SOURCES OF GRAVITATIONAL RADIATION.

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AXIALLY AND REFLECTION SYMMETRIC SOURCES (L.H,A. Di Prisco,J.Ibanez,J.Ospino, PRD89, 084034 (2014))

- Relationship between gravitational radiation and source properties. Gravitational radiation is an irreversible process, accordingly there must exist an entropy production factor in the equation of state (dissipation).
- We need to "break" the spherical symmetry in order to have gravitational radiation.
- Cylindrical symmetry is ruled out on physical grounds.
- The axial and reflection symmetries do not prevent the emission of gravitational radiation
- 1+3 formalism in a given coordinate system.
- Structure scalars.

BASIC EQUATIONS, CONVENTIONS AND NOTATION

- The line element:

$$ds^2 = -A^2 dt^2 + B^2 \left(dr^2 + r^2 d\theta^2\right) + C^2 d\phi^2 + 2G d\theta dt, \quad (1)$$

A, B, C, G are positive functions of t, r and θ.
$x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$.

- The energy momentum tensor:

$$T_{\alpha\beta} = (\mu + P) V_\alpha V_\beta + P g_{\alpha\beta} + \Pi_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha. \quad (2)$$

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$$\mu = T_{\alpha\beta} V^\alpha V^\beta, \quad q_\alpha = -\mu V_\alpha - T_{\alpha\beta} V^\beta, \quad (3)$$

$$P = \frac{1}{3} h^{\alpha\beta} T_{\alpha\beta}, \quad \Pi_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta \left(T_{\mu\nu} - P h_{\mu\nu}\right), \quad (4)$$

with $h_{\mu\nu} = g_{\mu\nu} + V_\nu V_\mu$.

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$$V^\alpha = (\frac{1}{A}, 0, 0, 0); \quad V_\alpha = (-A, 0, \frac{G}{A}, 0). \quad (5)$$

Next, let us introduce the unit, spacelike vectors **K**, **L**, **S**, with components

$$K_\alpha = (0, B, 0, 0); \quad L_\alpha = (0, 0, \frac{\sqrt{A^2B^2r^2 + G^2}}{A}, 0), \tag{6}$$

$$S_\alpha = (0, 0, 0, C), \tag{7}$$

satisfying the following relations:

$$V_\alpha V^\alpha = -K^\alpha K_\alpha = -L^\alpha L_\alpha = -S^\alpha S_\alpha = -1, \tag{8}$$

$$V_\alpha K^\alpha = V^\alpha L_\alpha = V^\alpha S_\alpha = K^\alpha L_\alpha = K^\alpha S_\alpha = S^\alpha L_\alpha = 0. \tag{9}$$

$$\begin{aligned} \Pi_{\alpha\beta} &= \frac{1}{3}(2\Pi_I + \Pi_{II})(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3}) + \frac{1}{3}(2\Pi_{II} + \Pi_I)(L_\alpha L_\beta - \frac{h_{\alpha\beta}}{3}) \\ &+ 2\Pi_{KL} K_{(\alpha} L_{\beta)}, \end{aligned} \tag{10}$$

with

$$\Pi_{KL} = K^\alpha L^\beta T_{\alpha\beta}, \quad , \tag{11}$$

$$\Pi_I = (2K^\alpha K^\beta - L^\alpha L^\beta - S^\alpha S^\beta) T_{\alpha\beta}, \tag{12}$$

$$\Pi_{II} = (2L^\alpha L^\beta - S^\alpha S^\beta - K^\alpha K^\beta) T_{\alpha\beta}. \tag{13}$$

$$q_\mu = q_I K_\mu + q_{II} L_\mu \tag{14}$$

$$q^\mu = (\frac{q_{II} G}{A\sqrt{A^2 B^2 r^2 + G^2}}, \frac{q_I}{B}, \frac{A q_{II}}{\sqrt{A^2 B^2 r^2 + G^2}}, 0), \tag{15}$$

$$q_\mu = \left(0, Bq_I, \frac{\sqrt{A^2 B^2 r^2 + G^2} q_{II}}{A}, 0\right). \tag{16}$$

Kinematical variables

$$
\begin{aligned}
a_\alpha &= V^\beta V_{\alpha;\beta} = a_I K_\alpha + a_{II} L_\alpha \\
&= \left(0, \frac{A_{,r}}{A}, \frac{G}{A^2}\left[-\frac{A_{,t}}{A} + \frac{G_{,t}}{G}\right] + \frac{A_{,\theta}}{A}, 0\right), \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\Theta &= V^\alpha_{;\alpha} \\
&= \frac{AB^2}{r^2A^2B^2 + G^2}\left[r^2\left(2\frac{B_{,t}}{B} + \frac{C_{,t}}{C}\right)\right. \\
&\left. + \frac{G^2}{A^2B^2}\left(\frac{B_{,t}}{B} - \frac{A_{,t}}{A} + \frac{G_{,t}}{G} + \frac{C_{,t}}{C}\right)\right], \qquad (18)
\end{aligned}
$$

$$
\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}. \qquad (19)
$$

$$
\begin{aligned}
\sigma_{11} &= -\frac{1}{3}\frac{1}{r^2A^2B^2+G^2}\frac{B^2}{A}\left[r^2A^2B^2\left(-\frac{B_{,t}}{B}+\frac{C_{,t}}{C}\right)\right. \\
&\left.+\,G^2\left(-2\frac{B_{,t}}{B}-\frac{A_{,t}}{A}+\frac{G_{,t}}{G}+\frac{C_{,t}}{C}\right)\right], \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{22} &= -\frac{1}{3}\frac{1}{A^3}\left[r^2A^2B^2\left(-\frac{B_{,t}}{B}+\frac{C_{,t}}{C}\right)\right. \\
&\left.+G^2\left(2\frac{A_{,t}}{A}+\frac{B_{,t}}{B}-2\frac{G_{,t}}{G}+\frac{C_{,t}}{C}\right)\right], \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{33} &= \frac{1}{3}\frac{1}{r^2A^2B^2+G^2}\frac{C^2}{A}\left[2r^2A^2B^2\left(-\frac{B_{,t}}{B}+\frac{C_{,t}}{C}\right)\right. \\
&\left.+G^2\left(2\frac{C_{,t}}{C}-\frac{B_{,t}}{B}-\frac{G_{,t}}{G}+\frac{A_{,t}}{A}\right)\right]. \qquad (22)
\end{aligned}
$$

$$\sigma_{\alpha\beta} = \frac{1}{3}(2\sigma_I + \sigma_{II})(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}) + \frac{1}{3}(2\sigma_{II} + \sigma_I)(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}). \tag{23}$$

