and initial rotation period of 0.5 ms placing it at Keplerian limit. Additionally, fittings of early optical V band lightcurve after G2 trigger indicate a transient nature of particle acceleration mechanism at work. Finally, afterglow of GRB 190114C, compared to the scaled Swift-XRT lightcurve of GRB 130427A (see Figure 5.9), shows a clear indication of the BdHN scenario.

Considerations of afterglow and its power-law decaying luminosity as coming from pulsar magnetic-braking radiation show that the \(\nu NS\) magnetic field has a complex dipole+quadrupole structure in which the quadrupole component is initially dominant. The strength of the dipole component is about \(10^{12}–10^{13} G\) while the one of the quadrupole can be of order \(10^{15} G\) (see Figure 5.6 and Table 5.3). Likewise, self-consistency checks of \(\nu NS\) parameters such as rotation period and magnetic field strength obtained for the explanation of the observed afterglow via the synchrotron radiation and the pulsar magnetic-braking emission are in agreement (see Sec. 5.2.4). Theoretically speaking, the activity of the higher multipoles is expected to be relevant for the stability of the star and to play an important role in its early evolution.

Further correspondence of GRB afterglow observations with BdHN model was demonstrated by examination of the \(\nu NS\) rotation period. BdHN paradigm postulates tidal locking of the binary components and the conservation of angular momentum
during the gravitational collapse of the iron core leading to the $\nu$NS. It was found that values of orbital period/separation of the binary, inferred from $\nu$NS, are in agreement with the BdHN numerical simulations describing the X-ray precursor and the time separation with the prompt emission \cite{[301]}. Additional consequence of the above cross-checks is that the identified CO\textsubscript{core} are almost identical in all the analyzed cases, independently on the BdHN type I and II.

This newly established connection of both GRBs and young SNe could certainly be of fundamental relevance for the understanding of UHECR and neutrino production. Likewise, further extensions in understanding of the latest phases of the BdHN afterglow and their possible connection with historical supernovae are clearly necessary. Major confirmation of BdHN model could come from the observations of still active afterglows of historical GRBs. Pulsar-like emission from the aged $\nu$NS as expected by the BdHN model could probably offer an explanation for the activity of some pulsars and associated PWNs. Additional extension in understanding are also observations of GRB afterglows in the GeV and TeV energy bands (e.g. MAGIC, CTA). TeV energy band observations of GRBs are clearly additional to the current observations of GeV radiation, which originate from the activity of Kerr-Newman BH and are unrelated to the physics and astrophysics of afterglows described in this thesis.
Bibliography


[9] AHARONIAN, F. A., KELNER, S. R., AND PROSEKIN, A. Y. Angular, spectral, and time distributions of highest energy protons and associated secondary gamma rays and neutrinos propagating through extragalactic magnetic and


[77] de Ugarte Postigo, A., ET AL. GRB 130427A: spectroscopic detection of the SN from the 10.4m GTC. GRB Coordinates Network, 14646 (2013), 1.


[281] Timokhin, A. N. and Arons, J. Current flow and pair creation at low altitude in rotation-powered pulsars’ force-free magnetospheres: space charge


[287] Usov, V. V. Millisecond pulsars with extremely strong magnetic fields as a cosmological source of γ-ray bursts. *Nature*, **357** (1992), 472. \[doi:10.1038/357472a0\]