$$2\sigma_I + \sigma_{II} = \frac{3}{A}\left(\frac{B_{,t}}{B} - \frac{C_{,t}}{C}\right), \tag{24}$$

$$2\sigma_{II} + \sigma_I = \frac{3}{A^2B^2r^2 + G^2}\left[AB^2r^2\left(\frac{B_{,t}}{B} - \frac{C_{,t}}{C}\right) + \frac{G^2}{A}\left(-\frac{A_{,t}}{A} + \frac{G_{,t}}{G} - \frac{C_{,t}}{C}\right)\right], \tag{25}$$

$$\omega_\alpha = \frac{1}{2}\,\eta_{\alpha\beta\mu\nu}\,V^{\beta;\mu}\,V^{\nu} = \frac{1}{2}\,\eta_{\alpha\beta\mu\nu}\,\Omega^{\beta\mu}\,V^{\nu}, \tag{26}$$

where $\Omega_{\alpha\beta} = V_{[\alpha;\beta]} + a_{[\alpha}V_{\beta]}$ and $\eta_{\alpha\beta\mu\nu}$ denote the vorticity tensor and the Levi-Civita tensor respectively;

$$\Omega_{\alpha\beta} = \Omega(L_\alpha K_\beta - L_\beta K_\alpha), \tag{27}$$

$$\omega_\alpha = -\Omega S_\alpha. \tag{28}$$

$$\Omega = \frac{G(\frac{G_{,r}}{G} - \frac{2A_{,r}}{A})}{2B\sqrt{A^2B^2r^2 + G^2}}. \tag{29}$$

Observe that from (29) and regularity conditions at the centre, it follows that: $G = 0 \Leftrightarrow \Omega = 0$.

Now, from the regularity conditions, necessary to ensure elementary flatness in the vicinity of the axis of symmetry, and in particular at the center we should require that as $r \approx 0$

$$\Omega = \sum_{n\geq 1} \Omega^{(n)}(t,\theta) r^n, \tag{30}$$

implying, because of (29) that in the neighborhood of the center

$$G = \sum_{n\geq 3} G^{(n)}(t,\theta) r^n. \tag{31}$$

Also, for the length of an orbit at t, θ constant, to be $2\pi r$, close to the origin (elementary flatness), we may write, as $r \rightarrow 0$,

$$C \approx r\gamma(t,\theta), \tag{32}$$

implying

$$C' \approx \gamma(t,\theta), \quad C_{,\theta} \approx r\gamma_{,\theta}, \tag{33}$$

where $\gamma(t,\theta)$ is an arbitrary function of its arguments, which as appears evident from the elementary flatness condition, cannot vanish anywhere within the fluid distribution.

The orthogonal splitting of Riemann Tensor and structure scalars

$$R^{\alpha\beta}{}_{\nu\delta} = R^{\alpha\beta}_{(F)\ \nu\delta} + R^{\alpha\beta}_{(Q)\ \nu\delta} + R^{\alpha\beta}_{(E)\ \nu\delta} + R^{\alpha\beta}_{(H)\ \nu\delta}, \tag{34}$$

with

$$R^{\alpha\beta}_{(F)\ \nu\delta} = \frac{16\pi}{3}(\mu + 3P) V^{[\alpha} V_{[\nu} h^{\beta]}_{\delta]} + \frac{16\pi}{3} \mu h^{\alpha}_{[\nu} h^{\beta}_{\delta]}, \tag{35}$$

$$R^{\alpha\beta}_{(Q)\ \nu\delta} = -16\pi V^{[\alpha} h^{\beta]}_{[\nu} q_{\delta]} - 16\pi V_{[\nu} h^{[\alpha}_{\delta]} q^{\beta]} - 16\pi V^{[\alpha} V_{[\nu} \Pi^{\beta]}_{\delta]} + 16\pi h^{[\alpha}_{[\nu} \Pi^{\beta]}_{\delta]} \tag{36}$$

$$R^{\alpha\beta}_{(E)\ \nu\delta} = 4 V^{[\alpha} V_{[\nu} E^{\beta]}_{\delta]} + 4 h^{[\alpha}_{[\nu} E^{\beta]}_{\delta]}, \tag{37}$$

$$R^{\alpha\beta}_{(H)\ \nu\delta} = -2\epsilon^{\alpha\beta\gamma} V_{[\nu} H_{\delta]\gamma} - 2\epsilon_{\nu\delta\gamma} V^{[\alpha} H^{\beta]\gamma}, \tag{38}$$

$$E_{\alpha\beta} = C_{\alpha\nu\beta\delta} V^{\nu} V^{\delta}, \qquad H_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\nu\epsilon\rho} C_{\beta\delta}{}^{\epsilon\rho} V^{\nu} V^{\delta}, \tag{39}$$

where $\epsilon_{\alpha\beta\rho} = \eta_{\nu\alpha\beta\rho} V^{\nu}$.

$$\begin{aligned} E_{\alpha\beta} &= \frac{1}{3}(2\mathcal{E}_I + \mathcal{E}_{II})(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}) + \frac{1}{3}(2\mathcal{E}_{II} + \mathcal{E}_I)(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}) \\ &+ \mathcal{E}_{KL}(K_\alpha L_\beta + K_\beta L_\alpha), \end{aligned} \tag{40}$$

$$H_{\alpha\beta} = H_1(S_\alpha K_\beta + S_\beta K_\alpha) + H_2(S_\alpha L_\beta + S_\beta L_\alpha). \tag{41}$$

$$Y_{\alpha\beta} = R_{\alpha\nu\beta\delta} V^{\nu} V^{\delta}, \tag{42}$$

$$X_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\nu}{}^{\epsilon\rho} R^{\star}_{\epsilon\rho\beta\delta} V^{\nu} V^{\delta}, \tag{43}$$

$$Z_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\epsilon\rho} R_{\delta\beta}{}^{\epsilon\rho} V^{\delta}, \tag{44}$$

where $R^{\star}_{\alpha\beta\nu\delta} = \frac{1}{2}\eta_{\epsilon\rho\nu\delta} R_{\alpha\beta}{}^{\epsilon\rho}$.

$$Y_{\alpha\beta} = \frac{1}{3} Y_T h_{\alpha\beta} + \frac{1}{3}(2Y_I + Y_{II})(K_\alpha K_\beta - \frac{1}{3} h_{\alpha\beta})$$
$$+\frac{1}{3}(2Y_{II} + Y_I)(L_\alpha L_\beta - \frac{1}{3} h_{\alpha\beta}) + Y_{KL}(K_\alpha L_\beta + K_\beta L_\alpha), \qquad (45)$$

with

$$Y_T = 4\pi(\mu + 3P), \qquad (46)$$
$$Y_I = \mathcal{E}_I - 4\pi\Pi_I, \qquad (47)$$
$$Y_{II} = \mathcal{E}_{II} - 4\pi\Pi_{II}, \qquad (48)$$
$$Y_{KL} = \mathcal{E}_{KL} - 4\pi\Pi_{KL}. \qquad (49)$$

$$X_{\alpha\beta} = \frac{1}{3} X_T h_{\alpha\beta} + \frac{1}{3}(2X_I + X_{II})(K_\alpha K_\beta - \frac{1}{3} h_{\alpha\beta})$$
$$+\frac{1}{3}(2X_{II} + X_I)(L_\alpha L_\beta - \frac{1}{3} h_{\alpha\beta}) + X_{KL}(K_\alpha L_\beta + K_\beta L_\alpha), \qquad (50)$$

$$X_T = 8\pi\mu, \tag{51}$$

$$X_I = -\mathcal{E}_I - 4\pi\Pi_I, \tag{52}$$

$$X_{II} = -\mathcal{E}_{II} - 4\pi\Pi_{II}, \tag{53}$$

$$X_{KL} = -\mathcal{E}_{KL} - 4\pi\Pi_{KL}. \tag{54}$$

$$Z_{\alpha\beta} = H_{\alpha\beta} + 4\pi q^{\rho}\epsilon_{\alpha\beta\rho}. \tag{55}$$

or

$$Z_{\alpha\beta} = Z_I K_\beta S_\alpha + Z_{II} K_\alpha S_\beta + Z_{III} L_\alpha S_\beta + Z_{IV} L_\beta S_\alpha \tag{56}$$

where

$$\begin{aligned} Z_I &= (H_1 - 4\pi q_{II}); \quad Z_{II} = (H_1 + 4\pi q_{II}); \\ Z_{III} &= (H_2 - 4\pi q_I); \quad Z_{IV} = (H_2 + 4\pi q_I). \end{aligned} \tag{57}$$

Variables: $\mu, P, \Pi_{I,II,KL}, q_{I,II}, a_{I,II}, \Theta, \sigma_{I,II}, \Omega, \mathcal{E}_{I,II,KL}, H_{I,II}$.

The super–Poynting vector

$$P_\alpha = \epsilon_{\alpha\beta\gamma}\left(Y^\gamma_\delta Z^{\beta\delta} - X^\gamma_\delta Z^{\delta\beta}\right); \quad P_\alpha = P_I K_\alpha + P_{II} L_\alpha \qquad (58)$$

$$\begin{aligned} P_I &= \frac{2H_2}{3}(2\mathcal{E}_{II} + \mathcal{E}_I) + 2H_1\mathcal{E}_{KL} + \frac{32\pi^2 q_I}{3}\left[3(\mu + P) + \Pi_I\right] \\ &+ 32\pi^2 q_{II}\Pi_{KL}, \\ P_{II} &= -\frac{2H_1}{3}(2\mathcal{E}_I + \mathcal{E}_{II}) + 2H_2\mathcal{E}_{KL} + \frac{32\pi^2 q_{II}}{3}\left[3(\mu + P) + \Pi_{II}\right] \\ &+ 32\pi^2 q_I\Pi_{KL}. \qquad (59) \end{aligned}$$

- A state of gravitational radiation is associated to a non–vanishing component of the super–Poynting vector. This is in agreement with the established link between the super–Poynting vector and the news functions, in the context of the Bondi–Sachs approach.

- Both components have terms not containing heat dissipative contributions. It is reasonable to associate these with gravitational radiation. Both components have contributions of both components of the heat flux vector.
- There is always a non-vanishing component of P^{μ}, on the plane orthogonal to a unit vector along which there is a non-vanishing component of vorticity (the $\theta - r$- plane). Inversely, P^{μ} vanishes along the ϕ-direction since there are no motions along this latter direction, because of the reflection symmetry.
- We can identify three different contributions in (59). On the one hand we have contributions from the heat transport process. These are independent of the magnetic part of the Weyl tensor, which explains why they remain in the spherically symmetric limit.

- On the other hand we have contributions from the magnetic part of the Weyl tensor. These are of two kinds:a) contributions associated with the propagation of gravitational radiation within the fluid, b) contributions of the flow of super–energy associated with the vorticity on the plane orthogonal to the direction of propagation of the radiation. Both are intertwined, and it appears to be impossible to disentangle them through two independent scalars.
- Both components of the four–vector **q**, appear in both components of **P**. Observe that this is achieved through the $X_{KL} + Y_{KL}$ terms in (59), or using (49, 54), through Π_{KL}. Thus, Π_{KL} couples the two components of the super–Poynting vector, with the two components of the heat flux vector.

The heat transport equation

$$\tau h^{\mu}_{\nu} q^{\nu}_{;\beta} V^{\beta} + q^{\mu} = -\kappa h^{\mu\nu}(T_{,\nu} + T a_{\nu}) - \frac{1}{2}\kappa T^2 \left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha} q^{\mu}, \quad (60)$$

where τ, κ, T denote the relaxation time, the thermal conductivity and the temperature, respectively. (92)

$$\begin{aligned} \frac{\tau}{A}(q_{II,t} + A q_I \Omega) + q_{II} &= -\frac{\kappa}{A}\left(\frac{G T_{,t} + A^2 T_{,\theta}}{\sqrt{A^2 B^2 r^2 + G^2}} + A T a_{II}\right) \\ &- \frac{\kappa T^2 q_{II}}{2}\left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha}, \end{aligned} \quad (61)$$

(93)

$$\frac{\tau}{A}(q_{I,t} - A q_{II}\Omega) + q_I = -\frac{\kappa}{B}(T_{,r} + B T a_I) - \frac{\kappa T^2 q_I}{2}\left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha}. \quad (62)$$

The vorticity acts as a coupling factor between the two components of the heat flux vector in the transport equation.

The basic equations

$$V_{\alpha;\beta} = \sigma_{\alpha\beta} + \Omega_{\alpha\beta} - a_\alpha V_\beta + \tfrac{1}{3} h_{\alpha\beta}\Theta \rightarrow V_{\alpha;\beta;\nu} - V_{\alpha;\nu;\beta} = R^{\mu}_{\alpha\beta\nu} V_\mu$$

$$\Theta_{;\alpha} V^\alpha + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \Omega^2) - a^\alpha_{;\alpha} + 4\pi(\mu + 3P) = 0 \qquad (63)$$

where $2\sigma^2 = \sigma_{\alpha\beta}\sigma^{\alpha\beta}$. (75)

$$h^\mu_\alpha h^\nu_\beta \sigma_{\mu\nu;\delta} V^\delta + \sigma^\mu_\alpha \sigma_{\beta\mu} + \frac{2}{3}\Theta\sigma_{\alpha\beta} - \frac{1}{3}\left(2\sigma^2 + \Omega^2 - a^\delta_{;\delta}\right) h_{\alpha\beta}$$
$$+ \quad \omega_\alpha\omega_\beta - a_\alpha a_\beta - h^\mu_{(\alpha} h^\nu_{\beta)} a_{\nu;\mu} + E_{\alpha\beta} - 4\pi\Pi_{\alpha\beta} = 0, \qquad (64)$$

(78)

$$h^\mu_\alpha h^\nu_\beta \Omega_{\mu\nu;\delta} V^\delta + \frac{2}{3}\Theta\Omega_{\alpha\beta} + 2\sigma_{\mu[\alpha}\Omega^{\mu}{}_{\beta]} - h^\mu_{[\alpha} h^\nu_{\beta]} a_{\mu;\nu} = 0. \qquad (65)$$

$$h^\beta_\alpha \left(\frac{2}{3}\Theta_{;\beta} - \sigma^\mu_{\beta;\mu} + \Omega_\beta{}^\mu{}_{;\mu}\right) + (\sigma_{\alpha\beta} + \Omega_{\alpha\beta})\, a^\beta = 8\pi q_\alpha, \qquad (66)$$

(79)

$$2\omega_{(\alpha} a_{\beta)} + h^\mu_{(\alpha} h_{\beta)\nu} \left(\sigma_{\mu\delta} + \Omega_{\mu\delta}\right)_{;\gamma} \eta^{\nu\kappa\gamma\delta} V_\kappa = H_{\alpha\beta}. \qquad (67)$$

(81)

The conservation law $T^{\alpha}_{\beta;\alpha}=0$ leads to the following equations:

$$\mu_{;\alpha}V^{\alpha}+(\mu+P)\Theta+\frac{1}{9}(2\sigma_{I}+\sigma_{II})\Pi_{I}+\frac{1}{9}(2\sigma_{II}+\sigma_{I})\Pi_{II}+q^{\alpha}_{;\alpha}+q^{\alpha}a_{\alpha}=0, \tag{68}$$

(92)

$$(\mu+P)a_{\alpha}+h^{\beta}_{\alpha}\left(P_{;\beta}+\Pi^{\mu}_{\beta;\mu}+q_{\beta;\mu}V^{\mu}\right)+\left(\frac{4}{3}\Theta h_{\alpha\beta}+\sigma_{\alpha\beta}+\Omega_{\alpha\beta}\right)q^{\beta}=0. \tag{69}$$

From the Bianchi identities and Einstein equations, the following set of equations are obtained:

$$
\begin{aligned}
& h^{\mu}_{(\alpha} h^{\nu}_{\beta)} E_{\mu\nu;\delta} V^{\delta} + \Theta E_{\alpha\beta} + h_{\alpha\beta} E_{\mu\nu}\sigma^{\mu\nu} - 3E_{\mu(\alpha}\sigma^{\mu}_{\beta)} \\
+ \quad & h^{\mu}_{(\alpha}\eta_{\beta)}{}^{\delta\gamma\kappa} V_{\delta} H_{\gamma\mu;\kappa} - E_{\delta(\alpha}\Omega_{\beta)}{}^{\delta} - 2H^{\mu}_{(\alpha}\eta_{\beta)\delta\kappa\mu} V^{\delta} a^{\kappa} = -4\pi(\mu + P)\sigma_{\alpha\beta} \\
- \quad & \frac{4\pi}{3}\Theta\Pi_{\alpha\beta} - 4\pi h^{\mu}_{(\alpha} h^{\nu}_{\beta)}\Pi_{\mu\nu;\delta} V^{\delta} - 4\pi\sigma_{\mu(\alpha}\Pi^{\mu}_{\beta)} - 4\pi\Omega^{\mu}{}_{(\alpha}\Pi_{\beta)\mu} \\
- \quad & 8\pi a_{(\alpha} q_{\beta)} + \frac{4\pi}{3}\left(\Pi_{\mu\nu}\sigma^{\mu\nu} + a_{\mu} q^{\mu} + q^{\mu}_{;\mu}\right) h_{\alpha\beta} - 4\pi h^{\mu}_{(\alpha} h^{\nu}_{\beta)} q_{\nu;\mu}, \qquad (70)
\end{aligned}
$$

(83)

$$
h^{\mu}_{\alpha} h^{\nu\beta} E_{\mu\nu;\beta} - \eta_{\alpha}{}^{\delta\nu\kappa} V_{\delta}\sigma^{\gamma}_{\nu} H_{\kappa\gamma} + 3H_{\alpha\beta}\omega^{\beta} =
$$

$$
\frac{8\pi}{3} h^{\beta}_{\alpha}\mu_{;\beta} - 4\pi h^{\beta}_{\alpha} h^{\mu\nu}\Pi_{\beta\nu;\mu} - 4\pi\left(\frac{2}{3}\Theta h^{\beta}_{\alpha} - \sigma^{\beta}_{\alpha} + 3\Omega_{\alpha}{}^{\beta}\right) q_{\beta}, \qquad (71)
$$

(87)

$$
\left(\sigma_{\alpha\delta} E^{\delta}_{\beta} + 3\Omega_{\alpha\delta} E^{\delta}_{\beta}\right)\epsilon_{\kappa}{}^{\alpha\beta} + a^{\nu} H_{\nu\kappa} - H^{\nu\delta}{}_{;\delta} h_{\nu\kappa} =
$$

$$
+4\pi(\mu + P)\Omega_{\alpha\beta}\epsilon_{\kappa}{}^{\alpha\beta} + 4\pi\left[q_{\alpha;\beta} + \Pi_{\nu\alpha}(\sigma^{\nu}{}_{\beta} + \Omega^{\nu}{}_{\beta})\right]\epsilon_{\kappa}{}^{\alpha\beta}, \qquad (72)
$$

(89)

$$
\begin{aligned}
& 2a_\beta E_{\alpha\kappa}\epsilon_\gamma{}^{\alpha\beta} - E_{\nu\beta;\delta}h^\nu_\kappa\epsilon_\gamma{}^{\delta\beta} + E^\delta_{\beta;\delta}\epsilon_{\gamma\kappa}{}^\beta + \frac{2}{3}\Theta H_{\kappa\gamma} + H^\mu_{\nu;\delta}V^\delta h^\nu_\kappa h_{\mu\gamma} \\
- & \left(\sigma_{\kappa\delta} + \Omega_{\kappa\delta}\right) H^\delta_\gamma + \left(\sigma_{\beta\delta} + \Omega_{\beta\delta}\right) H^\mu_\alpha \epsilon_\kappa{}^\delta{}_\mu \epsilon_\gamma{}^{\alpha\beta} + \frac{1}{3}\Theta H^\mu_\alpha \epsilon_\kappa{}^\delta{}_\mu \epsilon_\gamma{}^\alpha{}_\delta \\
= & \frac{4\pi}{3}\mu_{,\beta}\epsilon_{\gamma\kappa}{}^\beta + 4\pi\Pi_{\alpha\nu;\beta}h^\nu_\kappa\epsilon_\gamma{}^{\alpha\beta} \\
+ & 4\pi\left[q_\kappa\Omega_{\alpha\beta} + q_\alpha(\sigma_{\kappa\beta} + \Omega_{\kappa\beta} + \frac{1}{3}\Theta h_{\kappa\beta})\right]\epsilon_\gamma{}^{\alpha\beta}. \qquad (73)
\end{aligned}
$$

(90)

Scalar equations

Equation (63)

$$\Theta_{;\alpha}V^{\alpha}+\frac{1}{3}\Theta^2+2(\sigma^2-\Omega^2)-a^{\alpha}_{;\alpha}+Y_T=0. \tag{74}$$

Contracting (64) with **KK**, **KL** and **LL** we obtain respectively

$$\sigma_{I,\delta}V^{\delta}+\frac{1}{3}\sigma_I^2+\frac{2}{3}\Theta\sigma_I-(2\sigma^2+\Omega^2-a^{\delta}_{;\delta})-3(K^{\mu}K^{\nu}a_{\nu;\mu}+a_I^2)+Y_I=0, \tag{75}$$

$$\frac{1}{3}(\sigma_I-\sigma_{II})\Omega-a_Ia_{II}-K^{(\mu}L^{\nu)}a_{\nu;\mu}+Y_{KL}=0, \tag{76}$$

$$\sigma_{II,\delta}V^{\delta}+\frac{1}{3}\sigma_{II}^2+\frac{2}{3}\Theta\sigma_{II}-(2\sigma^2+\Omega^2-a^{\delta}_{;\delta})-3(L^{\mu}L^{\nu}a_{\nu;\mu}+a_{II}^2)+Y_{II}=0. \tag{77}$$

Contracting (65) with **KL**

$$\Omega_{,\delta}V^{\delta}+\frac{1}{3}(2\Theta+\sigma_I+\sigma_{II})\Omega+K^{[\mu}L^{\nu]}a_{\mu;\nu}=0. \tag{78}$$

Contracting (66) with **K** and **L** we obtain respectively

$$\frac{2}{3B}\Theta_{,r} - \Omega_{;\mu}L^{\mu} + \Omega(L_{\beta;\mu}K^{\mu}K^{\beta} - L^{\mu}_{;\mu}) + \frac{1}{3}\sigma_I a_I - \Omega a_{II} - \frac{1}{3}\sigma_{I;\mu}K^{\mu}$$
$$-\frac{1}{3}(2\sigma_I + \sigma_{II})(K^{\mu}_{;\mu} - \frac{a_I}{3}) - \frac{1}{3}(2\sigma_{II} + \sigma_I)(L_{\beta;\mu}L^{\mu}K^{\beta} - \frac{a_I}{3}) = 8\pi q_I, \quad (79)$$

$$\begin{aligned}
& \frac{1}{3\sqrt{A^2B^2r^2+G^2}}\left(\frac{2G}{A}\Theta_{,t} + 2A\Theta_{,\theta}\right) + \frac{a_{II}\sigma_{II}}{3} + \Omega_{;\mu}K^{\mu} - \frac{1}{3}\sigma_{II;\mu}L^{\mu} \\
+ \quad & \Omega(K^{\mu}_{;\mu} + L^{\mu}K^{\beta}L_{\beta;\mu}) + \Omega a_I + \frac{1}{3}(2\sigma_I + \sigma_{II})(L_{\beta;\mu}K^{\beta}K^{\mu} + \frac{a_{II}}{3}) \\
- \quad & \frac{1}{3}(2\sigma_{II} + \sigma_I)(L^{\mu}_{;\mu} - \frac{a_{II}}{3}) = 8\pi q_{II}. \qquad (80)
\end{aligned}$$

Contracting (67) with **KS** and **LS** we obtain respectively:

$$-\Omega a_I - \frac{1}{2}(K^\mu S_\nu + S^\mu K_\nu)(\sigma_{\mu\delta} + \Omega_{\mu\delta})_{;\gamma}\epsilon^{\nu\gamma\delta} = H_1, \qquad (81)$$

$$-\Omega a_{II} - \frac{1}{2}(L^\mu S_\nu + S^\mu L_\nu)(\sigma_{\mu\delta} + \Omega_{\mu\delta})_{;\gamma}\epsilon^{\nu\gamma\delta} = H_2. \qquad (82)$$

Contracting (70) with **KK**, **KL**, **LL** and **SS** we obtain:

$$\begin{aligned} & -\frac{1}{3}(X_I - 4\pi\mu)_{,\delta}V^\delta + \frac{1}{9}\mathcal{E}_I(3\Theta + \sigma_{II} - \sigma_I) + \frac{1}{9}(2\sigma_{II} + \sigma_I)\mathcal{E}_{II} \\ - & \; K_\nu\epsilon^{\nu\gamma\kappa}\left[H_{1,\kappa}S_\gamma + H_1 S_{\gamma;\kappa} + H_2(S_{\mu;\kappa}L_\gamma K^\mu + L_{\mu;\kappa}S_\gamma K^\mu)\right] + \Omega X_{KL} \\ & = 2a_{II}H_1 - \frac{4\pi}{3}(\mu + P + \frac{1}{3}\Pi_I)(\sigma_I + \Theta) - 8\pi a_I q_I - \frac{4\pi}{B}(q_I)_{,r} \\ - & \; \frac{4\pi q_{II}A}{\sqrt{A^2B^2r^2 + G^2}}\left(\frac{GB_{,t}}{A^2B} + \frac{B_{,\theta}}{B}\right), \qquad (83) \end{aligned}$$

$$
\begin{aligned}
- \quad & X_{KL,\delta}V^{\delta} + \frac{1}{6}\Omega\,(X_{II} - X_{I}) - \frac{1}{2}X_{KL}(2\Theta - \sigma_I - \sigma_{II}) + a_I H_1 - H_2 a_{II} \\
- \quad & \frac{1}{2}\Big[(H_{1,\kappa}S\gamma + H_1(S_{\gamma;\kappa} + S_{\mu;\kappa}K_{\gamma}K^{\mu}) + H_2 S_{\gamma}L_{\mu;\kappa}K^{\mu})\epsilon^{\beta\gamma\kappa}L_{\beta} \\
& -(H_1 K^{\mu}S_{\gamma} + H_2 S^{\mu}L_{\gamma})L_{\mu;\kappa}\epsilon^{\beta\gamma\kappa}K_{\beta}\Big] \\
- \quad & \frac{1}{2}(H_{2;\kappa}S_{\gamma} + H_2 S_{\gamma;\kappa})\epsilon^{\beta\gamma\kappa}K_{\beta} = \frac{8\pi}{3}\Pi_{KL}(\Theta - \sigma_I - \sigma_{II}) - 4\pi a_{II}q_I \\
- \quad & 2\pi(K^{\mu}L^{\nu} + K^{\nu}L^{\mu})q_{\nu;\mu} - 4\pi a_I q_{II}, \qquad (84)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{3}(-X_{II} + 4\pi\mu)_{;\delta}V^{\delta} + \frac{\mathcal{E}_{II}}{9}(3\Theta + \sigma_I - \sigma_{II}) + \frac{\mathcal{E}_I}{9}(2\sigma_I + \sigma_{II}) - \Omega X_{KL} \\
& -\left[H_{2;\kappa}S_{\gamma} - H_1(S_{\gamma}L_{\mu;\kappa}K^{\mu} + L_{\mu;\kappa}S^{\mu}K_{\gamma}) + H_2 S_{\gamma;\kappa}\right]\epsilon^{\beta\gamma\kappa}L_{\beta} + 2a_I H_2 \\
& = -\frac{4\pi}{3}(\mu + P + \frac{1}{3}\Pi_{II})(\sigma_{II} + \Theta) - 8\pi a_{II}q_{II} - 4\pi L^{\mu}L_{\nu}q^{\nu}_{;\mu}, \qquad (85)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{3}(X_I + X_{II} + 4\pi\mu)_{;\delta} V^{\delta} + \frac{1}{3}(X_I + X_{II})(\Theta + \sigma_I + \sigma_{II}) \\
+ \;& \frac{1}{9}(2\sigma_I + \sigma_{II})\mathcal{E}_I + \frac{1}{9}(2\sigma_{II} + \sigma_I)\mathcal{E}_{II} \\
- \;& (H_{1,\kappa} K_{\gamma} + H_{2,\kappa} L_{\gamma} + H_1 K_{\gamma;\kappa} + H_2 L_{\gamma;\kappa})\epsilon^{\beta\gamma\kappa} S_{\beta} + 2\,(H_1 a_{II} - H_2 a_I) \\
= \;& \frac{4\pi}{3}(\mu + P)(\sigma_I + \sigma_{II} - \Theta) - \frac{8\pi}{9}(\Theta + 2\sigma_I + 2\sigma_{II})(\Pi_I + \Pi_{II}) \\
- \;& 4\pi q_I \frac{C_{,r}}{BC} - \frac{4\pi q_{II} A}{\sqrt{A^2 B^2 r^2 + G^2}} \left(\frac{G C_{,t}}{A^2 C} + \frac{C_{,\theta}}{C} \right). \qquad (86)
\end{aligned}
$$

Contraction of (71) with **K** and **L** produces:

$$
\begin{aligned}
&-\frac{1}{3}X_{I,\beta}K^{\beta}-X_{KL,\beta}L^{\beta}-\frac{1}{3}(2X_I+X_{II})(K^{\beta}_{;\beta}-a_{\nu}K^{\nu})\\
-&\ \frac{1}{3}(X_I+2X_{II})L_{\mu;\beta}L^{\beta}K^{\mu}-X_{KL}(L_{\mu;\beta}K^{\mu}K^{\beta}+L^{\beta}_{;\beta}-a_{\beta}L^{\beta})\\
-&\ \frac{1}{3}H_2(\sigma_I+2\sigma_{II})-3\Omega H_1=\frac{8\pi}{3}\mu_{;\beta}K^{\beta}-\frac{4\pi}{3}q_I(2\Theta-\sigma_I)\\
+&\ 12\pi\Omega q_{II},
\end{aligned}
\tag{87}
$$

(95)

$$
\begin{aligned}
&-\frac{1}{3}X_{II,\beta}L^{\beta}-X_{KL,\beta}K^{\beta}-\frac{1}{3}(X_I+2X_{II})(L^{\beta}_{;\beta}-a_{\beta}L^{\beta})\\
-&\ \frac{1}{3}(2X_I+X_{II})K_{\mu;\beta}L^{\mu}K^{\beta}-X_{KL}(K_{\mu;\beta}L^{\mu}L^{\beta}+K^{\beta}_{;\beta}-a_{\beta}K^{\beta})\\
+&\ \frac{1}{3}H_1(2\sigma_I+\sigma_{II})-3\Omega H_2=\frac{8\pi}{3}\mu_{;\beta}L^{\beta}-12\pi\Omega q_I\\
-&\ \frac{4\pi q_{II}}{3}(2\Theta-\sigma_{II}).
\end{aligned}
\tag{88}
$$

Contraction of (72) with **S** yields:

$$
\begin{aligned}
& -\frac{1}{3}X_{KL}(\sigma_{II}-\sigma_I)+a_I H_1+a_{II}H_2-H_{1,\delta}K^\delta-H_{2,\delta}L^\delta \\
- & \quad H_1(K^\delta_{;\delta}+K^\nu_{;\delta}S^\delta S_\nu)-H_2(L^\delta_{;\delta}+S^\delta S_\nu L^\nu_{;\delta}) \\
= & \quad \left\{8\pi[\mu+P-\frac{1}{3}(\Pi_I+\Pi_{II})]-Y_I-Y_{II}\right\}\Omega-\frac{4\pi A(q_I B)_{,\theta}}{B\sqrt{A^2B^2r^2+G^2}} \\
+ & \quad \frac{4\pi A}{B\sqrt{A^2B^2r^2+G^2}}\left[\frac{q_{II}\sqrt{(A^2B^2r^2+G^2)}}{A}\right]_{,r},
\end{aligned}
\qquad (89)
$$

whereas by contracting (73) with **SK** and **SL** we obtain:

$$
\begin{aligned}
& -\frac{2}{3}a_{II}\mathcal{E}_I + 2a_I\mathcal{E}_{KL} - E^{\delta}_{2;\delta}L^2 - \frac{AY_{I,\theta}}{3\sqrt{A^2B^2r^2+G^2}} + \frac{Y_{KL,r}}{B} \\
- & \left[\frac{1}{3}(2Y_I + Y_{II})K_{\beta;\delta} + \frac{1}{3}(2Y_{II} + Y_I)K^{\nu}L_{\nu;\delta}L_{\beta}\right. \\
& \left. + Y_{KL}(L_{\nu;\delta}K^{\nu}K_{\beta} + L_{\beta;\delta})\right]\epsilon^{\gamma\delta\beta}S_{\gamma} \\
+ & H_{1,\delta}V^{\delta} + \frac{1}{3}H_1(3\Theta + \sigma_{II} - \sigma_I) + \Omega H_2 = -\frac{4\pi}{3}\mu_{,\theta}L^2 + 12\pi\Omega q_I \\
+ & \frac{4\pi q_{II}}{3}(\sigma_I + \Theta),
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
& \frac{2a_I}{3}\mathcal{E}_{II} - 2a_{II}\mathcal{E}_{KL} + E^{\delta}_{\beta;\delta}K^{\beta} + \frac{Y_{II,r}}{3B} - \frac{AY_{KL,\theta}}{\sqrt{A^2B^2r^2+G^2}} \\
- & \left[-\frac{1}{3}(2Y_I + Y_{II})L_{\nu;\delta}K^{\nu}K_{\beta} + \frac{1}{3}(2Y_{II}+Y_I)L_{\beta;\delta}\right. \\
& \left. +Y_{KL}(K_{\beta;\delta} - K^{\nu}L_{\beta}L_{\nu;\delta})\right]\epsilon^{\gamma\delta\beta}S_{\gamma} \\
+ & H_{2,\delta}V^{\delta} + \frac{1}{3}H_2(3\Theta + \sigma_I - \sigma_{II}) - \Omega H_1 = \frac{4\pi}{3}\mu_{,\beta}K^{\beta} \\
- & \frac{4\pi q_I}{3}(\sigma_{II} + \Theta) + 12\pi\Omega q_{II}. \qquad (91)
\end{aligned}
$$

The effective inertial mass density of the dissipative fluid

Combining the equations (69) and (60) we obtain

$$(\mu+P)\left[1-\frac{\kappa T}{\tau(\mu+P)}\right]a_\alpha = -h^\beta_\alpha \Pi^\mu_{\beta;\mu} - \nabla_\alpha P - (\sigma_{\alpha\beta}+\Omega_{\alpha\beta})q^\beta + \frac{\kappa}{\tau}\nabla_\alpha T + \left\{\frac{1}{\tau}+\frac{1}{2}D_t\left[ln(\frac{\tau}{\kappa T^2})\right]-\frac{5}{6}\Theta\right\}q_\alpha, \quad (92)$$

where $\nabla_\alpha P \equiv h^\beta_\alpha P_{,\beta}$ and $D_t f \equiv f_{,\beta}V^\beta$.

The factor multiplying the four acceleration vector represents the effective inertial mass density. Thus, the obtained expression for the e.i.m. density contains a contribution from dissipative variables, which reduces its value with respect to the non-dissipative situation.

From the equivalence principle it follows that the "passive" gravitational mass density should be reduced too, by the same factor. This in turn might lead, in some critical cases when such diminishing is significative, to a bouncing of the collapsing object.

Vorticity and heat transport

Let us assume that at some initial time (say $t = 0$) and before it, there is thermodynamic equilibrium in the θ direction, this implies $q_{II} = 0$. Then it follows at once from (61) that:

$$q_{II,t} = -A\Omega q_I, \tag{93}$$

implying that the propagation in time of the vanishing of the meridional flow, is subject to the vanishing of the vorticity and/or the vanishing of heat flow in the r- direction.
Inversely, repeating the same argument for (62)

$$q_{I,t} = A\Omega q_{II}. \tag{94}$$

In other words, time propagation of the thermal equilibrium condition, in either direction r, θ, is assured only in the absence of vorticity. Otherwise, it requires initial thermal equilibrium in both directions.
This result is a clear reminiscence of the von Zeipel's theorem.

The density inhomogeneity factors and their evolution

The density inhomogeneity factors (say Ψ_i),are such that their vanishing is sufficient and necessary condition for the homogeneity of energy density i.e.

$$\nabla_\alpha \mu \equiv h^\beta_\alpha \mu_{,\beta} = 0. \tag{95}$$

In the spherically symmetric case, in the absence of dissipation, the density inhomogeneity factor is the scalar associated to the trace–free part of $X_{\alpha\beta}$.
In the cylindrically symmetric case, the trace–free part of $X_{\alpha\beta}$, besides the magnetic part of the Weyl tensor and the dissipative flux determine the energy density inhomogeneity.
In the static axially symmetric case they are the structure scalars associated to the trace–free part of $X_{\alpha\beta}$.
In the present case it follows at once from (87) and (88) that the vanishing of $X_I, X_{II}, X_{KL}, Z_I, Z_{II}, Z_{III}, Z_{IV}$ implies the homogeneity of energy density. On the other hand, the evolution of the above mentioned scalars is determined by (61, 62, 83, 84, 85, 86, 90, 91).

The shear free case (L.H, A. Di Prisco, J.Ospino., PRD, 89, 127502 (2014)).

- For a general dissipative and anisotropic (shear free) fluid, vanishing vorticity, is a necessary and sufficient condition for the magnetic part of the Weyl tensor to vanish.
- Vorticity should necessarily appear if the system radiates gravitationally. This further reinforces the well established link between radiation and vorticity.
- In the geodesic case, the vorticity vanishes (and thereof the magnetic parts of the Weyl tensor). No gravitational radiation is produced. A similar result is obtained for the cylindrically symmetric case, suggesting a link between the shear of the source and the generation of gravitational radiation.
- In the geodesic (non-dissipative) case the models do not need to be FRW , however the system relaxes to the FRW spacetime (if $\Theta > 0$). Such tendency does not appear for dissipative fluids.

The perfect, geodesic fluid (L.H, A. Di Prisco, J.Ospino, J.Carot, PRD91,024010 (2015)).

- All possible models compatible with the line element (1) and a perfect fluid, are FRW, and accordingly non–radiating (gravitationally). Both, the geodesic and the non–dissipative, conditions, are quite restrictive, when looking for a source of gravitational waves.
- Not only in the case of dust, but also in the absence of dissipation in a perfect fluid, the system is not expected to radiate (gravitationally) due to the reversibility of the equation of state. Indeed, radiation is an irreversible process, this fact emerges at once if absorption is taken into account and/or Sommerfeld type conditions, which eliminate inward traveling waves, are imposed. Therefore, the irreversibility of the process of emission of gravitational waves, must be reflected in the equation of state through an entropy increasing (dissipative) factor.

- $$\Omega_{,\delta}V^{\delta}+\frac{1}{3}(2\Theta+\sigma_I+\sigma_{II})\Omega+K^{[\mu}L^{\nu]}a_{\mu;\nu}=0, \qquad (96)$$

  In the geodesic case, If at any given time, the vorticity vanishes, then it vanishes at any other time afterwards. Thus we should not expect gravitational radiation from a system, radiating for a finite period of time, for otherwise such a radiation will not be accompanied by the presence of vorticity.

- In the perfect (non–dissipative, non–geodesic) fluid, the condition of thermal equilibrium (absence of dissipative flux) reads

  $$a_{\mu}=-h^{\beta}_{\mu}\Gamma_{,\beta}, \qquad \Gamma=\ln T. \qquad (97)$$

  $$\Omega_{,\delta}V^{\delta}+\frac{1}{3}(2\Theta+\sigma_I+\sigma_{II}+V^{\mu}\Gamma_{,\mu})\Omega=0. \qquad (98)$$

  Thus, even if the fluid is not geodesic, but is non–dissipative, the situation is the same as in the geodesic case.

- This result is in full agreement with earlier works indicating that vorticity generation is sourced by entropy gradients. At the same time we confirm, by invoking the radiation–vorticity link, the Bondi's conjecture about the absence of radiation for non–dissipative systems.

- The role played by magnetic fields in the generation and survival of vorticity, has been been established in the past. This strongly suggest that the inclusion of magnetic fields in the discussion of gravitationally radiating sources, deserves further attention.
- Geodesic fluids not belonging to the class considered here (Szekeres) have also been shown not to produce gravitational radiation. This strengthens further the case of the non–radiative character of pure dust distributions.

The dissipative, geodesic fluid L.H, A. Di Prisco, J.Ospino, PRD,91, 124015 (2015)

- We consider separately the two possible subcases, namely: the fluid distribution is assumed, from the beginning, to be vorticity–free, or not.
- In the former case, it is shown that the vanishing vorticity implies the vanishing of the heat flux vector, and therefore, the resulting spacetime is FRW.
- In the latter case, it is shown that the enforcement of the regularity conditions at the center, implies the vanishing of the dissipative flux, leading also to a FRW spacetime.
- Thus all possible models, sourced by a dissipative geodesic dust fluid, belonging to the family of the line element considered here, do not radiate gravitational waves during their evolution, unless regularity conditions at the center of the distribution are relaxed.

The quasi–static approximation (L.H., A. Di Prisco, J. Ospino, J. Carot, IJMPD 2015

- Static regime: time like hypersurface Killing vector.
- Dynamic regime.
- Quasi–static regime: the time scale is much larger than the hydrostatic time scale ($\tau_{hydr.}$) and the thermal relaxation time (τ). The system evolves but is always in equilibrium, and the heat flux vector describes a steady heat flow.
- $\tau_{hydr.}$ of the Sun $\approx$ 27 min, white dwarf 4.5 sec., neutron star 10^{-4} sec.

- We have provided a general framework for describing the evolution of axially symmetric dissipative fluids in the QSA.
- The fluid distribution may split, under the effects of the vorticity, the shear and the dissipative flux, leading to a variety of very different structures.
- Finally it is worth mentioning that in the QSA the magnetic part of the Weyl tensor does not necessarily vanish (though it is of order $O(\epsilon)$), thereby implying that the "gravitational" part of the super-Poynting vector does not vanish either, meaning that gravitational radiation is not incompatible with the QSA.

However $\dot{H}_1 \approx \dot{H}_2 \approx O(\epsilon^2)$ (at least), and therefore are neglectable in the QSA. This in turn implies that if the magnetic part of the Weyl tensor vanishes at any given time, it will do so for any time afterwards. In other words, no state of radiation for a finite period of time is expected in the QSA. This result is in agreement with the one obtained by Bondi, for a more restricted case. However besides the "inductive" transfer of energy, mentioned by Bondi, we also have here, the transfer carried on by the dissipative flux.

The earliest stages of the non–equilibrium

- We evaluate the system immediately after departure from equilibrium, .
- "Immediately" means on the smallest time scale, at which we could observe the first signs of dynamical evolution. Such a time scale is assumed to be smaller than the thermal relaxation time, the hydrostatic time, and the thermal adjustment time.

The main issues we want to address here are:

1. what are the first signs of non–equilibrium?
2. what physical variables, do exhibit such signs?
3. what does control the onset of the dynamic regime, from an equlibrium initial configuration?

- The hydrostatic time is the typical time in which a fluid element reacts on a slight perturbation of hydrostatic equilibrium, it is basically of the order of magnitude of the time taken by a sound wave to propagate through the whole fluid distribution.
- The thermal relaxation time is the time taken by the system to return to the steady state in the heat flux (whether of thermodynamic equlibrium or not), after it has been removed from it.
- The thermal adjustment time is the time it takes a fluid element to adjust thermally to its surroundings. It is, essentially, of the order of magnitude of the time required for a significant change in the temperature gradients.

Conclusions

- A single function (f) controls the onset of non–equilibrium.
- Bondi's news function and f.
- Π_{KL} and $\dot{q}_{II}$ show the first signs of departure from equilibrium.
- No gravitational radiation is emitted at this stage of evolution.
- If the function f is different from zero until some time (always within the time scale under consideration), and vanishes afterwards, the system will turn back to equilibrium, without "remembering" to have been out of it previously.
- Both phenomena (radiation and vorticity) occur essentially at the same time scale.
- The decreasing of the effective inertial mass density appears already at this stage of evolution.

Open issues

- We have identified the subset of equations which should determine the density inhomogeneity factors and their evolution, but we were unable to isolate such factors in the general case. Is this possible?
- How could one describe the "cracking" (splitting) of the configurations ?
- We do not have an exact solution (written down in closed analytical form) describing gravitational radiation in vacuum, from bounded sources. Accordingly, any specific modeling of a source, and its matching to an exterior, should be done numerically.
- It should be useful to introduce the concept of the mass function, similar to the one existing in the spherically symmetric case. This could be relevant, in particular, in the matching of the source to a specific exterior.